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ON THE LIMIT STATE OF SOLUTIONS OF SOME SEMILINEAR DIFFUSION EQUATIONS

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Introduction. This paper is concerned with the behavior of solutions of the following Cauchy problem for the semilinear diffusion equation

$$(1)\qquad \begin{aligned} &\partial_t u = \Delta u + f(u)\,, \quad u = u(t, x)\,, \quad t>0, \quad x \in R^N\,, \\ &u(0, x) = u_0(x)\,, \quad x \in R^N\,, \end{aligned}$$

where ∂_t and Δ denote $\partial/\partial t$ and $\sum_{j=1}^{N} \partial^2/\partial x_j^2$.

The type of phenomena that occur to solutions depends of course on the type of the nonlinear term $f(u)$ in the equation. For the function of type $u^{1+\alpha}$, H. Fujita [1] dealt with the problem of blowing-up of solutions in a finite time (see also H. Fujita [2], the present author [3] and S. Sugitani [4]). On the other hand A.M.Kolmogorov-I.G.Petrovsky-N.S.Piscounov [5] and Y.I. Kaneli [6] investigated the behavior of solution $u(t, x)$ of (1) as $t \to \infty$ in the case when the function $f(u)$ is $u(1-u)$ as the typical instance.

Here we deal with the problem (1) for the function of the type $u^{1+\alpha}(1-u)$ and investigate the limit state of the solution $u(t, x)$ as $t \to \infty$. Results may be roughly spoken as follows. Whether all nontrivial solutions tend to 1 or not depends on the degree α of the increase of f near 0. In the latter case solutions tend to 0 or 1 according to the magnitude of the initial value. This seems parallel to results in [1].

Precisely, our results are the followings. We assume that the function $f(r)$ satisfies next conditions (i), (ii) and (iii).

(i) $f(r)$ is of class C^1 on the closed interval $[0, 1]$.

(ii) $f(r)>0$ on the open interval $(0,1)$ and $f(0)=f(1)=0$.

(iii) There exist positive constants C_0 and α, with which we have $f(r) \geqq C_0 r^{1+\alpha}$ for $0 \leqq r \leqq 1/2$.

Further, in Theorem 2, the assumption (iv) should be added.

(iv) $f(r) \leqq C_1 r^{1+\alpha}$ on $[0, 1]$ for some constant $C_1>0$.

For the initial data $u_0(x)$ we only consider such functions that are compatible to $f(r)$, *i.e.*, $0 \leqq u_0(x) \leqq 1$, and that are continuous only for the sake of simplicity.

Then we have the following theorems.

Theorem 1*). *Let the constant α satisfy $N\alpha \leqq 2$. Assume that the function $f(r)$ satisfies conditions (i), (ii) and (iii). Then, for any nontrivial initial data u_0, i.e., $u_0(x) \not\equiv 0$, we have $\lim_{t\to\infty} u(t, x)=1$, in which the convergence is uniform on any bounded set of x in R^N.*

Theorem 2. *Let the constant α satisfy $N\alpha > 2$. Assume that the function f satisfies conditions (i), (ii), (iii) and (iv). Take any real γ larger than $2^\alpha C_0^{-1}\alpha^{-1}$. Then there exist positive constants a_0 and a_1 having following properties:*
1) If $u_0(x)$ is less than the function $a_0 H(\gamma, x)$ all over R^N then the solution $u(t, x)$ starting from u_0 goes to 0 uniformly on R^N.
2) If $u_0(x)$ is larger than $a_1 H(\gamma, x)$ all over R^N. Then the solution goes to 1 uniformly on any bounded set of R^N. Here the function $H(t, x)$ denotes the fundamental solution $(2\pi t)^{-N/2} exp\, [-|x|^2/4t]$ of the heat equation.

These theorems will be proved in 3. 1. is devoted to preliminary lemmas used in the following parts. In 2. we shall prove a key theorem (Theorem 3).

1. Preliminary lemmas. We consider the Cauchy problem for the quasilinear diffusion equation

$$\text{(A)} \qquad \partial_t u = \Delta u + f(t, x, u)\,, \quad u(0, x) = u_0(x)\,.$$

Here, we assume that the function $f(t, x, r)$ is continuous in (t, x, r) and Lepschitz continuous in r (the Lipschitz constant is taken uniformly in (t, x, r)).

Definition 1.1. For the bounded cuntinuous initial data $u_0(x)$ the function $u(t, x)$ is called the solution of problem (A) in $[0, T)\times R^N$ if it satisfies following conditions a), b) and c) ([1]).
a) For any $T'<T$ $u(t, x)$ is the bounded continuous function of (t, x) on $[0, T']\times R^N$.
b) Initial condition in (A) is satisfied in the usual sence.
c) The differential equation in (A) is satisfied in the sence of distribution in $(0, T)\times R^N$.

In proving our results, we apply the comparison theorem in the next form. For two functions $f_1(t, x, r)$ and $f_2(t, x, r)$ which satisfy the same conditions as above, we consider two Cauchy problems

$$\text{(A}_1\text{)} \qquad \partial_t u^1 = \Delta u^1 + f_1(t, x, u^1)\,, \quad u^1(0, x) = u_0^1(x)\,,$$

*) Recently, T. Shirao, H. Tanaka and K. Kobayashi have reported to us that they got some generalized results based on our Theorems 1 and 2.

$$(\mathrm{A}_2) \qquad \partial_t u^2 = \Delta u^2 + f_2(t, x, u^2)\,, \quad u^2(0, x) = u_0^2(x)\,.$$

We denote by $u^k(t, x)$ the solution of (A_k) in $[0, T^k) \times R^N$ for $k=1, 2$.

Lemma 1.2. (see [5]). *If we have* $u_0^1(x) \leqq u_0^2(x)$ *for all* $x \in R^N$ *and* $f_1(t, x, r) \leqq f_2(t, x, r)$ *for all* (t, x, r), *we have* $u^1(t, x) \leqq u^2(t, x)$ *on* $[0, T') \times R^N$ $(T' \leqq T^1, T^2)$.

By using the Previous lemma we can show that under the assumptions (i) and (ii) the Cauchy problem (1) admits the unique solution $u(t, x)$ in $[0, \infty) \times R^N$ for the compatible initial data, and this $u(t, x)$ satisfies $0 \leqq u(t, x) \leqq 1$.

In the remaining part of this paper we assume that the function f satisfies conditions (i), (ii) and (iii).

Lemma 1.3. *For an arbitrary couple of real numbers* (A, B) *satisfying* $0<A<B<1$, *we can find such a positive real* $\delta_0 = \delta_0(A, B)$ *that has following properties*:

(P1) *For two couples* (A, B) *and* (A', B') *satisfying* $0<A' \leqq A<B \leqq B'<1$ *we have* $\delta_0(A', B') \geqq \delta_0(A, B)$.

(P2) *If the initial data* $u_0(x)$ *is less than* A *all over* R^N, *then the solution* $u(t, x)$ *of* (1) *is less than* B *on* $[0, \delta_0] \times R^N$.

Proof of Lemma 1.3. Let $\varphi(t)$ be the solution of the ordinary defferential equation $d\varphi(t)/dt = f(\varphi(t))$, $\varphi(0) = A$. We define the constant $\delta_0(A, B)$ by $\varphi(\delta_0) = B$. Then $\varphi(t)$ is less than B for $0 \leqq t \leqq \delta_0$. From Lemma 1.2, where we set $f_1 = f_2 = f$ and $u_0^1(x) \equiv A$ $u_0^2(x) = u_0(x)$, we have $u(t, x) \leqq \varphi(t) \leqq B$ for $0 \leqq t \leqq \delta_0(A, B)$.

DEFINITION 1.4. We define the function $\Phi(r)$ by

$$\Phi(r) = \inf\{C_0 \wedge (f(s)/s^{1+\alpha});\ 0<s \leqq r\}\,.$$

Here, $a \wedge b$ denotes min $\{a, b\}$ for any real couple $\{a, b\}$ and C_0 is the constant of the condition (iii).

Then we have

Lemma 1.5. 1) *$\Phi(r)$ is continuous, nonnegative and non-increasing on the closed interval* $[0, 1]$.
2) $\Phi(1) = 0$ *and* $\Phi(r)$ *goes to* 0 *if and only if* r *tends to* 1.

Simple calculation leads us to

Lemma 1.6. 1) *Let* A, γ, C *and* h *be positive constants satisfying*

$$(1.1) \qquad (1 \wedge e^{1-(N\alpha/2)})\alpha C A^{\alpha} \gamma \geqq e^h\,.$$

Then we have

(1.2) $$\frac{|x|^2}{4\gamma^2}-\frac{N}{2\gamma}+CA^{\alpha}\exp[-\alpha|x|^2/4\gamma]\geqq h/\alpha\gamma \quad \textit{for all } x\in R^N\,.$$

(2) $$1-\sigma r\leqq(1+r)^{-\sigma}\leqq 1-\sigma r/3 \quad \textit{for} \quad \sigma>0,\ 0\leqq r\leqq(\sigma+1)^{-1}\,.$$

2. Key theorem. Before going to Theorems 1 and 2 we shall show a Key theorem.

DEFINITION 2.1. A continuous function $u(x)$ of x is said to be of class $G[A, \gamma]$ for some positive constants A and γ, if it satisfies

(2.1) $$1\geqq u(x)\geqq A\exp[-|x|^2/4\gamma] \quad \text{for all } x \text{ in } R^N\,.$$

DEFINITION 2.2. A couple of real constants $[A, \gamma]$ is said to satisfy the condition (*) if the next inequalities are valid:

(*) $$1>A>0\,,\quad \gamma>0\,,\quad (1\wedge e^{1-(N\alpha/2)})\alpha A^{\alpha}\Phi(A)\gamma>1\,.$$

Theorem 3. *Let the initial data $u(x)$ be of class $G[A_0, \gamma_0]$ for some couple $[A_0, \gamma_0]$ satisfying (*). Then, for any constants A, γ with $0<A<1$, $\gamma>0$, we can find $T_0=T_0(A, \gamma; A_0, \gamma_0)>0$, so that the solution $u(t, x)$ of (1) exceeds the function $A\exp[-|x|^2/4\gamma]$ at any time $t\geqq T_0$.*

The proof of this theorem will be given in the last part of this paragraph. First we show the following poroposition.

Proposition 2.3. *Let $[A, \gamma]$ satisfy (*). If $u(t_0, x)$ is of class $G[A, \gamma]$ at some time $t_0\geqq 0$. Then there exist positive constants $\delta_1=\delta_1(A, \gamma)$ and $\varepsilon=\varepsilon(A, \gamma)$ such that*

(2.2) $$u(t_0+t, x)\geqq(1+\varepsilon t)A\exp[-|x|^2/4\gamma] \quad \textit{for } 0\leqq t\leqq\delta_1,\ x\in R^N\,.$$

Proof. As the equation in (1) is invarient under the translation of t, we can put $t_0=0$. Let $u^*(t, x)$ denote the solution of the Cauchy problem

(1*) $$\partial_t u^* = \Delta u^*+f(u^*)\,,\quad u^*(0, x) = u_0^*(x) = A\exp[-|x|^2/4\gamma]\,.$$

By Lemma 1.2. we have $u(t, x)\geqq u^*(t, x)$. We shall prove (2.2) for $u^*(t, x)$. We define the constant $h>0$ by the relation $e^{2h}=(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(A)A^{\alpha}\gamma$ and define A' by $e^h=(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(A')A^{\alpha}\gamma$ $(A<A'<1)$. The definitions of the function Φ and the constant $\delta_0=\delta_0(A, A')$ lead us to

(2.3) $$f(u^*(t, x))\geqq\Phi(A')u^*(t, x)^{1+\alpha}\,,\quad 0\leqq t\leqq\delta_0\,,\quad x\in R^N\,.$$

Let $u^{**}(t, x)$ denote the solution of the problem

(1**) $$\partial_t u^{**} = \Delta u^{**}+\Phi(A')u^{**1+\alpha}\,,\quad u^{**}(0, x) = u_0^{**}(x) = u_0^*(x)\,.$$

Again by Lemma 1.2, we have $u^*(t, x)\geqq u^{**}(t, x)$ for $0\leqq t\leqq\delta_0$. The Cauchy problem (1**) is equivarent to the next problem (2**) in the integral form.

$$(2^{**})\qquad u^{**}(t, x)=\int_{R^N} H(t, x-y)u_0^*(y)\mathrm{d}y+ \\ +\int_0^t\int_{R^N} H(t-\tau, x-y)\Phi(A')\{u^{**}(\tau, y)\}^{1+\alpha}\,dy\,d\tau\,.$$

From this equation we have

$$(2.4)\qquad u^{**}(\tau, y)\geqq\int_{R^N} H(\tau, y-z)u_0^{**}(z)\,dz= \\ =A\left(1+\frac{\tau}{\gamma}\right)^{-N/2}\exp\left[-|x|^2/(\tau+\gamma)\right].$$

Substituting (2.4) for the second term of the right hand side of (2**), we have

$$(2.5)\qquad \int_0^t\int_{R^N} H(t-\tau, x-y)\Phi(A')\{u^{**}(\tau, y)\}^{1+\alpha}\,dy\,d\tau \\ \geqq \Phi(A')A^{1+\alpha}\left(1+\frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\right)^{-N/2}\left(1+\frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \\ \exp\left[-|x|^2/4\left(\frac{\gamma+\tau}{1+\alpha}+t-\tau\right)\right] \\ \geqq \Phi(A')A^{1+\alpha}\left(1+\frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2}\exp\left[-(1+\alpha)|x|^2/4(\gamma+\tau)\right]+ \\ +\Phi(A')A^{1+\alpha}\left\{\left(1+\frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\right)^{-N/2}-1\right\}\left(1+\frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \\ \exp\left[-(1+\alpha)|x|^2/4(\gamma+t)\right].$$

Further, by the simple calculation, we have

$$(2.6)\qquad \int_{R^N} H(t, x-y)u_0^{**}(y)\,dy=u_0^{**}(x)+\int_0^t\frac{\partial}{\partial\tau}\int_{R^N} H(\tau, x-y)u_0^{**}(y)\,dy\,d\tau \\ =u_0^{**}(x)+\int_0^t A\left(1+\frac{\tau}{\gamma}\right)^{-N/2}\left\{\frac{|x|^2}{4(\gamma+\tau)^2}-\frac{N}{2(\gamma+\tau)}\right\} \\ \exp\left[-\alpha|x|^2/4(\gamma+\tau)\right]d\tau\,.$$

Substituting (2.5) and (2.6) in to the right hand side of (2**), we get

$$(2.7)\qquad u^{**}(t, x)\geqq u_0^{**}(x)+\int_0^t\left\{\frac{|x|^2}{4(\gamma+\tau)^2}-\frac{N}{2(\gamma+\tau)}\right. \\ \left.+\Phi(A')A^{\alpha}\exp\left[-\alpha|x|^2/4(\gamma+\tau)\right]\right\}AG(x, \tau; \gamma)\,d\tau+$$

$$+\int_0^t \Phi(A')A^\alpha\left\{\left(1+\frac{t(1+\alpha)}{\gamma+\tau}\right)^{-N/2}\left(1+\frac{\tau}{\gamma}\right)^{-N\alpha/2}-1\right\}$$

$$\exp[-\alpha|x|^2/4(\gamma+\tau)]AG(x,\tau;\gamma)d\tau\,,$$

where $G(x,\tau;\gamma)$ denotes $\left(1+\frac{\tau}{\gamma}\right)^{-N/2}\exp[-|x|^2/4(\tau+\gamma)]=(4\pi\gamma)^{N/2}H(x,\gamma+\tau)$. Applying 1) of Lemma 1.6 to the second term and 2) to the third term, we get

$$u^{**}(t,x)\geqq u_0^{**}(x)+\int_0^t\left\{\frac{h}{\alpha(\gamma+\tau)}-\frac{\Phi(A')A^\alpha N(1+2\alpha)}{\gamma+\tau}\right\}AG(x,\tau;\gamma)d\tau \tag{2.8}$$

$$\text{for}\quad t<\gamma(1+N)^{-1}(1+\alpha)^{-1}=d_1(\gamma).$$

Now we set $d_2(h,\gamma)=h/\{2\Phi(A')A^\alpha N(1+2\alpha)\}=he^{-h}(1\wedge e^{1-(N\alpha/2)})\gamma/N(1+2\alpha)$. For we have $AG(x,\tau;\gamma)\geqq\left(1+\frac{\tau}{\gamma}\right)^{-N/2}u_0^{**}(x)$, we get

$$u^{**}(t,x)\geqq\left(1+\frac{h}{4\alpha\gamma 2^{N/2}}t\right)u_0^{**}(x)\quad\text{for}\quad t\leqq d_1(\gamma)\wedge d_2(h,\gamma)\,. \tag{2.9}$$

Thus we have (2.2) for the constants $\delta_1=\delta_0(A,A')\wedge d_1\wedge d_2$ and $\varepsilon=h/(4\alpha\gamma 2^{N/2})$.

Before finishing this proposition we make a remark on the constants $\delta_1(A,\gamma)$ and $\varepsilon(A,\gamma)$.

REMARK 2.4. Let constants B_0, B_1, B_2, satisfy $0<B_0<B_1<B_2<1$ and $(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(B_2)B_0^\alpha\gamma>1$. Take a constant A between B_0 and B_1. Then we have following uniform estimates with respect to $\delta_1(A,\gamma)$ and $\varepsilon(A,\gamma)$.

$$\begin{aligned}&\delta_1(A,\gamma)\geqq\bar\delta_1(B_0,B_1,B_2,\gamma)=\delta_0(B_1,B_2)\wedge d_1(\gamma)\wedge d_2(h_1,\gamma)\,,\\&\varepsilon(A,\gamma)\geqq\bar\varepsilon(B_0,B_2,\gamma)=h_1/(4\alpha\gamma 2^{N/2})\,,\end{aligned} \tag{2.10}$$

where the constant h_1 denotes $1\wedge(1/2)\log\{(1\wedge e^{1-(N\alpha/2)})\alpha\Phi(B_2)B_0^\alpha\gamma$.

Proposition 2.5. *Let $[A_0,\gamma_0]$ satisfy the condition (*). If $u(t_0,x)$ is of class $G[A_0,\gamma_0]$ at some time $t_0\geqq 0$, the solution $u(t,x)$ remains in the class $G[A_0,\gamma_0]$ at all time after t_0.*

Proof. By Proposition 2.3 we have (2.2). Thus, at any time t between t_0 and $t_0+\delta_1(A_0,\gamma_0)$, we have

$$u(t,x)\geqq A_0\exp[-|x|^2/4\gamma_0]\quad\text{for all } x \text{ in } R^N\,. \tag{2.11}$$

By using this argument at the time $t_0+\delta_1(A_0,\gamma_0)$, we get (2.11) for $t\in[t_0+\delta_1(A_0,\gamma_0),t_0+2\delta_1(A_0,\gamma_0)]$. The repetition of this argument leads us to the same estimate (2.11) at any time t in $[t_0+n\delta_1(A_0,\gamma_0),t_0+(n+1)\delta_0(A_1^0,\gamma_0)]$ for $n=0,1,2,\cdots$. This proves our proposition.

Now we are going to prove Theorm 3. Chose constants A_0', h_0 just in the same way as A', h in the proof of Proposition 2.3. Take a real number $A_1 < A_0'$ sufficiently close to A_0', and put $t_0=0$. Then we get

Lemma 2.6. *Under the assumptions in Theorem* 3 *we have at some time* t_0+t_1''

$$(2.12) \qquad u(t_0+t'_1, x) \geqq A_1 \exp[-|x|^2/4\gamma_0] \quad \textit{for all } x \textit{ in } R^N .$$

Proof. When we have

$$(2.13) \qquad A_{0,1} = (1+\varepsilon(A_0, \gamma_0)\delta_1(A_0, \gamma_0))A_0 \geqq A_1 ,$$

we can take $t_1'=\delta_1(A_0, \gamma_0)$ by Proposition 2.3.

Else if (2.13) is false, we use Proposition 2.3 again, substituing $A_{0,1}$ for A_0. If the inequality (2.13) is true where A_0 is replaced by $A_{0,1}$, we can take $t_1'=\delta_1(A_0, \gamma_0)+\delta_1(A_{0,1}, \gamma_0)$. In the case when the inequality is false, we continue these steps defining $A_{0,k+1}=(1+\varepsilon(A_{0,k}, \gamma_0)\ \delta_1(A_{0,k}, \gamma_0))A_{0,k}$, until the constant $A_{0,n}$ exceeds A_1. On account of REMARK 2.4 we can stop this iteration in a finite step. So, at the time $t_0+t_1'=t_0+\delta_1(A_0, \gamma_0)+\delta_1(A_{0,1}, \gamma_0)+\cdots+\delta_1(A_{0,n}, \gamma_0)$ the estimate (2.12) holds.

By using Lemma 1.2, where we set $f_1(r)=f(r)$, $f_2(r)=0$ and $u_0^1(x)=A^* \exp[-|x|^2/4\gamma^*]$, we have

Lemma 2.7. *Let* A^*, γ^* *be some positive constants with* $A^*<1$. *Let the solution* $u(t, x)$ *of* (1) *be larger than the function* $A^* \exp[-|x|^2/4\gamma^*]$ *at some time* $t^*>0$. *Then we have*

$$(2.14) \qquad u(t+t^*, x) \geqq A^*\left(1+\frac{t}{\gamma^*}\right)^{-N/2} \exp[-|x|^2/4(\gamma^*+t)] \quad \textit{for all } x \textit{ and } t>0.$$

Using this lemma we have

$$(2.15) \qquad u(t_0+t_1'+t_1'', x) \geqq A_0 \exp[-|x|^2/4(\gamma_0+t_1'')] \quad \text{for all } x \text{ in } R^N ,$$

where t_1'' is defined by $A_0=\left(1+\dfrac{t_1''}{\gamma_0}\right)^{-N/2} A_1$.

Now we put $t_1=t_0+t_1'+t_1''$, $\gamma_1=\gamma_0+t_1''=(A_1/A_0)^{2/N}\gamma_0$.

By the fact that γ_1 is larger than γ_0, we can take the same process as the above argument, where t_0 and γ_0 are replaced by t_1 and γ_1. Thus we have constants $t_2>t_1$ and $\gamma_2=(A_1/A_0)^{2/N}\gamma_1$ such that

$$(2.16) \qquad u(t_2, x) \geqq A_0 \exp[-|x|^2/4\gamma_2] \quad \text{for all } x \text{ in } R^N .$$

Denote the constant $(A_1/A_0)^{2n/N}\gamma_0$ by γ_n for $n=0, 1, 2, \cdots$. By repetition of these arguments we have, at some time t_n, (2.17) $\quad u(t_n, x) \geqq A_0 \exp[-|x|^2/4\gamma_n]$ for all x in R^N.

We chose the integer n sufficiently large so that

(2.18) $\quad \gamma_n > \gamma,$

(2.19) $\quad (1 \wedge e^{1-(N\alpha/2)}) \alpha A_0^\alpha \Phi(A) \gamma_n > 1,$

where A and γ are constants considered in the conclusion part of Theorem 3.

Because the function $\Phi(A)$ is continuous, we can find $B > A$, so that the ineqality (2.19) remains true for B changed in place of A. By Lemma 2.6 we can take a constant $T_0 > 0$ such that we have at the time $t = T_0$

(2.20) $\quad u(t, x) \geqq A \exp[-|x|^2/4\gamma_n]$ for all x in R^N.

By the inequality $(1 \wedge e^{1-(N\alpha/2)}) \alpha A^\alpha \Phi(A) \gamma_n > 1$ and Proposition 2.5 we have the same estimate (2.20) for any time after T_0. Because of the fact (2.18), this proves Theorem 3.

3. Proofs of Theorem 1 and 2. To prove Theorem 1, it is enough to prove it in the case of $N\alpha = 2$.

Proposition 3.1. *Let α be equal to $2/N$. For any nontrivial solution $u(t, x)$ of* (1), *we can find real constants A_0 and γ_0, so that these constants satisfy* (*) *in Definition* 2.2, *and the solution $u(t, x)$ exceeds A_0 exp $[-|x|^2/4\gamma_0]$ at some time $t = t_0$.*

Proof. Lemma 2.7 shows that, at any time $t > 0$, the solution $u(t, x)$ of (1) starting from a nontrivial initial data is positive for all x. Thus we can take a positive number ε, so that we have the estimate $u(1, x) > \varepsilon$ for $|x| \leqq 1$. Using this lemma again, we get

(3.1) $$u(t+1, x) \geqq \int_{R^N} H(t, x-y) u(1, y) dy \geqq \varepsilon \int_{|y| \leqq 1} H(t, x-y) dy.$$

The last term of the above inequality is larger than $E(t)H(2t, x)$ where $E(t)$ denote $\varepsilon 2^{N/2} \omega_N e^{-1/2t}$ (ω_N is the volume of the unit ball in R^N). So, we can assume that the initial data is larger than $C \exp[-|x|^2/4\beta]$ for some positive constants C and β with $C < 1/2$. Now we define a function $v(t, x)$ by the integral equation

(3.2) $$v(t, x) = \int_{R^N} H(t, x-y) v_0(y) dy + \\ + \int_0^t \int_{R^N} H(t-\tau, x-y) C_0 \left\{\int_{R^N} H(\tau, y-z) v_0(z) dz\right\}^{1+\alpha} dy\, d\tau,$$

where $v_0(y)$ denotes the function $C \exp[-|x|^2/4\beta]$.

By some calculation we have

$$(3.3)\qquad v(t,x)\leqq C\Big(1+\frac{t}{\beta}\Big)^{-N/2}\Big(1+KC^{\alpha}\log\Big(1+\frac{t}{\beta}\Big)\Big),\quad K=C_0\beta(4\pi)^{N/2},$$

$$(3.4)\qquad v(t,x)\geqq A(t)\exp\left[-|x|^2/4\beta(t)\right],$$

where $A(t)=C_0C^{1+\alpha}\beta^{1+(N/2)}\Big(t+\frac{\beta}{1+\alpha}\Big)^{-N/2}\log\Big(1+\frac{t}{\beta}\Big)$ and $\gamma(t)=(\beta+t)/(1+\alpha)$.
By changing the constant C for the smaller one if necessary, we may assume that C^{α} is less than $N/2K$. From this assumption and the inequality (3.3) we have

$$(3.5)\qquad v(t,x)\leqq C<1/2\quad\text{for all } t\geqq 0 \text{ and all } x \text{ in } R^N.$$

Operating $\partial_t-\Delta$ to $v(t,x)$ of (3.2), we have

$$(3.6)\qquad \partial_t v-\Delta v=C_0\Big\{\int_{R^N}H(t,x-y)v_0(y)\,dy\Big\}^{1+\alpha}.$$

The right hand side of (3.6) is less than $C_0\{v(t,x)\}^{1+\alpha}$. So the condition (iii) for $f(r)$ and the inequality (3.5) lead us to

$$(3.7)\qquad \partial_t v(t,x)\leqq\Delta v(t,x)+f(v(t,x))\quad\text{for}\quad t>0,\ x\in R^N.$$

On the other hand we have $u(0,x)\geqq v(0,x)$. This shows that the solution $u(t,x)$ is larger than $v(t,x)$ on $[0,\infty[\times R^N$.
Chose a constant t_0 large enough so that the quantity

$$\alpha\Phi(A(t_0))A(t_0)^{\alpha}\gamma(t_0)=\alpha C_0^{1+\alpha}C^{\alpha(1+\alpha)}\beta^{1+\alpha}\Big(t_0+\frac{\beta}{1+\alpha}\Big)^{-1}\Big(\log\Big(1+\frac{t_0}{\beta}\Big)\Big)^{\alpha}(\beta+t_0)/(1+\alpha)$$

is larger than 1 and denote $A(t_0)$, $\gamma(t_0)$ by A_0, γ_0.
Thus we have $u(t_0,x)\geqq v(t_0,x)\geqq A_0\exp\left[-|x|^2/4\gamma_0\right]$ where $[A_0,\gamma_0]$ satisfies (*).
Theorem 1 is an immediate consequence of Proposition 3.1 and Theorem 3.

Proof of Theorem 2. The existence of the constant a_1 is obvious because we can take $a_1=(1/2)(4\pi\gamma)^{-N/2}$ on account of the fact that γ is larger than $2^{\alpha}C_0^{-1}\alpha^{-1}$. Taking this a_1 and denoting $a_1(4\pi\gamma)^{N/2}=1/2$ by A, we have (*) with this $[A,\gamma]$. Theorem 3 shows that this a_1 has the property in Theorem 2. The existence of the constant a_0 will be proved by using the next proposition, which was proved in [1].

Proposition 3.2. (Theorem 2 in [1]). *Let the function $f(r)$ of nonlinear* *term satisfy (iv) in addition to (i), (ii) and (iii) and let the constant α be larger than $2/N$. Take any positive number γ. Then there exists a positive number a_0 with the following property; if the initial data $u_0(x)$ is less than the function $a_0H(\gamma, x)$, then the solution of (1) is subject to*

$$(3.8) \qquad 0 \leqq u(t, x) \leqq MH(t+\gamma, x)\,, \quad t>0\,, \quad x \in R_N\,,$$

for some positive constant M.

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References

[1] H. Fujita: *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. I, **13** (1966), 109–124.

[2] ————: *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations*, Proc. Symposia in Pure Math. vol. 18, A.M.S., 105–113.

[3] K. Hayakawa: *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan. Acad. **49** (1973), 503–505.

[4] S. Sugitani: *On nonexistence of global solutions for some nonlinear integral equations*, Osaka J. Math. **12** (1975), 45–51.

[5] A.M. Kolmogorov, I.G. Petrovsky and N.S. Piscounov: *Étude de l'equation de la diffusion avec croissance de la quantité de matière et son application a un problème biologique*, Bull. de l'Univ. a Moscou, (1937).

[6] Y.I. Kaneli: *On the stability of solutions of the equation in the theory of burning for finite initial functions*, Mat. Sb. **65** (1964), 398–413. (Russian).