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This paper will introduce a concept of a cobordism theory, called $\mathcal{H}$-cobordism, between 3-dimensional homology handles. The set of the types of distinguished homology orientable handles modulo $\mathcal{H}$-cobordism relation will form an abelian group $\Omega(S^1 \times S^2)$, called the $\mathcal{H}$-cobordism group of homology orientable handles. As a basic property of the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^2)$ the following commutative triangle will be established:

$$
\begin{array}{ccc}
C^1 & \xrightarrow{e} & \Omega(S^1 \times S^2) \\
\downarrow\phi & & \downarrow\psi \\
G_- & & \\
\end{array}
$$

Here, $C^1$ is the Fox-Milnor's 1-knot cobordism group (See Fox-Milnor [3].), $G_-$ is the Levine's integral matrix cobordism group (See Levine [9].), $e$ is a homomorphism and $\phi, \psi$ are epimorphisms. In particular the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^2)$ will have an infinite rank. Analogously the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^3)$ of homology non-orientable handles will be also constructed. We shall show that the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^3)$ is isomorphic to the direct sum of infinitely many copies of the cyclic group of order two. Furthermore, it will be shown that the assignment $\tau: m \rightarrow m'$ of the type $m$ of any distinguished homology non-orientable handle to the type $m'$ of its 2-fold orientation-cover (which is a distinguished homology orientable handle) induces a well-defined homomorphism $\tau^*: \Omega(S^1 \times S^3) \rightarrow T \subset \Omega(S^1 \times S^3)$ from $\Omega(S^1 \times S^3)$ to the subgroup $T$ of $\Omega(S^1 \times S^3)$ consisting of elements of order two. As one consequence $T$ will be infinitely generated.

Section 1 will construct the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^2)$ of homology orientable handles. In Section 2 we will discuss the properties of the invariants of $\Omega(S^1 \times S^2)$ and compare $\Omega(S^1 \times S^2)$ with Fox-Milnor's 1-knot cobordism group $C^1$ and with the Levine's integral matrix cobordism group $G_-$. Section 3 will concern the zero element and the order-two-elements of the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^2)$. It will be shown that the type $m$ of a distinguished homology orientable
handle $M(\alpha, \iota)$ represents the zero element of $\Omega(S^1 \times S^2)$ (that is, $m$ is null-$\mathcal{H}$-cobordant) if $M(\alpha, \iota)$ is embeddable to a homology 4-sphere. To consider the order-two-elements of $\Omega(S^1 \times S^2)$, we will introduce the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^2)$ of homology non-orientable handles and determine its group structure and discuss the homomorphism $\tau^*: \Omega(S^1 \times S^2) \to T_2 \subset \Omega(S^1 \times S^2)$ in this section.

Throughout this paper, spaces and maps will be considered from the piecewise linear point of view.

1. A construction of the $\mathcal{H}$-cobordism group $\Omega(S^1 \times S^2)$

A 3-dimensional homology orientable handle $M$ is a compact 3-manifold having the integral homology group of the orientable handle $S^1 \times S^2$: $H_*(M; \mathbb{Z}) \cong H_*(S^1 \times S^2; \mathbb{Z})$. A homology orientable handle $M$ is said to be distinguished if generators $\alpha \in H_1(M; \mathbb{Z}) (\cong \mathbb{Z})$ and $\iota \in H_3(M; \mathbb{Z}) (\cong \mathbb{Z})$ are specified. In that case the notation $M(\alpha, \iota)$ will be used. Two distinguished homology orientable handles $M(\alpha, \iota), M'(\alpha', \iota')$ are said to have the same type if there is a piecewise-linear homeomorphism $h: M(a, \iota) \cong M'(\alpha', \iota')$ which induces an isomorphism $h_*: H_*(M(\alpha, \iota); \mathbb{Z}) \cong H_*(M'(\alpha', \iota'); \mathbb{Z})$ with $h_*(\alpha) = \alpha'$ and $h_*(\iota) = \iota'$. The class of distinguished homology orientable handles having the same type as $M(\alpha, \iota)$ is called the type of $M(\alpha, \iota)$. The set of all types is denoted by $\mathbb{C}_+(S^1 \times S^2)$. Let $m$ be a type of $M(\alpha, \iota)$. By $-m$ we denote the type of $M(\alpha, -\iota)$. It is easily checked that the four distinguished handles $S^1 \times S^2(\alpha, \iota), S^1 \times S^2(\alpha, -\iota), S^1 \times S^2(-\alpha, \iota)$ and $S^1 \times S^2(-\alpha, -\iota)$ of the orientable handle $S^1 \times S^2$ have the same type. We denote this type by $0$.

**Definition 1.1.** Two types $m_1, m_2$ in $\mathbb{C}_+(S^1 \times S^2)$ are $\mathcal{H}$-cobordant and denoted by $m_1 \sim m_2$ if for some representatives $M_1(\alpha_1, \iota_1) \in m_1, M_2(\alpha_2, \iota_2) \in m_2$, there exists a pair $(W, \varphi)$ where $W$ is a compact connected oriented 4-manifold with $\partial W = M_1(\alpha_1, \iota_1) + M_2(\alpha_2, -\iota_2)$ (disjoint union) and $\varphi$ is a cohomology class in $H^4(W; \mathbb{Z})$ whose restrictions $\varphi | M_i(\alpha_i, \iota_i) \in H^4(M_i(\alpha_i, \iota_i); \mathbb{Z})$ are dual to $\alpha_i$ for $i=1, 2$, and such that the infinite cyclic cover $W_\varphi$ associated with $\varphi$ has a finitely generated rational homology group $H_*(W_\varphi; \mathbb{Q})$ [that is, for each $i$, $H_i(W_\varphi; \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$]. As usual the triad $(W, M_1(\alpha_1, \iota_1), M_2(\alpha_2, \iota_2))$ is called an $\mathcal{H}$-cobordism.

It is easily seen that $m \sim 0$ if and only if for some representative $M(\alpha, \iota) \in m$, there exists a pair $(W^+, \varphi)$ where $W^+$ is a compact connected oriented 4-manifold with $\partial W^+ = M(\alpha, \iota)$ and $\varphi \in H^4(W^+; \mathbb{Z})$ with $\varphi | M(\alpha, \iota) \in H^4(M(\alpha, \iota); \mathbb{Z})$ dual to $\alpha$, and such that the infinite cyclic cover $W^+_\varphi$ associated with $\varphi$ has a finitely generated rational homology group $H_*(W^+_\varphi; \mathbb{Q})$. In this case the notation $(W^+, M(\alpha, \iota), \varphi)$ may be adopted as an $\mathcal{H}$-cobordism.
Lemma 1.2. The \( \mathcal{H} \)-cobordism relation \( \sim \) is an equivalence relation.

Proof. The relation \( \sim \) is reflexive, since the infinite cyclic cover \( \tilde{M} \) of any homology orientable handle \( M \) has a finitely generated rational homology group \( H_q(\tilde{M};Q) \). [To see this, notice that for any \( i \), \( i \neq 2 \), \( H_q(\tilde{M};Q) \) is finitely generated (See for example Kawauchi [6, Proposition 3.4] for \( i=1 \)). The partial Poincaré duality theorem (See Kawauchi [6].) then asserts a duality \( H^q(\tilde{M};Q) \approx H_q(\tilde{M};Q) \). So \( H_q(\tilde{M};Q) \approx Q \).] The relation is obviously symmetric. Further the use of the Mayer-Vietoris sequence easily yields that the relation is transitive. This completes the proof.

Definition 1.3. The set \( \Omega(S^1 \times S^2) \) is defined to be the set of \( \mathbb{C}_*(S^1 \times S^2) \) modulo the \( \mathcal{H} \)-cobordism relation \( \sim \).

For any \( m \in \mathbb{C}_*(S^1 \times S^2) \) the symbol \( [m] \) denotes the element of \( \Omega(S^1 \times S^2) \) having \( m \) as the representative.

Now we shall introduce a sum operation, called a circle union, in the set \( \Omega(S^1 \times S^2) \).

Let \( m_0, m_1 \in \mathbb{C}_*(S^1 \times S^2) \) and \( M_i(\alpha_i, \iota_i) \in m_i, i=0, 1 \). Choose for each \( i \) a polygonal oriented simple closed curve \( \omega_i \) in \( M_i(\alpha_i, \iota_i) \) which represents the homology class \( \alpha_i \). Then for each \( i \) there exists a closed connected orientable surface \( F_i \) in \( M_i(\alpha_i, \iota_i) \) which intersects \( \omega_i \) in a single point. [To see this, first note that the identity map \( \omega_0 \subset \omega_1 \) can be extended to a piecewise-linear map \( f_i: M_i(\alpha_i, \iota_i) \to \omega_i \) by means of the elementary obstruction theory. Second, note that there is a point \( p_i \in \omega_i \) such that the preimage \( f_i^{-1}(p_i) \) is a closed (not necessarily connected) orientable surface. Now choose as \( F_i \) the component of \( f_i^{-1}(p_i) \) containing \( p_i \).]

Consider the solid torus \( S^1 \times B^2 \) and choose piecewise-linear embeddings

\[
\begin{align*}
h_0: & S^1 \times B^2 \times 0 \to M_0(\alpha_0, \iota_0) \\
h_1: & S^1 \times B^2 \times 1 \to M_1(\alpha_1, \iota_1)
\end{align*}
\]

such that

1. there exist points \( s \in S^1, b \in \text{Int } B^2 \) with \( h_0(s \times B^2 \times 0) \subset F_0, h_0(S^1 \times b \times 0) = \omega_0, h_0(s \times B^2 \times 1) \subset F_1, \) and \( h_0(S^1 \times b \times 1) = \omega_1, \)

2. both \( h_0 \) and \( h_1 \) are orientation-reversing with respect to the orientations of \( S^1 \times B^2 \times 0 \) and \( S^1 \times B^2 \times 1 \) induced from some orientation of \( S^1 \times B^2 \times [0, 1], \)

3. \( \omega_0 \) and \( \omega_1 \) are homologous in the adjunction space \( M_0(\alpha_0, \iota_0) \cup h_0 S^1 \times B^2 \times [0, 1] \cup h_0 M_1(\alpha_1, \iota_1). \)

Then the manifold \( M = M_0(\alpha_0, \iota_0) \cup h_0 S^1 \times B^2 \times [0, 1] \cup h_1 M_1(\alpha_1, \iota_1) \) is a homology handle. [Proof. Let \( i=0 \) or 1. Consider the manifold \( M'_i = M_i - h_i(S^1 \times \text{Int } B^2 \times i) \). Let \( b' \subset \partial B^2 \) and the simple closed curve \( \omega_i' = h_i(S^1 \times b' \times i) \subset \partial M'_i \) be oriented so that \( \omega_i' \) is homologous to \( \omega_i \) in \( M_i \). Let \( \eta_i = h_i(s \times \partial B^2 \times i) \subset \partial M'_i \) be oriented suitably. It is easily checked that \( \omega_i' \)
represents a generator of \( H_t(M; \mathbb{Z}) \) and \( \eta_t \) represents the zero element of \( H_t(M; \mathbb{Z}) \) (since \( \eta_t \) bounds an orientable surface \( F_t - h_t(s \times \text{Int} B^3 \times t) \) in \( M_t \)) and that \( \omega_t, \eta_t \) represent a basis for \( H_t(\partial M_t; \mathbb{Z}) \). Then from consideration of the Mayer-Vietoris sequence we obtain that \( H_t(M; \mathbb{Z}) \cong H_t(S^1 \times S^2; \mathbb{Z}) \) by Poincaré duality.

From construction it can be seen that the homology classes \( \alpha \in H_t(M; \mathbb{Z}) \) are called a circle union of \( M_0(\alpha_0, \omega_0) \) and \( M_1(\alpha_1, \omega_1) \) and denoted by \( M_0(\alpha_0, \omega_0) \cup M_1(\alpha_1, \omega_1) \). Also, the type of \( M(\alpha, \omega) \) is called a circle union of the types \( m_0 \) and \( m_1 \) and denoted by \( m_0 \cup m_1 \).

Clearly the type of \( M_0(\alpha_0, -\omega_0) \cup M_1(\alpha_1, -\omega_1) \) is \( -(m_0 \cup m_1) = (-m_0) \cup (-m_1) \).

**Definition 1.4.** The distinguished homology orientable handle \( M(\alpha, \omega) \) is called a circle union of \( M_0(\alpha_0, \omega_0) \) and \( M_1(\alpha_1, \omega_1) \) and denoted by \( M_0(\alpha_0, \omega_0) \cup M_1(\alpha_1, \omega_1) \). Also, the type of \( M(\alpha, \omega) \) is called a circle union of the types \( m_0 \) and \( m_1 \) and denoted by \( m_0 \cup m_1 \).

1.5. Remark to Definition 1.4. It should be remarked that the circle union \( m_0 \cup m_1 \) depends upon the choices of \( \omega_0, \omega_1, h_0, h_1 \). Consider for example a distinguished orientable handle \( S^1 \times S^2(\alpha, \omega) \). Let \( \omega \subset S^1 \times S^2(\alpha, \omega) \) be an oriented simple closed curve representing \( \alpha \) of geometrical index \( \lambda \) and \( T(\omega) \) be the regular neighborhood of \( \omega \) in \( S^1 \times S^2(\alpha, \omega) \). If the circle union \( S^1 \times S^2(\alpha, \omega) \cup S^1 \times S^2(\alpha, -\omega) \) is defined to be the double of \( cl(S^1 \times S^2(\alpha, \omega) \cup T(\omega)) \), then \( S^1 \times S^2(\alpha, \omega) \cup S^1 \times S^2(\alpha, -\omega) \) has the same type as \( S^1 \times S^2(\alpha, \omega) \).

On the other hand, consider for example an oriented simple closed curve \( \omega' \subset S^1 \times S^2(\alpha, \omega) \) representing \( \alpha \) of geometrical index \( \lambda \) and algebraic index \( \lambda' \). Let \( T(\omega') \) be the regular neighborhood of \( \omega' \) in \( S^1 \times S^2(\alpha, \omega) \).

---

* A simple closed curve \( \omega \) in \( S^1 \times S^2 \) has **geometric index** \( \lambda \), if \( \lambda \) is the least number of intersections that a curve ambient isotopic to \( \omega \) can have with \( S^1 \times S^2 \) and has **algebraic index** \( \lambda' \), if \( \lambda' \) is the unique integer such that \( \omega \) is homologous to \( \lambda' \) times \( S^1 \times S_2 \) for a point \((s_1, s_2) \in S^1 \times S^2\).
If the circle union $S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota)$ is defined to be the double of $cl(S^1 \times S^2(\alpha, \iota) - \partial(S^1 \times S^2(\alpha, \iota)))$, then $S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota)$ does not have the same type as $S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota)$, because $\pi_1(S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota)) \approx \mathbb{Z}$, but $\pi_1(S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota))$ is non-abelian. [In fact, the natural injection $\partial \cdot T(\omega') \hookrightarrow S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota)$ induces a monomorphism $\pi_1(\partial \cdot T(\omega')) \rightarrow \pi_1(S^1 \times S^2(\alpha, \iota) \circ S^1 \times S^2(\alpha, -\iota))$ by the loop theorem.]

In spite of Remark 1.5 we can prove the following for arbitrary two circle unions $m_0 \circ m_1$, $m_0 \circ 'm_1$ of given two types $m_0$, $m_1$:

**Lemma 1.6.** $m_0 \circ m_1 \sim m_0 \circ 'm_1$.

**Proof.** Let $M_0(\alpha_0, \iota_0) \in m_0$ and $M_1(\alpha_1, \iota_1) \in m_1$. Assume $M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, \iota_1) \in m_0 \circ m_1$ and $M_0(\alpha_0, \iota_0) \circ 'M_1(\alpha_1, \iota_1) \in m_0 \circ 'm_1$ are given by the following:

$$M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, \iota_1) = M_0(\alpha_0, \iota_0) \times 0 \cup_{k_0} S^1 \times B^2 \times [0, 1] \cup_{k_1} M_1(\alpha_1, \iota_1) \times 0 - S^1 \times \text{Int } B^2 \times [0, 1]$$

$$M_0(\alpha_0, \iota_0) \circ 'M_1(\alpha_1, \iota_1) = M_0(\alpha_0, \iota_0) \times 1 \cup_{k_0'} S^1 \times B^2 \times [0, 1] \cup_{k_1'} M_1(\alpha_1, \iota_1) \times 1 - S^1 \times \text{Int } B^2 \times [0, 1].$$

Then we let

$$W = M_0(\alpha_0, \iota_0) \times [0, 1] \cup_{k_0} S^1 \times B^2 \times [0, 1] \cup_{k_1} M_1(\alpha_1, \iota_1) \times [0, 1].$$

(See figure 2.)

![Figure 2](image-url)
Clearly we have \( \partial W = M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, \iota_1) + M_0(\alpha_0, -\iota_0) \circ' M_1(\alpha_1, -\iota_1) \).

Note that \( \alpha_0, \alpha_1 \) represent the same element \( \alpha \) in \( H^1(W; \mathbb{Z}) \). Let \( H_\ell(W, Z) \) be dual to \( \alpha \) and \( W_\varphi \) be the infinite cyclic cover of \( W \) associated with \( \varphi \). Since \( W_\varphi \) is the union of \( \tilde{M}_0(\alpha_0, \iota_0) \times [0, 1], R^1 \times B^2 \times [0, 1], R^1 \times B^2 \times [0, 1] \) and \( \tilde{M}_1(\alpha_1, \iota_1) \times [0, 1] \), each two intersections of which is empty or homeomorphic to \( R^1 \times B^2 \), it follows from the Mayer-Vietoris sequence that \( H_4(W_\varphi; \mathbb{Q}) \) is finitely generated over \( \mathbb{Q} \), where \( \tilde{M}_i(\alpha_i, \iota_i) \) are the infinite cyclic covers of \( M_i(\alpha_i, \iota_i) \), \( i = 0, 1 \). Thus, the triad \( (W, M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, \iota_1), M_0(\alpha_0, \iota_0) \circ' M_1(\alpha_1, \iota_1)) \) gives an \( H \)-cobordism and hence \( m_0 \circ m_1 \sim m_0 \circ m \). This completes the proof.

**Lemma 1.7.** \( m_0 \sim m_1 \) is equivalent to \( m_0 \circ -m_1 \sim 0 \).

**Proof.** Assume \( m_0 \sim m_1 \). Then for some representatives \( M_0(\alpha_0, \iota_0) \in m_0, M_1(\alpha_1, \iota_1) \in m_1 \) there is an \( H \)-cobordism \( (W, M_0(\alpha_0, \iota_0), M_1(\alpha_1, \iota_1)) \). Note that there is a cohomology class \( \varphi \in H^1(W; \mathbb{Z}) \) such that for each \( i \varphi | M_i(\alpha_i, \iota_i) \in H^1(M_i(\alpha_i, \iota_i); \mathbb{Z}) \) is dual to \( \alpha_i \). Let \( M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, -\iota_1) = M_0(\alpha_0, \iota_0) \cup_{h_0} S^1 \times B^2 \times [0, 1] \cup_{h_1} M_1(\alpha_1, -\iota_1) - S^1 \times \text{Int} B^2 \times [0, 1] \) and \( W' = W \cup_{h_0} S^1 \times B^2 \times [0, 1] \). Clearly \( \partial W' = M_0(\alpha_0, -\iota_0) \circ M_1(\alpha_1, -\iota_1) \). The cohomology class \( \varphi \in H^1(W'; \mathbb{Z}) \) is easily extended to a cohomology class \( \varphi' \in H^1(W'; \mathbb{Z}) \) such that the restriction \( \varphi' | M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, -\iota_1) \in H^1(M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, -\iota_1); \mathbb{Z}) \) is dual to the specified generator of \( H_4(M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, -\iota_1); \mathbb{Z}) \). By applying the Mayer-Vietoris sequence, it is not difficult to see that the infinite cyclic cover \( W_\varphi \) of \( W' \) associated with \( \varphi' \) has a finitely generated rational homology group \( H_4(W_\varphi; \mathbb{Q}) \). [Use that \( H_4(W_\varphi; \mathbb{Q}) \) is finitely generated over \( \mathbb{Q} \).] So, \( m_0 \circ -m_1 \sim 0 \).

Conversely assume \( m_0 \circ -m_1 \sim 0 \). For \( M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, -\iota_1) \in m_0 \circ -m_1 \), there is an \( H \)-cobordism \( (W'', M_0(\alpha_0, \iota_0) \circ M_1(\alpha_1, -\iota_1), \psi) \). By the definition
of the circle union there is a natural injection \( j: S^1 \times \mathbb{D}^2 \times [0, 1] \to M_0(\alpha_0, \epsilon_0) \circ M_1(\alpha_1, -\epsilon_1) \). Let \( W'' = W'' \cup j \partial S^1 \times \mathbb{D}^2 \times [0, 1] \). It is easy to see that the boundary \( \partial W'' \) is equal to the disjoint union \( M_0(\alpha_0, \epsilon_0) + M_1(\alpha_1, -\epsilon_1) \) and that the triad \((W'', M_0(\alpha_0, \epsilon_0), M_1(\alpha_1, \epsilon_1))\) gives an \( H \)-cobordism between \( M_0(\alpha_0, \epsilon_0) \) and \( M_1(\alpha_1, \epsilon_1) \). This completes the proof.

**Lemma 1.8.** If \( m_0 \sim 0 \) and \( m_1 \sim 0 \), then \( m_0 \circ m_1 \sim 0 \).

Proof. For \( M_0(\alpha_0, \epsilon_0) \in m_0 \), \( M_1(\alpha_1, \epsilon_1) \in m_1 \), there are \( H \)-cobordisms \((W_0, M_0(\alpha_0, \epsilon_0), \phi)\) and \((W_1, M_1(\alpha_1, \epsilon_1), \phi)\). Let \( M_0(\alpha_0, \epsilon_0) \circ M_1(\alpha_1, \epsilon_1) = M_0(\alpha_0, \epsilon_0) \cup_{\partial_0} S^1 \times \mathbb{D}^2 \times [0, 1] \cup_{\partial_1} M_1(\alpha_1, \epsilon_1) - S^1 \times \text{Int} \mathbb{D}^2 \times [0, 1] \). If we let \( W = W_0 \cup_{\partial_0} S^1 \times \mathbb{D}^2 \times [0, 1] \cup_{\partial_1} W_1 \) (See figure 4.), then the triad \((W, M_0(\alpha_0, \epsilon_0) \circ M_1(\alpha_1, \epsilon_1), \phi)\) gives an \( H \)-cobordism. So, \( m_0 \circ m_1 \sim 0 \), which completes the proof.

![figure 4.](image-url)

Now we can derive the following theorem which is a main purpose of this section.

**Theorem 1.9.** The set \( \Omega(S^1 \times S^2) \) forms an abelian group under the sum \([m] + [n] = [m \circ n]\). The zero element of this group is \([0]\). The inverse of any element \([m]\) is the element \([-m]\).

Proof. To show that the sum \([m] + [n] = [m \circ n]\) is well-defined, let \( m_0 \sim m_0' \) and \( m_1 \sim m_1' \). By Lemma 1.7 \( m_0 \circ -m_0' \sim 0 \) and \( m_1 \circ -m_1' \sim 0 \). Then by Lemma 1.8 \((m_0 \circ -m_0') \circ (m_1 \circ -m_1') \sim 0 \). Since \((m_0 \circ m_1) \circ m_2 \sim m_0 \circ (m_1 \circ m_2)\) and \( m_0 \circ m_1 = m_1 \circ m_0 \) for all \( m_0, m_1, \) and \( m_2 \), we obtain \((m_0 \circ m_1) \circ -(m_0' \circ m_1') \sim (m_0 \circ -m_0') \circ (m_1 \circ -m_1')\). Hence again by Lemma 1.7 \( m_0 \circ m_1 \sim m_0 \circ m_1' \). Thus, \([m] = [m']\) and \([n] = [n']\) imply \([m] + [n] = [m'] + [n']\). It is clear that \([m] + [0] = [m] + (0 + [m])\) and \([m] + [n] = [m] + [n]\). Also, we have \([m] + [0] = [m \circ 0] = [m]\) and, by Lemma 1.7, \([m] + [-m] = [0]\). This completes the proof.
The group $\Omega(S^1 \times S^2)$ is called the $H$-cobordism group of 3-dimensional homology orientable handles. The zero element is denoted by 0 and the inverse of $[m]$ is $-m$.

2. Relating the $\tilde{H}$-cobordism group $\Omega(S^1 \times S^2)$ to the Fox-Milnor's group $C$ and the Levine's group $G$.

The purpose of this section is to prove the following theorem.

**Theorem 2.1.** There is a commutative triangle

$$
\begin{array}{ccc}
C^1 & \rightarrow & \Omega(S^1 \times S^2) \\
\downarrow \phi & & \downarrow \psi \\
& G_\ast & \\
\end{array}
$$

of groups and homomorphisms, where the homomorphisms $\phi: C^1 \rightarrow G_\ast$ and $\psi: \Omega(S^1 \times S^2) \rightarrow G_\ast$ are onto.

A knot $k \subset S^3$ is a polygonal oriented 1-sphere $k$ in the oriented piecewise-linear 3-sphere $S^3$. Two knots $k_i \subset S^3$, $k_i \subset S^3$ have the same knot type if there is a piecewise-linear homeomorphism $(S^3, k_i) \rightarrow (S^3, k_i)$ which is orientation-preserving as both the maps $S^3 \rightarrow S^3$ and $k_i \rightarrow k_i$. The knot type of a knot $k \subset S^3$ will mean the class of knots with the same knot type as $k \subset S^3$. The set of knot types is denoted by $\mathcal{K}$. Let $k$ be a knot type and $(k \subset S^3) \in \mathcal{K}$ be a representative knot. By $-k$, we denote the knot type of the knot $(-k \subset -S^3)$, where $-k$ and $-S^3$ are the same as $k$ and $S^3$ but have the opposite orientations, respectively.

Now we shall construct a function $e: \mathcal{K} \rightarrow C_+(S^1 \times S^2)$. Let $k$ be a knot type and $(k \subset S^3) \in \mathcal{K}$ be a knot. Consider the regular neighborhood $T(k) \subset S^3$ of the knot $k \subset S^3$. Then $T(k)$ is clearly piecewise-linear homeomorphic to the solid torus $S^1 \times B^2$. We note that the solid tours $T(k)$ in $S^3$ has unique meridian and longitude curves* (up to isotopies of $\partial T(k)$ and the orientations of curves).

*) A meridian curve of a solid (knotted) torus $T$ in $S^3$ is a simple closed curve $\omega$ in $\partial T$ such that $\omega$ is homologous to 0 in $T$ but not in $\partial T$. A longitude curve of $T$ in $S^3$ is a simple closed curve $\omega$ in $\partial T$ such that $\omega$ is homologous to 0 in $S^3-I\text{nt}T$ but not in $\partial T$. The uniqueness of the meridian and longitude curves follows from a more general principle: Let $X$ be a homology orientable circle i.e. $X$ is a compact 3-manifold with $H_*(X; Z) = H_*(S^1; Z)$ and $H_0(\partial X; Z) = H_{0}(S^1 \times S^1; Z)$. If $\omega, \omega' \subset \partial X$ are homologous to 0 in $X$ but not in $\partial X$, then with suitable orientations of $\omega, \omega'$, $\omega$ is isotopic to $\omega'$ in $\partial X$. [Proof. Take a simple closed curve $\omega^* \subset \partial X$, intersecting $\omega$ in single point. Using that $\omega$ represents the zero element of $H_1(X; Z)$ and that the natural homomorphism $H_1(\partial X; Z) \rightarrow H_1(X; Z)$ is onto, it follows that $\omega^*$ represents a generator of $H_1(X; Z)$. Let $f: \partial X \rightarrow \omega^*$ be a natural projection such that for some point $p^* \in \omega^*$, $f^{-1}(p^*) = \omega$. Then we may find an extension $f': X \rightarrow \omega^*$ of $f$ such that $(f')^{-1}(p^*) = F$ is a connected surface with $\partial F = \omega$. Since the infinite cyclic covering...
The orientation of the longitude curve should be chosen so that the longitude curve is homologous to \( k \) in \( T(k) \). The orientation of the meridian curve should be chosen so that the linking number of the meridian curve and the knot \( k \) in \( S^3 \) is +1. Let \( h: S^1 \times S^1 \to \partial T(k) \) be a piecewise-linear homeomorphism such that for some point \((s_1, s_2)\) in \( S^1 \times S^1 \) the curves \( h(s_1, s_2) \) are the meridian curve and the longitude curve of \( T(k) \), respectively. Define \( M \) to be the adjunction space \( S^3 - \text{Int}(T(k)) \cup hB^2, \partial B^2 \) is identified with \( S^1 \). By applying the Mayer-Vietoris sequence, we have \( H_1(M; Z) \approx Z \). Hence \( M \) is a homology orientable handle by Poincaré duality. Note that the oriented meridian curve of \( T(k) \) represents a generator \( \alpha \) of \( H_1(M; Z) \). We specify the orientation of \( M \) compatible with the orientation of \( S^3 - T(k) \) induced from that of \( S^3 \). So, a generator \( \iota \in H_1(M; Z) \) is specified.

**Definition 2.2.** The distinguished homology orientable handle \( M(\alpha, \iota) \) is called the distinguished homology orientable handle obtained from \( S^3 \) by the elementary surgery along the knot \( k \subset S^3 \).

By using the uniqueness of the meridian curve, the longitude curve and the regular neighborhood, it is easily checked that the type of \( M(\alpha, \iota) \) is uniquely determined by the knot type \( k \) of \( k \subset S^3 \). So we denote this type by \( e(k) \).

Thus, we have the following:

**Lemma 2.3.** There is a function \( e: \mathcal{K} \to \mathcal{C}_+(S^1 \times S^2) \).

For any two knot types \( k_1, k_2 \), one can construct a unique knot type \( k_1 \equiv k_2 \) well-known as the knot sum. Two knot types \( k_1, k_2 \) are cobordant if for a representative knot \( k \subset S^3 \) of the knot sum \( k_1 \equiv k_2 \) \( k \) bounds a locally flat 2-cell in the 4-cell \( B^4 \). The set \( \mathcal{K} \) modulo this knot cobordism relation forms an abelian group \( C^1 \), called the knot cobordism group. (See Fox-Milnor [3] for details.) The sum operation of \( C^1 \) is the usual knot sum operation.

**Lemma 2.4.** The function \( e: \mathcal{K} \to \mathcal{C}_+(S^1 \times S^2) \) induces a homomorphism \( C^1 \to \Omega(S^1 \times S^2) \) also denoted by \( e \).

**Proof.** For two knot types \( k_1, k_2 \), it is directly checked that \( e(k_1 \equiv k_2) \) is a circle union of \( e(k_1) \) and \( e(k_2) \) i.e. \( e(k_1 \equiv k_2) = e(k_1) \cup e(k_2) \). [Note that for \((K_1 \subset S^3) \equiv k_1, i=1, 2, \) the exterior of the knot sum \((K_1 \subset S^3) \equiv (K_2 \subset S^3) \) is the adjunction...
space of the exteriors of $K \subset S^3$ along uniquely specified annuli on the boundaries. Hence it suffices to show that if a knot type $k$ is cobordant to the trivial knot type, then $e(\delta) \sim 0$. According to Fox-Milnor [3], this knot type $k$ can be realized as a local knot type of a piecewise linear 2-sphere $S(\delta)$ in $S^4$ with just one locally knotted point. Let $N = N(S(\delta), S^3)$ be the regular neighborhood of $S(\delta)$ in $S^4$. Let $W = S^4 - \text{Int} N$ and $M = \partial W$. Notice that $H_*(\partial W; Z) \cong H_*(S^1; Z)$ by the Alex ander duality. By using the Mayer-Vietoris sequence of the triple $(S^4; W, N)$, we obtain that $H_*(M; Z) \cong Z$. Hence $M$ is a homology orientable handle. $M$ may be a distinguished homology orientable handle obtained from $S^3$ by the elementary surgery along a representative knot $(k \subset S^3) \in \delta$: $M = M(\alpha, \iota)$. [For $N$ is obtained from a 4-cell by attaching a 2-handle along a solid torus $T \subset S^3$ representing $\iota$. Using $H_*(M; Z) = Z$ and the unique longitude curve of $T \subset S^3$, $M$ with suitably chosen $\alpha \in H_1(M; Z)$ and $\iota \in H_2(M; Z)$ belongs to $e(\delta)$]. Since $W$ has the homology of a circle, it follows from Milnor [11, Assertion 5] that the rational homology group $H_*(W; Q)$ of any infinite cyclic cover $\tilde{W}$ is finitely generated over $Q$. This shows that the triad $(W; M(\alpha, \iota), \phi)$ gives an $\mathcal{H}$-cobordism. Therefore $e(\delta) \sim 0$. This completes the proof.

Usually any knot type cobordant to the trivial knot type is called a slice knot type.

In the proof of Lemma 2.4, we have also proved the following:

**Corollary 2.5 (Kato [5]).** If a knot type $k$ is a slice knot type, then any representative homology orientable handle of $e(\delta)$ is embeddable to the 4-sphere $S^4$.

A Seifert matrix $A$ (with sign $-1$) is an integral square matrix with $\det(A - A') = \pm 1$. ($A'$ is the transpose of $A$.) Two Seifert matrices $A_1$, $A_2$ are said to be cobordant if the block sum $A_1 \oplus -A_2$ is congruent (over $Z$) to a matrix of the form

$$
\begin{pmatrix}
O & B \\
C & D
\end{pmatrix}
$$

($B, C, D$ are square matrices of the same size.) The set of Seifert matrices modulo this cobordism relation forms an abelian group $G_-$, called the matrix cobordism group. (See Levine [9] for details. Note that only Seifert matrices with sign $-1$ are considered here.) In [10] Levine calculated that $G_-$ is isomorphic to the direct sum $\sum_{t=1}^\infty Z^t + \sum_{t=1}^\infty (Z/2Z)^t + \sum_{t=1}^\infty (Z/4Z)^t$.

For a while we would like to spare time for describing familiar algebraic invariants of a polygonal oriented 1-sphere in a piecewise linear oriented homology 3-sphere, called a homological knot. The arguments may proceed in the same way as the usual knot theory. Let $k \subset \mathbb{S}^3$ be a homological knot. $k$ bounds an oriented connected surface $F$, called a Seifert surface for $k$, by using a notion of the transverse regularity. We define a pairing $\theta: H_1(F; Z) \otimes H_1(F; Z) \to Z$ such that $\theta(\alpha \otimes \beta) = L(\alpha, \iota_*(\beta))$, where $L$ denotes the homological linking number in $\mathbb{S}^3$ and $\iota_*(\beta)$ denotes the translate of the cycle $\beta$ off $F$ in the positive normal direction. With a basis for $H_1(F; Z)$, $\theta$ represents an integral square matrix $A$,
called a Seifert matrix for \( k \subset S^3 \) associated with surface \( F \). Using a formula
\[
\partial (\alpha \otimes \beta) - \partial (\beta \otimes \alpha) = \alpha \cdot \beta,
\]
where \( \alpha \cdot \beta \) is the intersection number, we obtain
\[
det(A - A') = \pm 1.
\]
(See for example Levine [8].) So, \( A \) is in fact a Seifert matrix.

The integral polynomial
\[
A(t) = \det(tA - A')
\]
is called the Alexander polynomial of \( S^3 \). Let \( X = S^3 - \text{Int}(k) \) for the regular neighborhood \( T(k) \) of \( k \) in \( S^3 \) and \( \tilde{X} \) be the infinite cyclic cover of \( X \) associated with the Hurewicz homomorphism \( \pi_1(X) \to H_1(X; \mathbb{Z}) \). We choose an orientation of \( \tilde{X} \) induced by that of \( X \) and a generator \( t \) of the covering transformation group of \( \tilde{X} \) associated with a generator \( \alpha \) of \( H_1(X; \mathbb{Z}) \) with linking number \( L(\alpha, k) = -1 \).

By using the Mayer-Vietoris sequence, the matrix \( tA - A' \) is a relation matrix of \( H_1(X; \mathbb{Z}) \) as a \( \mathbb{Z}[t] \)-module. The Seifert surface \( F \) induces a generator \( \mu \) of \( H_2(X; \mathbb{Z}) \) called a finite fundamental class of \( \tilde{X} \). (See Kawauchi [6, Theorem 2.3] and also Erle [1].) By Kawauchi [6, Theorem 2.3] (See also Milnor [11, p 127],) there is a duality
\[
\cap: H^q(X; \mathbb{Q}) \cong H^q(X; \mathbb{Q}) \quad \text{and} \quad \cap: H^q(X; \mathbb{Q}) \cong H^q(X; \mathbb{Q})
\]
for all \( q \), since \( H^q(X; \mathbb{Q}) \) is finitely generated over \( \mathbb{Q} \). Hence using a canonical isomorphism \( H^1(X, \partial X; \mathbb{Q}) \approx H^1(X; \mathbb{Q}) \), the cup product
\[
H^1(X, \partial X; \mathbb{Q}) \times H^1(X, \partial X; \mathbb{Q}) \to H^2(X, \partial X; \mathbb{Q})
\]
is a non-singular skew-symmetric bilinear form. Define a symmetric bilinear form
\[
\langle \cdot, \cdot \rangle: H^1(X, \partial X; \mathbb{Q}) \times H^1(X, \partial X; \mathbb{Q}) \to H^2(X, \partial X; \mathbb{Q})
\]
by the equality
\[
\langle x, y \rangle = \langle x \cup ty \rangle \cap \mu + \langle ty \cup tx \rangle \cap \mu.
\]
This bilinear form is isometric on \( t: \langle tx, ty \rangle = \langle x, y \rangle \) and non-singular.

**Definition 2.6.** The pair \( \langle \cdot, \cdot \rangle, t \) is called the quadratic form of the homological knot \( k \subset S^3 \). (See Erle [1] and Milnor [11].)

**The signature** of \( k \subset S^3 \) is the signature of this form \( \langle \cdot, \cdot \rangle \).

The following proposition is essentially proved by Erle [1].

**Proposition 2.7.** Let \( A \) be any Seifert matrix for a homological knot \( k \subset S^3 \) associated with a Seifert surface. \( A \) is \( S \)-equivalent to a non-singular Seifert matrix \( A_* \) such that, with a suitable basis for \( H^1(X, \partial X; \mathbb{Q}) \), the linear isomorphism \( t: H^1(X, \partial X; \mathbb{Q}) \to H^1(X, \partial X; \mathbb{Q}) \) and the form \( \langle \cdot, \cdot \rangle: H^1(X, \partial X; \mathbb{Q}) \times H^1(X, \partial X; \mathbb{Q}) \to \mathbb{Q} \) represent the matrices \( A_*^{-1}A_* \) and \( A_* + A_* \), respectively. (In fact, Erle [1] proved this proposition for any usual knot \( k \subset S^3 \). Without difficulty, Erle's proof may be applied for homological knot \( k \subset S^3 \). See Trotter [13] for a concept of \( S \)-equivalences.)

By Proposition 2.7, the signature of \( k \subset S^3 \) is equal to the signature
\[
\sigma(A_* + A'_*) = \sigma(A + A')
\]
and \( m \subset C_*(S^1 \times S^2) \) and \( M(\alpha, \iota) \subset m \). We choose a polygonal oriented simple closed curve \( \omega \) in \( M(\alpha, \iota) \) representing \( \alpha \) and let \( T(\omega) \) be the regular neighborhood of \( \omega \) in \( M(\alpha, \iota) \). Also we choose polygonal oriented simple closed curves \( k \) and \( l \) in \( \partial T(\omega) \) intersecting in a single point such that \( k \) is oriented so as to be
Let \( L(k, \omega) := +1 \) and bounds a 2-cell in \( T(\omega) \) and such that \( l \) is homologous to \( \omega \) in \( T(\omega) \). (Note that in any case the choice of \( l \) is not unique.) Let \((s, s) \in S^1 \times S^1\) and define a piecewise-linear homeomorphism \( h: S^1 \times S^1 \to \partial T(\omega) \) such that \( h(s \times s^1) = k \) and \( h(s^1 \times s) = l \). Let \( \bar{S}^3 = M(\alpha, \iota) - \text{Int } T(\omega) \cup kB^2 \times S^1 \). It is easy to see that \( \bar{S}^3 \) is a homology 3-sphere. (Notice that \( k \) is homologous to 0 in \( M(\alpha, \iota) - \text{Int } T(\omega) \).) The orientation of \( \bar{S}^3 \) is chosen so as to coincide with that of \( M(\alpha, \iota) - \text{Int } T(\omega) \). Thus, we obtain a homological knot \( k \subset \bar{S}^3 \) from \( M(\alpha, \iota) \) (although the homeomorphism type of the pair \( (\bar{S}^3, k) \) is never uniquely determined by the type of \( M(\alpha, \iota) \)).

**Definition 2.8.** A Seifert matrix for the homological knot \( k \subset \bar{S}^3 \) associated with a Seifert surface is called a **Seifert matrix for** \( M(\alpha, \iota) \) (or the **type m**).

Accordingly if \( A \) is a Seifert matrix for a knot type \( \phi \), then \( A \) is also a Seifert matrix for the type \( \epsilon(\phi) \).

**Definition 2.9.** The Alexander polynomial \( A(t) = \det(tA - A') \) of \( k \subset \bar{S}^3 \) is called the **Alexander polynomial of** \( M(\alpha, \iota) \) (or the **type m**).

This definition coincides with that of Kawauchi [7, Definition 1.3], because the matrix \( tA - A' \) is a relation matrix of \( H_i(M(\alpha, \iota); \mathbb{Z}) \) by the canonical isomorphism \( H_i(\bar{X}; \mathbb{Z}) \approx H_i(\bar{M}(\alpha, \iota); \mathbb{Z}) \). Here \( \bar{X} \) denotes the infinite cyclic cover of \( X = M(\alpha, \iota) - \text{Int } T(\omega) \) with the uniquely specified generator \( t \) of the covering transformation group and with the associated orientation. \( \bar{M}(\alpha, \iota) \) denotes the infinite cyclic cover of \( M(\alpha, \iota) \) such that the covering projection \( \bar{M}(\alpha, \iota) \to M(\alpha, \iota) \) is an extension of the covering projection \( \bar{X} \to X \). \( \bar{M}(\alpha, \iota) \) has an orientation compatible with that of \( \bar{X} \). The generator of the covering transformation group of \( \bar{M}(\alpha, \iota) \) is an extension of \( t: \bar{X} \to \bar{X} \), also denoted by \( t \). Note that the finite fundamental class \( \mu \in H_2(\bar{X}, \partial \bar{X}; \mathbb{Z}) \) determined by a Seifert surface specifies a unique generator of \( H_2(\bar{M}(\alpha, \iota); \mathbb{Z}) \), also denoted by \( \mu \) by the canonical isomorphism \( H_2(\bar{X}, \partial \bar{X}; \mathbb{Z}) \approx H_2(\bar{M}(\alpha, \iota); \mathbb{Z}) \). This \( \mu \in H_2(\bar{M}(\alpha, \iota); \mathbb{Z}) \) is called the **finite fundamental class of** \( \bar{M}(\alpha, \iota) \). By using the canonical isomorphisms \( H^i(\bar{X}, \partial \bar{X}; \mathbb{Q}) \approx H^i(\bar{M}(\alpha, \iota); \mathbb{Q}) \), \( i = 1, 2 \), the bilinear form \( \langle , \rangle: H^1(\bar{X}, \partial \bar{X}; \mathbb{Q}) \times H^1(\bar{X}, \partial \bar{X}; \mathbb{Q}) \to \mathbb{Q} \) passes to the form \( ( , ) : H^1(\bar{M}(\alpha, \iota); \mathbb{Q}) \times H^1(\bar{M}(\alpha, \iota); \mathbb{Q}) \to \mathbb{Q} \) defined by the equality \( (x, y) = (x \cup ty) \cap \mu + (y \cup tx) \cap \mu \) for all \( x, y \) in \( H^1(\bar{M}(\alpha, \iota); \mathbb{Q}) \).

**Definition 2.10.** The pair \( ( , ) , t \) is called the **quadratic form of** \( M(\alpha, \iota) \) (or the **type m**).

The signature of \( M(\alpha, \iota) \) (or the **type m**), denoted by \( \sigma(M(\alpha, \iota)) \) (or \( \sigma(m) \)) is the signature of the homological knot \( k \subset \bar{S}^3 \). So, the signature of \( M(\alpha, \iota) \) coincides with the signature of the bilinear form \( ( , ) \). Easily \( \sigma(M(\alpha, \iota)) = \sigma(M(-\alpha, \iota)) \) and \( \sigma(M(\alpha, -\iota)) = -\sigma(M(\alpha, \iota)) \).

From Proposition 2.7, the following is immediately obtained:
Lemma 2.11. Let $A$ be a Seifert matrix for $M(\alpha, \iota)$. $A$ is $S$-equivalent to a non-singular Seifert matrix $A_*$ such that, with a suitable basis for $H^1(M(\alpha, \iota); \mathbb{Q})$, the linear isomorphism $t: H^1(M(\alpha, \iota); \mathbb{Q}) \to H^1(M(\alpha, \iota); \mathbb{Q})$ and the form $(,): H^1(M(\alpha, \iota); \mathbb{Q}) \times H^1(M(\alpha, \iota); \mathbb{Q}) \to \mathbb{Q}$ represent the matrices $A_*^{-1}A_*$ and $A_* + A_*'$, respectively.

Note that by Lemma 2.11 $\sigma(M(\alpha, \iota)) = \sigma(A_* + A_*') = \sigma(A + A')$.

For the quadratic form $(, t)$ of the type $m$ of $M(\alpha, \iota)$, if $H^1(M(\alpha, \iota); \mathbb{Q})$ contains a half-dimensional vector subspace $V$ with $tV = V$ and such that $(x, y) = 0$ for all $x, y$ in $V$, then the quadratic form $(, t)$ is said to be null-cobordant (See Levine [10]).

The following theorem is a basically important result.

Theorem 2.12. If $m = 0$, then the quadratic form $(, t)$ of $m$ is null-cobordant.

Proof. Since $m \sim 0$, for $M(\alpha, \iota) \in m$ there exists an $\tilde{H}$-cobordism $(W, M(\alpha, \iota), \phi)$. Hence for some $\varphi \in H^1(W; \mathbb{Z})$ with $\varphi| M(\alpha, \iota) \in H^1(M(\alpha, \iota); \mathbb{Z})$ dual to $\alpha$, the infinite cyclic cover $\tilde{W}_\varphi$ associated with $\varphi$ has a finitely generated rational homology group $H_*(\tilde{W}_\varphi; \mathbb{Q})$. Note that by Kawauchi [6, Theorem 2.3], the Poincaré dualities $\cap \overline{\mu}: H_*(\tilde{W}_\varphi; \mathbb{Q}) \approx H_{*-*}(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); \mathbb{Q})$ and $\cap \overline{\mu}: H_*(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); \mathbb{Q}) \approx H_{*-*}(\tilde{W}_\varphi, M(\alpha, \iota); \mathbb{Q})$ hold, where $\overline{\mu} \in H_{\delta}(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); \mathbb{Z})$ is a finite fundamental class determined from $\mu$ by the boundary-isomorphism $\partial: H_3(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); \mathbb{Z}) \approx H_2(\tilde{W}_\varphi; \mathbb{Q})$.

Now we consider the following commutative (up to sign) diagram:

$$
\begin{array}{cccccc}
\longrightarrow & H^1(\tilde{W}_\varphi; \mathbb{Q}) & \xrightarrow{i^*} & H^1(\tilde{M}(\alpha, \iota); \mathbb{Q}) & \xrightarrow{\delta} & H^1(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); \mathbb{Q}) \\
\cap \overline{\mu} & \downarrow & & \cap \overline{\mu} & \downarrow & \\
\longrightarrow & H_2(\tilde{W}_\varphi, \tilde{M}(\alpha, \iota); \mathbb{Q}) & \xrightarrow{\partial} & H_1(\tilde{M}(\alpha, \iota); \mathbb{Q}) & \xrightarrow{i_*} & H_1(\tilde{W}_\varphi; \mathbb{Q}) \\
\end{array}
$$

Here the top and bottom sequences are exact and the vertical homomorphisms are isomorphisms.

For all $u \in H^1(\tilde{W}_\varphi; \mathbb{Q})$, suppose $(i^*(u), y) = 0$. This situation is equivalent to $\delta(t-t^{-1})y = 0$ i.e. $(t-t^{-1})y \subset \text{Im}i^*$, because $(i^*u, y) = [i^*(u) \cap (t-t^{-1})y] \cap \mu = [u \cap \delta(t-t^{-1})y] \cap \overline{\mu}$. Using $(t-t^{-1})\text{Im}i^* \subset \text{Im}i^*$ and the isomorphism $t-t^{-1}: H^1(\tilde{M}(\alpha, \iota); \mathbb{Q}) \approx H^1(M(\alpha, \iota); \mathbb{Q})$, $(t-t^{-1})y \subset \text{Im}i^*$ is equivalent to $y \in \text{Im}i^*$. Thus we showed that the orthogonal complement of $\text{Im}i^*$ is $\text{Im}i^*$ itself. In particular, $\dim \mathbb{Q}\text{Im}i^* = \frac{1}{2} \dim \mathbb{Q}H^1(M(\alpha, \iota); \mathbb{Q})$. Since $t \text{Im}i^* \subset \text{Im}i^*$, the quad-

*) To prove this isomorphism, it suffices to check that the characteristic polynomial $A'(t)$ of $t: H^1(\tilde{M}(\alpha, \iota); \mathbb{Q}) \to H^1(\tilde{M}(\alpha, \iota); \mathbb{Q})$ satisfies $A'(\pm 1) \neq 0$, because $t-t^{-1} = t^{-1}(t-1)(t+1)$. For the Alexander polynomial $A(t)$ of $M(\alpha, \iota)$, $A'(t)$ equals to $A(t)$ up to units of $\mathbb{Q}[t]: A'(t) = A(t)$. (See [7, Lemma 2.6].) Since $A(\pm 1) \neq 0$, the result follows.
Lemma 2.13. There is a homomorphism \( \psi: \Omega(S^1 \times S^2) \rightarrow G_- \).

Proof. Let \( m \in \mathbb{C}_+(S^1 \times S^2) \) and \( A \) a Seifert matrix for \( m \). We define \( \psi[m] = [A] \). To prove the well-definedness, first we shall show that if \( m \sim 0 \), then \( A \) is null-cobordant. By Lemma 2.11, \( A \) is \( S \)-equivalent to a non-singular Seifert matrix \( A_\ast \) such that \( t \) represents \( A_\ast^{-1}A_\ast \) and the form \(( , )\) is null-cobordant. Since by Theorem 2.13 the quadratic form \(( , t)\) is null-cobordant, there exists a symplectic basis \( e_1, e_2, \cdots, e_s, e'_1, e'_2, \cdots, e'_{s'} \) of \( H^1(\tilde{M}(\alpha, t); \mathbb{Q}) \): \((e_i, e_j) = (e'_i, e'_j) = 0\), \((e_i, e'_j) = \delta_{ij}\) such that the vector subspace \( V \) spanned by \( e_1, e_2, \cdots, e_s \) is invariant under \( t \). (See for example Milnor-Husemoller [12, p 13].) Then there is a non-singular rational matrix \( P \) such that the matrix \( P^{-1}A_\ast^{-1}A_\ast P \) is of the form \( \begin{pmatrix} Q & R \\ O & S \end{pmatrix} \) (since \( tV = V \)), where \( Q, R, S \) are rational square matrices of the same size, and such that \( P'(A_\ast+A_\ast)P = \begin{pmatrix} O & I \\ I & O \end{pmatrix} \), \( I = \begin{pmatrix} O & 1 \\ 1 & O \end{pmatrix} \).

Using the equality \( P'A_\ast P = [P'(A_\ast+A_\ast)P(E+P^{-1}A_\ast^{-1}A_\ast P)^{-1}]' \) \((E = \text{the unit matrix.})\), it is not difficult to see that the matrix \( P'A_\ast P \) is of the form \( \begin{pmatrix} O & B \\ C & D \end{pmatrix} \). (\( B, C, D \) are rational square matrices of the same size.) [Note that \( \det(E+P^{-1}A_\ast^{-1}A_\ast P) \neq 0 \), since the Alexander polynomial \( A(t) \) satisfies \( A(-1) \neq 0 \).] Then by Levine [9, Lemma 8] \( A_\ast \) is null-cobordant. Since \( A \) is \( S \)-equivalent to \( A_\ast \), it follows that \( A \) is cobordant to \( A_\ast \). Hence \( A \) is null-cobordant. Let \( m_1, m_2 \in \mathbb{C}_+(S^1 \times S^2) \). Notice that if \( A_1, A_2 \) are Seifert matrices for \( m_1, m_2 \), respectively, then the block sum \( A_1 \oplus A_2 \) is a Seifert matrix for a circle union \( m_1 \circ m_2 \). [To see this, let \( M_i(\alpha_1, \iota_1) \subseteq S^2 \), \( i = 1, 2 \), and consider homological knots \( k_i \subseteq S^2 \) obtained from \( M_i(\alpha_1, \iota_1) \), \( i = 1, 2 \). Then one can verify that the homological knot sum \( (k_1 \subseteq S^2) \#(k_2 \subseteq S^2) \), defined to be analogous to the usual knot sum, is a homological knot obtained from some circle union \( M_i(\alpha_1, \iota_1) \circ M_i(\alpha_2, \iota_2) \). Now the desired result easily follows.] If \( m_1 \sim^* m_2 \), then \( m_1 \circ m_2 \sim 0 \). Hence the block sum \( A_1 \oplus -A_2 \) is null-cobordant, since \( A_1 \oplus -A_2 \) is a Seifert matrix for \( m_1 \circ m_2 \). Thus, \([m_1] = [m_2] \) implies \([A_1] = [A_2] \); that is, \( \psi[m_1] = [A_1] \) is well-defined. Further, \( \psi \) is a homomorphism, since for any \( m_1, m_2 \in \mathbb{C}_+(S^1 \times S^2) \)

\[
\psi([m_1] + [m_2]) = \psi[m_1 \circ m_2] = [A_1 \oplus A_2] = [A_1] + [A_2] = \psi[m_1] + \psi[m_2].
\]

This completes the proof.
2.14. Proof of Theorem 2.1. Levine [9] defined the homomorphism \( \phi: C^1 \to G_- \) sending any knot cobordism class to the matrix cobordism class of the corresponding Seifert matrices. By Lemma 2.4, the homomorphism \( e: C^1 \to \Omega(S^1 \times S^2) \) is obtained and by Lemma 2.13, the homomorphism \( \psi: \Omega(S^1 \times S^2) \to G_- \) is obtained. From construction, we have \( \psi e = \phi \). Since \( \phi \) is onto (See for example Levine [9],), \( \psi \) is onto. This proves Theorem 2.1.

Here are four corollaries to Theorem 2.1.

**Corollary 2.15.** The \( H \)-cobordism group \( \Omega(S^1 \times S^2) \) has the free part of infinite rank.

This follows from the facts that \( G_- \) has the free part of infinite rank and that the homomorphism \( \psi \) is onto.

The reduced Alexander polynomial \( \tilde{A}(t) \) of a type \( m \in \mathbb{C}_+(S^1 \times S^2) \) is the integral polynomial obtained from the Alexander polynomial \( A(t) \) of \( m \) by cancelling the factors of the type \( f(t) f(t^{-1}) \).

**Corollary 2.16.** If \( m \sim 0 \), then the Alexander polynomial \( A(t) \) splits as follows: \( A(t) = f(t) f(t^{-1}) \) for some integral polynomial \( f(t) \) and the signature \( \sigma(m) \) is 0. More generally, if \( m_1 \sim m_2 \), then the reduced Alexander polynomials \( \tilde{A}_1(t), \tilde{A}_2(t) \) are the same polynomial (up to \( \pm t^i \)): \( \tilde{A}_1(t) = \tilde{A}_2(t) \) and the signatures \( \sigma(m_1), \sigma(m_2) \) are equal: \( \sigma(m_1) = \sigma(m_2) \).

**Corollary 2.17.** For any \( [\xi] \in C^1 \), the equalities \( \sigma([\xi]) = \sigma([e(\xi)]) \) and \( \tilde{A}_{[\xi]}(t) = \tilde{A}_{[e(\xi)]}(t) \) hold.

**Corollary 2.18.** For any \( m \in \mathbb{C}_+(S^1 \times S^2) \), the signature \( \sigma(m) \) is even. For any integer \( i \), there exists \( m \in \mathbb{C}_+(S^1 \times S^2) \) with \( \sigma(m) = 2i \).

2.19. Addendum. Re-examination of the Seifert matrices. Let \( m \in \mathbb{C}_+(S^1 \times S^2) \) and \( M(\alpha, i) \subset m \). A Seifert matrix for \( M(\alpha, i) \) (or \( m \)) may be also defined as follows: Let \( f: M(\alpha, i) \to S^1 \) be a piecewise-linear map with \( f_\ast: H_i(M(\alpha, i); Z) \otimes H_i(S^1; Z) \) and such that for some point \( 0 \in S^1 \), \( F = f^{-1}(0) \) is a closed orientable connected surface (See Kawauchi [6, Corollary 1.3]). Using that \( [F] \in H_2(M(\alpha, i); Z) \) is a generator, we may orient \( F \) so that \( [F] = \varphi \cap \iota \), where \( \varphi \in H^1(M(\alpha, i); Z) \) is a dual element of \( H_1(M(\alpha, i); Z) \). Let \( M^* \) be the oriented manifold (with orientation induced by that of \( M(\alpha, i) \)) obtained from \( M(\alpha, i) \) by splitting along \( F \). Let \( \partial M^* = F \cup \overline{F} \). Here the component of \( \partial M^* \) with orientation coinciding with that of \( F \) is identified with \( F \). \( \overline{F} \) denotes the copy of \( F \) but with the opposite orientation. Let \( i': F \to F' \subset \partial M^* \subset M^* \) be the natural injection. If \( a \in H_i(F; Z) \), let \( a' \in H_i(M(\alpha, i), M(\alpha, i) - F; Z) \) be the image of \( a \) under the composite
By using a duality $\gamma_U\colon H_2(M(\alpha, \iota), M(\alpha, \iota)-F; Z) \cong H^1(F; Z)$, relating a slant product, where $U$ is the Thom class of $M(\alpha, \iota)$ corresponding to the fundamental class $\iota$, define a pairing

$$\theta': H_i(F; Z) \otimes H_1(F; Z) \to Z$$

by the equality $\theta'(a \otimes b) = \gamma_U(a') \cap b \in H_i(F; Z) = Z$.

It is checked that with a basis for $H_i(F; Z)$, $\theta'$ represents a Seifert matrix for $M(\alpha, \iota)$. The formula $\theta'(a \otimes b) - \theta'(b \otimes a) = a \cdot b$ is also obtained.

3. Elements of $\Omega(S^1 \times S^2)$ of order zero and two and the $\bar{H}$-cobordism group $\Omega(S^1 \times S^3)$ of homology non-orientable handles

A general problem of bringing about a better understanding of $\bar{H}$-cobordism between the types of distinguished homology orientable handles seems still difficult, but a partial answer is presented here.

**Theorem 3.1.** If a representative homology orientable handle $M(\alpha, \iota)$ of a type $m \in \mathbb{C}_i(S^1 \times S^2)$ is embeddable in a homology 4-sphere $\bar{S}^4$, then $m \sim 0$.

**Proof.** Assume $M(\alpha, \iota) \subset \bar{S}^4$. Then $M(\alpha, \iota)$ separates $\bar{S}^4$ into two manifolds, say, $W_1, W_2$ and, by easy computation of the homology, one of $W_1, W_2$ has the homology of a circle, say, $H_*(W_i; Z) \cong H_0(S^1; Z)$. Then the triad $(W_1, M(\alpha, \iota), \phi)$ gives an $\bar{H}$-cobordism. This completes the proof of Theorem 3.1.

Here are a few examples, whose somewhat analogous properties were also noticed by Kato [5, Theorems 5.1 and 5.5] in higher dimensions.

**Examples 3.2.** First we consider a (suitably oriented) trefoil $3_1$. (See figure 5.)
Using that $\sigma(\epsilon(3,))=\sigma(3,)=\pm 2\pm 0$ or that $A(t)=t^2-t+1$ is irreducible, $\epsilon(3,)=0$. Hence by Theorem 3.1, $\epsilon(3,)$ is not embeddable to the 4-sphere $S^4$. Note that $\epsilon(3,)$ is locally-flatly embeddable to the 5-sphere $S^5$, since according to Hirsch [4] every compact orientable 3-manifold is locally-flatly embeddable to $S^5$.

On the other hand, consider the stevedore's knot 6. (See figure 6.)

\[ \text{figure 6.} \]

Since this knot is a slice knot, by Corollary 2.5, $\epsilon(6,)$ is embeddable to $S^4$. Similar arguments also apply for the granny knot $3,\# 3, \# 3, \#$ and the square knot $3,\# -3^*$. (See figure 7.)

\[ \text{figure 7.} \]

In fact, $\epsilon(3,\# 3, \#)$ is not embeddable to $S^4$, although $\epsilon(3,\# -3^*)$ is embeddable to $S^4$, since $\sigma(\epsilon(3,\# 3, \#))=2\sigma(3,)=\pm 4\pm 0$ and $3,\# -3^*$ is a slice knot.

Next we would like to discuss order-two-elements of $\Omega(S^1 \times S^2)$. To do
this, we shall introduce the \( H \)-cobordism group of homology non-orientable handles.

A homology non-orientable handle \( M \) is a compact 3-manifold having the homology of the non-orientable handle \( S^1 \times \tau^2 \): \( H_*(M; \mathbb{Z}) \approx H_*(S^1 \times \tau^2; \mathbb{Z}) \), and is said to be distinguished if a generator \( \alpha \in H_1(M; \mathbb{Z}) \) is specified. If a homology non-orientable handle \( M \) is distinguished, then the notation \( M(\alpha) \) will be used. Two distinguished homology non-orientable handles \( M^1(\alpha^1), M^2(\alpha^2) \) have the same type if there is a piecewise-linear homeomorphism \( h: M^1(\alpha^1) \to M^2(\alpha^2) \) such that \( h_*(\alpha^1) = \alpha^2 \). The type of \( M(\alpha) \) is the class of distinguished homology non-orientable handles with the same type as \( M(\alpha) \). The set of the types is denoted by \( \mathbb{C}_+(S^1 \times \tau^2) \).

In \( \mathbb{C}_+(S^1 \times \tau^2) \) an \( H \)-cobordism relation is defined as an analogy of Definition 1.1.

**Definition 3.3.** Two types \( m^1, m^2 \) in \( \mathbb{C}_+(S^1 \times \tau^2) \) are \( H \)-cobordant and denoted by \( m^1 \sim m^2 \) if for \( M^1(\alpha^1) \in m^1, M^2(\alpha^2) \in m^2 \) there exists a pair \( (W, \phi) \), where \( W \) is a compact connected 4-manifold with \( \partial W = M^1(\alpha^1) + M^2(\alpha^2) \) (disjoint union) and \( \phi \in H^2(W; \mathbb{Z}) \) whose restrictions \( \phi|_{M^i(\alpha^i)} \in H^2(M^i(\alpha^i); \mathbb{Z}) \) are dual to \( \alpha^i, i = 1, 2 \), such that the infinite cyclic cover \( \tilde{W}_\phi \) associated with \( \phi \) is orientable and has a finitely generated rational homology group \( H_*(\tilde{W}_\phi; \mathbb{Q}) \). [Note that any infinite cyclic cover \( \tilde{M}(\alpha) \) is always orientable (See Kawauchi [7]).]

Let \( m^0, m^1 \in \mathbb{C}_+(S^1 \times \tau^2) \) and \( M^0(\alpha^0) \in m^0, M^1(\alpha^1) \in m^1 \). Choose polygonal oriented simple closed curves \( \omega^0 \subset M^0(\alpha^0), \omega^1 \subset M^1(\alpha^1) \) which represent \( \alpha^0, \alpha^1 \), respectively. It is not difficult to see that the regular neighborhoods \( T(\omega^0) \subset M^0(\alpha^0) \) of \( \omega^0 \) and \( T(\omega^1) \subset M^1(\alpha^1) \) of \( \omega^1 \) are both piecewise-linearly homeomorphic to the solid Klein bottle \( S^1 \times \tau^2 \). Note that there exists closed connected orientable surfaces \( F^0 \subset M^0(\alpha^0), F^1 \subset M^1(\alpha^1) \) transversally intersecting \( \omega^0, \omega^1 \) in single points, respectively.

Consider two piecewise-linear embeddings

\[
\begin{align*}
h_0 &: S^1 \times B^2 \times 0 \to M^0(\alpha^0) \\
h_1 &: S^1 \times B^2 \times 1 \to M^1(\alpha^1)
\end{align*}
\]

such that there exist points \( s \in S^1, b \in \text{Int } B^2 \) with \( h_0(S^1 \times b \times 0) = \omega^0, h_0(S^1 \times b \times 0) \subset F^0, h_1(S^1 \times b \times 1) = \omega^1, h_1(S^1 \times B^2 \times 1) \subset F^1 \) and such that \( \omega^0 \) and \( \omega^1 \) are homologous in the adjunction space \( M^0(\alpha^0) \cup h_0 S^1 \times B^2 \times [0, 1] \cup h_1 M^1(\alpha^1) \).

As an analogy of Definition 1.4, we may have Definition 3.4.

**Definition 3.4.** The homology non-orientable handle

\[
M_0(\alpha^0) \circ M^1(\alpha^1) = M_0(\alpha^0) \cup h_0 S^1 \times B^2 \times [0, 1] \cup h_1 M^1(\alpha^1) - S^1 \times \text{Int } B^2 \times [0, 1]
\]
distinguished naturally is called a circle union of $M_0(\alpha_0)$ and $M_1(\alpha_1)$. The type of $M_0(\alpha_0) \cup M_1(\alpha_1)$ is denoted by $m_0 \circ m_1$.

It is not difficult to check that for two circle unions $m_0 \circ m_1$, $m_0 \circ 'm_1$, $m_0 \circ m_1 \sim m_0 \circ 'm_1$. Further, we can prove that $m_0 \sim m_1$ if and only if $m_0 \circ m_1 \sim 0$ as an analogy of Lemma 1.7, where 0 is the type of $S^1 \times r, S^2$. [Note that $S^1 \times r, S^2(\alpha)$ has the same type as $S^1 \times r, S^2(-\alpha)$.] As a result, the set $\Omega(S^1 \times r, S^2)=C(S^1 \times r, S^2)/\sim$ forms an abelian group under the sum $[m_0]+[m_1]=[m_0 \circ m_1]$, called the H-cobordism group of homology non-orientable handles. Every non-zero element of $\Omega(S^1 \times r, S^2)$ has order 2, since $m \sim m$ implies $m \circ m \sim 1$ The zero element of $\Omega(S^1 \times r, S^2)$ is the H-cobordism class containing the type 0 of $S^1 \times r, S^2$.

**Theorem 3.5.** $\Omega(S^1 \times r, S^2)$ is the direct sum of infinite copies of the cyclic group of order 2.

To prove Theorem 3.5, the Alexander polynomial seems to be useful.

The Alexander polynomial $A(t)$ of $m \in C_+ (S^1 \times r, S^2)$ is the integral polynomial which is a generator of the smallest principal ideal containing the ideal associated with a relation matrix of $H_1(M(\alpha); Z)$ as a $Z[t]-$module (See Kawauchi [7] for details.). Here, $M(\alpha)$ denotes the infinite cyclic cover of $M(\alpha) \in m$ and $t$ denotes a generator of the covering transformation group of $M(\alpha)$, related to the generator $\alpha \in H_1 (M(\alpha); Z)$. $A(t)$ is the complete invariant of $M(\alpha)$ or the type $m$ up to units $\pm t^{\epsilon} \in Z(t) \cdot A(t)$ satisfies the properties that $A(t) \perp A(-t^{-1})$ and $A(1)|=1$; and, conversely, any integral polynomial with these properties is the Alexander polynomial of some $m \in C_+ (S^1 \times r, S^2)$. (See [7].) For characteristic polynomial $A'(t)$ of the linear isomorphism $t: H_1(M(\alpha); Q) \to H_1 (M(\alpha); Q)$ we have $A(t) \perp A'(t)$, that is, $A(t)$, $A'(t)$ are equal up to units $qt^{\epsilon} \in Q[t]$.

The following is an analogous result to Corollary 2.16.

**Lemma 3.6.** Let $m \in C_+ (S^1 \times r, S^2)$. If $m \sim 0$, then the Alexander polynomial $A(t)$ of $m$ has a type of $f(t)f(-t^{-1})$ for some integral polynomial $f(t)$.

Before showing Lemma 3.6 we shall show Theorem 3.5.

**3.7. Proof of Theorem 3.5.** Consider for example the irreducible integral polynomials $A_n(t)=nt^{n}+t-n$, $n=1, 2, 3, \cdots$. These $A_n(t)$ are realized as the Alexander polynomials of some $m_n \in C_+ (S^1 \times r, S^2)$, $n=1, 2, 3, \cdots$. Then it is easy to see that $m_1, m_2, m_3, \cdots$ represent a set of linearly independent elements of $\Omega (S^1 \times r, S^2)$ [Notice that if $A_i(t), A_j(t)$ are the Alexander polynomials of $m_i$, $m_j$, respectively, then the product $A_i(t)A_j(t)$ is the Alexander polynomial of any circle union $m_i \circ m_j$.]. This completes the proof.
3.8. Proof of Lemma 3.6. Since $m\sim 0$, for $M(\alpha)\subseteq m$ there exists a pair $(W, \varphi)$, where $W$ is a compact connected 4-manifold with $\partial W = M(\alpha)$ and $\varphi \in H^1(W; Z)$ with $\varphi | M(\alpha) \subseteq H^1(M(\alpha); Z)$ dual to $\alpha$, such that the infinite cyclic cover $W_\varphi$ is orientable and has a finitely generated rational homology group $H_*(W_\varphi; Q)$. Then from the exact sequence $H^1(W_\varphi; Q) \overset{i^*}{\to} H^1(M(\alpha); Q)$ $\delta \to H^2(W_\varphi, M(\alpha); Q)$ we obtain the short exact sequence $0 \to \text{Im} i^* \to \text{Im} \delta \to 0$. Then we have $A(t) = B(t) C(t)$, where $B(t), C(t)$ are the characteristic polynomials of $t$: $\text{Im} i^* \to \text{Im} i^*, t: \text{Im} \delta \to \text{Im} \delta$, respectively. Since the square

$$
\begin{array}{ccc}
H^1(M(\alpha); Q) & \overset{\delta}{\to} & H^1(W_\varphi, M(\alpha); Q) \\
\approx \cap \mu & & \approx \cap \bar{\mu} \\
H_{i}(M(\alpha); Q) & \overset{i_*}{\to} & H_i(W_\varphi; Q)
\end{array}
$$

is commutative, we obtain the Poincaré dual isomorphism $\cap \bar{\mu}: \text{Im} \delta \approx \text{Im} i_*$, where $\mu \in H_2(M(\alpha); Z)$ and $\bar{\mu} \in H_2(W_\varphi, M(\alpha); Z)$ are the finite fundamental classes such that $\bar{\mu}$ is mapped to $\mu$ by the boundary isomorphism $\partial: H_2(W_\varphi, M(\alpha); Z) \approx H_2(M(\alpha); Z) (\approx Z)$. (See Kawauchi [6, Theorem 2.3].) Notice that $t\mu = -\mu$. Using the identity $\text{Im} i^* = \text{Hom}(\text{Im} i_*; Q)$ and the equality $(tu) \cap \bar{\mu} = -t^{-1}(u \cap \bar{\mu})$, the Poincaré dual isomorphism $\cap \bar{\mu}: \text{Im} \delta \approx \text{Im} i_*$ gives the equality $C(-t^{-1}) = B(t)$. This proves Lemma 3.6.

Lemma 3.9. This is a well-defined function

$$
\tau: \mathbb{C}_+(S^1 \times S^3) \to \mathbb{C}_+(S^1 \times S^3)
$$

induced by the 2-fold orientation covering.

Proof. Let $m \equiv \mathbb{C}_+(S^1 \times S^3)$ and $M(\alpha) \subseteq m$. Consider the infinite cyclic covering $p: \tilde{M}(\alpha) \to M(\alpha)$ associated with the Hurewicz homomorphism. Let $t$ be the generator of the covering transformation group of $\tilde{M}(\alpha)$ related to $\alpha$. The 2-fold covering $\tau': M' \to M(\alpha)$ from the orbits space $M' = \tilde{M}(\alpha)/t^2$ to $M(\alpha)$ induced by the projection $p: \tilde{M}(\alpha) \to M(\alpha)$ is the 2-fold orientation covering, since $\tilde{M}(\alpha)$ is orientable.

We must prove that $M'$ is a homology orientable handle. Let $p': \tilde{M}(\alpha) \to M'$ be the natural projection. The short exact sequence $0 \to C_\ell(\tilde{M}(\alpha)) \overset{t^2-1}{\to} C_\ell(\tilde{M}(\alpha)) \overset{p'}{\to} C_\ell(M') \to 0$ of simplicial chain $Z[t]$-modules induces the following exact sequence
of \( Z[t^2] \)-modules, where \( H_1(M'; Z) \) and \( H_0(M(\alpha); Z) \) are regarded as trivial \( Z[t^2] \)-modules. Let \( \varepsilon: Z[t^2] \rightarrow Z \) be the augmentation homomorphism such that \( \varepsilon(t^2) = 1 \). By taking a tensor product, we obtain an exact sequence

\[
H_1(\tilde{M}(\alpha); Z) \otimes \varepsilon Z \xrightarrow{\rho^* \otimes 1} H_1(M'; Z) \otimes \varepsilon Z \rightarrow H_0(\tilde{M}(\alpha); Z) \otimes \varepsilon Z \rightarrow 0.
\]

**Sublemma 3.9.1.** \( H_1(\tilde{M}(\alpha); Z) \otimes \varepsilon Z = 0. \)

By assuming this sublemma, we obtain that \( H_1(M'; Z) \approx Z \). By the Poincaré duality, \( M' \) is a homology orientable handle. Let \( \alpha' \in H_1(M'; Z) \) be a generator determined by \( \alpha \) under the 2-fold orientation covering \( \tau: M' \rightarrow M(\alpha) \). Let \( i \in H_0(M'; Z) \) be any generator. The distinguished homology orientable handles \( M'(\alpha', \iota), M'(\alpha', -\iota) \) have the same type, because \( t \) of \( M(\alpha) \) induces a homeomorphism \( t': M' \rightarrow M' \) with \( t'(\alpha') = \alpha' \) and \( t'(\iota) = -\iota \). This type is denoted by \( \tau(m) \). Thus the function \( \tau: \mathbb{Z}(S^1 \times S^1) \rightarrow \mathbb{Z}(S^1 \times S^1) \) is obtained. This completes the proof.

### 3.10. Proof of Sublemma 3.9.1

Note that there exists a presentation square matrix \( S(t) \) of \( H_1(\tilde{M}(\alpha); Z) \) as a \( Z[t] \)-module i.e. \( Z[t]^{2g} \xrightarrow{S(t)} Z[t]^{2g} \rightarrow H_0(\tilde{M}(\alpha); Z) \rightarrow 0 \) is exact for some integer \( g \geq 0 \). [To see this, let \( F \subset M(\alpha) \) be a closed orientable connected 2-sided surface in \( M(\alpha) \) intersecting a simple closed curve representing \( \alpha \) in a single point, and \( M^* \) be the manifold obtained from \( M(\alpha) \) by splitting along \( F \). Since \( M^* \) is orientable, we have an isomorphism \( H_1(M^*; Z) \approx H_1(F; Z) \). Let \( i_1, i_2: F \rightarrow F_1 \cup F_2 = \partial M^* \subset M^* \) be two natural identifications. With suitable bases of \( H_1(F; Z), H_1(M^*; Z), i_1*, i_2*: H_1(F; Z) \rightarrow H_1(M^*; Z) \) represent square integral matrices \( S_1, S_2 \), respectively. By applying the Mayer-Vietoris sequence, we obtain an exact sequence

\[
H_1(F; Z) \otimes Z[t] \xrightarrow{i_1*} H_1(M^*; Z) \otimes Z[t] \rightarrow H_1(\tilde{M}(\alpha); Z) \rightarrow 0,
\]

where \( i_1*(x) = ti_1*(x) - i_2*(x) \). Thus, we can obtain an exact sequence

\[
Z[t]^{2g} \xrightarrow{S(t)} Z[t]^{2g} \rightarrow H_1(\tilde{M}(\alpha); Z) \rightarrow 0,
\]

where \( S(t) = ti_1 - S_2. \)] By taking a tensor product, we obtain an exact sequence

\[
Z[t]^{2g} \otimes Z \xrightarrow{Z[t]^{2g} \otimes S(t)} Z[t]^{2g} \otimes Z \rightarrow H_1(\tilde{M}(\alpha); Z) \otimes Z \rightarrow 0.
\]

\[
[Z[t]/(t^2 - 1)]^{2g} \rightarrow [Z[t]/(t^2 - 1)]^{2g}
\]
We shall show that \( A'(t) = \det S'(t) \) is a unit in the quotient ring \( \mathbb{Z}[t]/(t^2-1) \).
Note that \( A(t) = \det S(t) \) is the Alexander polynomial of \( M(\alpha) \). So, \( A(t) \) satisfies \( A(t) = A(-t+1) \) and \( |A(1)| = 1 \). We can write 
\[
t^m A(t) = A(1) \text{ and } A'(t) = t^n A'(1) \text{ is a unit in } \mathbb{Z}[t]/(t^2-1),
\]
where \( n(s) = 0 \) if \( s \) is even, 1 if \( s \) is odd. This implies that the homomorphism \( S'(t): \mathbb{Z}[t] \to \mathbb{Z}[t] \) is an isomorphism. Therefore \( H_1(M(\alpha); \mathbb{Z}) \otimes \mathbb{Z} = 0 \). This proves Sublemma 3.9.1.

**Lemma 3.11.** The function \( \tau: \mathbb{C}_+(S^1 \times S^2) \to \mathbb{C}_+(S^1 \times S^2) \) carries the Alexander polynomial \( A(t) \) of any \( m \in \mathbb{C}_+(S^1 \times S^2) \) to the Alexander polynomial \( A'(t) \) of \( \tau(m) \in \mathbb{C}_+(S^1 \times S^2) \) such that \( A'(t^2) = A(t)A(-t) \).

**Proof.** Let \( M(\alpha) \subset m \). With a basis for \( H_1(M(\alpha); \mathbb{Z}), t: H_1(M(\alpha); \mathbb{Z}) \to H_1(M(\alpha); \mathbb{Z}) \) represents a matrix \( B \). Then \( A(t) = \det(tE - B) \). For the linear isomorphism \( t^2 = t^2: H_1(M(\alpha); \mathbb{Z}) \to H_1(M(\alpha); \mathbb{Z}) \) representing \( B^2 \), we have \( A'(t^2) = \det(tE - B^2) \). Hence,
\[
A'(t^2) = \det(tE - B^2) = \det(tE - B) \det(tE + B) = A(t)A(-t).
\]
This completes the proof.

The reduced Alexander polynomial \( \tilde{A}(t) \) of \( m \in \mathbb{C}_+(S^1 \times S^2) \) is the integral polynomial obtained from the Alexander polynomial \( A(t) \) of \( m \) by cancelling the factors of the type \( f(t)f(-t^{-1}) \).

**Theorem 3.12.** The function \( \tau: \mathbb{C}_+(S^1 \times S^2) \to \mathbb{C}_+(S^1 \times S^2) \) induces a homomorphism \( \tau^*: \Omega(S^1 \times S^2) \to T_2 \subset \Omega(S^1 \times S^2) \) carrying the reduced Alexander polynomial \( \tilde{A}(t) \) to the reduced Alexander polynomial \( \tilde{A}'(t) \) such that \( A'(t^2) = A(t)A(-t) \), where \( T_2 \) is the subgroup of \( \Omega(S^1 \times S^2) \) consisting of elements of order 2.

**Proof.** For \( m_1, m_2 \in \mathbb{C}_+(S^1 \times S^2) \), the equality \( \tau(m_1 \circ m_2) = \tau(m_1) \circ \tau(m_2) \) is easily obtained. For \( m \in \mathbb{C}_+(S^1 \times S^2) \), assume \( m \sim 0 \). Then for \( M(\alpha) \in m \) there exists an \( \bar{H} \)-cobordism \( (W, M(\alpha), S^1 \times S^2) \). The 2-fold orientation cover \( (W', M', S^1 \times S^2) \) of \( (W, M(\alpha), S^1 \times S^2) \) gives an \( \bar{H} \)-cobordism. So, \( \tau(m) \sim 0 \). Therefore \( \tau^* \) is a homomorphism to \( T_2 \). The remainder follows from Corollary 2.16 and Lemmas 3.6 and 3.11. This completes the proof.

**Corollary 3.13.** \( T_2 \) is infinitely generated.

**Proof.** Consider for example \( m_n \in \mathbb{C}_+(S^1 \times S^2) \) with Alexander polynomial \( A_n(t) = nt^2 + t - n, n = 1, 2, 3, \ldots \), as in 3.7. Then the Alexander polynomial of the 2-fold orientation cover \( \tau(m_n) \) is \( A_n(t) = nt^2 - (2n^2 + 1)t + n^2 \). Since for
n=1, 2, 3,⋯ these Alexander polynomials $A_n(t)$ are irreducible and mutually distinct, the set \{τ($m_1$), τ($m_2$), τ($m_3$), ⋯\} gives a linearly independent subset of $T_2$, which completes the proof.

One may ask whether the subgroup $T_2'$ of order-2-elements of the Fox-Milnor's knot cobordism group $C^1$ is infinitely generated.

As a matter of fact, $T_2'$ is also infinitely generated, although it seems to be difficult to set up a general argument.

Claim. $T_2'$ is infinitely generated.

In fact, consider the knot $k_n \subset S^3$ with the numbers of crossings $2n$, $2n$, illustrated in figure 8. In the case $n=1$, this knot $k_1$ is called the figure eight knot: $k_1=4_1$ (See figure 8').

One can easily shown*) that each knot $k_n \subset S^3$ is -amphicheiral**) by an analogy of the method which is used for showing that the figure eight knot is -amphicheiral. Since the Alexander polynomial of $k_n \subset S^3$ is $A_n(t) = n^2t^2 - (2n^2 + 1)t + n^2$, which is irreducible, it follows that $T_2'$ is infinitely generated.

One can also derive the conclusion of Corollary 3.13 by using these knots.

In concluding this paper, the author would like to propose a few questions and one interesting conjecture.

Question. Is $\text{Im} \, \tau^* = T_2'$?


**) An oriented knot $k \subset S^3$ is said to be -amphicheiral, if $-k \subset S^3$ and $-k \subset -S^3$ belong to the same knot type. (See Fox [2, pl 143] for details.)
This question seems closely related to a question due to Fox and Milnor: Is an element of order 2 of $C_1$ necessarily determined by a $\alpha$-amphicheiral knot? One may also ask whether $\tau^*$ is injective, although the author expects a negative answer.

The following conjecture seems to be justified by Lemma 3.11.

**Conjecture.** The Alexander polynomial $A(t)$ of a $\alpha$-amphicheiral knot necessarily satisfies $A(t^2) = f(t)f(-t)$ for some integral polynomial $f(t)$ with $f(t) = f(-t^{-1})$.

One can easily check that any $\alpha$-amphicheiral knot in the Alexander and Briggs knot table satisfies this assertion. For example, the Alexander polynomial of the knot 812 which is known to be $\alpha$-amphicheiral is $A(t) = t^4 - 7t^3 + 13t^2 - 7t + 1$. Then,

$$A(t^2) = (t^4 + t^3 - 3t^2 - t + 1)(t^4 - t^3 - 3t^2 + t + 1).$$

**References**


