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THE CASSON-WALKER INVARIANT FOR BRANCHED CYCLIC COVERS OF S^3 BRANCHED OVER A DOUBLED KNOT

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0. Introduction

In 1985, A. Casson defined an invariant λ for oriented integral homology 3-spheres by using representations from their fundamental group into $SU(2)$ [1]. It was extended to an invariant for rational homology 3-spheres by K. Walker [11]. In 1993, C. Lescop [9] gave a formula to calculate this invariant for rational homology 3-spheres when they are presented by framed links and showed that it naturally extends to an invariant for all 3-manifolds.

Let L be a link in S^3 and let Σ_L^n be its n -fold cyclic branched cover. Define $\lambda_n(L) = \lambda(\Sigma_L^n)$. Then λ_n becomes an invariant of links. For doubles of knots, torus knots and iterated torus knots, A. Davidow (see [3], [4]) calculated Casson's integer invariant for n -fold branched covers, when Σ_K^n is an integral homology sphere. For any links, D. Mullins [10] have succeeded in calculating Casson-Walker's rational valued invariant for 2-fold branched covers, when Σ_L^2 is a rational homology sphere.

In this paper, using C. Lescop's formula and the result of D. Mullins, we will calculate the Casson-Walker invariant for branched cyclic covers of S^3 branched over the m -twisted double of a knot. We will show the following theorem and corollary.

Theorem 3.1. *Let K be a knot in S^3 and $D_m K$ its m -twisted double. Then $\lambda_n(D_m K)$ is determined by $d/dt V_{D_m K}(-1)$ and m where $d/dt V_{D_m K}(-1)$ is the derivative of the Jones polynomial of $D_m K$ at $t = -1$.*

Corollary 3.2. *$\lambda_n(D_m K)$ is determined by $a_1(K)$ and m where $a_1(K)$ is the coefficient of z^2 of the Conway polynomial of K .*

1. Preliminaries

DEFINITION 1.1. The Conway polynomial $\nabla_L(z)$ of an oriented link L is defined by

1. $\nabla_U(z) = 1$, where U is an unknot,

2. $\nabla_{L^+}(z) - \nabla_{L^-}(z) = -z\nabla_{L^o}(z)$, where L^+, L^-, L^o are oriented links identical except within a ball where they are as shown in Figure 1.

It is well known that the Conway polynomial is of the form

$$\nabla_L(z) = z^{\sharp L-1}(a_0(L) + a_1(L)z^2 + \dots + a_{d(L)}z^{2d(L)}).$$

This defines the coefficients $a_i(L)$ of $\nabla_L(z)$.

Let K be a knot in S^3 and D_mK its m -twisted double. It is easy to see that $\nabla_{D_mK}(z) = 1 - mz^2$. Thus $a_1(D_mK) = -m$ and $a_i(D_mK) = 0$ for $i \geq 2$.

DEFINITION 1.2. The Jones polynomial $V_L(t)$ of an oriented link L is defined by

1. $V_U(t) = 1$, where U is an unknot,
2. $t^{-1}V_{L^+}(t) - tV_{L^-}(t) = (t^{1/2} - t^{-1/2})V_{L^o}(t)$, where L^+, L^-, L^o are oriented links identical except within a ball where they are as shown in Figure 1.

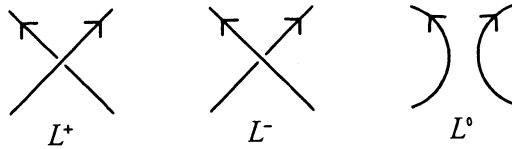


Fig. 1.

Let W_L be a Seifert matrix for an oriented link L .

DEFINITION 1.3. The signature $\sigma(L)$ of L is defined as

$$\sigma(L) = \text{signature}(W_L + W_L^T).$$

DEFINITIONS 1.4. Let $\mathcal{L} = \{(K_1, \mu_1), \dots, (K_n, \mu_n)\}$ be a framed link in S^3 , where each component K_i is oriented and μ_i gives integer framing. The manifold obtained by surgery on \mathcal{L} is denoted by $\chi(\mathcal{L})$. Let L denote the underlying link of \mathcal{L} . Let l_{ij} be the linking number $lk(K_i, K_j)$ of K_i and K_j if $i \neq j$ and μ_i if $i = j$.

- The linking matrix of \mathcal{L} is defined by

$$E(\mathcal{L}) = (l_{ij})_{1 \leq i, j \leq n}.$$

- The sign of \mathcal{L} , denoted by $\text{sign}(\mathcal{L})$, is equal to $(-1)^{b_-(\mathcal{L})}$ where $b_-(\mathcal{L})$ denotes the number of negative eigen values of $E(\mathcal{L})$.
- Restriction of a framed link.

If I is a subset of $N = \{1, \dots, n\}$, then \mathcal{L}_I (resp. L_I) denotes the framed link obtained by \mathcal{L} (resp. the link obtained by L) by forgetting the components whose subscripts do not belong to I .

- The circular linking of \mathcal{L}_I , denoted by $Lk_c(\mathcal{L}_I)$, is defined by

$$Lk_c(\mathcal{L}_I) = \sum_{\sigma \in \sigma_I} \left(\prod_{k \in I} l_{k\sigma(k)} \right),$$

where σ_I denotes the set of cyclic permutations of I .

- The θ -linking of \mathcal{L}_I .

Let $\theta_b(\mathcal{L}_I)$ be defined by

$$\theta_b(\mathcal{L}_I) = \sum_{\{(K, i, j, g) \mid K \subset I, (i, j) \in K^2, g \in S_{I \setminus K}\}} Lk_c(\mathcal{L}_K) l_{ig(1)} l_{g(1)g(2)} \cdots l_{g(\#I \setminus K - 1)g(\#I \setminus K)} l_{g(\#I \setminus K)j}.$$

($S_{I \setminus K}$ denotes the set of one to one maps from $\{1, \dots, \#I \setminus K\}$ to $I \setminus K$.)

This sum can be seen as the sum of the linking numbers of \mathcal{L}_I with respect to the edes of one of the graphs in Figure 2 for all combinatorial ways of constructing such graphs.

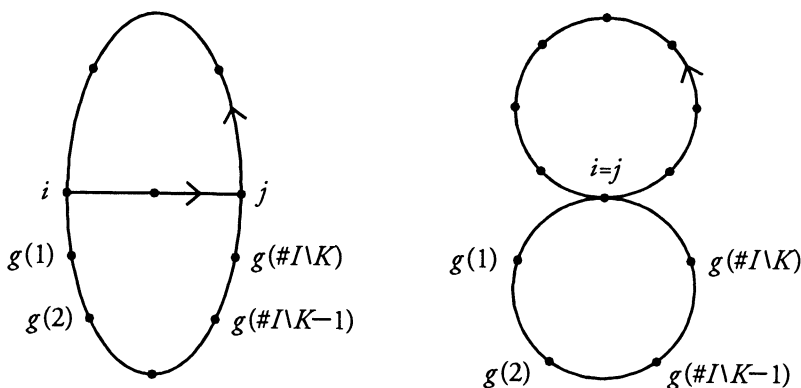


Fig. 2.

Then, the θ -linking of \mathcal{L}_I , denoted by $\theta(\mathcal{L}_I)$, is defined by

$$\theta(\mathcal{L}_I) = \begin{cases} \theta_b(\mathcal{L}_I) & \text{if } \#I > 2 \\ \theta_b(\mathcal{L}_I) - 2l_{ij} & \text{if } I = \{i, j\} \\ \theta_b(\mathcal{L}_I) + 2 & \text{if } I = \{i\}. \end{cases}$$

- The modified linking matrix $E(\mathcal{L}_{N \setminus I}; I)$ is defined by

$$E(\mathcal{L}_{N \setminus I}; I) = (l_{ijI})_{i, j \in N \setminus I}$$

with

$$l_{ijI} = \begin{cases} l_{ij} & \text{if } i \neq j \\ l_{ii} + \sum_{k \in I} l_{ki} & \text{if } i = j. \end{cases}$$

We state C. Lescop’s formula for the Casson-Walker invariant.

Proposition 1.5 ([9]). *Let \mathcal{L} and $\chi(\mathcal{L})$ be as above. Then the Casson-Walker invariant λ of $\chi(\mathcal{L})$ is given by*

$$\begin{aligned} & \lambda(\chi(\mathcal{L})) \\ &= \text{sign}(\mathcal{L}) \sum_{\{J|J \neq \emptyset, J \subset N\}} \left(\det(E(\mathcal{L}_{N \setminus J}; J)) a_1(L_J) + \frac{\det(E(\mathcal{L}_{N \setminus J}))(-1)^{\#J} \theta(\mathcal{L}_J)}{24} \right) \\ & \quad + \text{sign}(\mathcal{L}) \det(E(\mathcal{L})) \frac{\text{signature}(E(\mathcal{L}))}{8}, \end{aligned}$$

where the determinant of an empty matrix equals to one.

REMARK 1.6. We follow C. Lescop’s normalization of the Casson-Walker invariant. If λ_w denotes the Walker invariant as described in [11],

$$\lambda(M) = \frac{|H_1(M; \mathbf{Z})|}{2} \lambda_w(M).$$

Finally, we state the result of D. Mullins for two-fold branched covers.

Proposition 1.7 ([10]). *Let L be a link in S^3 . Suppose the two-fold branched cover of L , Σ_L^2 , is a rational homology sphere. Then*

$$\lambda_2(L) = -\frac{i^{\sigma(L)}}{12} \frac{d}{dt} V_L(-1) + |H_1(\Sigma_L^2; \mathbf{Z})| \frac{\sigma(L)}{8}.$$

Note that if L is a knot, then Σ_L^2 is a rational homology sphere.

2. Surgery description of cyclic branched covers

Let $D_m K$ be the m -twisted double of a knot K in S^3 . If we introduce one surgery curve C which have the framing 1 for the crossing that must be changed to obtain an unknot, we may arrive at a surgery description of $D_m K$ as shown in Figure 3, where $\overline{D_m K}$ is an unknot corresponding to $D_m K$ in other version of S^3 .

Applying an isotopy to S^3 , we can exchange the position of $\overline{D_m K}$ with that of C (see Figure 4).

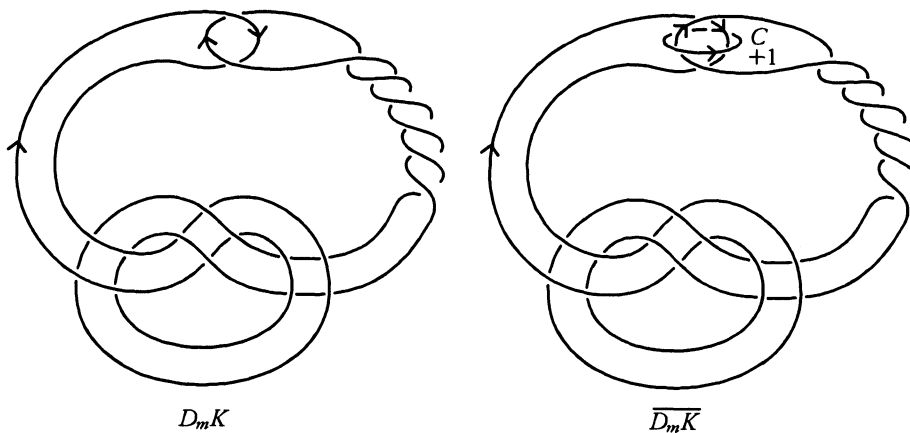


Fig. 3.

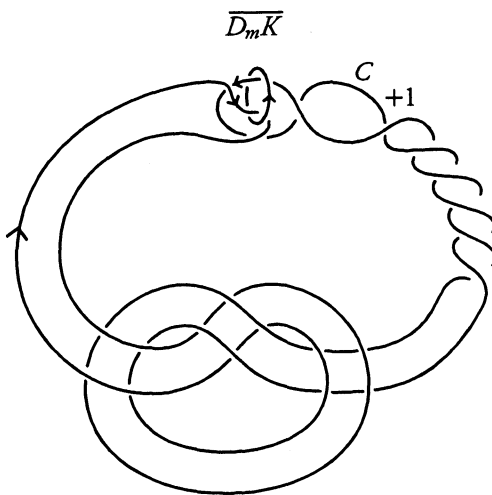


Fig. 4.

Let $T_{D_m K}$ denote the tangle which is obtained by cutting C (at two points) by a spanning 2-disk for $\overline{D_m K}$ as in Figure 5.

Note that $T_{D_m K}$ has two arcs. By joining n -copies of $T_{D_m K}$ cyclically, we obtain an n -component link $L_{D_m K}^n = \{K_1, \dots, K_n\}$ as in Figure 6.

Then the n -fold cyclic branched cover of S^3 branched over $D_m K$, $\Sigma_{D_m K}^n$, is obtained by a surgery on the link $L_{D_m K}^n$. Note that $lk(K_1, K_2) = lk(K_2, K_3) = \dots = lk(K_n, K_1)$ and the framing of the component K_i is equal to $-2lk(K_1, K_2) + 1$ if $n \geq 3$ and is equal to $-lk(K_1, K_2) + 1$ if $n = 2$.

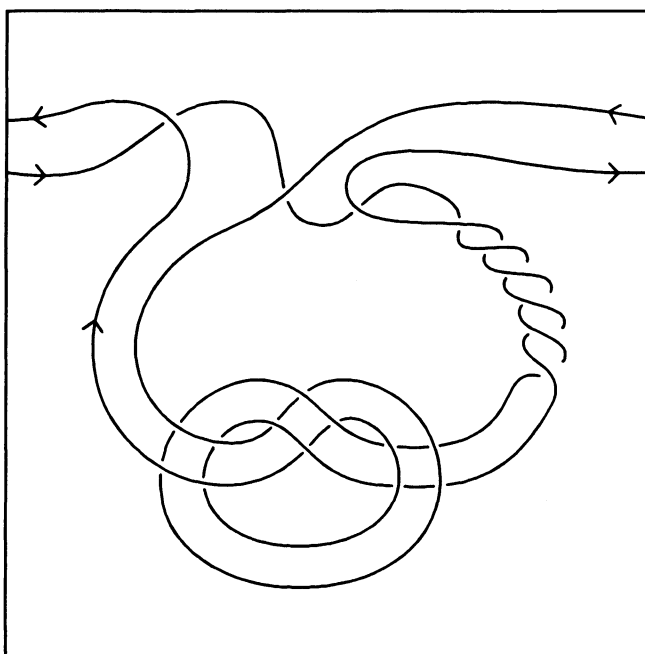
tangle $T_{D_m K}$

Fig. 5.

Lemma 2.1.

$$lk(K_1, K_2) = \begin{cases} -m & \text{if } n \geq 3 \\ -2m & \text{if } n = 2. \end{cases}$$

Proof. Consider the crossing of $D_m K$ that must be changed to obtain an unknot. From the skein relation of the Conway polynomial, we get $a_1(D_m K) = -a_0(K^\circ)$ where K° is the 2-component link obtained by splicing the crossing of $D_m K$. On the other hand, $-a_0(K^\circ)$ is equal to the linking number of the components of K° (see [5]). But this is equal to $lk(K_1, K_2)$ if $n \geq 3$ and is equal to $1/2lk(K_1, K_2)$ if $n = 2$. Noting that $a_1(D_m K) = -m$, we get the conclusion. \square

Then from Lemma 2.1, we can express the framing in terms of the twisting number m . Each K_i has the framing $1 + 2m$. Thus $\mathcal{L}_{D_m K}^n = \{(K_1, 1 + 2m), \dots, (K_n, 1 + 2m)\}$ is a framed link for $\Sigma_{D_m K}^n$.

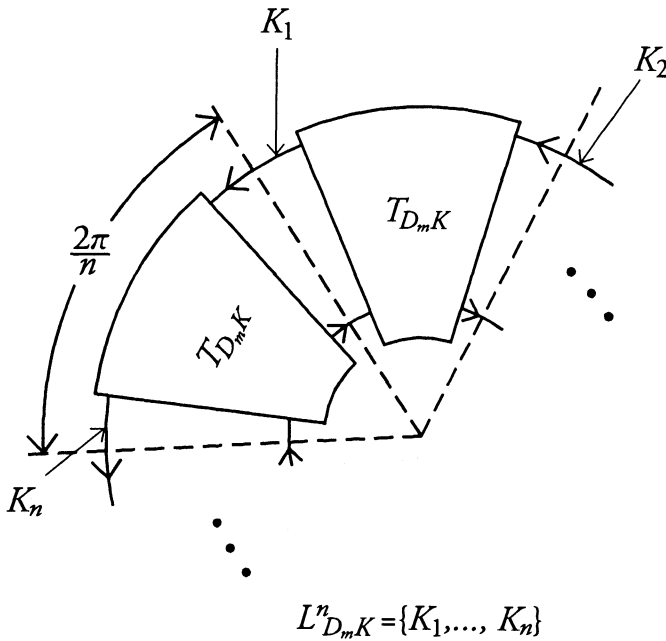


Fig. 6.

3. Calculation of λ_n

In this section, we will prove Theorem 3.1 and Corollary 3.2 and give a formula for λ_n .

Theorem 3.1. *Let K be a knot in S^3 and $D_m K$ its m -twisted double. Then $\lambda_n(D_m K)$ is determined by $d/dt V_{D_m K}(-1)$ and m where $d/dt V_{D_m K}(-1)$ is the derivative of the Jones polynomial of $D_m K$ at $t = -1$.*

Corollary 3.2. *$\lambda_n(D_m K)$ is determined by $a_1(K)$ and m where $a_1(K)$ is the coefficient of z^2 of the Conway polynomial of K .*

Proof of Theorem 3.1 and Corollary 3.2

The case of $n = 2$ follows from Proposition 1.7 and the fact that $|H_1(\Sigma^2_{D_m K}; \mathbf{Z})| = |1 - 4a_1(D_m K)| = |1 + 4m|$. So, assume that $n \geq 3$. We use Proposition 1.5 and the surgery description of $\Sigma^n_{D_m K}$. Let $\mathcal{L}^n_{D_m K} = \{(K_1, 1 + 2m), \dots, (K_n, 1 + 2m)\}$ be the surgery description for $\Sigma^n_{D_m K}$ as in section 2. The linking matrix is determined

by m as follows;

$$E(\mathcal{L}_{D_m K}^n) = \begin{pmatrix} 1 + 2m & -m & & & -m \\ -m & 1 + 2m & -m & & \\ & & \ddots & \ddots & \ddots \\ & & & -m & 1 + 2m & -m \\ -m & & & & -m & 1 + 2m \end{pmatrix}.$$

Then $sign(\mathcal{L}_{D_m K}^n)$, $\det(E((\mathcal{L}_{D_m K}^n)_{N \setminus J}; J))$, $\det(E((\mathcal{L}_{D_m K}^n)_{N \setminus J}))$, $\theta((\mathcal{L}_{D_m K}^n)_J)$, $\det(E(\mathcal{L}_{D_m K}^n))$ and $signature(E(\mathcal{L}_{D_m K}^n))$ are also determined by m . So we want to know whether or not $a_1((L_{D_m K}^n)_J)$ can be expressed in terms of the original data of $D_m K$.

To do this we introduce the following notations and proposition. For given tangles A and B , the tangle $A + B$ is defined as in Figure 7. Also, there are two operations that associate knots and links to a given tangle A . These are denoted $N(A)$ and $D(A)$ as in Figure 7.

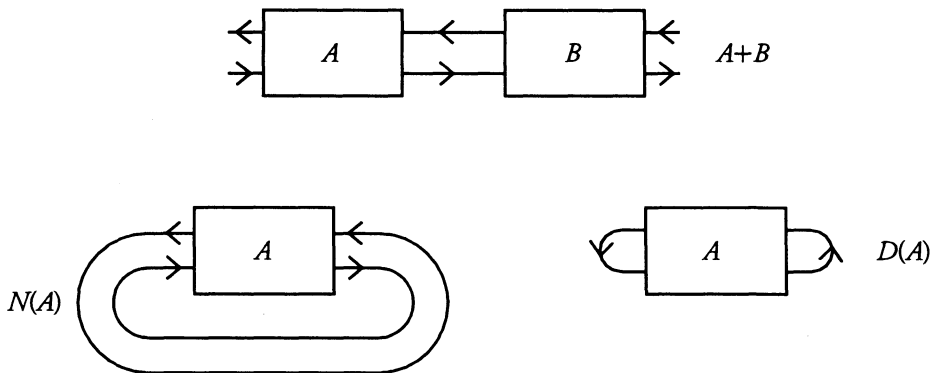


Fig. 7.

Proposition 3.3 ([2], [7]). *Let A and B be tangles. Then*

$$\begin{aligned} \nabla_{N(A+B)}(z) &= \nabla_{N(A)}(z)\nabla_{D(B)}(z) + \nabla_{D(A)}(z)\nabla_{N(B)}(z), \\ \nabla_{D(A+B)}(z) &= \nabla_{D(A)}(z)\nabla_{D(B)}(z). \end{aligned}$$

Let $T'_{D_m K}$ be the tangle which is obtained from $T_{D_m K}$ by splitting two arcs of $T_{D_m K}$ as in Figure 8.

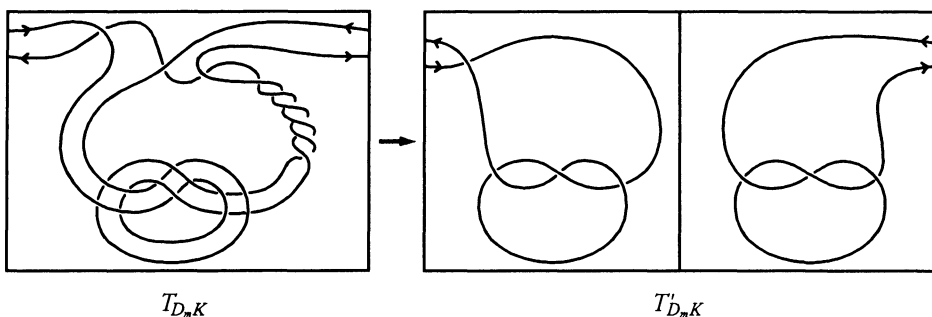


Fig. 8.

Since the Conway polynomial of a split link is zero, we only consider the case that $(L_{D_m K}^n)_J$ is not split. Then $(L_{D_m K}^n)_J = N(\overbrace{T_{D_m K} + \cdots + T_{D_m K}}{\#J-1} + T'_{D_m K})$ if $J \neq N$ and $(L_{D_m K}^n)_N = N(\overbrace{T_{D_m K} + \cdots + T_{D_m K}}{n})$. Then it follows from Proposition 3.3 that

$$(1) \quad \nabla_{(L_{D_m K}^n)_J}(z) = \begin{cases} \nabla_{D(T_{D_m K})}(z)^{\#J-1} \nabla_{N(T'_{D_m K})}(z) & \text{if } J \neq N \\ n \nabla_{D(T_{D_m K})}(z)^{n-1} & \text{if } J = N. \end{cases}$$

(Note that $D(T'_{D_m K})$ is a split link and $N(T_{D_m K})$ is an unknot.)

Hence

$$(2) \quad a_1((L_{D_m K}^n)_J) = \begin{cases} a_0(D(T_{D_m K}))^{\#J-2} \{a_0(D(T_{D_m K}))a_1(N(T'_{D_m K})) \\ \quad + (\#J - 1)a_1(D(T_{D_m K}))\} & \text{if } J \neq N \\ n(n-1)a_0(D(T_{D_m K}))^{n-2}a_1(D(T_{D_m K})) & \text{if } J = N. \end{cases}$$

Note that $D_m K$, $\overline{D_m K}$ (= unknot) and $D(T_{D_m K})$ are related by single crossing changes as indicated in Figure 9.

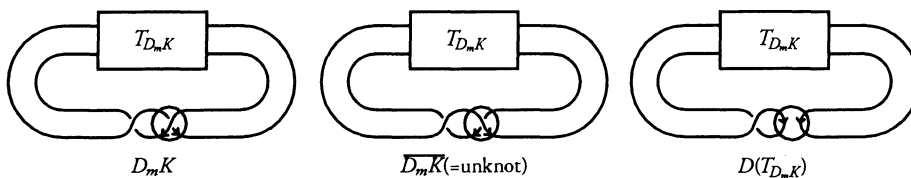


Fig. 9.

Then from the skein relation of the Conway polynomial, we get

$$(3) \quad a_0(D(T_{D_m K})) = -a_1(D_m K) = m$$

and

$$(4) \quad a_1(D(T_{D_m K})) = -a_2(D_m K) = 0.$$

(Recall that $\nabla_{D_m K}(z) = 1 - mz^2$.)

Thus only $a_1(N(T'_{D_m K}))$ has not been expressed in terms of the original data of $D_m K$ yet. To do this, we will calculate $\lambda_2(D_m K)$ in two ways using Proposition 1.5 and Proposition 1.7.

The two-fold branched cover $\Sigma^2_{D_m K}$ is presented by the surgery description $\mathcal{L}^2_{D_m K} = \{(K_1, 1 + 2m), (K_2, 1 + 2m)\}$. Note that $lk(K_1, K_2) = -2m$. The linking matrix is

$$E(\mathcal{L}^2_{D_m K}) = \begin{pmatrix} 1 + 2m & -2m \\ -2m & 1 + 2m \end{pmatrix}.$$

The eigen values of $E(\mathcal{L}^2_{D_m K})$ are 1 and $1 + 4m$. So,

$$\begin{aligned} \det(E(\mathcal{L}^2_{D_m K})) &= 1 + 4m, \\ \text{sign}(\mathcal{L}^2_{D_m K}) &= \text{sign}(\det(E(\mathcal{L}^2_{D_m K}))) = \text{sign}(1 + 4m), \end{aligned}$$

and

$$\text{signature}(E(\mathcal{L}^2_{D_m K})) = 1 + \text{sign}(1 + 4m).$$

Moreover we get

$$\det(E((\mathcal{L}^2_{D_m K})_{\{1,2\} \setminus J}; J)) = 1$$

and

$$\det(E((\mathcal{L}^2_{D_m K})_{\{1,2\} \setminus J})) = \begin{cases} 1 + 2m & \text{if } \#J = 1 \\ 1 & \text{if } \#J = 2. \end{cases}$$

Note that from (2)

$$a_1((\mathcal{L}^2_{D_m K})_{\{j\}}) = a_1(K_j) = a_1(N(T'_{D_m K})) \quad (j = 1, 2)$$

and from (2) and (4)

$$a_1((\mathcal{L}^2_{D_m K})_{\{1,2\}}) = a_1(\{K_1, K_2\}) = -2a_2(D_m K) = 0.$$

By considering all graphs appearing in Figure 2, we get

$$\theta((\mathcal{L}^2_{D_m K})_J) = \begin{cases} 4m^2 + 4m + 3 & \text{if } \#J = 1 \\ 4m(2m + 1)^2 & \text{if } \#J = 2. \end{cases}$$

Then according to Proposition 1.5, we get

$$(5) \quad \lambda_2(D_m K) = \text{sign}(1 + 4m) \left(2a_1(N(T'_{D_m K})) - \frac{(2m + 1)(2m + 3)}{12} + \frac{(1 + 4m)(1 + \text{sign}(1 + 4m))}{8} \right).$$

On the other hand, from Proposition 1.7, we get

$$(6) \quad \lambda_2(D_m K) = -\frac{i^{\sigma(D_m K)}}{12} \frac{d}{dt} V_{D_m K}(-1) + \text{sign}(1 + 4m)(1 + 4m) \frac{\sigma(D_m K)}{8}.$$

(Note that $|H_1(\Sigma^2_{D_m K}; \mathbf{Z})| = |\det(E(\mathcal{L}^2_{D_m K}))| = \text{sign}(1 + 4m)(1 + 4m)$.)

Using (5), (6) and the fact that $\sigma(D_m K) = 0$ if $m \geq 0$ and $\sigma(D_m K) = -2$ if $m < 0$, we can express $a_1(N(T'_{D_m K}))$ in terms of m and $d/dt V_{D_m K}(-1)$ as follows;

$$(7) \quad a_1(N(T'_{D_m K})) = -\frac{1}{24} \frac{d}{dt} V_{D_m K}(-1) + \frac{1}{6} m(m - 1).$$

Thus in the case of $\mathcal{L}^n_{D_m K}$, all terms appering in Proposition 1.5 are expressed in terms of the original data $d/dt V_{D_m K}(-1)$ of $D_m K$ and m . This completes the proof of Theorem 3.1.

To prove Collorary 3.2, note that $N(T'_{D_m K})$ is isotopic to $K\sharp(-K)$. Since the Conway polynomial is multiplicative under connected sum, we have

$$(8) \quad a_1(N(T'_{D_m K})) = a_1(K\sharp(-K)) = 2a_1(K).$$

This proves Corollary 3.2.

REMARK 3.4. From equations (7) and (8), we can get a relation between the Conway polynomial of K and the Jones polynomial of $D_m K$ as follows;

$$2a_1(K) = -\frac{1}{24} \frac{d}{dt} V_{D_m K}(-1) + \frac{1}{6} m(m - 1)$$

or equivalently

$$\frac{d}{dt} V_{D_m K}(-1) = -48a_1(K) + 4m^2 - 4m.$$

A formula for λ_n

Note that the linking matrix $E(\mathcal{L}^n_{D_m K})$ can be diagonalized to the following matrix;

$$E_n(D_m K)$$

and

$$\det(E((\mathcal{L}_{D_m K}^n)_{N \setminus J})) = \det(B_{n-\#J}(D_m K)).$$

By considering all graphs appearing in Figure 2, we can calculate $\theta((\mathcal{L}_{D_m K}^n)_J)$ as follows;

$$\theta((\mathcal{L}_{D_m K}^n)_J) = \begin{cases} 4m^2 + 4m + 3 & \text{if } \#J = 1 \\ 2m(3m^2 + 2m + 1) & \text{if } \#J = 2 \\ 6m^4 & \text{if } \#J = 3, \#N = 3 \\ 2m^4 & \text{if } \#J = 3, \#N > 3 \\ 0 & \text{if } 4 \leq \#J \leq n - 1 \\ 2n(-m)^n(2 + m) & \text{if } \#J = n. \end{cases}$$

Then $\lambda_n(D_m K)$ can be expressed as a combination of m and $d/dtV_{D_m K}(-1)$ as in the following theorem.

Theorem 3.5. *Let K be a knot in S^3 and $D_m K$ its m -twisted double. Then*

$$\begin{aligned} \lambda_n(D_m K) = S_n(m)n \left(\varphi(D_m K) \sum_{j=1}^{n-1} m^{j-1} \det(A_{n-j}(m)) \right. \\ \left. + \frac{1}{24} \sum_{j=1}^n (-1)^j \det(B_{n-j}(m)) \psi_j(m) \right) \\ + S_n(m) \frac{\det(E_n(m)) \text{signature}(E_n(m))}{8} \end{aligned}$$

with

$$S_n(m) = \begin{cases} 1 & n: \text{ odd or } n: \text{ even, } m \geq 0 \\ -1 & n: \text{ even, } m < 0, \end{cases}$$

the $j \times j$ matrix

$$A_j(m) = \begin{pmatrix} 1+m & -m & & & & & & & & \mathbf{0} \\ & -m & 1+2m & -m & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & -m & 1+2m & -m & & & \\ \mathbf{0} & & & & & & -m & 1+m & & \end{pmatrix},$$

the $j \times j$ matrix

$$B_j(m) = \begin{pmatrix} 1+2m & -m & & & & \mathbf{0} \\ -m & 1+2m & -m & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -m & 1+2m & -m \\ \mathbf{0} & & & & -m & 1+2m \end{pmatrix},$$

$$\varphi(D_m K) = -\frac{1}{24} \frac{d}{dt} V_{D_m K}(-1) + \frac{1}{6} m(m-1),$$

$$\psi_j(m) = \begin{cases} 4m^2 + 4m + 3 & \text{if } j = 1 \\ 2m(3m^2 + 2m + 1) & \text{if } j = 2 \\ 6m^4 & \text{if } j = 3, n = 3 \\ 2m^4 & \text{if } j = 3, n > 3 \\ 0 & \text{if } 4 \leq j \leq n-1 \\ 2(-m)^n(2+m) & \text{if } j = n, \end{cases}$$

and the $n \times n$ diagonal matrix

$$E_n(m) = \begin{pmatrix} 1 & & & & & \mathbf{0} \\ & 1 - 2m \left(\cos \frac{2\pi}{n} - 1 \right) & & & & \\ & \dots & \dots & & & \\ & & & 1 - 2m \left(\cos \frac{2k\pi}{n} - 1 \right) & & \\ & & & \dots & \dots & \\ \mathbf{0} & & & & & 1 - 2m \left(\cos \frac{2(n-1)\pi}{n} - 1 \right) \end{pmatrix}.$$

REMARK 3.6. In case of $m = 0$ (untwisted double), the n -fold cyclic branched cover $\Sigma_{D_0 K}^n$ is an integral homology sphere. J. Hoste [6] has calculated $\lambda_n(D_0 K)$ in terms of $a_1(K)$ as follows;

$$\lambda_n(D_0 K) = 2na_1(K).$$

Note that $\varphi(D_m K) = 2a_1(K)$. Therefore Theorem 3.5 is a generalization of this formula.

On the other hand, $\lambda_n(D_0 K)$ is expressed in terms of $d/dt V_{D_0 K}(-1)$ as follows;

$$\lambda_n(D_0 K) = -\frac{n}{24} \frac{d}{dt} V_{D_0 K}(-1).$$

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