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ON THE LIMIT STATE OF SOLUTIONS OF SOME SEMILINEAR DIFFUSION EQUATIONS

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Introduction. This paper is concerned with the behavior of solutions of the following Cauchy problem for the semilinear diffusion equation

$$(1) \quad \begin{aligned} \partial_t u &= \Delta u + f(u), \quad u = u(t, x), \quad t > 0, \quad x \in R^N, \\ u(0, x) &= u_0(x), \quad x \in R^N, \end{aligned}$$

where ∂_t and Δ denote $\partial/\partial t$ and $\sum_{j=1}^N \partial^2/\partial x_j^2$.

The type of phenomena that occur to solutions depends of course on the type of the nonlinear term $f(u)$ in the equation. For the function of type $u^{1+\alpha}$, H. Fujita [1] dealt with the problem of blowing-up of solutions in a finite time (see also H. Fujita [2], the present author [3] and S. Sugitani [4]). On the other hand A.M.Kolmogorov-I.G.Petrovsky-N.S.Piscounov [5] and Y.I. Kaneli [6] investigated the behavior of solution $u(t, x)$ of (1) as $t \rightarrow \infty$ in the case when the function $f(u)$ is $u(1-u)$ as the typical instance.

Here we deal with the problem (1) for the function of the type $u^{1+\alpha}(1-u)$ and investigate the limit state of the solution $u(t, x)$ as $t \rightarrow \infty$. Results may be roughly spoken as follows. Whether all nontrivial solutions tend to 1 or not depends on the degree α of the increase of f near 0. In the latter case solutions tend to 0 or 1 according to the magnitude of the initial value. This seems parallel to results in [1].

Precisely, our results are the followings. We assume that the function $f(r)$ satisfies next conditions (i), (ii) and (iii).

- (i) $f(r)$ is of class C^1 on the closed interval $[0, 1]$.
- (ii) $f(r) > 0$ on the open interval $(0, 1)$ and $f(0) = f(1) = 0$.
- (iii) There exist positive constants C_0 and α , with which we have $f(r) \geq C_0 r^{1+\alpha}$ for $0 \leq r \leq 1/2$.

Further, in Theorem 2, the assumption (iv) should be added.

- (iv) $f(r) \leq C_1 r^{1+\alpha}$ on $[0, 1]$ for some constant $C_1 > 0$.

For the initial data $u_0(x)$ we only consider such functions that are compatible to $f(r)$, i.e., $0 \leq u_0(x) \leq 1$, and that are continuous only for the sake of simplicity.

Then we have the following theorems.

Theorem 1*^o. *Let the constant α satisfy $N\alpha \leq 2$. Assume that the function $f(r)$ satisfies conditions (i), (ii) and (iii). Then, for any nontrivial initial data u_0 , i.e., $u_0(x) \not\equiv 0$, we have $\lim_{t \rightarrow \infty} u(t, x) = 1$, in which the convergence is uniform on any bounded set of x in R^N .*

Theorem 2. *Let the constant α satisfy $N\alpha > 2$. Assume that the function f satisfies conditions (i), (ii), (iii) and (iv). Take any real γ larger than $2^\alpha C_0^{-1} \alpha^{-1}$. Then there exist positive constants a_0 and a_1 having following properties:*

- 1) *If $u_0(x)$ is less than the function $a_0 H(\gamma, x)$ all over R^N then the solution $u(t, x)$ starting from u_0 goes to 0 uniformly on R^N .*
- 2) *If $u_0(x)$ is larger than $a_1 H(\gamma, x)$ all over R^N . Then the solution goes to 1 uniformly on any bounded set of R^N . Here the function $H(t, x)$ denotes the fundamental solution $(2\pi t)^{-N/2} \exp[-|x|^2/4t]$ of the heat equation.*

These theorems will be proved in 3. 1. is devoted to preliminary lemmas used in the following parts. In 2. we shall prove a key theorem (Theorem 3).

1. Preliminary lemmas. We consider the Cauchy problem for the quasilinear diffusion equation

$$(A) \quad \partial_t u = \Delta u + f(t, x, u), \quad u(0, x) = u_0(x).$$

Here, we assume that the function $f(t, x, r)$ is continuous in (t, x, r) and Lipschitz continuous in r (the Lipschitz constant is taken uniformly in (t, x, r)).

DEFINITION 1.1. For the bounded continuous initial data $u_0(x)$ the function $u(t, x)$ is called the solution of problem (A) in $[0, T] \times R^N$ if it satisfies following conditions a), b) and c) ([1]).

- a) For any $T' < T$ $u(t, x)$ is the bounded continuous function of (t, x) on $[0, T'] \times R^N$.
- b) Initial condition in (A) is satisfied in the usual sense.
- c) The differential equation in (A) is satisfied in the sense of distribution in $(0, T) \times R^N$.

In proving our results, we apply the comparison theorem in the next form. For two functions $f_1(t, x, r)$ and $f_2(t, x, r)$ which satisfy the same conditions as above, we consider two Cauchy problems

$$(A_1) \quad \partial_t u^1 = \Delta u^1 + f_1(t, x, u^1), \quad u^1(0, x) = u_0^1(x),$$

*) Recently, T. Shirao, H. Tanaka and K. Kobayashi have reported to us that they got some generalized results based on our Theorems 1 and 2.

$$(A_2) \quad \partial_t u^2 = \Delta u^2 + f_2(t, x, u^2), \quad u^2(0, x) = u_0^2(x).$$

We denote by $u^k(t, x)$ the solution of (A_k) in $[0, T^k) \times R^N$ for $k=1, 2$.

Lemma 1.2. (see [5]). *If we have $u_0^1(x) \leq u_0^2(x)$ for all $x \in R^N$ and $f_1(t, x, r) \leq f_2(t, x, r)$ for all (t, x, r) , we have $u^1(t, x) \leq u^2(t, x)$ on $[0, T^1) \times R^N$ ($T^1 \leq T^2$).*

By using the Previous lemma we can show that under the assumptions (i) and (ii) the Cauchy problem (1) admits the unique solution $u(t, x)$ in $[0, \infty) \times R^N$ for the compatible initial data, and this $u(t, x)$ satisfies $0 \leq u(t, x) \leq 1$.

In the remaining part of this paper we assume that the function f satisfies conditions (i), (ii) and (iii).

Lemma 1.3. *For an arbitrary couple of real numbers (A, B) satisfying $0 < A < B < 1$, we can find such a positive real $\delta_0 = \delta_0(A, B)$ that has following properties:*

(P1) *For two couples (A, B) and (A', B') satisfying $0 < A' \leq A < B \leq B' < 1$ we have $\delta_0(A', B') \geq \delta_0(A, B)$.*

(P2) *If the initial data $u_0(x)$ is less than A all over R^N , then the solution $u(t, x)$ of (1) is less than B on $[0, \delta_0] \times R^N$.*

Proof of Lemma 1.3. Let $\varphi(t)$ be the solution of the ordinary differential equation $d\varphi(t)/dt = f(\varphi(t))$, $\varphi(0) = A$. We define the constant $\delta_0(A, B)$ by $\varphi(\delta_0) = B$. Then $\varphi(t)$ is less than B for $0 \leq t \leq \delta_0$. From Lemma 1.2, where we set $f_1 = f_2 = f$ and $u_0^1(x) \equiv A$, $u_0^2(x) = u_0(x)$, we have $u(t, x) \leq \varphi(t) \leq B$ for $0 \leq t \leq \delta_0(A, B)$.

DEFINITION 1.4. We define the function $\Phi(r)$ by

$$\Phi(r) = \inf \{C_0 \wedge (f(s)/s^{1+\alpha}); 0 < s \leq r\}.$$

Here, $a \wedge b$ denotes $\min \{a, b\}$ for any real couple $\{a, b\}$ and C_0 is the constant of the condition (iii).

Then we have

Lemma 1.5. 1) $\Phi(r)$ is continuous, nonnegative and non-increasing on the closed interval $[0, 1]$.

2) $\Phi(1) = 0$ and $\Phi(r)$ goes to 0 if and only if r tends to 1.

Simple calculation leads us to

Lemma 1.6. 1) Let A, γ, C and h be positive constants satisfying

$$(1.1) \quad (1 \wedge e^{1-(N\alpha/2)}) \alpha C A^\alpha \gamma \geq e^h.$$

Then we have

$$(1.2) \quad \frac{|x|^2}{4\gamma^2} - \frac{N}{2\gamma} + CA^\alpha \exp[-\alpha|x|^2/4\gamma] \geq h/\alpha\gamma \quad \text{for all } x \in R^N.$$

$$(2) \quad 1 - \sigma r \leq (1+r)^{-\sigma} \leq 1 - \sigma r/3 \quad \text{for } \sigma > 0, 0 \leq r \leq (\sigma+1)^{-1}.$$

2. Key theorem. Before going to Theorems 1 and 2 we shall show a Key theorem.

DEFINITION 2.1. A continuous function $u(x)$ of x is said to be of class $G[A, \gamma]$ for some positive constants A and γ , if it satisfies

$$(2.1) \quad 1 \geq u(x) \geq A \exp[-|x|^2/4\gamma] \quad \text{for all } x \text{ in } R^N.$$

DEFINITION 2.2. A couple of real constants $[A, \gamma]$ is said to satisfy the condition (*) if the next inequalities are valid:

$$(*) \quad 1 > A > 0, \quad \gamma > 0, \quad (1 \wedge e^{1-(N\alpha/2)})\alpha A^\alpha \Phi(A)\gamma > 1.$$

Theorem 3. Let the initial data $u(x)$ be of class $G[A_0, \gamma_0]$ for some couple $[A_0, \gamma_0]$ satisfying (*). Then, for any constants A, γ with $0 < A < 1, \gamma > 0$, we can find $T_0 = T_0(A, \gamma; A_0, \gamma_0) > 0$, so that the solution $u(t, x)$ of (1) exceeds the function $A \exp[-|x|^2/4\gamma]$ at any time $t \geq T_0$.

The proof of this theorem will be given in the last part of this paragraph. First we show the following proposition.

Proposition 2.3. Let $[A, \gamma]$ satisfy (*). If $u(t_0, x)$ is of class $G[A, \gamma]$ at some time $t_0 \geq 0$. Then there exist positive constants $\delta_1 = \delta_1(A, \gamma)$ and $\varepsilon = \varepsilon(A, \gamma)$ such that

$$(2.2) \quad u(t_0 + t, x) \geq (1 + \varepsilon t) A \exp[-|x|^2/4\gamma] \quad \text{for } 0 \leq t \leq \delta_1, x \in R^N.$$

Proof. As the equation in (1) is invariant under the translation of t , we can put $t_0 = 0$. Let $u^*(t, x)$ denote the solution of the Cauchy problem

$$(1^*) \quad \partial_t u^* = \Delta u^* + f(u^*), \quad u^*(0, x) = u_0^*(x) = A \exp[-|x|^2/4\gamma].$$

By Lemma 1.2. we have $u(t, x) \geq u^*(t, x)$. We shall prove (2.2) for $u^*(t, x)$. We define the constant $h > 0$ by the relation $e^{2h} = (1 \wedge e^{1-(N\alpha/2)})\alpha\Phi(A)A^\alpha\gamma$ and define A' by $e^h = (1 \wedge e^{1-(N\alpha/2)})\alpha\Phi(A')A'^\alpha\gamma$ ($A < A' < 1$). The definitions of the function Φ and the constant $\delta_0 = \delta_0(A, A')$ lead us to

$$(2.3) \quad f(u^*(t, x)) \geq \Phi(A')u^*(t, x)^{1+\alpha}, \quad 0 \leq t \leq \delta_0, \quad x \in R^N.$$

Let $u^{**}(t, x)$ denote the solution of the problem

$$(1^{**}) \quad \partial_t u^{**} = \Delta u^{**} + \Phi(A')u^{**1+\alpha}, \quad u^{**}(0, x) = u_0^{**}(x) = u_0^*(x).$$

Again by Lemma 1.2, we have $u^*(t, x) \geq u^{**}(t, x)$ for $0 \leq t \leq \delta_0$. The Cauchy

problem (1**) is equivalent to the next problem (2**) in the integral form.

$$(2^{**}) \quad u^{**}(t, x) = \int_{R^N} H(t, x-y) u_0^*(y) dy + \\ + \int_0^t \int_{R^N} H(t-\tau, x-y) \Phi(A') \{u^{**}(\tau, y)\}^{1+\alpha} dy d\tau.$$

From this equation we have

$$(2.4) \quad u^{**}(\tau, y) \geq \int_{R^N} H(\tau, y-z) u_0^*(z) dz = \\ = A \left(1 + \frac{\tau}{\gamma}\right)^{-N/2} \exp[-|x|^2/(\tau+\gamma)].$$

Substituting (2.4) for the second term of the right hand side of (2**), we have

$$(2.5) \quad \int_0^t \int_{R^N} H(t-\tau, x-y) \Phi(A') \{u^{**}(\tau, y)\}^{1+\alpha} dy d\tau \\ \geq \Phi(A') A^{1+\alpha} \left(1 + \frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\right)^{-N/2} \left(1 + \frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \\ \exp\left[-|x|^2/4\left(\frac{\gamma+\tau}{1+\alpha} + t-\tau\right)\right] \\ \geq \Phi(A') A^{1+\alpha} \left(1 + \frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \exp[-(1+\alpha)|x|^2/4(\gamma+\tau)] + \\ + \Phi(A') A^{1+\alpha} \left\{ \left(1 + \frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\right)^{-N/2} - 1 \right\} \left(1 + \frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \\ \exp[-(1+\alpha)|x|^2/4(\gamma+t)].$$

Further, by the simple calculation, we have

$$(2.6) \quad \int_{R^N} H(t, x-y) u_0^*(y) dy = u_0^*(x) + \int_0^t \frac{\partial}{\partial \tau} \int_{R^N} H(\tau, x-y) u_0^*(y) dy d\tau \\ = u_0^*(x) + \int_0^t A \left(1 + \frac{\tau}{\gamma}\right)^{-N/2} \left\{ \frac{|x|^2}{4(\gamma+\tau)^2} - \frac{N}{2(\gamma+\tau)} \right\} \\ \exp[-\alpha|x|^2/4(\gamma+\tau)] d\tau.$$

Substituting (2.5) and (2.6) in to the right hand side of (2**), we get

$$(2.7) \quad u^{**}(t, x) \geq u_0^*(x) + \int_0^t \left\{ \frac{|x|^2}{4(\gamma+\tau)^2} - \frac{N}{2(\gamma+\tau)} \right. \\ \left. + \Phi(A') A^\alpha \exp[-\alpha|x|^2/4(\gamma+\tau)] \right\} A G(x, \tau; \gamma) d\tau +$$

$$\begin{aligned}
 & + \int_0^t \Phi(A')A^\alpha \left\{ \left(1 + \frac{t(1+\alpha)}{\gamma+\tau} \right)^{-N/2} \left(1 + \frac{\tau}{\gamma} \right)^{-N\alpha/2} - 1 \right\} \\
 & \quad \exp [-\alpha |x|^2/4(\gamma+\tau)] AG(x, \tau; \gamma) d\tau,
 \end{aligned}$$

where $G(x, \tau; \gamma)$ denotes $\left(1 + \frac{\tau}{\gamma} \right)^{-N/2} \exp [-|x|^2/4(\tau+\gamma)] = (4\pi\gamma)^{N/2} H(x, \gamma+\tau)$.

Applying 1) of Lemma 1.6 to the second term and 2) to the third term, we get

$$\begin{aligned}
 (2.8) \quad u^{**}(t, x) & \geq u_0^{**}(x) + \int_0^t \left\{ \frac{h}{\alpha(\gamma+\tau)} - \frac{\Phi(A')A^\alpha N(1+2\alpha)}{\gamma+\tau} \right\} AG(x, \tau; \gamma) d\tau \\
 & \quad \text{for } t < \gamma(1+N)^{-1}(1+\alpha)^{-1} = d_1(\gamma).
 \end{aligned}$$

Now we set $d_2(h, \gamma) = h / \{2\Phi(A')A^\alpha N(1+2\alpha)\} = h e^{-h} (1 \wedge e^{1-(N\alpha/2)}) \gamma / N(1+2\alpha)$.

For we have $AG(x, \tau; \gamma) \geq \left(1 + \frac{\tau}{\gamma} \right)^{-N/2} u_0^{**}(x)$, we get

$$(2.9) \quad u^{**}(t, x) \geq \left(1 + \frac{h}{4\alpha\gamma 2^{N/2}} t \right) u_0^{**}(x) \quad \text{for } t \leq d_1(\gamma) \wedge d_2(h, \gamma).$$

Thus we have (2.2) for the constants $\delta_1 = \delta_0(A, A') \wedge d_1 \wedge d_2$ and $\varepsilon = h / (4\alpha\gamma 2^{N/2})$.

Before finishing this proposition we make a remark on the constants $\delta_1(A, \gamma)$ and $\varepsilon(A, \gamma)$.

REMARK 2.4. Let constants B_0, B_1, B_2 , satisfy $0 < B_0 < B_1 < B_2 < 1$ and $(1 \wedge e^{1-(N\alpha/2)}) \alpha \Phi(B_2) B_0^\alpha \gamma > 1$. Take a constant A between B_0 and B_1 . Then we have following uniform estimates with respect to $\delta_1(A, \gamma)$ and $\varepsilon(A, \gamma)$.

$$\begin{aligned}
 (2.10) \quad \delta_1(A, \gamma) & \geq \bar{\delta}_1(B_0, B_1, B_2, \gamma) = \delta_0(B_1, B_2) \wedge d_1(\gamma) \wedge d_2(h_1, \gamma), \\
 \varepsilon(A, \gamma) & \geq \bar{\varepsilon}(B_0, B_2, \gamma) = h_1 / (4\alpha\gamma 2^{N/2}),
 \end{aligned}$$

where the constant h_1 denotes $1 \wedge (1/2) \log \{ (1 \wedge e^{1-(N\alpha/2)}) \alpha \Phi(B_2) B_0^\alpha \gamma \}$.

Proposition 2.5. *Let $[A_0, \gamma_0]$ satisfy the condition (*). If $u(t_0, x)$ is of class $G[A_0, \gamma_0]$ at some time $t_0 \geq 0$, the solution $u(t, x)$ remains in the class $G[A_0, \gamma_0]$ at all time after t_0 .*

Proof. By Proposition 2.3 we have (2.2). Thus, at any time t between t_0 and $t_0 + \delta_1(A_0, \gamma_0)$, we have

$$(2.11) \quad u(t, x) \geq A_0 \exp [-|x|^2/4\gamma_0] \quad \text{for all } x \text{ in } R^N.$$

By using this argument at the time $t_0 + \delta_1(A_0, \gamma_0)$, we get (2.11) for $t \in [t_0 + \delta_1(A_0, \gamma_0), t_0 + 2\delta_1(A_0, \gamma_0)]$. The repetition of this argument leads us to the same estimate (2.11) at any time t in $[t_0 + n\delta_1(A_0, \gamma_0), t_0 + (n+1)\delta_1(A_0, \gamma_0)]$ for $n = 0, 1, 2, \dots$. This proves our proposition.

Now we are going to prove Theorem 3. Chose constants A_0', h_0 just in the

same way as A' , h in the proof of Proposition 2.3. Take a real number $A_1 < A'_0$ sufficiently close to A'_0 , and put $t_0 = 0$. Then we get

Lemma 2.6. *Under the assumptions in Theorem 3 we have at some time $t_0 + t'_1$*

$$(2.12) \quad u(t_0 + t'_1, x) \geq A_1 \exp[-|x|^2/4\gamma_0] \quad \text{for all } x \text{ in } R^N.$$

Proof. When we have

$$(2.13) \quad A_{0,1} = (1 + \varepsilon(A_0, \gamma_0) \delta_1(A_0, \gamma_0)) A_0 \geq A_1,$$

we can take $t'_1 = \delta_1(A_0, \gamma_0)$ by Proposition 2.3.

Else if (2.13) is false, we use Proposition 2.3 again, substituing $A_{0,1}$ for A_0 . If the inequality (2.13) is true where A_0 is replaced by $A_{0,1}$, we can take $t'_1 = \delta_1(A_0, \gamma_0) + \delta_1(A_{0,1}, \gamma_0)$. In the case when the inequality is false, we continue these steps defining $A_{0,k+1} = (1 + \varepsilon(A_{0,k}, \gamma_0) \delta_1(A_{0,k}, \gamma_0)) A_{0,k}$, until the constant $A_{0,n}$ exceeds A_1 . On account of REMARK 2.4 we can stop this iteration in a finite step. So, at the time $t_0 + t'_1 = t_0 + \delta_1(A_0, \gamma_0) + \delta_1(A_{0,1}, \gamma_0) + \dots + \delta_1(A_{0,n}, \gamma_0)$ the estimate (2.12) holds.

By using Lemma 1.2, where we set $f_1(r) = f(r)$, $f_2(r) = 0$ and $u_0^1(x) = A^* \exp[-|x|^2/4\gamma^*]$, we have

Lemma 2.7. *Let A^* , γ^* be some positive constants with $A^* < 1$. Let the solution $u(t, x)$ of (1) be larger than the function $A^* \exp[-|x|^2/4\gamma^*]$ at some time $t^* > 0$. Then we have*

$$(2.14) \quad u(t + t^*, x) \geq A^* \left(1 + \frac{t}{\gamma^*}\right)^{-N/2} \exp[-|x|^2/4(\gamma^* + t)] \quad \text{for all } x \text{ and } t > 0.$$

Using this lemma we have

$$(2.15) \quad u(t_0 + t'_1 + t''_1, x) \geq A_0 \exp[-|x|^2/4(\gamma_0 + t''_1)] \quad \text{for all } x \text{ in } R^N,$$

where t''_1 is defined by $A_0 = \left(1 + \frac{t''_1}{\gamma_0}\right)^{-N/2} A_1$.

Now we put $t_1 = t_0 + t'_1 + t''_1$, $\gamma_1 = \gamma_0 + t''_1 = (A_1/A_0)^{2/N} \gamma_0$.

By the fact that γ_1 is larger than γ_0 , we can take the same process as the above argument, where t_0 and γ_0 are replaced by t_1 and γ_1 . Thus we have constants $t_2 > t_1$ and $\gamma_2 = (A_1/A_0)^{2/N} \gamma_1$ such that

$$(2.16) \quad u(t_2, x) \geq A_0 \exp[-|x|^2/4\gamma_2] \quad \text{for all } x \text{ in } R^N.$$

Denote the constant $(A_1/A_0)^{2n/N} \gamma_0$ by γ_n for $n = 0, 1, 2, \dots$. By repetition of these arguments we have, at some time t_n ,

$$(2.17) \quad u(t_n, x) \geq A_0 \exp[-|x|^2/4\gamma_n] \quad \text{for all } x \text{ in } R^N.$$

We chose the integer n sufficiently large so that

$$(2.18) \quad \gamma_n > \gamma,$$

$$(2.19) \quad (1 \wedge e^{1-(N\alpha/2)}) \alpha A_0^\alpha \Phi(A) \gamma_n > 1,$$

where A and γ are constants considered in the conclusion part of Theorem 3.

Because the function $\Phi(A)$ is continuous, we can find $B > A$, so that the inequality (2.19) remains true for B changed in place of A . By Lemma 2.6 we can take a constant $T_0 > 0$ such that we have at the time $t = T_0$

$$(2.20) \quad u(t, x) \geq A \exp[-|x|^2/4\gamma_n] \quad \text{for all } x \text{ in } R^N.$$

By the inequality $(1 \wedge e^{1-(N\alpha/2)}) \alpha A^\alpha \Phi(A) \gamma_n > 1$ and Proposition 2.5 we have the same estimate (2.20) for any time after T_0 . Because of the fact (2.18), this proves Theorem 3.

3. Proofs of Theorem 1 and 2. To prove Theorem 1, it is enough to prove it in the case of $N\alpha = 2$.

Proposition 3.1. *Let α be equal to $2/N$. For any nontrivial solution $u(t, x)$ of (1), we can find real constants A_0 and γ_0 , so that these constants satisfy (*) in Definition 2.2, and the solution $u(t, x)$ exceeds $A_0 \exp[-|x|^2/4\gamma_0]$ at some time $t = t_0$.*

Proof. Lemma 2.7 shows that, at any time $t > 0$, the solution $u(t, x)$ of (1) starting from a nontrivial initial data is positive for all x . Thus we can take a positive number ε , so that we have the estimate $u(1, x) > \varepsilon$ for $|x| \leq 1$. Using this lemma again, we get

$$(3.1) \quad u(t+1, x) \geq \int_{R^N} H(t, x-y) u(1, y) dy \geq \varepsilon \int_{|y| \leq 1} H(t, x-y) dy.$$

The last term of the above inequality is larger than $E(t)H(2t, x)$ where $E(t)$ denote $\varepsilon 2^{N/2} \omega_N e^{-1/2t}$ (ω_N is the volume of the unit ball in R^N). So, we can assume that the initial data is larger than $C \exp[-|x|^2/4\beta]$ for some positive constants C and β with $C < 1/2$. Now we define a function $v(t, x)$ by the integral equation

$$(3.2) \quad v(t, x) = \int_{R^N} H(t, x-y) v_0(y) dy + \int_0^t \int_{R^N} H(t-\tau, x-y) C_0 \left\{ \int_{R^N} H(\tau, y-z) v_0(z) dz \right\}^{1+\alpha} dy d\tau,$$

where $v_0(y)$ denotes the function $C \exp[-|x|^2/4\beta]$.

By some calculation we have

$$(3.3) \quad v(t, x) \leq C \left(1 + \frac{t}{\beta}\right)^{-N/2} \left(1 + KC^\alpha \log \left(1 + \frac{t}{\beta}\right)\right), \quad K = C_0 \beta (4\pi)^{N/2},$$

$$(3.4) \quad v(t, x) \geq A(t) \exp[-|x|^2/4\beta(t)],$$

where $A(t) = C_0 C^{1+\alpha} \beta^{1+(N/2)} \left(t + \frac{\beta}{1+\alpha}\right)^{-N/2} \log \left(1 + \frac{t}{\beta}\right)$ and $\gamma(t) = (\beta+t)/(1+\alpha)$.

By changing the constant C for the smaller one if necessary, we may assume that C^α is less than $N/2K$. From this assumption and the inequality (3.3) we have

$$(3.5) \quad v(t, x) \leq C < 1/2 \quad \text{for all } t \geq 0 \text{ and all } x \text{ in } R^N.$$

Operating $\partial_t - \Delta$ to $v(t, x)$ of (3.2), we have

$$(3.6) \quad \partial_t v - \Delta v = C_0 \left\{ \int_{R^N} H(t, x-y) v_0(y) dy \right\}^{1+\alpha}.$$

The right hand side of (3.6) is less than $C_0 \{v(t, x)\}^{1+\alpha}$. So the condition (iii) for $f(r)$ and the inequality (3.5) lead us to

$$(3.7) \quad \partial_t v(t, x) \leq \Delta v(t, x) + f(v(t, x)) \quad \text{for } t > 0, x \in R^N.$$

On the other hand we have $u(0, x) \geq v(0, x)$. This shows that the solution $u(t, x)$ is larger than $v(t, x)$ on $[0, \infty[\times R^N$.

Chose a constant t_0 large enough so that the quantity

$$\begin{aligned} \alpha \Phi(A(t_0)) A(t_0)^\alpha \gamma(t_0) &= \alpha C_0^{1+\alpha} C^{\alpha(1+\alpha)} \beta^{1+\alpha} \left(t_0 + \frac{\beta}{1+\alpha}\right)^{-1} \\ &\quad \left(\log \left(1 + \frac{t_0}{\beta}\right)\right)^\alpha (\beta+t_0)/(1+\alpha) \end{aligned}$$

is larger than 1 and denote $A(t_0), \gamma(t_0)$ by A_0, γ_0 .

Thus we have $u(t_0, x) \geq v(t_0, x) \geq A_0 \exp[-|x|^2/4\gamma_0]$ where $[A_0, \gamma_0]$ satisfies (*).

Theorem 1 is an immediate consequence of Proposition 3.1 and Theorem 3.

Proof of Theorem 2. The existence of the constant a_1 is obvious because we can take $a_1 = (1/2) (4\pi\gamma)^{-N/2}$ on account of the fact that γ is larger than $2^\alpha C_0^{-1} \alpha^{-1}$. Taking this a_1 and denoting $a_1 (4\pi\gamma)^{N/2} = 1/2$ by A , we have (*) with this $[A, \gamma]$. Theorem 3 shows that this a_1 has the property in Theorem 2. The existence of the constant a_0 will be proved by using the next proposition, which was proved in [1].

Proposition 3.2. (Theorem 2 in [1]). *Let the function $f(r)$ of nonlinear*

term satisfy (iv) in addition to (i), (ii) and (iii) and let the constant α be larger than $2/N$. Take any positive number γ . Then there exists a positive number a_0 with the following property; if the initial data $u_0(x)$ is less than the function $a_0 H(\gamma, x)$, then the solution of (1) is subject to

$$(3.8) \quad 0 \leq u(t, x) \leq MH(t + \gamma, x), \quad t > 0, \quad x \in R_N,$$

for some positive constant M .

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