Kodama, A.
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ON THE EQUIVALENCE PROBLEM FOR A CERTAIN CLASS OF GENERALIZED SIEGEL DOMAINS, II

AKIO KODAMA

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Introduction. Let $\mathfrak{D}$ be a generalized Siegel domain in C^N with exponent 1/2 and $\mathfrak{g}(\mathcal{D})$ the Lie algebra consisting of all complete holomorphic vector fields on $\mathfrak{D}$. In [3], Kaup, Matsushima and Ochiai studied the structure of $\mathfrak{g}(\mathfrak{D})$ and applied the results to the equivalence problem for Siegel domain of the second kind. They showed that every biholomorphic isomorphism of a Siegel domain of the second kind onto another one is birational. Moreover, using this fact they showed also that two Siegel domains of the second kind are holomorphically equivalent only if they are linearly equivalent. Motivated by these results, in [5] we studied the equivalence problem for a certain class of generalized Siegel domains.

The purpose of this note is to generalize our previous results in [5]. After some preparations in section 1, we show the following theorems in section 2.

Theorem 1. *Every biholomorphic isomorphism between two generalized Siegel domains in* $C\times C^m$ *with exponent* 1/2 *is birational.*

By means of this theorem and our result in [5], we obtain

Theorem 2. *Let* $\mathcal{D}$ *and* $\mathfrak{D}'$ *be generalized Siegel domains in* $C\times C^m$ *with exponent* 1/2. *Then* $\mathcal{D}$ *and* $\mathfrak{D}'$ *are holomorphically equivalent only if they are linearly equivalent, that is, there exists a non-singular linear mapping* $\mathcal{L}$: $C\times C^m\rightarrow C\times C^m$ *such that* $\mathcal{L}(\mathcal{D})=\mathcal{D}'$.

Throughout this note we use the same notations as in [4], unless otherwise stated.

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1. Preliminaries

According to Kaup, Matsushima and Ochiai [3], we say that a domain $\mathfrak{D}$ in $C^n\times C^m$ is a *generalized Siegel domain with exponent* 1/2 if it satisfies the following conditions:

(1) $\mathfrak{D}$ is holomorphically equivalent to a bounded domain in C^{n+m} and $\mathcal{D}$ contains a point of the form $(z, 0)$ where $z\in C^n$ and 0 denotes the origin of C^m.

(2) $\mathfrak{D}$ is invariant by the holomorphic transformations of C^{n+m} of the following types:

(a) $(z, w) \mapsto (z+a, w)$ for all $a\in \boldsymbol{R}^n$;

(b) $(z, w) \mapsto (z, e^{\sqrt{-1}t}w)$ for all $t\in \boldsymbol{R}$;

(c) $(z, w) \mapsto (e^t z, e^{(1/2)t}w)$ for all $t\in \boldsymbol{R}$.

Let Aut($\mathcal{D}$) be the group of all holomorphic transformations of $\mathfrak{D}$. Then it is known that Aut($\mathfrak{D}$) is a real Lie group and its Lie algebra is canonically identified with the Lie algebra $\mathfrak{g}(\mathfrak{D})$ consisting of all complete holomorphic vector fields on $\mathcal{D}$. We know that the following holomorphic vector fields on $\mathcal{D}$ are contained in $\mathfrak{g}(\mathcal{D})$:

(a) $\dfrac{\partial}{\partial z_k}$ for $k = 1, 2, \cdots, n$;

(b) $I = \sqrt{-1}\sum_{\alpha=1}^{m} w_\alpha \dfrac{\partial}{\partial w_\alpha}$;

(c) $E = \sum_{k=1}^{n} z_k \dfrac{\partial}{\partial z_k} + \dfrac{1}{2}\sum_{\alpha=1}^{m} w_\alpha \dfrac{\partial}{\partial w_\alpha}$,

where $(z_1, z_2, \cdots, z_n, w_1, \cdots, w_m)$ is the natural coordinate system in $C^n \times C^m$. Now, we have the following theorems on generalized Siegel domains with exponent 1/2.

Theorem A (Kaup, Matsushima and Ochiai [3]). *Let $\mathfrak{D}$ be a generalized Siegel domain in $C^n \times C^m$ with exponent* 1/2. *Then we have*

$$(1.1)\quad \mathfrak{g}(\mathcal{D}) = \mathfrak{g}_{-1}+\mathfrak{g}_{-1/2}+\mathfrak{g}_0+\mathfrak{g}_{1/2}+\mathfrak{g}_1$$
$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu]\subset \mathfrak{g}_{\lambda+\mu}, \ \textit{where}\ \ \mathfrak{g}_\lambda = \{X\in \mathfrak{g}(\mathcal{D}) \mid [E, X] = \lambda X\}\ .$$

$$(1.2)\quad dim_{\boldsymbol{R}}\, \mathfrak{g}_{-1/2} = 2k \ \textit{ for some }\ 0\leqq k\leqq m\ .$$

Theorem B (Kodama [4]). *Let $\mathcal{D}$ be a generalized Siegel domain in $C\times C^m$ with exponent* 1/2 *and $dim_{\boldsymbol{R}}\, \mathfrak{g}_{-1/2}=2k$, $0\leqq k\leqq m$. Let $Aut_0(\mathcal{D})$ denote the identity component of $Aut(\mathcal{D})$. Then there exists a non-singular linear mapping $\varphi\colon C\times C^m\to C\times C^m$ such that the image $\tilde{\mathcal{D}}=\varphi(\mathcal{D})$ is also a generalized Siegel domain with exponent* 1/2 *and, by choosing a suitable coordinate system $(z, w_1, \cdots, w_m)$ in $C\times C^m$,*

(1.3) *the orbit $\tilde{\mathcal{D}}_0$ of $Aut_0(\tilde{\mathcal{D}})$ containing the point $(\sqrt{-1}, 0, \cdots, 0)\in\tilde{\mathcal{D}}$ is the elementary Siegel domain*

$$\tilde{\mathcal{D}}_0 = \{(z, w_1, \cdots, w_k, 0, \cdots, 0)\in C\times C^m \mid Im.\ z-\sum_{\alpha=1}^{k}|w_\alpha|^2>0\}\ .$$

(1.4) *if we put*

$$\tilde{\mathscr{D}}_{\sqrt{-1}} = \{(w_{k+1}, \cdots, w_m) \in \boldsymbol{C}^{m-k} \mid (\sqrt{-1}, 0, \cdots, 0, w_{k+1}, \cdots, w_m) \in \tilde{\mathscr{D}}\} ,$$

then $\tilde{\mathscr{D}}_{\sqrt{-1}}$ *is a circular domain in* $\boldsymbol{C}^{m-k}$ *containing the origin* o *of* $\boldsymbol{C}^{m-k}$.

(1.5) *Let* $\mathfrak{g}(\tilde{\mathscr{D}}) = \sum \mathfrak{g}_\lambda$ *be the decomposition of* $\mathfrak{g}(\tilde{\mathscr{D}})$ *as in Theorem A. Then we have*

$$\mathfrak{g}_{-1/2} = \{2\sqrt{-1}F(w', C)\frac{\partial}{\partial z} + \sum_{\alpha=1}^{k} c^\alpha \frac{\partial}{\partial w_\alpha} \mid C = (c^\alpha) \in \boldsymbol{C}^k\}$$

where $w' = (w_1, \cdots, w_k)$ *and* $F\colon \boldsymbol{C}^k \times \boldsymbol{C}^k \to \boldsymbol{C}$ *is a hermitian form given by*

$$F(u, v) = \sum_{\alpha=1}^{k} u^\alpha \overline{v^\alpha} \qquad \textit{for} \ \ u = (u^\alpha),\ v = (v^\alpha) \in \boldsymbol{C}^k .$$

Let $(z_1, \cdots, z_N)$ be a coordinate system in $\boldsymbol{C}^N$ and D a domain in $\boldsymbol{C}^N$. For a holomorphic mapping $f = (f_1, \cdots, f_N)\colon D \to \boldsymbol{C}^N$, we denote by $J_f(p)$ the Jacobi matrix $(\partial f_i / \partial z_j)$ of f at a point $p \in D$.

Theorem C. *Let D be a domain in $\boldsymbol{C}^N$ which is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^N$ and f a holomorphic mapping of D into itself. Suppose that there exists a point $p \in D$ such that $f(p) = p$ and $J_f(p) = \mathbf{1}_N$. Then f is the identity transformation of D.*

Proof. This is immediate from Théorème VII, Chap. II in [1]. q.e.d.

Theorem D. *Let D and D' be two circular domains in $\boldsymbol{C}^N$ with centers o, the origin of $\boldsymbol{C}^N$. We suppose that at least one of these domains is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^N$. Let $f\colon D \to D'$ be a biholomorphic isomorphism such that $f(o) = o$. Then f is linear.*

Proof. By using Theorem C we can prove this theorem in the same way as in Théorème VI, Chap. II in [1]. q.e.d.

2. Proof of Theorems

To prove Theorem 1 we need few preparations. Let $\tilde{\mathscr{D}}$ and $(z, w_1, \cdots, w_m)$ be a generalized Siegel domain in $\boldsymbol{C} \times \boldsymbol{C}^m$ with exponent $1/2$, $\dim_{\boldsymbol{R}} \mathfrak{g}_{-1/2} = 2k$ and a coordinate system in $\boldsymbol{C} \times \boldsymbol{C}^m$ as in Theorem B. We consider a mapping $\tilde{\phi}\colon \{z \in \boldsymbol{C} \mid \mathrm{Im}. z > 0\} \times \boldsymbol{C}^m \to \boldsymbol{C}^{m+1}$ defined by

$$(2.1) \quad z^1 = (z - \sqrt{-1})(z + \sqrt{-1})^{-1},\ z^j = 2w_{j-1}(z + \sqrt{-1})^{-1}$$

for $j = 2, 3, \cdots, m+1$. Then, as is shown in the proof of Theorem 2 in [4], $\tilde{\phi}$ defines a biholomorphic isomorphism of $\tilde{\mathscr{D}}$ onto the image domain $\tilde{\mathscr{B}} = \tilde{\phi}(\tilde{\mathscr{D}})$ in $\boldsymbol{C}^{m+1}$. Under these notations we have the following

Lemma 1. *The domain $\tilde{\mathcal{B}}$ is a circular domain in $\boldsymbol{C}^{m+1}$ with center o which is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{m+1}$.*

Proof. Since $(\sqrt{-1}, 0)\in\tilde{\mathcal{D}}$ and $\tilde{\phi}(\sqrt{-1}, 0)=o$, it is clear that $o\in\tilde{\mathcal{B}}$. Put

$$U(k+1, 1)=\left\{g\in GL(k+2, \boldsymbol{C})\;\middle|\; {}^t\bar{g}\cdot\begin{pmatrix}\mathbf{1}_{k+1} & 0\\ 0 & -1\end{pmatrix}\cdot g=\begin{pmatrix}\mathbf{1}_{k+1} & 0\\ 0 & -1\end{pmatrix}\right\}$$

and

$$SU(k+1, 1)=U(k+1, 1)\cap SL(k+2, \boldsymbol{C}).$$

Then from Remark 3 of section 4, [4], we know that $\mathrm{Aut}_0(\tilde{\mathcal{B}})=\{\tilde{\Psi}_{\gamma,K}\mid\gamma\in SU(k+1, 1), K\in K^0_{\sqrt{-1}}\}$, where $K^0_{\sqrt{-1}}$ is the identity component of the isotropy subgroup of $\mathrm{Aut}(\tilde{\mathcal{D}}_{\sqrt{-1}})$ at the origin $O\in\tilde{\mathcal{D}}_{\sqrt{-1}}$, and moreover $\mathrm{Aut}_0(\tilde{\mathcal{B}})$ operates on $\tilde{\mathcal{B}}$ as follows. For $\gamma=\begin{pmatrix}A & \mathfrak{b}\\ \mathfrak{c} & d\end{pmatrix}\in SU(k+1, 1)$ and $K\in K^0_{\sqrt{-1}}\subset GL(m-k, \boldsymbol{C})$, $\tilde{\Psi}_{\gamma,K}$ acts on $\tilde{\mathcal{B}}$ by the holomorphic transformation

$$(2.2)\quad \tilde{\Psi}_{\gamma,K}\colon \begin{cases}\mathfrak{z}\mapsto(A\mathfrak{z}+\mathfrak{b})(\mathfrak{c}\mathfrak{z}+d)^{-1}\\ \mathfrak{z}'\mapsto K\cdot(\mathfrak{c}\mathfrak{z}+d)^{-1}\cdot\mathfrak{z}'\end{cases}$$

where $\mathfrak{z}={}^t(z^1, \cdots, z^{k+1})$ and $\mathfrak{z}'={}^t(z^{k+2}, \cdots, z^{m+1})$. If we set now, for any $\theta\in\boldsymbol{R}$

$$\gamma_\theta:=\begin{pmatrix}e^{\sqrt{-1}\theta} & 0 & \\ 0 & \ddots & 0\\ & e^{\sqrt{-1}\theta} & \\ 0\ \cdots & 0 & e^{-\sqrt{-1}(k+1)\theta}\end{pmatrix}\in SL(k+2, \boldsymbol{C})$$

and

$$k_\theta:=\begin{pmatrix}e^{\sqrt{-1}\theta} & & 0\\ & \ddots & \\ 0 & & e^{\sqrt{-1}\theta}\end{pmatrix}\in GL(m-k, \boldsymbol{C})$$

then $\gamma_\theta\in SU(k+1, 1)$ and $k_\theta\in K^0_{\sqrt{-1}}$, since $\tilde{\mathcal{D}}_{\sqrt{-1}}$ is a circular domain in $\boldsymbol{C}^{m-k}$ with center O by Theorem B. Thus, by (2.2) we see that

$$\tilde{\Psi}_{\gamma_\theta,k_\theta}\colon \begin{cases}\mathfrak{z}\mapsto e^{\sqrt{-1}(k+2)\theta}\cdot\mathfrak{z}\\ \mathfrak{z}'\mapsto e^{\sqrt{-1}(k+2)\theta}\cdot\mathfrak{z}'\end{cases},\quad \theta\in\boldsymbol{R}$$

is a one-parameter subgroup of $\mathrm{Aut}_0(\tilde{\mathcal{B}})$. This implies that $\tilde{\mathcal{B}}$ is a circular domain with center O. Since $\tilde{\mathcal{D}}$ is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{m+1}$, so is $\tilde{\mathcal{B}}$. q.e.d.

As in the case of bounded Reinhardt domains [6] [7], we can show the following lemma.

Lemma 2. *Let $\tilde{\mathcal{D}}$ be a generalized Siegel domain in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent*

$1/2$ *and put* $dim_{\boldsymbol{R}}\, \mathfrak{g}_{-1/2}=2k$ *as before. Then we have*

$$dim\,(Aut_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0))<dim\,(Aut_0(\tilde{\mathcal{D}})\cdot(z,w))$$

for any $(z,w)\in\tilde{\mathcal{D}}$ *not belonging to the orbit* $Aut_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0)$.

Proof. First we remark that if $k=m$, $\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0)=\tilde{\mathcal{D}}$ by Theorem B. So we assume in the following that $k<m$. By using the concrete expression of $\Psi_{\gamma,K}\in\mathrm{Aut}_0(\tilde{\mathcal{D}})$ as in section 2 of [4], we can show that for any $(z,w)\in\tilde{\mathcal{D}}$ there exists a point $(w^0_{k+1},\cdots,w^0_m)\in\tilde{\mathcal{D}}_{\sqrt{-1}}$ such that

$$\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(z,w)=\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0,\cdots,0,w^0_{k+1},\cdots,w^0_m)\,.$$

On the other hand, we know from Theorem B that a point $(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m)$ of $\tilde{\mathcal{D}}$ does not belong to the orbit $\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0)$ only if $(w_{k+1},\cdots,w_m)\neq(0,\cdots,0)$. Thus, to prove Lemma 2 it is enough to show that

$$\dim\,(\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0))<\dim\,(\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m))$$

for any $(w_{k+1},\cdots,w_m)\neq(0,\cdots,0)$. For this let G be the one-parameter subgroup $\{\psi_{1,k_\theta}\,|\,\theta\in\boldsymbol{R}\}$ of $\mathrm{Aut}_0(\tilde{\mathcal{D}})$ defined by the identity element $\mathbf{1}$ of $SU(k+1,1)$ and $k_\theta\in K^0_{\sqrt{-1}}$ as in the proof of lemma 1. For a given point $(z,w)\in\tilde{\mathcal{D}}$ we denote by $K_{(z,w)}$ the isotropy subgroup of $\mathrm{Aut}_0(\tilde{\mathcal{D}})$ at (z,w). Now, take a point $(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m)\in\tilde{\mathcal{D}}$ with $(w_{k+1},\cdots,w_m)\neq(0,\cdots,0)$. Then it is easy to check by using Theorem 2 in [4] that $K_{(\sqrt{-1},0)}\supset K_{(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m)}$ and the one-parameter subgroup G is contained in $K_{(\sqrt{-1},0)}$ but not in $K_{(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m)}$, since $(w_{k+1},\cdots,w_m)\neq(0,\cdots,0)$. This implies that $\dim K_{(\sqrt{-1},0)}>\dim K_{(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m)}$, and hence we have as a result that

$$\begin{aligned}
&\dim\,(\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0))=\dim\,(\mathrm{Aut}_0(\tilde{\mathcal{D}})/K_{(\sqrt{-1},0)})\\
&=\dim \mathrm{Aut}_0(\tilde{\mathcal{D}})-\dim K_{(\sqrt{-1},0)}\\
&<\dim \mathrm{Aut}_0(\tilde{\mathcal{D}})-\dim K_{(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m)}\\
&=\dim\,(\mathrm{Aut}_0(\tilde{\mathcal{D}})/K_{(\sqrt{-1},0,\cdots,w_{k+1},\cdots,w_m)})\\
&=\dim\,(\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0,\cdots,0,w_{k+1},\cdots,w_m))\,. \qquad \text{q.e.d.}
\end{aligned}$$

Proof of Theorem 1. Let $\mathcal{D}$ (resp. $\mathcal{D}'$) be a generalized Siegel domain in in $\boldsymbol{C}\times\boldsymbol{C}^m$ with exponent $1/2$ and $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k$ (resp. $\dim_{\boldsymbol{R}}\mathfrak{g}'_{-1/2}=2k'$). Let Φ: $\mathcal{D}\to\mathcal{D}'$ be a given biholomorphic isomorphism. From Theorem B there exists a non-singular linear mapping φ: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ (resp. φ': $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$) such that $\tilde{\mathcal{D}}=\varphi(\mathcal{D})$ (resp. $\tilde{\mathcal{D}}'=\varphi'(\mathcal{D}')$). Therefore, in order to prove Theorem 1 it is sufficient to show that the biholomorphic isomorphism $\tilde{\Phi}:=\varphi'\cdot\Phi\cdot\varphi^{-1}$ of $\tilde{\mathcal{D}}$ onto $\tilde{\mathcal{D}}'$ is birational. First we suppose that $k<m$. We claim now that $\tilde{\Phi}(\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0))=\mathrm{Aut}_0(\tilde{\mathcal{D}}')\cdot(\sqrt{-1},0)$, and so it follows in particular that

$k=k'$. Indeed, by Lemma 2 the orbit $\tilde{\Phi}(\mathrm{Aut}_0(\tilde{\mathcal{D}})\cdot(\sqrt{-1},0))$ is of lowest dimension, it must coinside with the orbit $\mathrm{Aut}_0(\tilde{\mathcal{D}}')\cdot(\sqrt{-1},0)$. Thus we can choose an element $g\in\mathrm{Aut}_0(\tilde{\mathcal{D}})$ in such a way that $(\tilde{\Phi}\cdot g)\cdot(\sqrt{-1},0)=(\sqrt{-1},0)$. Put $\bar{\Phi}=\tilde{\Phi}\cdot g$. Once it is shown that $\bar{\Phi}$: $\tilde{\mathcal{D}}\to\tilde{\mathcal{D}}'$ is birational, our proof can be completed, since g: $\tilde{\mathcal{D}}\to\tilde{\mathcal{D}}$ is birationaly by Theorem 2 in [4]. To show this we consider again the biholomorphic isomorphism $\tilde{\phi}$: $\tilde{\mathcal{D}}\to\tilde{\mathcal{B}}$ defined in (2.1). Let $\tilde{\phi}'$: $\tilde{\mathcal{D}}'\to\tilde{\mathcal{B}}'$ be the corresponding isomorphism of $\tilde{\mathcal{D}}'$ onto the image domain $\tilde{\mathcal{B}}'$. Then, by Lemma 1 $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}'$ are both circular domains in $\boldsymbol{C}^{m+1}$ with the origin O of $\boldsymbol{C}^{m+1}$ as their centers. Moreover, putting $\bar{\bar{\Phi}}$: $=\tilde{\phi}'\cdot\bar{\Phi}\cdot\tilde{\phi}^{-1}$, we get a biholomorphic isomorphism $\bar{\bar{\Phi}}$: $\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}'$ satisfying the condition that $\bar{\bar{\Phi}}(O)=O$. Hence it follows from Theorem D that $\bar{\bar{\Phi}}$: $\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}'$ is linear. Noting that $\tilde{\phi}$ and $\tilde{\phi}'$ are birational from (2.1), we conclude that $\bar{\Phi}$ is also birational. It remains the case where $k=m$. But, in this case the domain $\mathcal{D}$ (and so $\mathcal{D}'$) is necessarily a Siegel domain of the second kind by Corollary 1 in [4]. Thus our theorem follows from [3]. q.e.d.

The proof of Theorem 2 is now an immediate consequence of Theorem 1 and our previous result [5], but we give a proof here for completeness.

Proof of Theorem 2. Since it is trivial that $\mathcal{D}$ and $\mathcal{D}'$ are holomorphically equivalent if they are linearly equivalent, we have only to show the converse. Let $\mathfrak{g}(\mathcal{D})=\sum\mathfrak{g}_\lambda$ (resp. $\mathfrak{g}(\mathcal{D}')=\sum\mathfrak{g}'_\lambda$) be the decomposition of $\mathfrak{g}(\mathcal{D})$ (resp. of $\mathfrak{g}(\mathcal{D}')$) due to Kaup, Matsushima and Ochiai as in Theorem A. Put $\dim_{\boldsymbol{R}}\mathfrak{g}_{-1/2}=2k$ and $\dim_{\boldsymbol{R}}\mathfrak{g}'_{-1/2}=2k'$. Suppose that there exists a biholomorphic isomorphism Φ: $\mathcal{D}\to\mathcal{D}'$. Then, by Theorem 1 Φ is a birational holomorphic mapping, and moreover $k=k'$ as we showed in the proof of Theorem 1. In the following, for the domain $\mathcal{D}'$ we employ the notation A' for denoting the object corresponding to an object A for the domain $\mathcal{D}$. Let φ: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ be a non-singular linear mapping as in Theorem B such that $\tilde{\mathcal{D}}=\varphi(\mathcal{D})$. We claim:

(*) there exists a non-singular linear mapping $\tilde{\mathcal{L}}$: $\boldsymbol{C}\times\boldsymbol{C}^m\to\boldsymbol{C}\times\boldsymbol{C}^m$ of the form

$$\tilde{\mathcal{L}}:\begin{pmatrix} z' \\ w'_1 \\ \vdots \\ w'_m \end{pmatrix}=\begin{pmatrix} a & 0 & 0 \\ 0 & A & * \\ 0 & 0 & B \end{pmatrix}\cdot\begin{pmatrix} z \\ w_1 \\ \vdots \\ w_m \end{pmatrix}$$

such that $\tilde{\mathcal{L}}(\tilde{\mathcal{D}})=\tilde{\mathcal{D}}'$, where $a\in\boldsymbol{R}$ and A (resp. B) is a $k\times k$ (resp. $(m-k)\times(m-k)$) matrix.

If (*) is valid, we obtain our proof by putting $\mathcal{L}=\varphi'^{-1}\cdot\tilde{\mathcal{L}}\cdot\varphi$. We shall show that (*) is really true. Let $\tilde{\Phi}$ be a biholomorphic isomorphism of $\tilde{\mathcal{D}}$ onto $\tilde{\mathcal{D}}'$ defined by $\tilde{\Phi}=\varphi'\cdot\Phi\cdot\varphi^{-1}$. Put $\Psi=\tilde{\Phi}^{-1}$. Since $\mathfrak{g}(\tilde{\mathcal{D}})$ (and also $\mathfrak{g}(\tilde{\mathcal{D}}')$)

has the graded structure as in Theorem A and since $\tilde{\Phi}$ is birational, it can be shown in the same way as the proof of Theorem 11 in [3] that we may assume the mappings $\tilde{\Phi}\colon \tilde{\mathscr{D}}\to\tilde{\mathscr{D}}'$ and $\tilde{\Psi}\colon \tilde{\mathscr{D}}'\to\tilde{\mathscr{D}}$ are both affine transformations of the forms

$$(2.3)\qquad \tilde{\Phi}\colon \begin{pmatrix} z' \\ w_1' \\ \vdots \\ w_m' \end{pmatrix} = \begin{pmatrix} \Theta_1^1 & \Theta_2^1 \cdots\cdots & \Theta_{m+1}^1 \\ 0 & \Theta_2^2 & \Theta_{m+1}^2 \\ \vdots & \vdots \cdots\cdots\cdots & \vdots \\ 0 & \Theta_2^{m+1} \cdots & \Theta_2^{m+1} \end{pmatrix} \cdot \begin{pmatrix} z \\ w_1 \\ \vdots \\ w_m \end{pmatrix} + \begin{pmatrix} C^1 \\ C^2 \\ \vdots \\ C^{m+1} \end{pmatrix},\ \Theta_1^1\in \boldsymbol{R}$$

and

$$(2.4)\qquad \tilde{\Psi}\colon \begin{pmatrix} z \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} \Lambda_1^1 & \Lambda_2^1 \cdots\cdots & \Lambda_{m+1}^1 \\ 0 & \Lambda_2^2 & \Lambda_{m+1}^2 \\ \vdots & \vdots \cdots\cdots\cdots & \vdots \\ 0 & \Lambda_2^{m+1} \cdots & \Lambda_{m+1}^{m+1} \end{pmatrix} \cdot \begin{pmatrix} z' \\ w_1' \\ \vdots \\ w_m' \end{pmatrix} + \begin{pmatrix} D^1 \\ D^2 \\ \vdots \\ D^{m+1} \end{pmatrix},\ \Lambda_1^1\in \boldsymbol{R}.$$

We consider now the vector field $E=z\dfrac{\partial}{\partial z}+\dfrac{1}{2}\sum_{\alpha=1}^{m} w_\alpha\dfrac{\partial}{\partial w_\alpha}$ of $\mathfrak{g}(\tilde{\mathscr{D}})$. By direct computations we see

$$\begin{aligned}\tilde{\Phi}_*E = &\left(\Theta_1^1\Lambda_1^1 z'\frac{\partial}{\partial z'}+\frac{1}{2}\sum_{\alpha,\mu,\lambda=1}^{m}\Theta_{\alpha+1}^{\lambda+1}\Lambda_{\mu+1}^{\alpha+1}w_\mu'\frac{\partial}{\partial w_\lambda'}\right)\\ &+\left\{\sum_{\mu=1}^{m}\left(\Theta_1^1\Lambda_{\mu+1}^1+\frac{1}{2}\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^1\Lambda_{\mu+1}^{\alpha+1}\right)w_\mu'\frac{\partial}{\partial z'}+\frac{1}{2}\sum_{\alpha,\lambda=1}^{m}\Theta_{\alpha+1}^{\lambda+1}D^{\alpha+1}\frac{\partial}{\partial w_\lambda'}\right\}\\ &+\left(\Theta_1^1 D^1+\frac{1}{2}\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^1 D^{\alpha+1}\right)\frac{\partial}{\partial z'}\end{aligned}$$

where $\tilde{\Phi}_*\colon \mathfrak{g}(\tilde{\mathscr{D}})\to\mathfrak{g}(\tilde{\mathscr{D}}')$ is the differential of $\tilde{\Phi}$. Since $(\Lambda_j^i)\cdot(\Theta_k^l)=(\Theta_k^l)\cdot(\Lambda_j^i)=\mathbf{1}_{m+1}$, we have

$$\begin{cases}\Theta_1^1\Lambda_1^1=1\\ \displaystyle\sum_{\alpha,\mu,\lambda=1}^{m}\Theta_{\alpha+1}^{\lambda+1}\Lambda_{\mu+1}^{\alpha+1}w_\mu'\frac{\partial}{\partial w_\lambda'}=\sum_{\alpha=1}^{m}w_\alpha'\frac{\partial}{\partial w_\alpha'}\\ \displaystyle\Theta_1^1\Lambda_{\mu+1}^1+\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^1\Lambda_{\mu+1}^{\alpha+1}=0\end{cases}$$

and hence $\Theta_1^1\Lambda_{\mu+1}^1+\dfrac{1}{2}\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^1\Lambda_{\mu+1}^{\alpha+1}=\dfrac{1}{2}\Theta_1^1\Lambda_{\mu+1}^1$. As a result we get

$$\begin{aligned}\tilde{\Phi}_*E = &\, z'\frac{\partial}{\partial z'}+\frac{1}{2}\sum_{\alpha=1}^{m}w_\alpha'\frac{\partial}{\partial w_\alpha'}\\ &+\frac{1}{2}\left\{\left(\sum_{\mu=1}^{m}\Theta_1^1\Lambda_{\mu+1}^1 w_\mu'\right)\frac{\partial}{\partial z'}+\sum_{\lambda=1}^{m}\left(\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^{\lambda+1}D^{\alpha+1}\right)\frac{\partial}{\partial w_\lambda'}\right\}\end{aligned}$$

$$+\left(\Theta_1^1 D^1+\frac{1}{2}\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^1 D^{\alpha+1}\right)\frac{\partial}{\partial z'}.$$

Put

$$X=\left(\Theta_1^1 D^1+\frac{1}{2}\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^1 D^{\alpha+1}\right)\frac{\partial}{\partial z'} \qquad \text{and}$$

$$Y=\left(\sum_{\mu=1}^{m}\Theta_1^1\Lambda_{\mu+1}^1 w'_\mu\right)\frac{\partial}{\partial z'}+\sum_{\lambda=1}^{m}\left(\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^{\lambda+1}D^{\alpha+1}\right)\frac{\partial}{\partial w'_\lambda}.$$

Since $\tilde{\Phi}_*E$ and $E'=z'\frac{\partial}{\partial z'}+\frac{1}{2}\sum_{\alpha=1}^{m}w'_\alpha\frac{\partial}{\partial w'_\alpha}$ belong to $\mathfrak{g}(\tilde{\mathscr{D}}')$, $X+\frac{1}{2}Y$ belongs also to $\mathfrak{g}(\tilde{\mathscr{D}}')$. Then, from the concrete expression of holomorphic vector fields belonging to $\mathfrak{g}(\tilde{\mathscr{D}}')$ (see [3], (3.1) and (3.2)), we have $X\in\mathfrak{g}'_{-1}$ and $Y\in\mathfrak{g}'_{-1/2}$. Recall that

$$\mathfrak{g}'_{-1/2}=\left\{2\sqrt{-1}F(w',C)\frac{\partial}{\partial z'}+\sum_{\alpha=1}^{k}c^\alpha\frac{\partial}{\partial w'_\alpha}\,|\,C=(c^\alpha)\in\boldsymbol{C}^k\right\} \tag{2.5}$$

where $w'=(w'_1,\cdots,w'_k)$. By comparing the components of Y with (2.5) we see that

$$\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^{\lambda+1}D^{\alpha+1}=0 \qquad \text{for } k+1\leqq\lambda\leqq m\,; \tag{2.6}$$

$$\Theta_1^1\Lambda_{\mu+1}^1=0 \qquad \text{for } k+1\leqq\mu\leqq m\,; \tag{2.7}$$

$$\Theta_1^1\Lambda_{\mu+1}^1=2\sqrt{-1}\,\overline{\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^{\mu+1}D^{\alpha+1}} \qquad \text{for } 1\leqq\mu\leqq k\,. \tag{2.8}$$

On the other hand, since $\tilde{\Phi}\cdot\tilde{\Psi}$ is the identity mapping, it follows from (2.3) and (2.4) that

$$\sum_{\alpha=1}^{m}\Theta_{\alpha+1}^{\lambda+1}D^{\alpha+1}+C^{\lambda+1}=0 \qquad \text{for } 1\leqq\lambda\leqq m\,. \tag{2.9}$$

Then, from (2.6) and (2.9) we get

$$C^{\lambda+1}=0 \qquad \text{for } k+1\leqq\lambda\leqq m\,.$$

Thus we have shown that $\tilde{\Phi}$ is of the form

$$\tilde{\Phi}:\begin{cases} z'=\Theta_1^1 z+\sum_{\lambda=1}^{m}\Theta_{\lambda+1}^1 w_\lambda+C^1 & \\ w'_\alpha=\sum_{\lambda=1}^{m}\Theta_{\lambda+1}^{\alpha+1}w_\lambda+C^{\alpha+1} & \text{for } 1\leqq\alpha\leqq k \\ w'_\beta=\sum_{\lambda=1}^{m}\Theta_{\lambda+1}^{\beta+1}w_\lambda & \text{for } k+1\leqq\beta\leqq m\,. \end{cases}$$

Since the group $\mathrm{Aut}(\tilde{\mathscr{D}}')$ contains the affine transformations

$$l_a\colon (z', w', w'') \mapsto (z'+a, w', w'') \quad (a\in \boldsymbol{R})$$

and

$$p_c\colon (z', w', w'') \mapsto (z'+2\sqrt{-1}F(w', C)+\sqrt{-1}F(C, C), w'+C, w'') \quad (C\in \boldsymbol{C}^k)$$

where $w'=(w'_1,\cdots,w'_k)$ and $w''=(w'_{k+1},\cdots,w'_m)$, changing $\tilde{\Phi}$ by a suitable affine transformation $l_a\cdot p_c\cdot\tilde{\Phi}$ if necessary, we may assume that $\tilde{\Phi}$ is of the form

$$\tilde{\Phi}\colon \begin{cases} z' = \Theta^1_1 z+\sum_{\lambda=1}^m \Theta^1_{\lambda+1}w_\lambda+C^1,\ \Theta^1_1\in \boldsymbol{R} \text{ and } C^1\in\sqrt{-1}\boldsymbol{R} \\ w'_\alpha=\sum_{\lambda=1}^m \Theta^{\alpha+1}_{\lambda+1}w_\lambda \qquad \text{for } 1\leqq\alpha\leqq m\,. \end{cases}$$

Now, for $I'=\sqrt{-1}\sum_{\alpha=1}^m w'_\alpha\frac{\partial}{\partial w'_\alpha}$, we have

$$\begin{aligned}\tilde{\Psi}_*I' &= \sqrt{-1}\sum_{\lambda=1}^m\Big(\sum_{\alpha=1}^m \Theta^{\alpha+1}_{\lambda+1}\Lambda^1_{\alpha+1}\Big)w_\lambda\frac{\partial}{\partial z}+\sqrt{-1}\sum_{\alpha=1}^m w_\alpha\frac{\partial}{\partial w_\alpha} \\ &= -\sqrt{-1}\sum_{\lambda=1}^m \Lambda^1_1\Theta^1_{\lambda+1}w_\lambda\frac{\partial}{\partial z}+\sqrt{-1}\sum_{\alpha=1}^m w_\alpha\frac{\partial}{\partial w_\alpha}\,,\end{aligned}$$

because $\Lambda^1_1\Theta^1_{\lambda+1}+\sum_{\alpha=1}^m \Lambda^1_{\alpha+1}\Theta^{\alpha+1}_{\lambda+1}=0$ for $\lambda\geqq 1$. Since $\tilde{\Psi}_*I'$ and $I=\sqrt{-1}\sum_{\alpha=1}^m w_\alpha\frac{\partial}{\partial w_\alpha}$ belong to $\mathfrak{g}(\tilde{\mathscr{D}})$, so does $Z\colon=-\sqrt{-1}\sum_{\lambda=1}^m \Lambda^1_1\Theta^1_{\lambda+1}w_\lambda\frac{\partial}{\partial z}$. We have then $\sqrt{-1}Z=[I, Z]\in\mathfrak{g}(\tilde{\mathscr{D}})$. By H. Cartan's principle for bounded domains, we see $Z=0$. This shows that

$$\Theta^1_{\lambda+1}=0 \qquad \text{for } 1\leqq\lambda\leqq m\,, \tag{2.10}$$

since $\Lambda^1_1\neq 0$. It remains to show that $C^1=0$, but this can be proved with the same arguments as in the proof of Theorem 11 in [3]. Finally we have shown that $\tilde{\Phi}$ is a linear mapping. Moreover, as is shown in the proof of Theorem 1 we have

$$\tilde{\Phi}(\tilde{\mathscr{D}}_0)=\tilde{\Phi}(\mathrm{Aut}_0(\tilde{\mathscr{D}})\cdot(\sqrt{-1}, 0))=\mathrm{Aut}_0(\tilde{\mathscr{D}}')\cdot(\sqrt{-1}, 0)=\tilde{\mathscr{D}}'_0\,.$$

Obviously these facts imply that (*) is valid. We have thus completed the proof of Theorem 2. q.e.d.

AKITA UNIVERSITY

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