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## ON THE LIMIT STATE OF SOLUTIONS OF SOME SEMILINEAR DIFFUSION EQUATIONS

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**Introduction.** This paper is concerned with the behavior of solutions of the following Cauchy problem for the semilinear diffusion equation

$$(1) \quad \begin{aligned} \partial_t u &= \Delta u + f(u), \quad u = u(t, x), \quad t > 0, \quad x \in R^N, \\ u(0, x) &= u_0(x), \quad x \in R^N, \end{aligned}$$

where  $\partial_t$  and  $\Delta$  denote  $\partial/\partial t$  and  $\sum_{j=1}^N \partial^2/\partial x_j^2$ .

The type of phenomena that occur to solutions depends of course on the type of the nonlinear term  $f(u)$  in the equation. For the function of type  $u^{1+\alpha}$ , H. Fujita [1] dealt with the problem of blowing-up of solutions in a finite time (see also H. Fujita [2], the present author [3] and S. Sugitani [4]). On the other hand A.M.Kolmogorov-I.G.Petrovsky-N.S.Piscounov [5] and Y.I. Kaneli [6] investigated the behavior of solution  $u(t, x)$  of (1) as  $t \rightarrow \infty$  in the case when the function  $f(u)$  is  $u(1-u)$  as the typical instance.

Here we deal with the problem (1) for the function of the type  $u^{1+\alpha}(1-u)$  and investigate the limit state of the solution  $u(t, x)$  as  $t \rightarrow \infty$ . Results may be roughly spoken as follows. Whether all nontrivial solutions tend to 1 or not depends on the degree  $\alpha$  of the increase of  $f$  near 0. In the latter case solutions tend to 0 or 1 according to the magnitude of the initial value. This seems parallel to results in [1].

Precisely, our results are the followings. We assume that the function  $f(r)$  satisfies next conditions (i), (ii) and (iii).

- (i)  $f(r)$  is of class  $C^1$  on the closed interval  $[0, 1]$ .
- (ii)  $f(r) > 0$  on the open interval  $(0, 1)$  and  $f(0) = f(1) = 0$ .
- (iii) There exist positive constants  $C_0$  and  $\alpha$ , with which we have  $f(r) \geq C_0 r^{1+\alpha}$  for  $0 \leq r \leq 1/2$ .

Further, in Theorem 2, the assumption (iv) should be added.

- (iv)  $f(r) \leq C_1 r^{1+\alpha}$  on  $[0, 1]$  for some constant  $C_1 > 0$ .

For the initial data  $u_0(x)$  we only consider such functions that are compatible to  $f(r)$ , i.e.,  $0 \leq u_0(x) \leq 1$ , and that are continuous only for the sake of simplicity.

Then we have the following theorems.

**Theorem 1\***<sup>\*)</sup>. *Let the constant  $\alpha$  satisfy  $N\alpha \leq 2$ . Assume that the function  $f(r)$  satisfies conditions (i), (ii) and (iii). Then, for any nontrivial initial data  $u_0$ , i.e.,  $u_0(x) \not\equiv 0$ , we have  $\lim_{t \rightarrow \infty} u(t, x) = 1$ , in which the convergence is uniform on any bounded set of  $x$  in  $R^N$ .*

**Theorem 2.** *Let the constant  $\alpha$  satisfy  $N\alpha > 2$ . Assume that the function  $f$  satisfies conditions (i), (ii), (iii) and (iv). Take any real  $\gamma$  larger than  $2^\alpha C_0^{-1} \alpha^{-1}$ . Then there exist positive constants  $a_0$  and  $a_1$  having following properties:*

- 1) *If  $u_0(x)$  is less than the function  $a_0 H(\gamma, x)$  all over  $R^N$  then the solution  $u(t, x)$  starting from  $u_0$  goes to 0 uniformly on  $R^N$ .*
- 2) *If  $u_0(x)$  is larger than  $a_1 H(\gamma, x)$  all over  $R^N$ . Then the solution goes to 1 uniformly on any bounded set of  $R^N$ . Here the function  $H(t, x)$  denotes the fundamental solution  $(2\pi t)^{-N/2} \exp[-|x|^2/4t]$  of the heat equation.*

These theorems will be proved in 3. 1. is devoted to preliminary lemmas used in the following parts. In 2. we shall prove a key theorem (Theorem 3).

**1. Preliminary lemmas.** We consider the Cauchy problem for the quasilinear diffusion equation

$$(A) \quad \partial_t u = \Delta u + f(t, x, u), \quad u(0, x) = u_0(x).$$

Here, we assume that the function  $f(t, x, r)$  is continuous in  $(t, x, r)$  and Lipschitz continuous in  $r$  (the Lipschitz constant is taken uniformly in  $(t, x, r)$ ).

**DEFINITION 1.1.** For the bounded continuous initial data  $u_0(x)$  the function  $u(t, x)$  is called the solution of problem (A) in  $[0, T] \times R^N$  if it satisfies following conditions a), b) and c) ([1]).

- a) For any  $T' < T$   $u(t, x)$  is the bounded continuous function of  $(t, x)$  on  $[0, T'] \times R^N$ .
- b) Initial condition in (A) is satisfied in the usual sense.
- c) The differential equation in (A) is satisfied in the sense of distribution in  $(0, T) \times R^N$ .

In proving our results, we apply the comparison theorem in the next form. For two functions  $f_1(t, x, r)$  and  $f_2(t, x, r)$  which satisfy the same conditions as above, we consider two Cauchy problems

$$(A_1) \quad \partial_t u^1 = \Delta u^1 + f_1(t, x, u^1), \quad u^1(0, x) = u_0^1(x),$$

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\*) Recently, T. Shirao, H. Tanaka and K. Kobayashi have reported to us that they got some generalized results based on our Theorems 1 and 2.

$$(A_2) \quad \partial_t u^2 = \Delta u^2 + f_2(t, x, u^2), \quad u^2(0, x) = u_0^2(x).$$

We denote by  $u^k(t, x)$  the solution of  $(A_k)$  in  $[0, T^k) \times R^N$  for  $k=1, 2$ .

**Lemma 1.2.** (see [5]). *If we have  $u_0^1(x) \leq u_0^2(x)$  for all  $x \in R^N$  and  $f_1(t, x, r) \leq f_2(t, x, r)$  for all  $(t, x, r)$ , we have  $u^1(t, x) \leq u^2(t, x)$  on  $[0, T') \times R^N$  ( $T' \leq T^1, T^2$ ).*

By using the Previous lemma we can show that under the assumptions (i) and (ii) the Cauchy problem (1) admits the unique solution  $u(t, x)$  in  $[0, \infty) \times R^N$  for the compatible initial data, and this  $u(t, x)$  satisfies  $0 \leq u(t, x) \leq 1$ .

In the remaining part of this paper we assume that the function  $f$  satisfies conditions (i), (ii) and (iii).

**Lemma 1.3.** *For an arbitrary couple of real numbers  $(A, B)$  satisfying  $0 < A < B < 1$ , we can find such a positive real  $\delta_0 = \delta_0(A, B)$  that has following properties:*

(P1) *For two couples  $(A, B)$  and  $(A', B')$  satisfying  $0 < A' \leq A < B \leq B' < 1$  we have  $\delta_0(A', B') \geq \delta_0(A, B)$ .*

(P2) *If the initial data  $u_0(x)$  is less than  $A$  all over  $R^N$ , then the solution  $u(t, x)$  of (1) is less than  $B$  on  $[0, \delta_0] \times R^N$ .*

Proof of Lemma 1.3. Let  $\varphi(t)$  be the solution of the ordinary differential equation  $d\varphi(t)/dt = f(\varphi(t))$ ,  $\varphi(0) = A$ . We define the constant  $\delta_0(A, B)$  by  $\varphi(\delta_0) = B$ . Then  $\varphi(t)$  is less than  $B$  for  $0 \leq t \leq \delta_0$ . From Lemma 1.2, where we set  $f_1 = f_2 = f$  and  $u_0^1(x) \equiv A$ ,  $u_0^2(x) = u_0(x)$ , we have  $u(t, x) \leq \varphi(t) \leq B$  for  $0 \leq t \leq \delta_0(A, B)$ .

DEFINITION 1.4. We define the function  $\Phi(r)$  by

$$\Phi(r) = \inf \{C_0 \wedge (f(s)/s^{1+\alpha}); 0 < s \leq r\}.$$

Here,  $a \wedge b$  denotes  $\min \{a, b\}$  for any real couple  $\{a, b\}$  and  $C_0$  is the constant of the condition (iii).

Then we have

**Lemma 1.5.** 1)  $\Phi(r)$  is continuous, nonnegative and non-increasing on the closed interval  $[0, 1]$ .

2)  $\Phi(1) = 0$  and  $\Phi(r)$  goes to 0 if and only if  $r$  tends to 1.

Simple calculation leads us to

**Lemma 1.6.** 1) Let  $A, \gamma, C$  and  $h$  be positive constants satisfying

$$(1.1) \quad (1 \wedge e^{1-(N\alpha/2)}) \alpha C A^\alpha \gamma \geq e^h.$$

Then we have

$$(1.2) \quad \frac{|x|^2}{4\gamma^2} - \frac{N}{2\gamma} + CA^\alpha \exp[-\alpha|x|^2/4\gamma] \geq h/\alpha\gamma \quad \text{for all } x \in R^N.$$

$$(2) \quad 1 - \sigma r \leq (1+r)^{-\sigma} \leq 1 - \sigma r/3 \quad \text{for } \sigma > 0, 0 \leq r \leq (\sigma+1)^{-1}.$$

**2. Key theorem.** Before going to Theorems 1 and 2 we shall show a Key theorem.

DEFINITION 2.1. A continuous function  $u(x)$  of  $x$  is said to be of class  $G[A, \gamma]$  for some positive constants  $A$  and  $\gamma$ , if it satisfies

$$(2.1) \quad 1 \geq u(x) \geq A \exp[-|x|^2/4\gamma] \quad \text{for all } x \text{ in } R^N.$$

DEFINITION 2.2. A couple of real constants  $[A, \gamma]$  is said to satisfy the condition (\*) if the next inequalities are valid:

$$(*) \quad 1 > A > 0, \quad \gamma > 0, \quad (1 \wedge e^{1-(N\alpha/2)})\alpha A^\alpha \Phi(A)\gamma > 1.$$

**Theorem 3.** *Let the initial data  $u(x)$  be of class  $G[A_0, \gamma_0]$  for some couple  $[A_0, \gamma_0]$  satisfying (\*). Then, for any constants  $A, \gamma$  with  $0 < A < 1, \gamma > 0$ , we can find  $T_0 = T_0(A, \gamma; A_0, \gamma_0) > 0$ , so that the solution  $u(t, x)$  of (1) exceeds the function  $A \exp[-|x|^2/4\gamma]$  at any time  $t \geq T_0$ .*

The proof of this theorem will be given in the last part of this paragraph. First we show the following proposition.

**Proposition 2.3.** *Let  $[A, \gamma]$  satisfy (\*). If  $u(t_0, x)$  is of class  $G[A, \gamma]$  at some time  $t_0 \geq 0$ . Then there exist positive constants  $\delta_1 = \delta_1(A, \gamma)$  and  $\varepsilon = \varepsilon(A, \gamma)$  such that*

$$(2.2) \quad u(t_0 + t, x) \geq (1 + \varepsilon t) A \exp[-|x|^2/4\gamma] \quad \text{for } 0 \leq t \leq \delta_1, x \in R^N.$$

Proof. As the equation in (1) is invariant under the translation of  $t$ , we can put  $t_0 = 0$ . Let  $u^*(t, x)$  denote the solution of the Cauchy problem

$$(1^*) \quad \partial_t u^* = \Delta u^* + f(u^*), \quad u^*(0, x) = u_0^*(x) = A \exp[-|x|^2/4\gamma].$$

By Lemma 1.2. we have  $u(t, x) \geq u^*(t, x)$ . We shall prove (2.2) for  $u^*(t, x)$ . We define the constant  $h > 0$  by the relation  $e^{2h} = (1 \wedge e^{1-(N\alpha/2)})\alpha\Phi(A)A^\alpha\gamma$  and define  $A'$  by  $e^h = (1 \wedge e^{1-(N\alpha/2)})\alpha\Phi(A')A'^\alpha\gamma$  ( $A < A' < 1$ ). The definitions of the function  $\Phi$  and the constant  $\delta_0 = \delta_0(A, A')$  lead us to

$$(2.3) \quad f(u^*(t, x)) \geq \Phi(A')u^*(t, x)^{1+\alpha}, \quad 0 \leq t \leq \delta_0, \quad x \in R^N.$$

Let  $u^{**}(t, x)$  denote the solution of the problem

$$(1^{**}) \quad \partial_t u^{**} = \Delta u^{**} + \Phi(A')u^{**1+\alpha}, \quad u^{**}(0, x) = u_0^{**}(x) = u_0^*(x).$$

Again by Lemma 1.2, we have  $u^*(t, x) \geq u^{**}(t, x)$  for  $0 \leq t \leq \delta_0$ . The Cauchy

problem (1\*\*) is equivalent to the next problem (2\*\*) in the integral form.

$$(2^{**}) \quad u^{**}(t, x) = \int_{R^N} H(t, x-y) u_0^*(y) dy + \\ + \int_0^t \int_{R^N} H(t-\tau, x-y) \Phi(A') \{u^{**}(\tau, y)\}^{1+\alpha} dy d\tau.$$

From this equation we have

$$(2.4) \quad u^{**}(\tau, y) \geq \int_{R^N} H(\tau, y-z) u_0^*(z) dz = \\ = A \left(1 + \frac{\tau}{\gamma}\right)^{-N/2} \exp[-|x|^2/(\tau+\gamma)].$$

Substituting (2.4) for the second term of the right hand side of (2\*\*), we have

$$(2.5) \quad \int_0^t \int_{R^N} H(t-\tau, x-y) \Phi(A') \{u^{**}(\tau, y)\}^{1+\alpha} dy d\tau \\ \geq \Phi(A') A^{1+\alpha} \left(1 + \frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\right)^{-N/2} \left(1 + \frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \\ \exp\left[-|x|^2/4\left(\frac{\gamma+\tau}{1+\alpha} + t-\tau\right)\right] \\ \geq \Phi(A') A^{1+\alpha} \left(1 + \frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \exp[-(1+\alpha)|x|^2/4(\gamma+\tau)] + \\ + \Phi(A') A^{1+\alpha} \left\{ \left(1 + \frac{(t-\tau)(1+\alpha)}{\tau+\gamma}\right)^{-N/2} - 1 \right\} \left(1 + \frac{\tau}{\gamma}\right)^{-(1+\alpha)N/2} \\ \exp[-(1+\alpha)|x|^2/4(\gamma+t)].$$

Further, by the simple calculation, we have

$$(2.6) \quad \int_{R^N} H(t, x-y) u_0^*(y) dy = u_0^*(x) + \int_0^t \frac{\partial}{\partial \tau} \int_{R^N} H(\tau, x-y) u_0^*(y) dy d\tau \\ = u_0^*(x) + \int_0^t A \left(1 + \frac{\tau}{\gamma}\right)^{-N/2} \left\{ \frac{|x|^2}{4(\gamma+\tau)^2} - \frac{N}{2(\gamma+\tau)} \right\} \\ \exp[-\alpha|x|^2/4(\gamma+\tau)] d\tau.$$

Substituting (2.5) and (2.6) in to the right hand side of (2\*\*), we get

$$(2.7) \quad u^{**}(t, x) \geq u_0^*(x) + \int_0^t \left\{ \frac{|x|^2}{4(\gamma+\tau)^2} - \frac{N}{2(\gamma+\tau)} \right. \\ \left. + \Phi(A') A^\alpha \exp[-\alpha|x|^2/4(\gamma+\tau)] \right\} A G(x, \tau; \gamma) d\tau +$$

$$\begin{aligned}
 & + \int_0^t \Phi(A')A^\alpha \left\{ \left( 1 + \frac{t(1+\alpha)}{\gamma+\tau} \right)^{-N/2} \left( 1 + \frac{\tau}{\gamma} \right)^{-N\alpha/2} - 1 \right\} \\
 & \quad \exp [-\alpha |x|^2/4(\gamma+\tau)] AG(x, \tau; \gamma) d\tau,
 \end{aligned}$$

where  $G(x, \tau; \gamma)$  denotes  $\left( 1 + \frac{\tau}{\gamma} \right)^{-N/2} \exp [-|x|^2/4(\tau+\gamma)] = (4\pi\gamma)^{N/2} H(x, \gamma+\tau)$ .

Applying 1) of Lemma 1.6 to the second term and 2) to the third term, we get

$$\begin{aligned}
 (2.8) \quad u^{**}(t, x) & \geq u_0^{**}(x) + \int_0^t \left\{ \frac{h}{\alpha(\gamma+\tau)} - \frac{\Phi(A')A^\alpha N(1+2\alpha)}{\gamma+\tau} \right\} AG(x, \tau; \gamma) d\tau \\
 & \quad \text{for } t < \gamma(1+N)^{-1}(1+\alpha)^{-1} = d_1(\gamma).
 \end{aligned}$$

Now we set  $d_2(h, \gamma) = h / \{2\Phi(A')A^\alpha N(1+2\alpha)\} = he^{-h}(1 \wedge e^{1-(N\alpha/2)})\gamma / N(1+2\alpha)$ .

For we have  $AG(x, \tau; \gamma) \geq \left( 1 + \frac{\tau}{\gamma} \right)^{-N/2} u_0^{**}(x)$ , we get

$$(2.9) \quad u^{**}(t, x) \geq \left( 1 + \frac{h}{4\alpha\gamma 2^{N/2}} t \right) u_0^{**}(x) \quad \text{for } t \leq d_1(\gamma) \wedge d_2(h, \gamma).$$

Thus we have (2.2) for the constants  $\delta_1 = \delta_0(A, A') \wedge d_1 \wedge d_2$  and  $\varepsilon = h / (4\alpha\gamma 2^{N/2})$ .

Before finishing this proposition we make a remark on the constants  $\delta_1(A, \gamma)$  and  $\varepsilon(A, \gamma)$ .

REMARK 2.4. Let constants  $B_0, B_1, B_2$ , satisfy  $0 < B_0 < B_1 < B_2 < 1$  and  $(1 \wedge e^{1-(N\alpha/2)})\alpha\Phi(B_2)B_0^\alpha\gamma > 1$ . Take a constant  $A$  between  $B_0$  and  $B_1$ . Then we have following uniform estimates with respect to  $\delta_1(A, \gamma)$  and  $\varepsilon(A, \gamma)$ .

$$\begin{aligned}
 (2.10) \quad \delta_1(A, \gamma) & \geq \bar{\delta}_1(B_0, B_1, B_2, \gamma) = \delta_0(B_1, B_2) \wedge d_1(\gamma) \wedge d_2(h_1, \gamma), \\
 \varepsilon(A, \gamma) & \geq \bar{\varepsilon}(B_0, B_2, \gamma) = h_1 / (4\alpha\gamma 2^{N/2}),
 \end{aligned}$$

where the constant  $h_1$  denotes  $1 \wedge (1/2) \log \{(1 \wedge e^{1-(N\alpha/2)})\alpha\Phi(B_2)B_0^\alpha\gamma\}$ .

**Proposition 2.5.** *Let  $[A_0, \gamma_0]$  satisfy the condition (\*). If  $u(t_0, x)$  is of class  $G[A_0, \gamma_0]$  at some time  $t_0 \geq 0$ , the solution  $u(t, x)$  remains in the class  $G[A_0, \gamma_0]$  at all time after  $t_0$ .*

Proof. By Proposition 2.3 we have (2.2). Thus, at any time  $t$  between  $t_0$  and  $t_0 + \delta_1(A_0, \gamma_0)$ , we have

$$(2.11) \quad u(t, x) \geq A_0 \exp [-|x|^2/4\gamma_0] \quad \text{for all } x \text{ in } R^N.$$

By using this argument at the time  $t_0 + \delta_1(A_0, \gamma_0)$ , we get (2.11) for  $t \in [t_0 + \delta_1(A_0, \gamma_0), t_0 + 2\delta_1(A_0, \gamma_0)]$ . The repetition of this argument leads us to the same estimate (2.11) at any time  $t$  in  $[t_0 + n\delta_1(A_0, \gamma_0), t_0 + (n+1)\delta_1(A_0, \gamma_0)]$  for  $n=0, 1, 2, \dots$ . This proves our proposition.

Now we are going to prove Theorem 3. Chose constants  $A_0', h_0$  just in the

same way as  $A'$ ,  $h$  in the proof of Proposition 2.3. Take a real number  $A_1 < A_0'$  sufficiently close to  $A_0'$ , and put  $t_0 = 0$ . Then we get

**Lemma 2.6.** *Under the assumptions in Theorem 3 we have at some time  $t_0 + t_1''$*

$$(2.12) \quad u(t_0 + t_1', x) \geq A_1 \exp[-|x|^2/4\gamma_0] \quad \text{for all } x \text{ in } R^N.$$

Proof. When we have

$$(2.13) \quad A_{0,1} = (1 + \varepsilon(A_0, \gamma_0) \delta_1(A_0, \gamma_0)) A_0 \geq A_1,$$

we can take  $t_1' = \delta_1(A_0, \gamma_0)$  by Proposition 2.3.

Else if (2.13) is false, we use Proposition 2.3 again, substituing  $A_{0,1}$  for  $A_0$ . If the inequality (2.13) is true where  $A_0$  is replaced by  $A_{0,1}$ , we can take  $t_1' = \delta_1(A_0, \gamma_0) + \delta_1(A_{0,1}, \gamma_0)$ . In the case when the inequality is false, we continue these steps defining  $A_{0,k+1} = (1 + \varepsilon(A_{0,k}, \gamma_0) \delta_1(A_{0,k}, \gamma_0)) A_{0,k}$ , until the constant  $A_{0,n}$  exceeds  $A_1$ . On account of REMARK 2.4 we can stop this iteration in a finite step. So, at the time  $t_0 + t_1' = t_0 + \delta_1(A_0, \gamma_0) + \delta_1(A_{0,1}, \gamma_0) + \dots + \delta_1(A_{0,n}, \gamma_0)$  the estimate (2.12) holds.

By using Lemma 1.2, where we set  $f_1(r) = f(r)$ ,  $f_2(r) = 0$  and  $u_0^1(x) = A^* \exp[-|x|^2/4\gamma^*]$ , we have

**Lemma 2.7.** *Let  $A^*$ ,  $\gamma^*$  be some positive constants with  $A^* < 1$ . Let the solution  $u(t, x)$  of (1) be larger than the function  $A^* \exp[-|x|^2/4\gamma^*]$  at some time  $t^* > 0$ . Then we have*

$$(2.14) \quad u(t + t^*, x) \geq A^* \left(1 + \frac{t}{\gamma^*}\right)^{-N/2} \exp[-|x|^2/4(\gamma^* + t)] \quad \text{for all } x \text{ and } t > 0.$$

Using this lemma we have

$$(2.15) \quad u(t_0 + t_1' + t_1'', x) \geq A_0 \exp[-|x|^2/4(\gamma_0 + t_1'')] \quad \text{for all } x \text{ in } R^N,$$

where  $t_1''$  is defined by  $A_0 = \left(1 + \frac{t_1''}{\gamma_0}\right)^{-N/2} A_1$ .

Now we put  $t_1 = t_0 + t_1' + t_1''$ ,  $\gamma_1 = \gamma_0 + t_1'' = (A_1/A_0)^{2/N} \gamma_0$ .

By the fact that  $\gamma_1$  is larger than  $\gamma_0$ , we can take the same process as the above argument, where  $t_0$  and  $\gamma_0$  are replaced by  $t_1$  and  $\gamma_1$ . Thus we have constants  $t_2 > t_1$  and  $\gamma_2 = (A_1/A_0)^{2/N} \gamma_1$  such that

$$(2.16) \quad u(t_2, x) \geq A_0 \exp[-|x|^2/4\gamma_2] \quad \text{for all } x \text{ in } R^N.$$

Denote the constant  $(A_1/A_0)^{2n/N} \gamma_0$  by  $\gamma_n$  for  $n = 0, 1, 2, \dots$ . By repetition of these arguments we have, at some time  $t_n$ ,



$$(2.17) \quad u(t_n, x) \geq A_0 \exp[-|x|^2/4\gamma_n] \quad \text{for all } x \text{ in } R^N.$$

We chose the integer  $n$  sufficiently large so that

$$(2.18) \quad \gamma_n > \gamma,$$

$$(2.19) \quad (1 \wedge e^{1-(N\alpha/2)}) \alpha A_0^\alpha \Phi(A) \gamma_n > 1,$$

where  $A$  and  $\gamma$  are constants considered in the conclusion part of Theorem 3.

Because the function  $\Phi(A)$  is continuous, we can find  $B > A$ , so that the inequality (2.19) remains true for  $B$  changed in place of  $A$ . By Lemma 2.6 we can take a constant  $T_0 > 0$  such that we have at the time  $t = T_0$

$$(2.20) \quad u(t, x) \geq A \exp[-|x|^2/4\gamma_n] \quad \text{for all } x \text{ in } R^N.$$

By the inequality  $(1 \wedge e^{1-(N\alpha/2)}) \alpha A^\alpha \Phi(A) \gamma_n > 1$  and Proposition 2.5 we have the same estimate (2.20) for any time after  $T_0$ . Because of the fact (2.18), this proves Theorem 3.

**3. Proofs of Theorem 1 and 2.** To prove Theorem 1, it is enough to prove it in the case of  $N\alpha = 2$ .

**Proposition 3.1.** *Let  $\alpha$  be equal to  $2/N$ . For any nontrivial solution  $u(t, x)$  of (1), we can find real constants  $A_0$  and  $\gamma_0$ , so that these constants satisfy (\*) in Definition 2.2, and the solution  $u(t, x)$  exceeds  $A_0 \exp[-|x|^2/4\gamma_0]$  at some time  $t = t_0$ .*

**Proof.** Lemma 2.7 shows that, at any time  $t > 0$ , the solution  $u(t, x)$  of (1) starting from a nontrivial initial data is positive for all  $x$ . Thus we can take a positive number  $\varepsilon$ , so that we have the estimate  $u(1, x) > \varepsilon$  for  $|x| \leq 1$ . Using this lemma again, we get

$$(3.1) \quad u(t+1, x) \geq \int_{R^N} H(t, x-y) u(1, y) dy \geq \varepsilon \int_{|y| \leq 1} H(t, x-y) dy.$$

The last term of the above inequality is larger than  $E(t)H(2t, x)$  where  $E(t)$  denote  $\varepsilon 2^{N/2} \omega_N e^{-1/2t}$  ( $\omega_N$  is the volume of the unit ball in  $R^N$ ). So, we can assume that the initial data is larger than  $C \exp[-|x|^2/4\beta]$  for some positive constants  $C$  and  $\beta$  with  $C < 1/2$ . Now we define a function  $v(t, x)$  by the integral equation

$$(3.2) \quad v(t, x) = \int_{R^N} H(t, x-y) v_0(y) dy + \int_0^t \int_{R^N} H(t-\tau, x-y) C_0 \left\{ \int_{R^N} H(\tau, y-z) v_0(z) dz \right\}^{1+\alpha} dy d\tau,$$

where  $v_0(y)$  denotes the function  $C \exp[-|x|^2/4\beta]$ .

By some calculation we have

$$(3.3) \quad v(t, x) \leq C \left(1 + \frac{t}{\beta}\right)^{-N/2} \left(1 + KC^\alpha \log \left(1 + \frac{t}{\beta}\right)\right), \quad K = C_0 \beta (4\pi)^{N/2},$$

$$(3.4) \quad v(t, x) \geq A(t) \exp \left[-|x|^2/4\beta(t)\right],$$

where  $A(t) = C_0 C^{1+\alpha} \beta^{1+(N/2)} \left(t + \frac{\beta}{1+\alpha}\right)^{-N/2} \log \left(1 + \frac{t}{\beta}\right)$  and  $\gamma(t) = (\beta+t)/(1+\alpha)$ .

By changing the constant  $C$  for the smaller one if necessary, we may assume that  $C^\alpha$  is less than  $N/2K$ . From this assumption and the inequality (3.3) we have

$$(3.5) \quad v(t, x) \leq C < 1/2 \quad \text{for all } t \geq 0 \text{ and all } x \text{ in } R^N.$$

Operating  $\partial_t - \Delta$  to  $v(t, x)$  of (3.2), we have

$$(3.6) \quad \partial_t v - \Delta v = C_0 \left\{ \int_{R^N} H(t, x-y) v_0(y) dy \right\}^{1+\alpha}.$$

The right hand side of (3.6) is less than  $C_0 \{v(t, x)\}^{1+\alpha}$ . So the condition (iii) for  $f(r)$  and the inequality (3.5) lead us to

$$(3.7) \quad \partial_t v(t, x) \leq \Delta v(t, x) + f(v(t, x)) \quad \text{for } t > 0, x \in R^N.$$

On the other hand we have  $u(0, x) \geq v(0, x)$ . This shows that the solution  $u(t, x)$  is larger than  $v(t, x)$  on  $[0, \infty[ \times R^N$ .

Chose a constant  $t_0$  large enough so that the quantity

$$\alpha \Phi(A(t_0)) A(t_0)^\alpha \gamma(t_0) = \alpha C_0^{1+\alpha} C^{\alpha(1+\alpha)} \beta^{1+\alpha} \left(t_0 + \frac{\beta}{1+\alpha}\right)^{-1} \left(\log \left(1 + \frac{t_0}{\beta}\right)\right)^\alpha (\beta+t_0)/(1+\alpha)$$

is larger than 1 and denote  $A(t_0), \gamma(t_0)$  by  $A_0, \gamma_0$ .

Thus we have  $u(t_0, x) \geq v(t_0, x) \geq A_0 \exp \left[-|x|^2/4\gamma_0\right]$  where  $[A_0, \gamma_0]$  satisfies (\*).

Theorem 1 is an immediate consequence of Proposition 3.1 and Theorem 3.

**Proof of Theorem 2.** The existence of the constant  $a_1$  is obvious because we can take  $a_1 = (1/2) (4\pi\gamma)^{-N/2}$  on account of the fact that  $\gamma$  is larger than  $2^\alpha C_0^{-1} \alpha^{-1}$ . Taking this  $a_1$  and denoting  $a_1 (4\pi\gamma)^{N/2} = 1/2$  by  $A$ , we have (\*) with this  $[A, \gamma]$ . Theorem 3 shows that this  $a_1$  has the property in Theorem 2. The existence of the constant  $a_0$  will be proved by using the next proposition, which was proved in [1].

**Proposition 3.2.** (Theorem 2 in [1]). *Let the function  $f(r)$  of nonlinear*

term satisfy (iv) in addition to (i), (ii) and (iii) and let the constant  $\alpha$  be larger than  $2/N$ . Take any positive number  $\gamma$ . Then there exists a positive number  $a_0$  with the following property; if the initial data  $u_0(x)$  is less than the function  $a_0 H(\gamma, x)$ , then the solution of (1) is subject to

$$(3.8) \quad 0 \leq u(t, x) \leq MH(t + \gamma, x), \quad t > 0, \quad x \in R_N,$$

for some positive constant  $M$ .

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