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THE CASSON-WALKER INVARIANT FOR BRANCHED CYCLIC COVERS OF S^3 BRANCHED OVER A DOUBLED KNOT

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0. Introduction

In 1985, A. Casson defined an invariant λ for oriented integral homology 3-spheres by using representations from their fundamental group into $SU(2)$ [1]. It was extended to an invariant for rational homology 3-spheres by K. Walker [11]. In 1993, C. Lescop [9] gave a formula to calculate this invariant for rational homology 3-spheres when they are presented by framed links and showed that it naturally extends to an invariant for all 3-manifolds.

Let L be a link in S^3 and let Σ_L^n be its n-fold cyclic branched cover. Define $\lambda_n(L) = \lambda(\Sigma_L^n)$. Then λ_n becomes an invariant of links. For doubles of knots, torus knots and iterated torus knots, A. Davidow (see [3], [4]) calculated Casson's integer invariant for n-fold branched covers, when Σ_K^n is an integral homology sphere. For any links, D. Mullins [10] have succeeded in calculating Casson-Walker's rational valued invariant for 2-fold branched covers, when Σ_L^2 is a rational homology sphere.

In this paper, using C. Lescop's formula and the result of D. Mullins, we will calculate the Casson-Walker invariant for branched cyclic covers of S^3 branched over the m-twisted double of a knot. We will show the following theorem and corollary.

Theorem 3.1. *Let K be a knot in S^3 and D_mK its m-twisted double. Then $\lambda_n(D_mK)$ is determined by $d/dtV_{D_mK}(-1)$ and m where $d/dtV_{D_mK}(-1)$ is the derivative of the Jones polynomial of D_mK at $t=-1$.*

Corollary 3.2. *$\lambda_n(D_mK)$ is determined by $a_1(K)$ and m where $a_1(K)$ is the coefficient of z^2 of the Conway polynomial of K.*

1. Preliminaries

DEFINITION 1.1. The Conway polynomial $\nabla_L(z)$ of an oriented link L is defined by

1. $\nabla_U(z) = 1$, where U is an unknot,

2. $\nabla_{L^+}(z) - \nabla_{L^-}(z) = -z\nabla_{L^o}(z)$, where L^+, L^-, L^o are oriented links identical except within a ball where they are as shown in Figure 1.

It is well known that the Conway polynomial is of the form

$$\nabla_L(z) = z^{\sharp L-1}(a_0(L) + a_1(L)z^2 + \cdots + a_{d(L)}z^{2d(L)}).$$

This defines the coefficients $a_i(L)$ of $\nabla_L(z)$.

Let K be a knot in S^3 and D_mK its m-twisted double. It is easy to see that $\nabla_{D_mK}(z) = 1 - mz^2$. Thus $a_1(D_mK) = -m$ and $a_i(D_mK) = 0$ for $i \geq 2$.

DEFINITION 1.2. The Jones polynomial $V_L(t)$ of an oriented link L is defined by

1. $V_U(t) = 1$, where U is an unknot,
2. $t^{-1}V_{L^+}(t) - tV_{L^-}(t) = (t^{1/2} - t^{-1/2})V_{L^o}(t)$, where L^+, L^-, L^o are oriented links identical except within a ball where they are as shown in Figure 1.

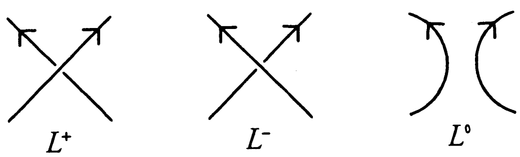


Fig. 1.

Let W_L be a Seifert matrix for an oriented link L.

DEFINITION 1.3. The signature $\sigma(L)$ of L is defined as

$$\sigma(L) = signature(W_L + W_L^T).$$

DEFINITIONS 1.4. Let $\mathcal{L} = \{(K_1, \mu_1), \ldots, (K_n, \mu_n)\}$ be a framed link in S^3, where each component K_i is oriented and μ_i gives integer framing. The manifold obtained by surgery on $\mathcal{L}$ is denoted by $\chi(\mathcal{L})$. Let L denote the underlying link of $\mathcal{L}$. Let l_{ij} be the linking number $lk(K_i, K_j)$ of K_i and K_j if $i \neq j$ and μ_i if $i = j$.

- The linking matrix of $\mathcal{L}$ is defined by

$$E(\mathcal{L}) = (l_{ij})_{1 \leq i,j \leq n}.$$

- The sign of $\mathcal{L}$, denoted by $sign(\mathcal{L})$, is equal to $(-1)^{b_-(\mathcal{L})}$ where $b_-(\mathcal{L})$ denotes the number of negative eigen values of $E(\mathcal{L})$.
- Restriction of a framed link.

If I is a subset of $N = \{1, \dots, n\}$, then $\mathcal{L}_I$ (resp. L_I) denotes the framed link obtained by $\mathcal{L}$ (resp. the link obtained by L) by forgetting the components whose subscripts do not belong to I.

- The circular linking of $\mathcal{L}_I$, denoted by $Lk_c(\mathcal{L}_I)$, is defined by

$$Lk_c(\mathcal{L}_I) = \sum_{\sigma \in \sigma_I} \left(\prod_{k \in I} l_{k\sigma(k)} \right),$$

where σ_I denotes the set of cyclic permutations of I.

- The θ-linking of $\mathcal{L}_I$.

Let $\theta_b(\mathcal{L}_I)$ be defined by

$$\theta_b(\mathcal{L}_I) = \sum_{\{(K,i,j,g) \mid K \subset I, (i,j) \in K^2, g \in S_{I \setminus K}\}} Lk_c(\mathcal{L}_K) l_{ig(1)} l_{g(1)g(2)} \cdots l_{g(\sharp(I\setminus K)-1)g(\sharp(I\setminus K))} l_{g(\sharp(I\setminus K))j}.$$

($S_{I \setminus K}$ denotes the set of one to one maps from $\{1, \dots, \sharp(I \setminus K)\}$ to $I \setminus K$.) This sum can be seen as the sum of the linking numbers of $\mathcal{L}_I$ with respect to the edes of one of the graphs in Figure 2 for all combinatorial ways of constructing such graphs.

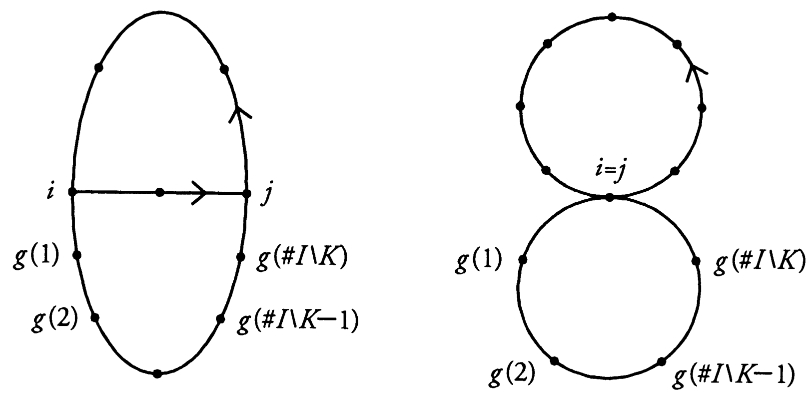


Fig. 2.

Then, the θ-linking of $\mathcal{L}_I$, denoted by $\theta(\mathcal{L}_I)$, is defined by

$$\theta(\mathcal{L}_I) = \begin{cases} \theta_b(\mathcal{L}_I) & \text{if } \sharp I > 2 \\ \theta_b(\mathcal{L}_I) - 2l_{ij} & \text{if } I = \{i,j\} \\ \theta_b(\mathcal{L}_I) + 2 & \text{if } I = \{i\}. \end{cases}$$

- The modified linking matrix $E(\mathcal{L}_{N\setminus I}; I)$ is defined by

$$E(\mathcal{L}_{N\setminus I}; I) = (l_{ijI})_{i,j \in N\setminus I}$$

with

$$l_{ijI}=\begin{cases} l_{ij} & \text{if } i\neq j \\ l_{ii}+\sum_{k\in I} l_{ki} & \text{if } i=j. \end{cases}$$

We state C. Lescop's formula for the Casson-Walker invariant.

Proposition 1.5 ([9]). *Let $\mathcal{L}$ and $\chi(\mathcal{L})$ be as above. Then the Casson-Walker invariant λ of $\chi(\mathcal{L})$ is given by*

$$\begin{aligned}&\lambda(\chi(\mathcal{L}))\\ &= sign(\mathcal{L}) \sum_{\{J\mid J\neq\phi, J\subset N\}} \left(\det(E(\mathcal{L}_{N\setminus J}; J)) a_1(L_J) + \frac{\det(E(\mathcal{L}_{N\setminus J}))(-1)^{\sharp J}\theta(\mathcal{L}_J)}{24} \right) \\ &\qquad + sign(\mathcal{L}) \det(E(\mathcal{L})) \frac{signature(E(\mathcal{L}))}{8},\end{aligned}$$

where the determinant of an empty matrix equals to one.

REMARK 1.6. We follow C. Lescop's normalization of the Casson-Walker invariant. If λ_w denotes the Walker invariant as described in [11],

$$\lambda(M)=\frac{|H_1(M;\mathbf{Z})|}{2}\lambda_w(M).$$

Finally, we state the result of D. Mullins for two-fold branched covers.

Proposition 1.7 ([10]). *Let L be a link in S^3. Suppose the two-fold branched cover of L, Σ_L^2, is a rational homology sphere. Then*

$$\lambda_2(L)=-\frac{i^{\sigma(L)}}{12}\frac{d}{dt}V_L(-1)+|H_1(\Sigma_L^2;\mathbf{Z})|\frac{\sigma(L)}{8}.$$

Note that if L is a knot, then Σ_L^2 is a rational homology sphere.

2. Surgery description of cyclic branched covers

Let D_mK be the m-twisted double of a knot K in S^3. If we introduce one surgery curve C which have the framing 1 for the crossing that must be changed to obtain an unknot, we may arrive at a surgery description of D_mK as shown in Figure 3, where $\overline{D_mK}$ is an unknot corresponding to D_mK in other version of S^3.

Applying an isotopy to S^3, we can exchange the position of $\overline{D_mK}$ with that of C (see Figure 4).

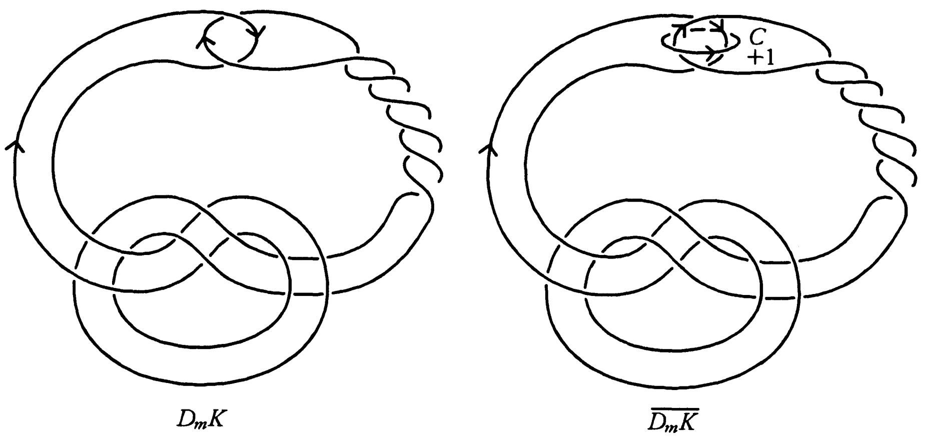


Fig. 3.

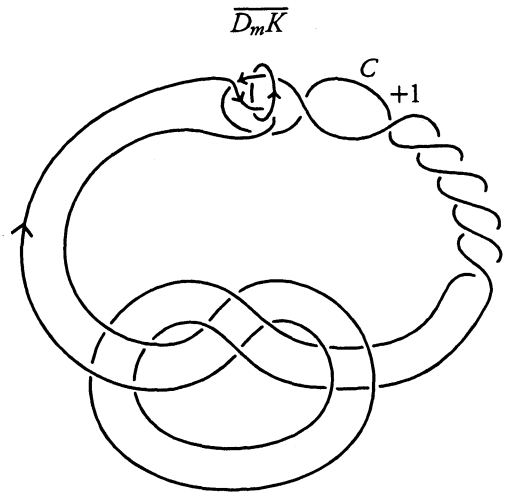
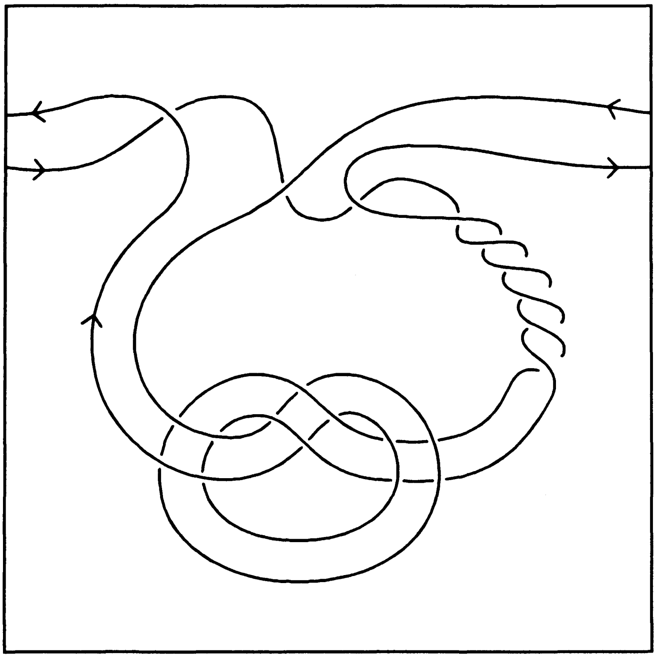


Fig. 4.

Let T_{D_mK} denote the tangle which is obtained by cutting C (at two points) by a spanning 2-disk for $\overline{D_mK}$ as in Figure 5.

Note that T_{D_mK} has two arcs. By joining n-copies of T_{D_mK} cyclically, we obtain an n-component link $L^n_{D_mK} = \{K_1, \ldots, K_n\}$ as in Figure 6.

Then the n-fold cyclic branched cover of S^3 branched over D_mK, $\Sigma^n_{D_mK}$, is obtained by a surgery on the link $L^n_{D_mK}$. Note that $lk(K_1, K_2) = lk(K_2, K_3) = \cdots = lk(K_n, K_1)$ and the framing of the component K_i is equal to $-2lk(K_1, K_2)+1$ if $n \geq 3$ and is equal to $-lk(K_1, K_2) + 1$ if $n = 2$.



tangle T_{D_mK}

Fig. 5.

Lemma 2.1.

$$lk(K_1, K_2) = \begin{cases} -m & \text{if } n \geq 3 \\ -2m & \text{if } n = 2. \end{cases}$$

Proof. Consider the crossing of D_mK that must be changed to obtain an unknot. From the skein relation of the Conway polynomial, we get $a_1(D_mK) = -a_0(K^o)$ where K^o is the 2-component link obtained by splicing the crossing of D_mK. On the other hand, $-a_0(K^o)$ is equal to the linking number of the components of K^o (see [5]). But this is equal to $lk(K_1, K_2)$ if $n \geq 3$ and is equal to $1/2lk(K_1, K_2)$ if $n = 2$. Noting that $a_1(D_mK) = -m$, we get the conclusion. □

Then from Lemma 2.1, we can express the framing in terms of the twisting number m. Each K_i has the framing $1+2m$. Thus $\mathcal{L}^n_{D_mK} = \{(K_1, 1+2m), \ldots, (K_n, 1+2m)\}$ is a framed link for $\Sigma^n_{D_mK}$.

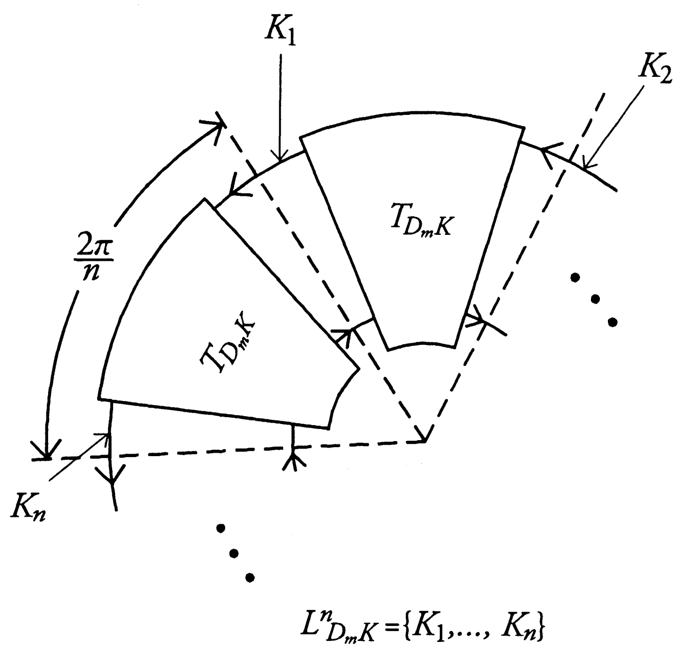


Fig. 6.

3. Calculation of λ_n

In this section, we will prove Theorem 3.1 and Corollary 3.2 and give a formula for λ_n.

Theorem 3.1. *Let K be a knot in S^3 and D_mK its m-twisted double. Then $\lambda_n(D_mK)$ is determined by $d/dtV_{D_mK}(-1)$ and m where $d/dtV_{D_mK}(-1)$ is the derivative of the Jones polynomial of D_mK at $t=-1$.*

Corollary 3.2. *$\lambda_n(D_mK)$ is determined by $a_1(K)$ and m where $a_1(K)$ is the coefficient of z^2 of the Conway polynomial of K.*

Proof of Theorem 3.1 and Corollary 3.2

The case of $n=2$ follows from Proposition 1.7 and the fact that $|H_1(\Sigma^2_{D_mK};\mathbf{Z})| = |1-4a_1(D_mK)| = |1+4m|$. So, assume that $n \geq 3$. We use Proposition 1.5 and the surgery description of $\Sigma^n_{D_mK}$. Let $\mathcal{L}^n_{D_mK} = \{(K_1, 1+2m), \ldots, (K_n, 1+2m)\}$ be the surgery description for $\Sigma^n_{D_mK}$ as in section 2. The linking matrix is determined by m as follows;

$$E(\mathcal{L}^n_{D_mK}) = \begin{pmatrix} 1+2m & -m & & & -m \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ -m & & & -m & 1+2m \end{pmatrix}.$$

Then $sign(\mathcal{L}^n_{D_mK})$, $\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J}; J))$, $\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J}))$, $\theta((\mathcal{L}^n_{D_mK})_J)$, $\det(E(\mathcal{L}^n_{D_mK}))$ and $signature(E(\mathcal{L}^n_{D_mK}))$ are also determined by m. So we want to know whether or not $a_1((L^n_{D_mK})_J)$ can be expressed in terms of the original data of D_mK.

To do this we introduce the following notations and proposition. For given tangles A and B, the tangle $A+B$ is defined as in Figure 7. Also, there are two operations that associate knots and links to a given tangle A. These are denoted $N(A)$ and $D(A)$ as in Figure 7.

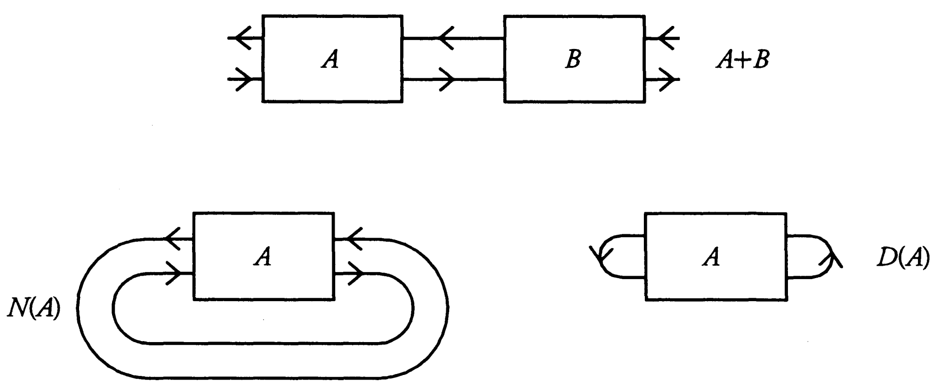


Fig. 7.

Proposition 3.3 ([2], [7]). *Let A and B be tangles. Then*

$$\nabla_{N(A+B)}(z) = \nabla_{N(A)}(z)\nabla_{D(B)}(z) + \nabla_{D(A)}(z)\nabla_{N(B)}(z),$$
$$\nabla_{D(A+B)}(z) = \nabla_{D(A)}(z)\nabla_{D(B)}(z).$$

Let T'_{D_mK} be the tangle which is obtained from T_{D_mK} by splitting two arcs of T_{D_mK} as in Figure 8.

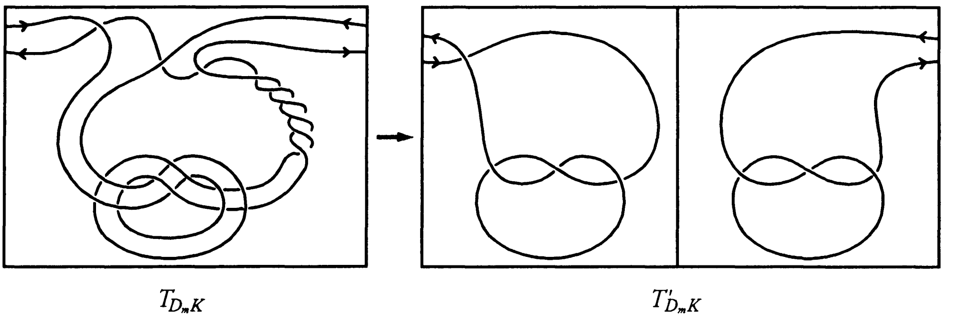


Fig. 8.

Since the Conway polynomial of a split link is zero, we only consider the case that $(L^n_{D_mK})_J$ is not split. Then $(L^n_{D_mK})_J = N(\overbrace{T_{D_mK} + \cdots + T_{D_mK}}^{\sharp J-1} + T'_{D_mK})$ if $J \neq N$ and $(L^n_{D_mK})_N = N(\overbrace{T_{D_mK} + \cdots + T_{D_mK}}^{n})$. Then it follows from Proposition 3.3 that

$$\nabla_{(L^n_{D_mK})_J}(z) = \begin{cases} \nabla_{D(T_{D_mK})}(z)^{\sharp J-1}\nabla_{N(T'_{D_mK})}(z) & \text{if } J \neq N \\ n\nabla_{D(T_{D_mK})}(z)^{n-1} & \text{if } J = N. \end{cases} \tag{1}$$

(Note that $D(T'_{D_mK})$ is a split link and $N(T_{D_mK})$ is an unknot.)
Hence

$$a_1((L^n_{D_mK})_J) = \begin{cases} a_0(D(T_{D_mK}))^{\sharp J-2}\{a_0(D(T_{D_mK}))a_1(N(T'_{D_mK})) \\ \qquad + (\sharp J - 1)a_1(D(T_{D_mK}))\} & \text{if } J \neq N \\ n(n-1)a_0(D(T_{D_mK}))^{n-2}a_1(D(T_{D_mK})) & \text{if } J = N. \end{cases} \tag{2}$$

Note that D_mK, $\overline{D_mK}$(= unknot) and $D(T_{D_mK})$ are related by single crossing changes as indicated in Figure 9.

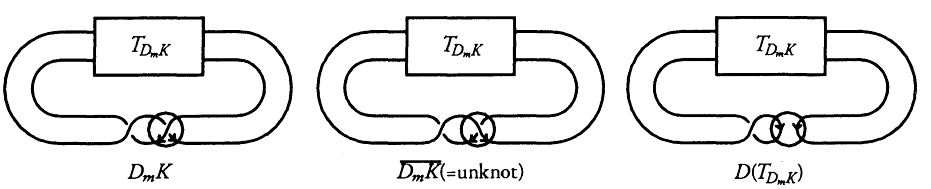


Fig. 9.

Then from the skein relation of the Conway polynomial, we get

$$a_0(D(T_{D_mK})) = -a_1(D_mK) = m \tag{3}$$

and

$$(4) \qquad a_1(D(T_{D_mK})) = -a_2(D_mK) = 0.$$

(Recall that $\nabla_{D_mK}(z) = 1 - mz^2$.)

Thus only $a_1(N(T'_{D_mK}))$ has not been expressed in terms of the original data of D_mK yet. To do this, we will caluculate $\lambda_2(D_mK)$ in two ways using Proposition 1.5 and Proposition 1.7.

The two-fold branched cover $\Sigma^2_{D_mK}$ is presented by the surgery description $\mathcal{L}^2_{D_mK} = \{(K_1, 1+2m), (K_2, 1+2m)\}$. Note that $lk(K_1, K_2) = -2m$. The linking matrix is

$$E(\mathcal{L}^2_{D_mK}) = \begin{pmatrix} 1+2m & -2m \\ -2m & 1+2m \end{pmatrix}.$$

The eigen values of $E(\mathcal{L}^2_{D_mK})$ are 1 and $1+4m$. So,

$$\det(E(\mathcal{L}^2_{D_mK})) = 1 + 4m,$$

$$sign(\mathcal{L}^2_{D_mK}) = sign(\det(E(\mathcal{L}^2_{D_mK}))) = sign(1+4m),$$

and

$$signature(E(\mathcal{L}^2_{D_mK})) = 1 + sign(1+4m).$$

Moreover we get

$$\det(E((\mathcal{L}^2_{D_mK})_{\{1,2\}\setminus J}; J)) = 1$$

and

$$\det(E((\mathcal{L}^2_{D_mK})_{\{1,2\}\setminus J})) = \begin{cases} 1+2m & \text{if } \sharp J = 1 \\ 1 & \text{if } \sharp J = 2. \end{cases}$$

Note that from (2)

$$a_1((L^2_{D_mK})_{\{j\}}) = a_1(K_j) = a_1(N(T'_{D_mK})) \quad (j = 1, 2)$$

and from (2) and (4)

$$a_1((L^2_{D_mK})_{\{1,2\}}) = a_1(\{K_1, K_2\}) = -2a_2(D_mK) = 0.$$

By considering all graphs appearing in Figure 2, we get

$$\theta((\mathcal{L}^2_{D_mK})_J) = \begin{cases} 4m^2 + 4m + 3 & \text{if } \sharp J = 1 \\ 4m(2m+1)^2 & \text{if } \sharp J = 2. \end{cases}$$

Then according to Proposition 1.5, we get

$$\lambda_2(D_mK) = sign(1+4m)\left(2a_1(N(T'_{D_mK})) - \frac{(2m+1)(2m+3)}{12} + \frac{(1+4m)(1+sign(1+4m))}{8}\right). \quad (5)$$

On the other hand, from Proposition 1.7, we get

$$(6)\quad \lambda_2(D_mK) = -\frac{i^{\sigma(D_mK)}}{12}\frac{d}{dt}V_{D_mK}(-1) + sign(1+4m)(1+4m)\frac{\sigma(D_mK)}{8}.$$

(Note that $|H_1(\Sigma^2_{D_mK};\mathbf{Z})| = |\det(E(\mathcal{L}^2_{D_mK}))| = sign(1+4m)(1+4m)$.)

Using (5), (6) and the fact that $\sigma(D_mK) = 0$ if $m \geq 0$ and $\sigma(D_mK) = -2$ if $m < 0$, we can express $a_1(N(T'_{D_mK}))$ in terms of m and $d/dtV_{D_mK}(-1)$ as follows;

$$(7)\quad a_1(N(T'_{D_mK})) = -\frac{1}{24}\frac{d}{dt}V_{D_mK}(-1) + \frac{1}{6}m(m-1).$$

Thus in the case of $\mathcal{L}^n_{D_mK}$, all terms appering in Proposition 1.5 are expressed in terms of the original data $d/dtV_{D_mK}(-1)$ of D_mK and m. This completes the proof of Theorem 3.1.

To prove Collorary 3.2, note that $N(T'_{D_mK})$ is isotopic to $K\sharp(-K)$. Since the Conway polynomial is multiplicative under connected sum, we have

$$(8)\quad a_1(N(T'_{D_mK})) = a_1(K\sharp(-K)) = 2a_1(K).$$

This proves Corollary 3.2.

REMARK 3.4. From equations (7) and (8), we can get a relation between the Conway polynomial of K and the Jones polynomial of D_mK as follows;

$$2a_1(K) = -\frac{1}{24}\frac{d}{dt}V_{D_mK}(-1) + \frac{1}{6}m(m-1)$$

or equivalently

$$\frac{d}{dt}V_{D_mK}(-1) = -48a_1(K) + 4m^2 - 4m.$$

A formula for λ_n

Note that the linking matrix $E(\mathcal{L}^n_{D_mK})$ can be diagonalized to the following matrix;

$E_n(D_mK)$

$$= \begin{pmatrix} 1 & & & & \mathbf{0} \\ & 1-2m\left(\cos\frac{2\pi}{n}-1\right) & & & \\ & \cdots & \cdots & & \\ & & 1-2m\left(\cos\frac{2k\pi}{n}-1\right) & & \\ & & \cdots & \cdots & \\ \mathbf{0} & & & & 1-2m\left(\cos\frac{2(n-1)\pi}{n}-1\right) \end{pmatrix}.$$

Then

$$sign(\mathcal{L}^n_{D_mK}) = sign(\det(E_n(D_mK))) = \begin{cases} 1 & n\text{: odd} \\ sign(1+4m) & n\text{: even.} \end{cases}$$

Let J be a subset of $N=\{1,\dots,n\}$ such that $(L^n_{D_mK})_J$ is not split. We only consider such J since $a_1((L^n_{D_mK})_J)=0$ and $\theta((L^n_{D_mK})_J)=0$ if $(L^n_{D_mK})_J$ is split.

Then from (2), (3), (4) and (7) we get

$$a_1((L^n_K)_J) = \begin{cases} m^{\sharp J-1}\left(-\frac{1}{24}V_{D_mK}(-1)+\frac{1}{6}m(m-1)\right) & \text{if } 1\le \sharp J\le n-1 \\ 0 & \text{if } \sharp J=n. \end{cases}$$

Let $A_j(D_mK)$ be the $j\times j$ matrix

$$\begin{pmatrix} 1+m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+m \end{pmatrix}$$

and $B_j(D_mK)$ be the $j\times j$ matrix

$$\begin{pmatrix} 1+2m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+2m \end{pmatrix}.$$

Then

$$\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J};J)) = \det(A_{n-\sharp J}(D_mK))$$

and

$$\det(E((\mathcal{L}^n_{D_mK})_{N\setminus J})) = \det(B_{n-\sharp J}(D_mK)).$$

By considering all graphs appearing in Figure 2, we can calculate $\theta((\mathcal{L}^n_{D_mK})_J)$ as follows;

$$\theta((\mathcal{L}^n_{D_mK})_J) = \begin{cases} 4m^2+4m+3 & \text{if } \sharp J = 1 \\ 2m(3m^2+2m+1) & \text{if } \sharp J = 2 \\ 6m^4 & \text{if } \sharp J = 3, \sharp N = 3 \\ 2m^4 & \text{if } \sharp J = 3, \sharp N > 3 \\ 0 & \text{if } 4 \le \sharp J \le n-1 \\ 2n(-m)^n(2+m) & \text{if } \sharp J = n. \end{cases}$$

Then $\lambda_n(D_mK)$ can be expressed as a combination of m and $d/dtV_{D_mK}(-1)$ as in the following theorem.

Theorem 3.5. *Let K be a knot in S^3 and D_mK its m-twisted double. Then*

$$\lambda_n(D_mK) = S_n(m)n\left(\varphi(D_mK)\sum_{j=1}^{n-1} m^{j-1}\det(A_{n-j}(m)) + \frac{1}{24}\sum_{j=1}^{n}(-1)^j\det(B_{n-j}(m))\psi_j(m)\right) + S_n(m)\frac{\det(E_n(m))signature(E_n(m))}{8}$$

with

$$S_n(m) = \begin{cases} 1 & n\text{: odd or } n\text{: even, } m \ge 0 \\ -1 & n\text{: even, } m < 0, \end{cases}$$

the $j \times j$ matrix

$$A_j(m) = \begin{pmatrix} 1+m & -m & & & \mathbf{0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \mathbf{0} & & & -m & 1+m \end{pmatrix},$$

the $j \times j$ matrix

$$B_j(m) = \begin{pmatrix} 1+2m & -m & & & \text{\huge 0} \\ -m & 1+2m & -m & & \\ & \ddots & \ddots & \ddots & \\ & & -m & 1+2m & -m \\ \text{\huge 0} & & & -m & 1+2m \end{pmatrix},$$

$$\varphi(D_m K) = -\frac{1}{24}\frac{d}{dt}V_{D_m K}(-1) + \frac{1}{6}m(m-1),$$

$$\psi_j(m) = \begin{cases} 4m^2 + 4m + 3 & \textit{if } j = 1 \\ 2m(3m^2 + 2m + 1) & \textit{if } j = 2 \\ 6m^4 & \textit{if } j = 3, n = 3 \\ 2m^4 & \textit{if } j = 3, n > 3 \\ 0 & \textit{if } 4 \le j \le n-1 \\ 2(-m)^n(2+m) & \textit{if } j = n, \end{cases}$$

and the $n \times n$ diagonal matrix

$$E_n(m) = \begin{pmatrix} 1 & & & & \text{\huge 0} \\ & 1 - 2m\left(\cos\frac{2\pi}{n} - 1\right) & & & \\ & \cdots & \cdots & & \\ & & 1 - 2m\left(\cos\frac{2k\pi}{n} - 1\right) & & \\ & & \cdots & \cdots & \\ \text{\huge 0} & & & & 1 - 2m\left(\cos\frac{2(n-1)\pi}{n} - 1\right) \end{pmatrix}.$$

REMARK 3.6. In case of $m = 0$ (untwisted double), the n-fold cyclic branched cover $\Sigma^n_{D_0 K}$ is an integral homology sphere. J. Hoste [6] has caluculated $\lambda_n(D_0 K)$ in terms of $a_1(K)$ as follows;

$$\lambda_n(D_0 K) = 2na_1(K).$$

Note that $\varphi(D_m K) = 2a_1(K)$. Therefore Theorem 3.5 is a generalization of this formula.

On the other hand, $\lambda_n(D_0 K)$ is expressed in terms of $d/dt V_{D_0 K}(-1)$ as follows;

$$\lambda_n(D_0 K) = -\frac{n}{24}\frac{d}{dt}V_{D_0 K}(-1).$$

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