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FOUR-GENUS AND UNKNOTTING NUMBER OF POSITIVE KNOTS AND LINKS

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1. Introduction

A *link* is a closed 1-manifold embedded in the 3-sphere S^3 and a *knot* is a link with one connected component. The *unknotting number* of a knot K, denoted by $u(K)$, is the minimum number of crossing changes needed to create an unknotted diagram, where the minimum should be taken over all possible sets of changes in all possible diagrams representing K. The *4-genus* (*3-genus* resp.) of a link L, denoted by $g^*(L)$ (by $g(L)$ resp.), is the minimum number of genera of all smooth compact connected and orientable surfaces bounded by $L \subset \partial B^4 = S^3$ in B^4 (in S^3 resp.). These are very intuitive invariants of knot and link types, but hard to compute for a given knot or link. L. Rudolph has studied the links which appear as the intersection of $S^3 = \{(z, w) \mid |z|^2 + |w|^2 = 1,\ z, w \in \mathbf{C}\}$ and an algebraic curve in the 2-dimensional complex plane $\mathbf{C}^2$ (cf. [10]). He obtained a formula for computing the 4-genera of *quasipositive links*, which are algebraic curves. In this paper, by using his results, we give a formula to compute directly the 4-genera of *positive links* from any positive diagrams.

Theorem 1.1. *Let L be a positive link with μ components, and D any non-split positive diagram of L. Then we have*

$$g(L) = g^*(L) = \frac{2 - \mu - s(D) + c(D)}{2},$$

where $s(D)$ is the number of Seifert circles, $c(D)$ is the number of crossings of D.

We remark that we can compute the genus of a splittable positive link from a split positive diagram by applying Theorem 1.1 to each connected component of diagrams and by summing up their genera.

In order to prove Theorem 1.1 we make a precise observation on Yamada's algorithm concerned with braid representations of knots and links, and apply Rudolph's theorem.

Using Theorem 1.1, we determine the positive knots with unknotting number one (Theorem 5.1). This gives an alternate proof of Przytycki-Taniyama's theorem [9].

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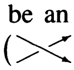
2. Positive Knots and Links

In this section, we recall the definition of positive knots and links.

DEFINITION 2.1. Let K be an oriented knot or link, and D a diagram of K. If D has only positive crossings (), then we say that D is a *positive diagram* for K and K is a *positive knot* or *link*.

For positive knots and links, their polynomial invariants and ordinary knot signatures which could be computed with those figures (in the sense of diagrams) have been studied. On the other hand, there is not a complete criterion whether a given knot or link is positive or not, except that we can describe a positive diagram or not. But for 249 prime knots with 10 crossings or less, we can check their positivity by using the following three results, so that there are 42 positive knots among them.

Theorem 2.1 ([1]). *A positive link has a positive Conway polynomial* $\nabla_L(z)$.

Theorem 2.2 ([8]). *Let K be a nontrivial positive knot, then its signature $\sigma(K)$ is negative.*

Theorem 2.3 ([6]). *Let K be a knot, D a diagram of K, and $V_K(t)$ the Jones polynomial of K. Then we have*

$$\operatorname{mindeg} V_K(t) \geq -c_-(D) - \frac{1}{2}\sigma(K),$$

where $c_-(D)$ is the number of negative crossings of D.

Besides we have only eight *closed positive braids* in the 249 prime knots with 10 crossings or less. We can choose eight closed positive braids from the 42 positive knots using the following theorem.

Theorem 2.4 ([15]). *Let K be a closed positive braid with* $\operatorname{maxdeg} \nabla_K(z) = 2m$ *(m : a positive integer). Then we have*

$$\nabla_K(z) = z^{2m} + a_{2m-2}z^{2m-2} + \cdots + a_4z^4 + a_2z^2 + 1,$$

where $\binom{m}{k} \leq a_{2(m-k)} \leq \binom{2m-k}{k}$.

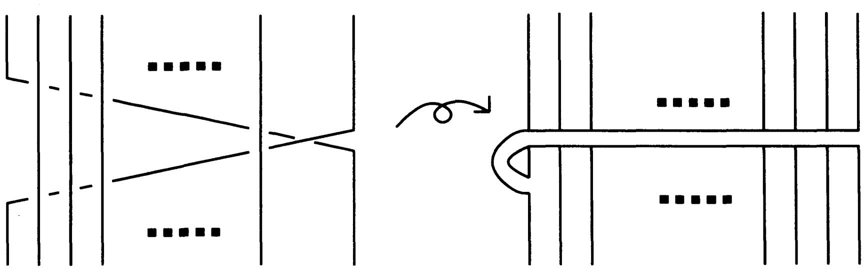


Fig. 1.

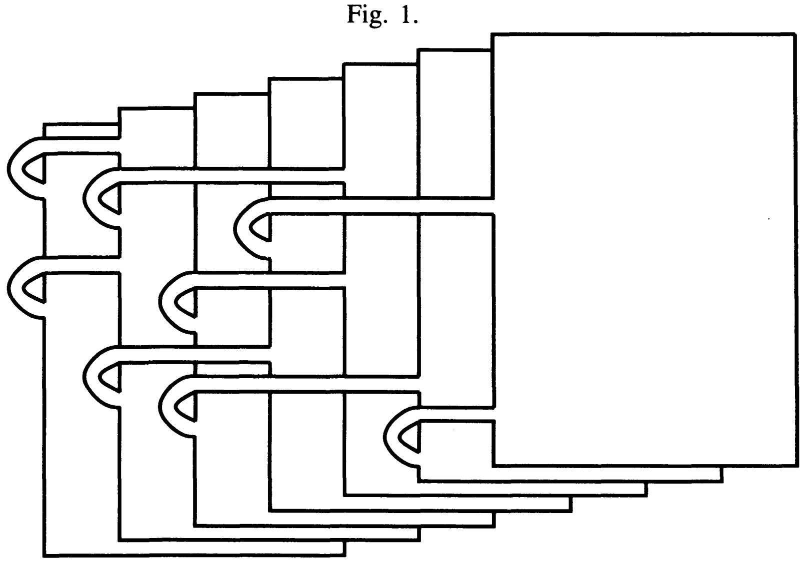


Fig. 2.

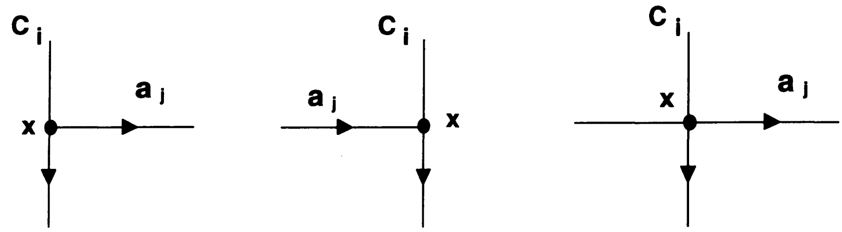
3. Rudolph's Theorem

In this section we briefly survey a work of L. Rudolph concerned with the genera of a *quasipositive link* defined as follows:

DEFINITION 3.1. For a generator σ_i in the n-braid group B_n, a *positive band* is a conjugate $w\sigma_i w^{-1}$ ($w \in B_n$). And a *positive embedded band* $\sigma_{i,j}$ is one of the positive band defined as $\sigma_{i,j} := (\sigma_i\sigma_{i+1}\cdots\sigma_{j-2})\sigma_{j-1}(\sigma_i\sigma_{i+1}\cdots\sigma_{j-2})^{-1}$ ($1 \leq i < j \leq n$). Fig. 1 illustrates a positive embedded band, and we can easily see a *band* by turning over the left side of Fig. 1 and deforming it like as in the right side.

A (*strongly*) *quasipositive braid* is a product of positive (embedded) bands, and a (*strongly*) *quasipositive link* is a closure of that braid.

A strongly quasipositive link is a product of positive embedded bands as in the



(a) (b)

Fig. 3.

right side of Fig. 1, so it can be easily seen that this link bounds a surface which called a *quasipositive Seifert surface* [11] in S^3 like as in Fig. 2.

The following theorem is proved by L. Rudolph in [12], where $\chi(L)$ ($\chi_s(L)$ resp.) is the maximal Euler characteristic number of an oriented 2-manifold without closed components smoothly embedded in S^3 (in B^4 resp.) with boundary L.

Theorem 3.1 ([12]). *Let L be a strongly quasipositive link, $\beta \in B_n$ a strongly quasipositive braid with L as the closure, where $\beta = w_1\sigma_{i_1}w_1{}^{-1}\cdots w_k\sigma_{i_k}w_k{}^{-1}$. Let S be a quasipositive Seifert surface bounded by L. Then we have*

$$\chi(L) = \chi_s(L) = \chi(S) = n - k,$$

where k is the number of positive embedded bands, n is the number of strings of β.

REMARK. This is a restatement of [12, Corollary] in present terminology.

4. Bunching Operation

We show the following lemma in Section 5 to prove Theorem 1.1 using Yamada's algorithm, bunching operation, concerned with braid representations of knots and links.

Lemma 4.1. *Every positive link is a strongly quasipositive link.*

In this section, we define bunching operations on a *Seifert circle system*.

DEFINITION 4.1. A Seifert circle system S is defined as a system which consists of disjoint oriented circles $C_1, C_2, \ldots, C_n$ and disjoint oriented simple arcs a_1, $a_2, \ldots, a_r$ on S^2 such that each C_i is weighted with a positive integer $w(C_i)$, and an intersection of a circle and an arc in a system is diffeomorphic to one of the situations as in Fig. 3. Then we denote $S = \{C_1, C_2, \ldots, C_n; a_1, a_2, \ldots, a_r\}$.

Now, we compose a link diagram from a Seifert circle system. At first we replace

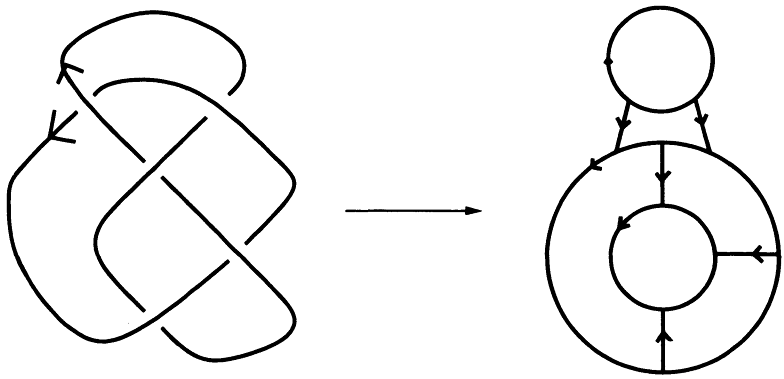


Fig. 4.

every C_i with an arbitrary $w(C_i)$-string closed braid which is parallel to C_i. Secondly we replace every a_i with a braid whose number of strings is a sum of weights of all circles which intersect a_i, and whose direction is according to that of circles which transversally intersect a_i. When the system consists of only one circle, we call it a *trivial system* ($S = \{C;\quad\}$). This presents a $w(C)$-string closed braid. If a Seifert circle system is given, a set of link diagrams which are presented by that system is determined.

Conversely we can obtain a Seifert circle system by smoothing for the crossings in a link diagram D on S^2 and inserting oriented arcs instead of the crossings. The orientation of the arcs is according to that of an arc which is one of the situations in Fig. 3. We call this system the *Seifert circle system derived from D*. (See Fig. 4.)

We define two kinds of operations for two circles in a Seifert circle system. For that purpose we define a relation among circles which are operated.

DEFINITION 4.2. Let C_1, C_2 be oriented circles on S^2. C_1 and C_2 are coherent (anti-coherent resp.) if and only if $[C_1] = [C_2]$ ($-[C_2]$ resp.) $\in \mathrm{H}_1(A)$ for the annulus A bounded by C_1, C_2 on S^2.

Now, we define the bunching operations to a Seifert circle system

$$S = \{C_1, C_2, \dots, C_n; a_1, a_2, \dots, a_r\}$$

as follows:

DEFINITION 4.3. If there are two coherent circles C_i and C_j such that $\mathrm{Int}(A) \cap \{C_1, \dots, C_n\} = \phi$ for the annulus A bounded by C_i, C_j on S^2. Let C be a core circle of A and $f : A \to C$ a continuous map such that $f|_{C_i}$ and $f|_{C_j}$ are homeomorphisms, such that if $A \cap a_k \neq \phi$ then $f(A \cap a_k)$ is one point on C, and such that if $C_i \cap a_k \neq \phi$,

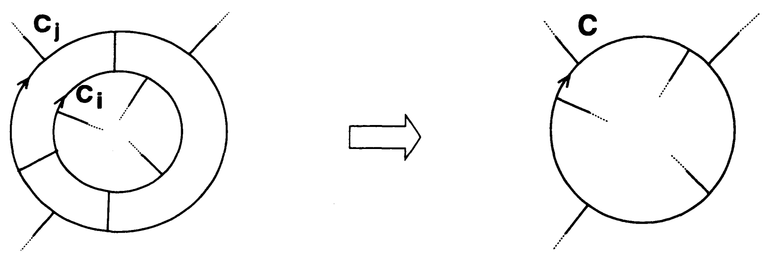


Fig. 5.

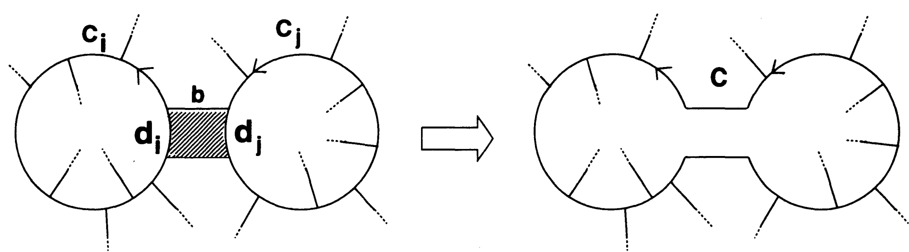


Fig. 6.

$C_j \cap a_l \neq \phi$ and $a_k \neq a_l$ then $f(C_i \cap a_k) \neq f(C_j \cap a_l)$. Then the quotient space $(S^2 \cup C)/\langle f(x) \sim x\rangle$ is a 2-sphere. We define the new system S' to be $(S \cup C)/\langle f(x) \sim x\rangle$ with $w(C) = w(C_i)+w(C_j)$. (See Fig. 5.) We say that S' is derived from S by applying a *bunching operation of type I*, denoted by bI, to C_i and C_j.

DEFINITION 4.4. If there are two anti-coherent circles C_i and C_j, and a band b on S^2 such that $S \cap b = \partial b \cap (C_i \cup C_j) = d_i \cup d_j$ and $b \cap \{a_1 \cup \cdots \cup a_r\} = \phi$, where d_i and d_j are subarcs of C_i and C_j respectively, then let $S' = \{C_1, \ldots, \check{C}_i, \ldots, \check{C}_j, \ldots, C_n, C; a_1, \ldots, a_r\}$. Here $\check{C}_i, \check{C}_j$ means the deleting of these circles, $C = (C_i \cup C_j \cup \partial b) - \mathrm{Int}(d_i \cup d_j)$, and the orientation of C is determined from those of C_i and C_j naturally with $w(C) = w(C_i) + w(C_j)$. (See Fig. 6.) We say that S' is derived from S by applying a *bunching operation of type II*, denoted by bII, to C_i and C_j.

The following theorem is concerned with these operations and braid representations of knots and links.

Theorem 4.1 ([16]). *Any Seifert circle system S can be deformed to a trivial system by a finite sequence of bI,bII without changing of the writhe and the number of Seifert circles of the link diagram D from S.*

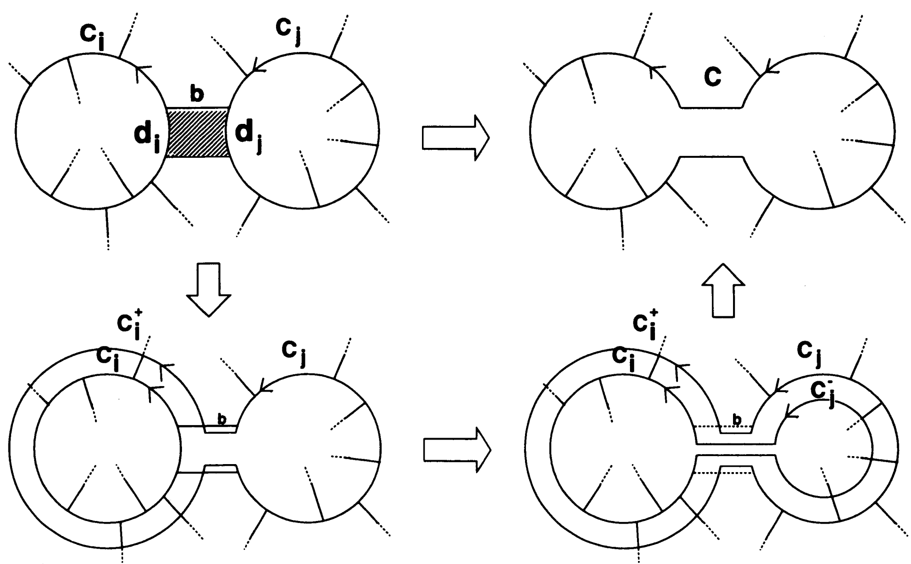


Fig. 7.

5. Proofs

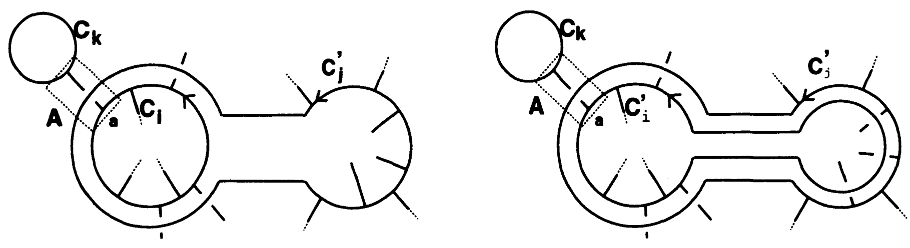
Now, we prove Lemma 4.1.

Proof of Lemma 4.1. We claim that we can deform any positive diagram to a strongly quasipositive braid representation by bunching operations. For that purpose we show that operations bI and bII preserve the positive embedded band.

Let $S(D_+)$ be a Seifert circle system derived from a positive diagram D_+; $S(D_+) := \{C_1, \dots, C_n; a_1, \dots, a_r\}$ $(n \geq 2, r \geq 1)$, $w(C_i) = 1$ for any i, and each a_i presents a positive crossing in D_+. Before operating, this system $S(D_+)$ has no intersection of a circle and an arc like as in Fig. 3(b). In this situation, each arc corresponds to a positive embedded band.

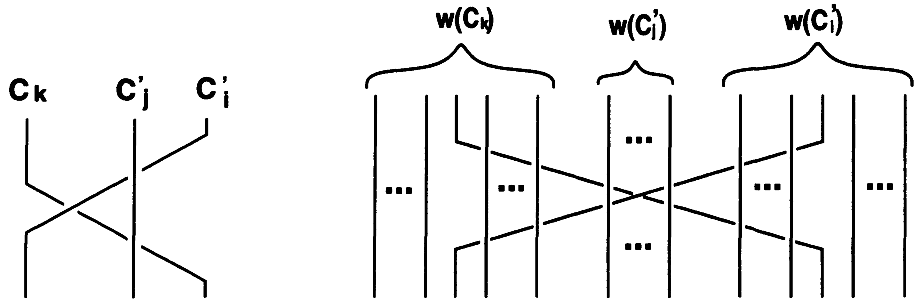
At first we consider an operation bI. This operation changes only a representation of a system, and does not change the diagram. Secondly we consider an operation bII with a precise observation of an influence on the diagram. We can regard an operation bII as a following sequence. (See Fig. 7.)

Let C_i, C_j be circles of $S(D_+)$ to which an operation bII is applied, and A_i, A_j tubular-neighborhood of C_i, C_j on S^2 respectively. The circle C_i divided S^2 into two regions. Let C_i^+ be one of components of ∂A_i which lies on the same region as C_j, and C_j^- one of components of ∂A_j which lies on a different region from C_i^+. Then we have a new circle C_j' which is a fusion of C_j and C_i^+ on S^2. We make a same argument over C_j', C_j^- and C_i as C_i, C_i^+ and C_j respectively. Then we have a new circle C_i' which is a fusion of C_i and C_j^- on the region bounded by C_j'. We can regard these



(a) (b)

Fig. 8.



(a) (b)

Fig. 9.

operations as deforming the subarc d_j (d_i resp.) of C_j (C_i resp.) along band b by an isotopy. (See Fig. 8.) And we apply an operation bI to C_i', C_j'. Then this completes the operation bII.

These deformations have an influence on the diagram as follows. We choose the deforming of d_j (d_i resp.) such that we put d_j (d_i resp.) *over* all arcs connected to C_i (C_j resp.) for every deformations until we get a trivial system.

Let C_k be one of the circles connected to C_i except C_j, and a one of the arcs connecting C_i and C_k. We observe the domain A, bounded by dotted rectangle, with C_k, C_j', C_i' and a on the diagram. (See Fig. 8.)

If the deformation of an operation bII to $S(D_+)$ is the first deformation, then the domain A on the diagram is presented by Fig. 9(a). This is a positive embedded band. Assume that the positive embedded band is preserved after several deformations. If we do deformation on arc a (we use the same figure and the same marks unless it makes confusion), then we get Fig. 9(b). This is also a positive embedded band.

Now we choose the deformation such that we put a circle *over* arcs for every operation bII. Then positive embedded bands are inductively preserved. By Theorem 4.1, we get a trivial system with a product of positive embedded bands. The proof is completed. □

Now we can prove Theorem 1.1 by using Rudolph's theorem.

Proof of Theorem 1.1. Let L be a positive link with μ components, D_+ a non-split positive diagram of L, and β a strongly quasipositive braid with L as the closure. By Lemma 4.1 and Theorem 4.1, the numbers of strings and positive embedded bands of β equal to the numbers of Seifert circles and crossings of D_+, denoted by $s(D_+)$ and $c(D_+)$, respectively. By Rudolph's theorem, we have

$$g(L) = g^*(L) = \frac{2 - \mu - s(D_+) + c(D_+)}{2}.$$

This completes the proof. □

By Theorem 1.1, we obtain the following theorem.

Theorem 5.1. *Let K be a nontrivial positive knot. It has the unknotting number one if and only if it is a twist knot.*

Proof. The sufficient condition is obvious. Now we prove a necessary condition. Let K be an unknotting number one positive knot. By Theorem 2.2, the 4-genus $g^*(K)$ of K is one because of the relation $|\sigma(K)/2| \leq g^*(K) \leq u(K)$ (cf. [5]). Let D be a reduced positive diagram of K. By Theorem 1.1, we have

$$g^*(L) = \frac{2 - 1 - s(D) + c(D)}{2} = 1,$$

therefore we get the following equation.

$$c(D) - s(D) = 1. \tag{1}$$

Now, we consider a *Seifert graph* which is the spine of a Seifert surface F obtained from D by Seifert's algorithm. The vertices of the graph correspond to the discs of F which are bounded by Seifert circles of D. The edges of the graph correspond to the twisted rectangles of F, hence to the crossings in D. We note that it is a planar graph. Since D is a reduced knot diagram, the Seifert graph G of D is a connected graph without degree-one vertices. From the equation (5.1), we have that the first Betti number of G is two. Hence the topological type of G is one of the following; *theta-curve*, *one vertex and two loops*, *one bridge and two loops*. But one bridge and two loops contradicts that D is a reduced diagram. So we have two types of the G as in Fig. 10. Note that each cycle of the G has even edges, because a Seifert surface is orientable.

Thus we see that D is a pretzel knot diagram with three strands which have odd crossings. Then by Kobayashi [4], 3-genus one and unknotting number one knots are doubled knots of certain knots. (By Theorem 1.1, 3-genus is equal to 4-genus on a

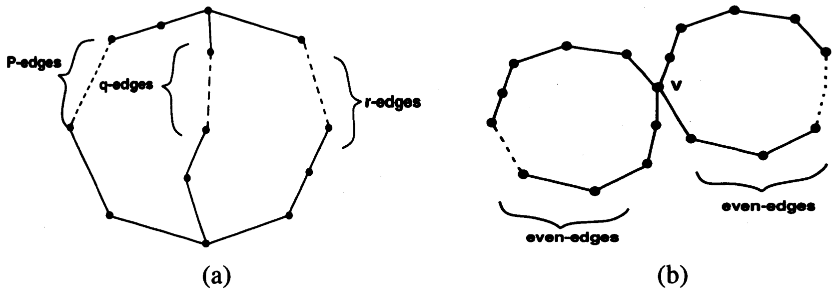


Fig. 10.

positive link.) And by Kawauchi [3], every pretzel knot is *simple*. Therefore K is a twist knot. □

Addendum. This paper is a part of the author's Master thesis at Keio university in February 1998 [7]. More recently, L. Rudolph showed the same result as Lemma 4.1 in [13] independently.

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