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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A CLASS OF ALLEN–CAHN EQUATIONS IN $\mathbb{R}^2$

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Abstract

In this paper we study the entire solutions of a class of periodic Allen–Cahn equations

$$-\Delta u(x, y) + a(x)W^{'}(u(x, y)) = 0, \quad (x, y) \in \mathbb{R}^2, \tag{0.1}$$

where $a(x)\colon \mathbb{R} \to \mathbb{R}^+$ is a periodic, positive function and $W \in C^2(\mathbb{R}, \mathbb{R})$ is a double-well potential. We look for the entire solutions of the above equation with asymptotic conditions $u(x, y) \to \sigma_\pm$ as $x \to \pm\infty$ uniformly with respect to $y \in \mathbb{R}$. Via variational methods we find infinitely many solutions.

1. Introduction

In this paper we consider a class of Allen–Cahn equation

$$\begin{cases} -\Delta u(x, y) + a(x)W^{'}(u(x, y)) = 0, & (x, y) \in \mathbb{R}^2, \\ \lim\limits_{x\to\pm\infty} u(x, y) = \sigma_\pm & \text{uniformly w.r.t. } y \in \mathbb{R}, \end{cases} \tag{1.1}$$

where we assume
(H_1): $a(x) \in C(\mathbb{R})$ is T periodic and positive;
(H_2): $W(t)$ is a non-negative C^2 function with two zeros $\sigma_\pm$ and $W'(\sigma_\pm) = 0$, and there exists a $R_0 > 0$ such that $W'(s)s \geq 0$ for any $|s| \geq R_0$.

Potentials satisfying the assumption (H_2) are widely used in physical models. For example, the Ginzbrug–Laudau potential $W(s) = (s^2 - 1)^2$ and the Sine–Gordon potential $W(s) = 1 + \cos(\pi s)$ are introduced to study various problems in phase transitions and condensed state physics. Function u represents the mixed state of material and the global minima of W represents pure phase. The introduction of an oscillatory factor $a(x)$ can be used to describe inhomogeneity of the material.

For autonomous case, i.e., $a(x)$ is identically a constant, Ghoussoub and Gui first proved a long standing conjecture by De Giorgi in $\mathbb{R}^2$ (see [11]). L. Ambrosio and X. Cabré in [10] proved the conjecture in $\mathbb{R}^n$ when $n \leq 3$. For $4 \leq n \leq 8$, assuming an additional limiting condition on u, O. Savin proved that this conjecture is also true

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(see [12]). These results tell us that the solutions reduces to one dimensional solutions q_0 modulo space transition, and the problem (1.1) is in fact one dimensional.

In [5], F. Alessio, L. Jeanjean and P. Montecchiari studied (1.1) under the same conditions with $|u| \le 1$. Given $\sigma_-, \sigma_+ \in \{\sigma_1, \sigma_2, \dots, \sigma_m\}$, $\sigma_- \ne \sigma_+$, where σ_i is zero of $W(t)$, they got the existence of multiple layered solutions depending on both x and y. Firstly, they discussed some features of the one dimensional problem associated to (1.1), i.e.,

$$\begin{cases} -\ddot{q}(x) + a(x)W^{'}(q(x)) = 0, & x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} q(x) = \sigma_\pm & \text{uniformly w.r.t. } y \in \mathbb{R}. \end{cases} \tag{1.2}$$

Then they considered the functional $F(q) = \int_{\mathbb{R}} (1/2)|\dot{q}(x)|^2 + a(x)W(q(x))\,dx$ on the Hilbert space $E = \left\{q \in H^1_{\mathrm{loc}}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} |\dot{q}(x)|^2\,dx < +\infty\right\}$ endowed with the norm $\|q\|^2 := |q(0)|^2 + \int_{\mathbb{R}} |\dot{q}(x)|^2\,dx$. They showed that, given any $i \in \{1, \dots, m\}$, there exist some $j(i) \in \{1, \dots, m\} \setminus \{i\}$ such that the functional F attains its minimum on the set $\Gamma^i = \{q \in \{F < +\infty\} \mid \lim_{x \to -\infty} q(t) = \sigma_i, \lim_{x \to +\infty} q(t) = \sigma_{j(i)}\}$. Setting $c(i) := \min_{\Gamma^i} F(q)$, the set $\mathcal{K}^i = \{q \in \Gamma^i \mid F(q) = c(i)\}$ is considered. Finally the critical discreteness assumption on $\mathcal{K}^i$ was verified, i.e.,

$(*)_i$: There exists $\emptyset \ne \mathcal{K}_0 \subset \mathcal{K}^i$, setting $\mathcal{K}_j = \{q(\cdot - jT) \mid q \in \mathcal{K}_0\}$ for $j \in \mathbb{Z}$, such that
(i) $\mathcal{K}_0$ is compact with respect to the $H^1(\mathbb{R})$ topology;
(ii) $\mathcal{K}^i = \bigcup_{j \in \mathbb{Z}} \mathcal{K}_j$ and there exists some $d_0 > 0$ such that if $j \ne j'$ then $\mathrm{d}(\mathcal{K}_j, \mathcal{K}_{j'}) \ge d_0$.

Here $\mathrm{d}(A, B) = \inf\{\|q_1(x) - q_2(x)\|_{L^2(\mathbb{R})} / q_1 \in A,\ q_2 \in B\}$, $A, B \subset \Gamma$. They obtained the following result.

Theorem 1.1 ([5]). *Let* (H_1)–(H_2) *be satisfied, then for any* $i \in \{1, \dots, m\}$ *for which* $(*)_i$ *holds, there exist* $\xi_1, \dots \xi_l \in \mathbb{Z} \setminus \{0\}$ *such that* $\left\{\sum_{\iota=1}^{l} n_\iota \xi_\iota \mid n_\iota \in \mathbb{N} \cup \{0\}\right\} = \mathbb{Z}$, *and for which for any* $\iota \in \{1, \dots, l\}$ *there exists a solution* $u_\iota \in C^2(\mathbb{R}^2)$ *to* (1.2) *with* $\sigma_- = \sigma_i$, $\sigma_+ = \sigma_{j(i)}$, *satisfying*

$$\lim_{y \to -\infty} \mathrm{dist}(u_\iota(x, y), \mathcal{K}^i_0) = \lim_{y \to +\infty} \mathrm{dist}(u_\iota(x, y), \mathcal{K}^i_{\xi_\iota}) = 0. \tag{1.3}$$

In fact, the assumption $(*)_i$ excludes the autonomous case, i.e., $(*)_i$-(ii) cannot hold when a is a constant. In [5], the authors checked the $(*)_i$ by perturbations analysis.

In [6], Alessio and Montecchiari extended the results in [5] and proved the existence of infinitely many periodic solutions to (1.1) of the brake orbits type.

Theorem 1.2 ([6]). *Let* (H_1)–(H_2) *be satisfied, and assume that condition* $(*)_i$ *holds true. If* $c_p \in (c, c^*)$ *is a regular value of* F, *then there exist* $T_p > 0$, $j_p \in \mathbb{Z} \setminus \{0\}$ *and a solution* $v_p \in C^2(\mathbb{R}^2)$ *to the problem* (1.1) *such that*

i) $E_{v_p}(y) = -(1/2)\|\partial_y v_p(\cdot, y)\|^2_{L^2(\mathbb{R})} + F(v_p(\cdot, y)) = c_p$ *for any* $y \in \mathbb{R}$;

ii) $F(v_p(\cdot, 0)) = F(v_p(\cdot, T_p)) = c_p, v_p(\cdot, 0) \in \Gamma_0,\ v_p(\cdot, T_p) \in \Gamma_{j_p}$ *and* $F(v_p(\cdot, y)) > c_p$ *for any* $y \in (0, T_p)$;
iii) $v_p(\cdot, -y) = v_p(\cdot, y)$ *and* $v_p(\cdot, y + T_p) = v_p(\cdot, T_p - y)$ *for any* $y \in \mathbb{R}$, *in particular,* $v_p(\cdot, y + 2T_p) = v_p(\cdot, y)$ *for any* $(x, y) \in \mathbb{R}^2$.

Due to conservation of energy, the solution v_p satisfies the Neumann boundary conditions $\partial_y v_p(x, 0) = \partial_y v_p(x, T_p)$ for any $x \in \mathbb{R}$, thus the solution in $\mathbb{R} \times [0, T_p]$ can be extended to an entire one. Theorem 1.2 guarantees the existence of a brake orbits type solution at level c_p whenever $c_p \in (c, c^*)$ is a regular value of F. By Sard Smale theorem and local compactness properties of F, they proved the set of regular values of F is open and dense in $[c, c^*]$. Then Theorem 1.2 provides in fact the existence of an uncountable set of geometrically distinct two dimensional solutions to (1.1) of the brake orbits type.

Inspired by [7], we will show the existence of infinitely many layered solutions of (1.1).

For problem (1.2), we define the action functional

$$F(q) := \int_{\mathbb{R}} \frac{1}{2}|\dot{q}(x)|^2 + a(x)W(q(x))\, dx$$

on the space

$$E := \left\{ q \in H^1_{\mathrm{loc}}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} |\dot{q}(x)|^2\, dx < +\infty \right\}.$$

Moreover one can consider the minima of F on the subclass Γ of E

$$\Gamma := \left\{ q \in E \;\middle|\; \lim_{x \to -\infty} q(x) = \sigma_-, \quad \lim_{x \to +\infty} q(x) = \sigma_+ \right\}.$$

For problem (1.1), we also define the corresponding action functional

$$\varphi(u) := \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{2}|\nabla u(x, y)|^2 + a(x)W(u(x, y))\, dx - c \right] dy$$

on

$$\mathcal{H} := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^2) \mid u(\,\cdot\,, y) \in \Gamma \text{ for a.e. } y \in \mathbb{R}\}.$$

Note that the solutions of (1.2) are the minimizers of $\varphi(u)$, i.e., $u(x, y) = q(x)$ is one dimensional, symmetric solution of (1.1). We set $\mathcal{K} := \{q \in \Gamma \mid F(q) = \min_\Gamma F(q')\}$. We write $z_1 \sim_x z_2$ if $z_1(x) = z_2(x + jT)$ for some $j \in \mathbb{Z}$.

If $\mathcal{K}/\sim_x$ is finite, then $\mathcal{K}$ is constituted by isolated points, which takes an essential role like $(*)_i$ in [5]. Taking a similar argument, we get the minimizer u_η on $\mathcal{H}_\xi :=$

$\{u \in \mathcal{H} \mid \lim_{y\to-\infty} \mathrm{d}(u(\,\cdot\,,y),q_0) = \lim_{y\to+\infty} \mathrm{d}(u(\,\cdot\,,y),q_\xi) = 0\}$, $\xi \in \mathbb{Z}$, such that $\varphi(u_\eta) = \min_{\xi\neq 0} m_\xi$. Here $m_\xi = \inf_{u\in\mathcal{H}_\xi} \varphi(u)$ and $\mathcal{K}_\xi = \{u \in \mathcal{H}_\xi \mid \varphi(u) = m_\xi\}$, $\xi \in \mathbb{Z}$.

We mean $u_1 \sim_y u_2$ if there exists some $j \in \mathbb{Z}$ such that $u_1(x, y + j) = u_2(x, y)$.

If $\mathcal{K}/{\sim_x}$ and $\mathcal{K}_\eta/{\sim_y}$ are finite, we set

$$\mathcal{U}_0 := \{u \in \mathcal{H} \mid D(u(\,\cdot\,,-L),q_0) \le \delta,\ D(u(\,\cdot\,,L),q_\eta) \le \delta\}$$

and

$$\mathcal{U}_\eta := \{u \in \mathcal{H} \mid D(u(\,\cdot\,,-L),q_\eta) \le \delta,\ D(u(\,\cdot\,,L),q_0) \le \delta\}.$$

Let us define an odd number $N \in \mathbb{N}$, $p = (p_1,\dots,p_N) \in \mathbb{Z}^N$, $\sigma = (\sigma_1,\dots,\sigma_N) \in \{0,\eta\}^N$ with $\sigma_i \neq \sigma_{i-1}$ for all $i = 2,\dots,N$. We also define

$$\mathcal{H}_{N,p,\sigma} := \{u \in \mathcal{H} \mid u(x, y - p_i) \in \mathcal{U}_\sigma \text{ for a.e. } (x,y) \in \mathbb{R}^2,\ i = 1,\dots,N\}$$

and look for multibump solutions on it.

Note that there are only two zeros $\sigma_\pm$ for $W(t)$. Similar as in [5], here we need the following assumption

$(*)$: There exists $\emptyset \neq \mathcal{K}_0 \subset \mathcal{K}$, setting $\mathcal{K}_j = \{q(\cdot - jT) \mid q \in \mathcal{K}_0\}$ for $j \in \mathbb{Z}$, there result

(i) $\mathcal{K}_0$ is compact with respect to the $H^1(\mathbb{R})$ topology;

(ii) $\mathcal{K} = \bigcup_{j\in\mathbb{Z}} \mathcal{K}_j$ and there exists some $d_0 > 0$ such that if $j \neq j'$ then $\mathrm{d}(\mathcal{K}_j, \mathcal{K}_{j'}) \ge d_0$.

Following the procedure of [7], we get the existence of infinitely many heteroclinic solutions of multibump type.

Theorem 1.3. *Let* (H_1)–(H_2) *be satisfied, then* (1.1) *admits infinitely many solutions distinct up to periodic transitions. More precisely we have*

(i) *the set* $\mathcal{K}$ *of periodic solutions of* (1.2) *is not empty*;

(ii) *if the set* $\mathcal{K}/{\sim_x}$ *is finite, then there exists some* $\eta \in \mathbb{Z}$ *such that the set* $\mathcal{K}_\eta$ *of heteroclinic type solutions of* (1.1) *is not empty*;

(iii) *if the set* $\mathcal{K}_\eta^0$ *is finite, then for every odd number* $N \in \mathbb{N}$, $p = (p_1,\dots,p_N) \in \mathbb{Z}^N$ *and* $\sigma = (\sigma_1,\dots,\sigma_N) \in \{0,\eta\}^N$ *with* $p_i - p_{i-1} \ge 4L$ *and* $\sigma_i \neq \sigma_{i-1}$ *for all* $i = 2,\dots,N$, *the set* $\mathcal{K}_{N,p,\sigma}$ *of multibump type solutions of* (1.1) *is not empty.*

2. One dimensional symmetric solutions

In this section, we look for one dimensional symmetric solutions of equation (1.1). We consider the action functional

$$F(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot q(x)|^2 + a(x)W(q(x))\,dx$$

on the space

$$E := \left\{ q \in H^1_{\rm loc}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} |\dot q(x)|^2 \, dx < +\infty \right\},$$

which is endowed with the Hilbert norm $\|q\| = \left(|q(0)|^2 + \int_{\mathbb{R}} |\dot q(x)|^2 \, dx\right)^{1/2}$. It's standard to show that F is weakly lower semicontinuous on E (see also [5]). Now one can consider the subclass of E, i.e.,

$$\Gamma := \left\{ q \in E \;\middle|\; \lim_{x\to-\infty} q(x) = \sigma_-, \quad \lim_{x\to+\infty} q(x) = \sigma_+ \right\}.$$

There exists a minimizer $\bar q \in \Gamma$ such that $F(\bar q) = \min_\Gamma F(q) := c$, i.e., the class $\mathcal{K} := \{q \in \Gamma \mid F(q) = c\}$ is not empty. Moreover, each element in $\mathcal{K}$ is a classic solution to (1.2).

Finally, by the definition of Γ and quadratical behavior of W around $\sigma_\pm$, then the following L^2 metric is well defined on Γ

$$\mathrm{d}(q_1, q_2) = \left(\int_{\mathbb{R}} |q_1(x) - q_2(x)|^2 \, dx \right)^{1/2}, \quad \forall q_1, q_2 \in \Gamma.$$

Note that the metric space (Γ, d) is not complete and we will denote by $\bar\Gamma$ its completion. We also need to define another metric on Γ

$$D(q_1, q_2) = \|q_1 - q_2\|_{H^1(\mathbb{R})}, \quad \forall q_1, q_2 \in \Gamma.$$

REMARK 2.1. As in [5], if $q_n \in \Gamma$ such that $F(q_n) \to c$, then there exists $\bar q \in \mathcal{K}$ such that, along a subsequence, $\|q_n - \bar q\|_{H^1(\mathbb{R})} \to 0$.

Clearly, for any $r > 0$ there exists some $h_r > 0$ such that

$$\text{if} \quad q \in \Gamma \quad \text{and} \quad \inf_{\bar q \in \mathcal{K}} \|q - \bar q\|_{H^1(\mathbb{R})} \ge r \quad \text{then} \quad F(q) \ge c + h_r. \tag{2.1}$$

Let

$$\mathcal{H} := \{u \in H^1_{\rm loc}(\mathbb{R}^2) \mid u(\,\cdot\,, y) \in \Gamma \text{ for a.e. } y \in \mathbb{R}\}.$$

We note that if $u \in \mathcal{H}$, then the function $y \to \int_{\mathbb{R}} (1/2)|\nabla u(x, y)|^2 + a(x)W(u(x, y))\, dx$ is measurable and greater than or equal to c for a.e. $y \in \mathbb{R}$. Therefore the functional $\varphi \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi(u) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{2} |\nabla u(x, y)|^2 + a(x)W(u(x, y))\, dx - c \right] dy, \quad u \in \mathcal{H},$$

is well defined and it can be rewritten in the more enlightening form

$$\varphi(u) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{2} |\partial_y u(x, y)|^2 \, dx + F(u(\cdot, y)) - c \right] dy, \quad u \in \mathcal{H}.$$

We see that $\varphi(u) \geq 0$ when $u \in \mathcal{H}$, if $q \in \mathcal{K}$, then the function $u(x, y) = q(x)$ belongs to $\mathcal{H}$ and $\varphi(u) = 0$, i.e., the one dimensional solution of (1.1) is global minimal of φ on $\mathcal{H}$. If there are infinite elements in $\mathcal{K}$ distinct up to periodic transitions, then Theorem 1.3 is true. Otherwise, we will analyze the case where
$(*)$ $\mathcal{K}$ is finite distinct up to periodic transitions.

3. Two dimensional heteroclinic solutions

In this section, we assume $(*)$, $\mathcal{K}$ is constituted by isolated points that we will enumerate by q_ξ, $\xi \in \mathbb{Z}$. Also, setting

$$\inf_{\xi \neq \eta} D(q_\xi, q_\eta) := \alpha = 3r_0, \tag{3.1}$$

we have $\alpha > 0$, since by $(*)$ $\mathcal{K}$ is locally finite.

REMARK 3.1. As in [5], if $(y_1, y_2) \subset \mathbb{R}$ and $u \in \mathcal{H}$ are such that $\inf_{q \in \mathcal{K}} \|u(\cdot, y) - q\|_{H^1(\mathbb{R})} \geq r > 0$ for a.e. $y \in (y_1, y_2)$, then

$$\varphi(u) \geq \sqrt{2h_r}\, \mathrm{d}(u(\cdot, y_1), u(\cdot, y_2)). \tag{3.2}$$

Especially, corresponding to $r_0 = \alpha/3$, let us fix $h_0 > 0$ such that

$$\text{if} \quad q \in \Gamma \quad \text{and} \quad \inf_{\bar{q} \in \mathcal{K}} \|q - \bar{q}\|_{H^1(\mathbb{R})} \geq \frac{r_0}{2}, \quad \text{then} \quad F(q) \geq c + h_0. \tag{3.3}$$

If $u \in \mathcal{H}$, we obtain for $y_1, y_2 \in \mathbb{R}$ that

$$\begin{aligned}
\int_{\mathbb{R}} |u(x, y_2) - u(x, y_1)|^2 \, dx &= \int_{\mathbb{R}} \left| \int_{y_1}^{y_2} \partial_y u(x, y) \, dy \right|^2 dx \\
&\leq |y_2 - y_1| \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_y u(x, y) \, dy|^2 \, dy \, dx \\
&\leq 2\varphi(u) |y_2 - y_1|.
\end{aligned}$$

If $\varphi(u) < +\infty$, then the function $y \to u(\cdot, y)$ is Holder continuous from a dense subset of $\mathbb{R}$. Following the procedure introduced by Alessio, Jeanjean and Montecchiari in [5] (see also [12]), we look for solutions to (1.1) depending on both the variables x and y.

Lemma 3.1. *For any $C > 0$ there exists $C' > 0$ such that if $u \in \mathcal{H} \cap \{\varphi \leq C\}$, then* $\mathrm{d}(u(\cdot, y_1), u(\cdot, y_2)) \leq C'$ *for any $y_1, y_2 \in \mathbb{R}$.*

Proof. Denoting $\gamma(y) = u(\cdot, y), y \in \mathbb{R}$, we can consider γ as a path in $\bar{\mathcal{H}}$. For any $y_1, y_2 \in \mathbb{R}$, by compactness $\gamma([y_1, y_2])$ intersects only a finite number of sets $B_{r_0}(q_\xi)$, $\xi \in \mathbb{Z}$. Let $\{B_i \mid i = 1, \dots, k\}$ be the family in $\{B_{r_0}(q_\xi) \mid B_{r_0}(q_\xi) \cap \gamma([y_1, y_2]) \neq \emptyset, \xi \in \mathbb{Z}\}$ such that if $\gamma(y) \notin \bigcup_{i=1}^k B_i$, $y \in [y_1, y_2]$, by (3.1), then $\mathrm{d}(u(\cdot, \mathcal{K})) \geq r_0$, and $\mathrm{dist}(B_i, B_{i+1}) \geq r_0$ for $i \in \{1, \dots, k-1\}$. Moreover, we have $\max_i(\mathrm{diam}(B_i)) \leq 2r_0$. From (2.1) and (3.2) one obtains that

$$C \geq \varphi(u) \geq \sqrt{2h_0} \max\{\mathrm{d}(\gamma(y_1), \gamma(y_2)) - 2kr_0, (k-1)r_0\},$$

hence $\mathrm{d}(\gamma(y_1), \gamma(y_2) \leq 3C/\sqrt{2h_0} + 2r_0 := C'$. □

The consequence of (2.1) and Lemma 3.1 is that they provide information on the asymptotic behavior of the functions in the sublevels of φ as $y \to \pm\infty$.

Lemma 3.2. *If $u \in \mathcal{H} \cap \{\varphi < +\infty\}$, then there exist $\xi_\pm \in \mathbb{Z}$ such that $\mathrm{d}(u(\cdot, y), q_{\xi_\pm}) \to 0$ as $y \to \pm\infty$.*

Proof. If $\varphi(u) < +\infty$, by the definition of $\varphi(u)$, we have $F(u(\cdot, y)) \to c$ as $y \to \pm\infty$, i.e., $\liminf_{y\to\pm\infty} \mathrm{d}(u(\cdot, y), \mathcal{K}) = 0$. Since by Lemma 3.1 the path $y \to u(\cdot, y)$ is bounded in $\bar{\mathcal{H}}$, there exist $\xi_\pm \in \mathbb{Z}$ such that $\liminf_{y\to\pm\infty} \mathrm{d}(u(\cdot, y), q_{\xi_\pm}) = 0$. Or else we assume by contradiction that $\liminf_{y\to+\infty} \mathrm{d}(u(\cdot, y), q_{\xi_+}) \geq r > 0$, then there exist infinite many intervals $(p_i, s_i) \subset \mathbb{R}$, $i \in \mathbb{N}$ such that $\mathrm{d}(u(\cdot, y), \mathcal{K}) \geq r/2$ for any $y \in (p_i, s_i)$, by the definition of α and (3.2) we have

$$\varphi(u) \geq \sum_{i=1}^{\infty} \sqrt{2h_r} \cdot \frac{r}{2} = +\infty,$$

which is a contradiction. Similarly, one can prove $\lim_{y\to-\infty} \mathrm{d}(u(\cdot, y), q_{\xi_-}) = 0$. □

By Lemma 3.2 we can restrict ourselves to consider the elements in $\mathcal{H}$ which have prescribed limits as $y \to \pm\infty$. By periodicity it is sufficient to consider, for $\xi \in \mathbb{Z}$, the classes

$$\mathcal{H}_\xi = \left\{u \in \mathcal{H} \;\middle|\; \lim_{y\to-\infty} \mathrm{d}(u(\cdot, y), q_0) = \lim_{y\to+\infty} \mathrm{d}(u(\cdot, y), q_\xi) = 0\right\}$$

and

$$m_\xi = \inf_{u\in\mathcal{H}_\xi} \varphi(u) \quad \text{and} \quad \mathcal{K}_\xi = \{u \in \mathcal{H}_\xi \mid \varphi(u) = m_\xi\}, \quad \xi \in \mathbb{Z}.$$

Using suitable test functions, one can prove that $m_\xi < +\infty$ for any $\xi \in \mathbb{Z}$. Moreover, we have the following lemma.

Lemma 3.3. *There holds $m_\xi \geq \sqrt{2h_0}r_0$ for any $\xi \neq 0$ and $m_\xi \to +\infty$ as $|\xi| \to +\infty$.*

Proof. It's easy to see that $D(q_0, q_\xi) \to +\infty$ as $|\xi| \to +\infty$, by the definition of $\mathcal{H}_\xi$ and Lemma 3.1, It follows that $m_\xi \to +\infty$ as $|\xi| \to +\infty$. To prove the first estimate, let $\xi \neq 0$ and $u \in \mathcal{H}_\xi$, we have $D(u(\,\cdot\,, y), q_0) \to 0$ as $y \to -\infty$ while $\liminf_{y\to+\infty} D(u(\,\cdot\,, y), q_0) \geq D(q_0, q_\xi) \geq \alpha$. By the continuity of $u(\,\cdot\,, y)$ there exists $(y_1, y_2) \subset \mathbb{R}$ such that $r_0 \leq \mathrm{d}(u(\,\cdot\,, y), q_0) \leq 2r_0$ for any $y \in (y_1, y_2)$, by $(*)$, $r_0 \leq \mathrm{d}(u(\,\cdot\,, y), \mathcal{K})$, and using (3.2) we have $\varphi(u) \geq \sqrt{2h_0}r_0$ and the lemma follows. □

By Lemma 3.3, there exists some $\eta \in \mathbb{Z}$ such that

$$m_\eta = \min_{\xi\neq 0} m_\xi. \tag{3.4}$$

As we will see in the next lemma, the minimality property of η allows us to further characterize the functions in $\mathcal{H}_\eta$ whose action is close to m_η.

REMARK 3.2. We define

$$\chi^+_{\xi,y_0}(u)(x, y) = \begin{cases} q_\xi(x) & \text{if} \quad y \geq y_0 + 1, \\ u(x, y)(y_0 + 1 - y) + q_\xi(x)(y - y_0) & \text{if} \quad y_0 \leq y < y_0 + 1, \\ u(x, y) & \text{if} \quad y < y_0, \end{cases}$$

$$\chi^-_{\xi,y_0}(u)(x, y) = \begin{cases} u(x, y) & \text{when} \quad y \geq y_0 + 1, \\ u(x, y)(y - y_0) + q_\xi(x)(y_0 + 1 - y) & \text{when} \quad y_0 \leq y < y_0 + 1, \\ q_\xi(x) & \text{when} \quad y < y_0, \end{cases}$$

and set

$$\varphi_{(s,p)}(u) = \varphi_s^p(u) := \int_s^p \left[\int_{\mathbb{R}} \frac{1}{2}|\partial_y u|^2\, dx + F(u(\,\cdot\,, y)) - c\right] dy.$$

Lemma 3.4. *There exists a $\delta_0 \in (0, r_0/2)$, and for any $\delta \in (0, \delta_0)$ such that if $u \in \mathcal{H}_\xi$ and $\varphi(u) \leq m_\eta + \lambda_\delta$, then*
(i) *if $D(u(\,\cdot\,, y), \mathcal{K}) \geq \delta$ for all $y \in (s, p)$, then $p - s \leq l_\delta$;*
(ii) *if $D(u(\,\cdot\,, y_0), q_0) \leq \delta$ then $D(u(\,\cdot\,, y), q_0) \leq r_0$ for all $y \leq y_0$;*
(iii) *if $D(u(\,\cdot\,, y_0), q_\eta) \leq \delta$ then $D(u(\,\cdot\,, y), q_\eta) \leq r_0$ for all $y \geq y_0$;*
(iv) *if $\xi \in \mathbb{Z} \setminus \{0, \eta\}$, then $D(u(\,\cdot\,, y), q_\xi) > \delta$ for all $y \in \mathbb{R}$.*

Proof. By (2.1), $F(u(\,\cdot\,, y)) \geq c + h_\delta$ for $y \in (s, p)$, by (3.2) we have $m_\eta + \lambda_\eta \geq \varphi(u) \geq h_\delta(p - s)$, thus $p - s \leq (1/h_\delta)(m_\eta + \lambda_\eta) := l_\delta$.

To prove (ii), we first fix some notations. Note Remark 2.1, for any $\delta > 0$, by continuity there exists a λ_δ with $\lambda_\delta \to 0$ as $\delta \to 0$, and $\delta \in (0, \sqrt{2\lambda_\delta})$, $\lambda_\delta \leq$

$\min\{(r_0/5)\sqrt{h_0/2}, (1/4)m_\eta\}$ such that for any $u \in \mathcal{H}$,

$$\text{if} \quad D(u(\,\cdot\,, y), \mathcal{K}) \le \delta, \quad \text{then} \quad F(u(\,\cdot\,, y)) \le c + \lambda_\delta. \tag{3.5}$$

Let $u \in \mathcal{H}_\eta$ be such that $\varphi(u) \le m_\eta + \lambda_\delta$ and assume that $y_0 \in \mathbb{R}$ is such that $D(u(\,\cdot\,, y_0), q_0) \le \delta_0$, we define $\tilde{u}(x, y) = \chi^-_{\xi, y_0-1}(u)(x, y)$, note that $\tilde{u} \in \mathcal{H}_\eta$ and so $\varphi(\tilde{u}) \ge m_\eta$, then

$$m_\eta \le \varphi(\tilde{u}) = \varphi(u) - \varphi^{y_0}_{-\infty}(u) + \varphi^{y_0}_{y_0-1}(\tilde{u})$$

and

$$\varphi^{y_0}_{-\infty}(u) \le \lambda_\delta + \int_{y_0-1}^{y_0} \left[\int_{\mathbb{R}} \frac{1}{2} |u(x, y_0) - q_0(x)|^2 \, dx + F(\tilde{u}(\,\cdot\,, y)) - c \right] dy.$$

Since $D(\tilde{u}(\cdot, y), q_0) \le D(u(\cdot, y_0), q_0) \le \delta$ for $y \in (y_0 - 1, y_0)$, then $F(\tilde{u}(\cdot, y)) - c \le \lambda_\delta$, and

$$\varphi^{y_0}_{-\infty}(u) \le \lambda_\delta + \frac{1}{2}\delta^2 + \lambda_\delta \le 3\lambda_\delta.$$

Assume by contradiction that there exists $y_1 \le y_0$ such that $D(u(\,\cdot\,, y_1), q_0) \ge r_0$, then by continuity there exists $(y_1', y_0') \subset (y_1, y_0)$ such that $D(u(\,\cdot\,, y), q_0(\,\cdot\,)) \in (r_0/2, r_0)$ for a.e. $y \in (y_1', y_0')$ and $D(u(\,\cdot\,, y_1'), u(\,\cdot\,, y_0')) \ge r_0/2$. Hence by (3.2)

$$3\lambda_\delta \ge \varphi^{y_0}_{-\infty}(u) \ge \sqrt{2h_0}\,\frac{r_0}{2} \ge 5\lambda_\delta,$$

which is a contradiction. Similarly one can show (iii).

To prove (iv), we assume by contradiction that there exists $y_0 \in \mathbb{R}$ and $\xi \in \mathbb{R} \setminus \{0, \eta\}$ such that $D(u(\cdot, y_0), q_\xi) \le \delta$. Let $u_1 = \chi^+_{\xi, y_0}(u)(x, y)$, $u_2 = \chi^-_{\xi, y_0}(u)(x, y)$ and note that $u_1 \in \mathcal{H}_\xi$ while $u_2(\,\cdot - \xi,\ \cdot\,) \in \mathcal{H}_{\eta-\xi}$. Since $m_\eta = \min_{\xi \ne 0} m_\xi$, then $\varphi(u_1) + \varphi(u_2) \ge 2m_\eta$, and we have

$$\begin{aligned}
\varphi(u_1) + \varphi(u_2) &= \varphi(u) - \varphi^{y_0+1}_{y_0}(u) + \varphi^{y_0+1}_{y_0}(u_1) + \varphi^{y_0+1}_{y_0}(u_2) \\
&\le \varphi(u) + \varphi^{y_0+1}_{y_0}(u_1) + \varphi^{y_0+1}_{y_0}(u_2) \\
&\le m_\eta + \lambda_\delta + \varphi^{y_0+1}_{y_0}(u_1) + \varphi^{y_0+1}_{y_0}(u_2).
\end{aligned}$$

Since $D(u_1(\,\cdot\,, y), q_\xi) \le D(u(\,\cdot\,, y_0), q_\xi) \le \delta$ and $D(u_2(\,\cdot\,, y), q_\xi) \le D(u(\,\cdot\,, y_0), q_\xi) \le \delta$ for $y \in [y_0, y_0 + 1]$, then $\varphi^{y_0+1}_{y_0}(u_1) + \varphi^{y_0+1}_{y_0}(u_2) \le 2\lambda_\delta$ by (3.4), which leads to that $m_\eta \le 3\lambda_\delta$, it's a contradiction. $\square$

We are now able to prove the following compactness property of the minimizing sequence of φ in $\mathcal{H}_\eta$. It will be sufficient to use the direct method of the calculus of variation to show that the functional φ admits a minimum in class $\mathcal{H}_\eta$.

Lemma 3.5. *Let* $(u_n) \subset \mathcal{H}_\eta$, $\varphi(u_n) \to m_\eta$ *be such that* $D(u_n(\cdot, 0), \mathcal{K}) \geq \delta$ *for any* $n \in \mathbb{N}$, *then there exists* $u_\eta \in \mathcal{K}_\eta$ *such that up to a subsequence* $u_n \to u_\eta$ *as* $n \to \infty$ *weakly in* $H^1_{\text{loc}}(\mathbb{R}^2)$. *Moreover,* $D(u_n(\cdot\,, y), u_\eta(\cdot\,, y)) \to 0$ *for a.e.* $y \in \mathbb{R}$ *as* $n \to \infty$.

Proof. Let $(u_n) \subset \mathcal{H}_\eta$ be such that $\varphi(u_n) \leq m_\eta + \lambda_\delta$ for any $n \in \mathbb{N}$. Assume that $\|u_n\|_{L^\infty} \leq R_0$, indeed otherwise we can consider the minimizing sequence $\tilde{u}_n = \max\{\min\{u_n, R_0\}, -R_0\}$. Since $D(u_n(\cdot, 0), \mathcal{K}) \geq \delta$, by Lemma 3.4 we have $D(u_n(\cdot, y), q_0) \leq r_0$ for $y \leq -l_\delta$ and $D(u_n(\cdot\,, y), q_\eta) \leq r_0$ for $y \geq l_\delta$.

Since $\varphi(u_n) \leq m_\eta + \lambda_\delta$, there exists a function $u_\eta \in \mathcal{K}_\eta$ such that along a subsequence $u_n \to u_\eta$ in $H^1(\Omega)$ for every $\Omega \Subset \mathbb{R}^2$ (refer to [5]), and $D(u_\eta(\cdot\,, y), q_0) \leq r_0$ for $y \leq -l_\delta$ and $D(u_\eta(\cdot\,, y), q_\eta) \leq r_0$ for $y \geq l_\delta$. By Lemma 3.2 we conclude that $D(u_\eta(\cdot\,, y), q_0) \to 0$ as $y \to -\infty$ and $D(u_\eta(\cdot\,, y), q_\eta) \to 0$ as $y \to +\infty$, i.e., $u_\eta \in \mathcal{H}_\eta$. By weak semi-continuity of φ (see[5]), we have $\varphi \leq m_\eta$, thus $u_\eta \in \mathcal{K}_\eta$.

To prove the last argument, we first claim that $\varphi_{y_1}^{y_2}(u_n) \to \varphi_{y_1}^{y_2}(u_\eta)$ for $\forall y_1 < y_2 \in \mathbb{R}$.

Indeed by semicontinuity, $\varphi_{y_1}^{y_2}(u_\eta) \leq \liminf_{n\to\infty} \varphi_{y_1}^{y_2}(u_n)$. Assume by contradiction that there exists a interval (y_1, y_2) such that $\limsup_{n\to\infty}(\varphi_{y_1}^{y_2}(u_n) - \varphi_{y_1}^{y_2}(u_\eta)) \geq \varepsilon_0 > 0$. By the continuity of $y \to u(\cdot\,, y)$, there exists $B_{h_0}(y_0) \subset (y_1, y_2)$ and a subsequence $(u_{n_j}) \subset (u_n)$ such that $\varphi_{B_{h_0}(y_0)}(u_{n_j}) - \varphi_{B_{h_0}(y_0)}(u_\eta) \geq \varepsilon_0/2$ as $j \to \infty$. Then

$$\begin{aligned} \varphi_{y_1}^{y_2}(u_{n_j}) - \varphi_{y_1}^{y_2}(u_\eta) &= \varphi_{(y_1,y_2)\setminus B_{h_0}(y_0)}(u_{n_j}) - \varphi_{(y_1,y_2)\setminus B_{h_0}(y_0)}(u_\eta) \\ &\quad + \varphi_{B_{h_0}(y_0)}(u_{n_j}) - \varphi_{B_{h_0}(y_0)}(u_\eta) \end{aligned}$$

and $\liminf_{j\to\infty}(\varphi_{y_1}^{y_2}(u_{n_j}) - \varphi_{y_1}^{y_2}(u_\eta)) \geq \varepsilon_0/2$, it's a contradiction.

Since $y_1 < y_2$ is arbitrary, let $y_1 \to y_2$, by the definition of φ we have $F(u_n(\cdot\,, y)) \to F(u_\eta(\cdot\,, y))$ for every fixed $y \in \mathbb{R}$ as $n \to \infty$. Let

$$X_y := X(u(\cdot\,, y)) = \sup\{x \in \mathbb{R} \mid \min|u(x, y) \mp \sigma_\pm| \geq \rho_0\},$$

where $\rho_0 = (1/6)|\sigma_+ - \sigma_-|$. Since $F(u_n(\cdot\,, y)), F(u_\eta(\cdot\,, y)) < +\infty$, for $\forall \varepsilon > 0$, there exists a constant $l_{y,\varepsilon} > 0$ related to y and ε such that

$$\int_{|x-X_y|>l_{y,\varepsilon}} \frac{1}{2}|\partial_x u_n|^2 + a(x)W(u_n(x, y))\,dx < \underline{a}\omega_0\varepsilon \tag{3.6}$$

and

$$\int_{|x-X_y|>l_{y,\varepsilon}} \frac{1}{2}|\partial_x u_\eta|^2 + a(x)W(u_\eta(x, y))\,dx < \underline{a}\omega_0\varepsilon. \tag{3.7}$$

Observing that $W(u(x, y)) \geq \omega_0|u(x, y) - Q(x - X_y)|^2$ for any $|x - X_y| > l_{y,\varepsilon}$, then we have

$$\int_{|x-X_y|>l_{y,\varepsilon}} |u_n(x, y) - Q(x - X_y)|^2\, dx < \varepsilon \tag{3.8}$$

and

$$\int_{|x-X_y|>l_{y,\varepsilon}} |u_\eta(x, y) - Q(x - X_y)|^2\, dx < \varepsilon. \tag{3.9}$$

Since $u_n(\,\cdot\,, y) \to u_\eta(\,\cdot\,, y)$ in $L^\infty(\mathbb{R})$ and by (3.8), (3.9) we have $u_n(\,\cdot\,, y) \to u_\eta(\,\cdot\,, y)$ in $L^2(\mathbb{R})$. By (3.6), (3.7) we have $\int_{\mathbb{R}} W(u_n(x, y))\, dx \to \int_{\mathbb{R}} W(u_\eta(x, y))\, dx$, and since $F(u_n(\cdot, y)) \to F(u_\eta(\cdot, y))$, we obtain $\int_{\mathbb{R}} |\partial_x u_n(x, y)|^2\, dx \to \int_{\mathbb{R}} |\partial_x u_\eta(x, y)|^2\, dx$. Together we have $\|u_n(\,\cdot\,, y) - u_\eta(\,\cdot\,, y)\|_{H^1(\mathbb{R})} \to 0$, i.e., $D(u_n(\,\cdot\,, y), u_\eta(\,\cdot\,, y)) \to 0$ as $n \to \infty$. □

In fact, $u_\eta \in \mathcal{C}^2(\mathbb{R}^2)$ is a classical solution to (1.1) with $\|u_\eta\|_{L^\infty(\mathbb{R}^2)} \le R_0$ (see [5]). Lemma 3.5 admits two dimensional heteroclinic solutions to (1.1), if the set $\mathcal{K}_\eta$ is infinite distinct up to transitions, then Theorem 1.3 holds. Otherwise, we will analyze the case where
$(**)$ $\mathcal{K}_\eta$ is finite distinct up to transitions.

4. Multibump type solutions

In this section, we assume $(*)$ and $(**)$. Since $u_\eta(x, y + \theta)$, $\forall\theta \in \mathbb{R}$ are solutions to (1.1), and $\mathrm{d}(q_\xi, q_\eta) \ge 3r_0$ for $q_\xi, q_\eta \in \mathcal{K}$, $\xi \ne \eta$, we define

$$\mathcal{K}_\eta^0 = \left\{ u \in \mathcal{K}_\eta \;\middle|\; D(u(\,\cdot\,, 0), q_0) = \frac{3}{2} r_0 \right\}.$$

REMARK 4.1. Let $J = \{D(u(\,\cdot\,, i), q_0) \mid u \in \mathcal{K}_\eta^0,\ i \in \mathbb{Z}\}$, by $(**)$ the set J is countable and so the set $\triangle = (0, r_0) \setminus J$ is non countable, dense subset of $(0, r_0)$.

Lemma 4.1. *For all $\delta \in \triangle$ there exists a $\Lambda \in (0, r_0)$ such that if $u \in \mathcal{H}_\eta$ satisfies $D(u(\,\cdot\,, \xi), q_0) = \delta$ for some $\xi \in \mathbb{Z}$, then $\varphi(u) \ge m_\eta + \Lambda$.*

Proof. By contradiction assume that there exists a sequence $u_n \in \mathcal{H}_\eta$ such that $\varphi(u_n) \to m_\eta$ and $D(u_n(\,\cdot\,, \xi), q_0) = \delta$ for some $\xi \in \mathbb{Z}$. Since $\delta < r_0$, by Lemma 3.5 there exists a $u \in \mathcal{K}_\eta$ such that up to a subsequence $u_n(\cdot, y) \to u(\cdot, y)$ in $H^1(\mathbb{R})$. Then $D(u(\,\cdot\,, \xi), q_0) = \delta$ and $u \in \mathcal{K}_\eta$, which contradicts with the assumption $\delta \in \triangle$. □

REMARK 4.2. For $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$, let $\overline{u}(x, y) = u(x, -y), (x, y) \in \mathbb{R}^2$, and set $\overline{\mathcal{H}}_\eta = \{u \in \mathcal{H} \mid \bar{u} \in \mathcal{H}_\eta\}$. We have $\varphi(u) = \varphi(\bar{u})$ and so that $\inf_{\bar{\mathcal{H}}_\eta} \varphi = m_\eta$. Setting $\bar{\mathcal{K}}_\eta = \{u \in \bar{\mathcal{H}}_\eta \mid \varphi(u) = m_\eta\} = \{u \in \mathcal{H} \mid u \in \mathcal{K}\}$, $\bar{J} = \{D(u(\,\cdot\,, i), q_\eta) \mid u \in \bar{\mathcal{K}}_\eta,\ i \in \mathbb{Z}\}$ and

$\bar{\Delta} = (0, r_0) \setminus \bar{J}$, and arguing as in Lemma 4.1 one can prove that for all $\delta \in \bar{\Delta}$ there exists a $\Lambda \in (0, 4_0)$ such that if $u \in \bar{\mathcal{H}}_\eta$ satisfies $D(u(\,\cdot\,, \xi), q_\eta) = \delta$ for some $\xi \in \mathbb{Z}$, then $\varphi(u) \geq m_\eta + \Lambda$.

If $(**)$ holds, we can choose $\delta \in \triangle \cap \bar{\Delta}$ and $\Lambda \in (0, r_0)$ such that if $u \in \mathcal{H}_\eta$ and $D(u(\,\cdot\,, \xi), q_0) = \delta$ for some $\xi \in \mathbb{Z}$, or $u \in \mathcal{H}_\eta$ and $D(u(\,\cdot\,, \xi), q_0) = \delta$ then

$$\varphi(u) \geq m_\eta + \Lambda. \tag{4.1}$$

Now let us fix some constants, let $\tilde{\Lambda} > 0$ be such that $\tilde{\Lambda} < \Lambda/2$. Let $\varepsilon \in (0, \delta)$ be such that $\lambda_\varepsilon < \Lambda/8$. Let $L \in \mathbb{N}$ be such that $L > (m_\eta + \tilde{\Lambda})/h_\varepsilon$, where h_ε is given by (2.1) and such that there exists a $u_\eta \in \mathcal{K}_\eta$ with

$$D(u_\eta(\,\cdot\,, -L), q_0) \leq \delta, \quad D(u_\eta(\,\cdot\,, L), q_\eta) \leq \delta.$$

We define

$$\mathcal{U}_0 = \{u \in \mathcal{H} \mid D(u(\,\cdot\,, -L), q_0) \leq \delta,\ D(u(\,\cdot\,, L), q_\eta) \leq \delta\}$$

and

$$\mathcal{U}_\eta = \{u \in \mathcal{H} \mid D(u(\,\cdot\,, -L), q_\eta) \leq \delta,\ D(u(\,\cdot\,, L), q_0) \leq \delta\}.$$

Lemma 4.2. *If $u \in \mathcal{U}_\sigma$, $\sigma \in \{0, \eta\}$ satisfies $\varphi_{-2L}^{2L}(u) \leq m_\eta + \tilde{\Lambda}$, then there exist $l^- \in [-2L, -L]$ and $l^+ \in [L, 2L]$ such that*

$$D(u(\,\cdot\,, l^-), q_0) \leq \varepsilon \quad \textit{and} \quad D(u(\,\cdot\,, l^-), q_\eta) \leq \varepsilon \quad \textit{if} \quad \sigma = 0$$

or

$$D(u(\,\cdot\,, l^+), q_\eta) \leq \varepsilon \quad \textit{and} \quad D(u(\,\cdot\,, l^+), q_0) \leq \varepsilon \quad \textit{if} \quad \sigma = \eta.$$

Proof. We consider the case $\sigma = 0$, the other case follows. By contradiction, assume that there exist a $u \in \mathcal{U}_0$ such that $\varphi_{-2L}^{2L}(u) \leq m_\eta + \tilde{\Lambda}$ and $D(u(\,\cdot\,, y), q_0) > \varepsilon$ for all $y \in [-2L, -L]$. Then, by (2.1) we have $F(u(\,\cdot\,, y)) > h_\varepsilon + c$ for all $y \in [-2L, -L]$, therefore

$$m_\eta + \tilde{\Lambda} \geq \varphi_{-2L}^{2L}(u) \geq \varphi_{-2L}^{-L}(u) > Lh_\varepsilon,$$

which is a contradiction with the choice of L. In the same way, we can prove the existence of $l^+ \in [L, 2L]$ such that $D(u(\,\cdot\,, l^+), q_\eta) \leq \varepsilon$. □

Then we obtain

Lemma 4.3. *If $u \in \mathcal{U}_\sigma$, $\sigma \in \{0, \eta\}$ satisfies $\varphi_{-2L}^{2L}(u) \le m_\eta + \tilde{\Lambda}$, then*

$$D(u(\,\cdot\,, -L), q_0) < \delta \quad \text{and} \quad D(u(\,\cdot\,, L), q_\eta) < \delta \quad \text{if} \quad \sigma = 0$$

or

$$D(u(\,\cdot\,, -L), q_\eta) < \delta \quad \text{and} \quad D(u(\,\cdot\,, L), q_0) < \delta \quad \text{if} \quad \sigma = \eta.$$

Proof. We consider the case $\sigma = 0$, the other case follows similarly.

Let $u \in \mathcal{U}_0$ with $\varphi_{-2L}^{2L}(u) \le m_\eta + \tilde{\Lambda}$ and let $l^- \in [-2L, -L]$ and $l^+ \in [L, 2L]$ be given by Lemma 4.2, we set $\tilde{u} = \chi_{0,l^-}^- \circ \chi_{\eta,l^+}^+(u)$ then $\tilde{u} \in \mathcal{H}_\eta$. Since $D(\tilde{u}(\,\cdot\,, y), q_\eta) \le D(u(\,\cdot\,, l^+), q_\eta) \le \varepsilon$ when $y \in (l^+, l^+ + 1)$, then $\varphi_{l^+}^{l^+ + 1}(\tilde{u}) \le \lambda_\varepsilon$ and also $\varphi_{l^-}^{l^- + 1}(\tilde{u}) \le \lambda_\varepsilon$, we obtain

$$\varphi(\tilde{u}) \le \varphi_{-2L}^{2L}(u) + 2\lambda_\varepsilon \le m_\eta + \tilde{\Lambda} + 2\lambda_\varepsilon < m_\eta + \Lambda,$$

by the choice of ε and $\tilde{\Lambda}$.

Setting $\overline{u}(x, y) = \tilde{u}(x, y + L)$, $\overline{u} \in \mathcal{H}_\eta$ with $\varphi(\overline{u}) = \varphi(\tilde{u}) < m_\eta + \Lambda$ and $D(\overline{u}(\cdot, 0), q_0) \le \delta$, by (4.1) we exclude the case $D(\overline{u}(\cdot, 0), q_0) = \delta$, i.e., $D(\overline{u}(\cdot, 0), q_0) < \delta$. So we conclude $D(u(\,\cdot\,, -L), q_0) < \delta$. $\square$

One can see that Lemma 4.3 excludes the case that minimizers might be on the border of the set. We now define the classes of functions in which we look for multibump solutions. Let us define $N \in \mathbb{N}$, N is odd, $p = (p_1, \dots, p_N) \in \mathbb{Z}^N$, $\sigma = (\sigma_1, \dots, \sigma_N) \in \{0, \eta\}^N$ with $\sigma_i \ne \sigma_{i-1}$ for all $i = 2, \dots, N$. We set

$$\mathcal{H}_{N,p,\sigma} = \{u \in \mathcal{H} \mid u(x, y - p_i) \in \mathcal{U}_\sigma \text{ for a.e. } (x, y) \in \mathbb{R}^2,\ i = 1, \dots, N\}$$

and $m_{N,p,\sigma} = \inf_{\mathcal{H}_{N,p,\sigma}} \varphi$.

REMARK 4.3. Given any $i \in \{1, \dots, N\}$, we define

$$w_i(x, y) = \begin{cases} u_\eta(x, y - p_i) & \text{if} \quad \sigma_i = 0, \\ u_\eta(x, p_i - y) & \text{if} \quad \sigma_i = \eta. \end{cases}$$

By Lemma 4.2, there exist $l_i^- \in [p_i - 2L, p_i - L]$ and $l_i^+ \in [p_i + L, p_i + 2L]$ such that

$$D(w_i(\,\cdot\,, l_i^-), q_{\sigma_i}) \le \varepsilon$$

and

$$D(w_i(\,\cdot\,, l_i^+), q_{\sigma_{i+1}}) \le \varepsilon.$$

Setting $\tilde{w}_i = \chi^-_{\sigma_i, l_i^-} \circ \chi^+_{\sigma_{i+1}, l_i^+}(w_i)$, then we have $\varphi(\tilde{w}_i) \le m_\eta + 2\lambda_\varepsilon$.

Lemma 4.4. *If $u \in \mathcal{H}_{N,p,\sigma}$ satisfies $\varphi(u) = m_{N,p,\sigma}$, then $\varphi^{p_i+2L}_{p_i-2L}(u) \le m_\eta + \tilde{\Lambda}$, $\forall i = 1, \ldots, N$.*

Proof. We define

$$I := \left\{ i \in \{1, \ldots, N\} \;\middle|\; \varphi^{p_i+2L}_{p_i-2L}(u) > m_\eta + \tilde{\Lambda} \right\},$$
$$I^+ = \{ i \in \{1, \ldots, N\} \setminus I \mid i + 1 \in I \},$$

and

$$I^- = \{ i \in \{1, \ldots, N\} \setminus I \mid i - 1 \in I \}.$$

It's obvious that $\operatorname{Card}(I^+ \cup I^-) \le 2 \operatorname{Card} I$.

Applying Lemma 4.2 for every $i \in I$, we can let $l_i^- \in [p_i - 2L, p_i - L]$ and $l_i^+ \in [p_i + L, p_i + 2L]$ be the corresponding real numbers. Let us consider the sets of consecutive intervals

$$J_1 = \left\{ y \in \mathbb{R} \mid y \le \frac{p_1 + p_2}{2} \right\}, \quad J_N = \left\{ y \in \mathbb{R} \mid y \ge \frac{p_{N-1} + p_N}{2} \right\},$$
$$J_i = \left\{ y \in \mathbb{R} \mid \frac{p_{i-1} + p_i}{2} \le y \le \frac{p_i + p_{i+1}}{2} \right\}, \quad i = 2, \ldots, N-1,$$

and note that $l_i^\pm \in J_i$, $i = 1, \ldots, N$.

For any $i \in \{i, \ldots, N\}$, we replace u in J_i with the function

$$\tilde{u} = \begin{cases} \tilde{w}_i & \text{if} \quad i \in I, \\ \chi^-_{\sigma_i, l_i^-} \circ \chi^+_{\sigma_{i+1}, l_i^+}(u) & \text{if} \quad i \in I^- \cap I^+, \\ \chi^-_{\sigma_i, l_i^-}(u) & \text{if} \quad i \in I^- \setminus I^+, \\ \chi^+_{\sigma_{i+1}, l_i^+}(u) & \text{if} \quad i \in I^+ \setminus I^-, \\ u & \text{if otherwise.} \end{cases}$$

Note that $\tilde{u} \in \mathcal{H}_{N,p,\sigma}$ and so $\varphi(\tilde{u}) \ge \varphi(u)$. Moreover $\tilde{u} = u$ if and only if $I = \emptyset$, i.e., the lemma holds if and only $I = \emptyset$.

If $i \in I$, then

$$\varphi^{p_i+2L}_{p_i-2L}(\tilde{u}) = \varphi^{p_i+2L}_{p_i-2L}(\tilde{w}_i) \le m_\eta + 2\lambda_\varepsilon \le \varphi^{p_i+2L}_{p_i-2L}(u) + 2\lambda_\varepsilon - \tilde{\Lambda}.$$

If $i \in I^- \cup I^+$, then

$$\varphi^{p_i+2L}_{p_i-2L}(\tilde{u}) \le \varphi^{p_i+2L}_{p_i-2L}(u) + 2\lambda_\varepsilon.$$

Finally, letting $I^c = \{1, \dots, N\} \setminus (I \cup I^- \cup I^+)$.

If $1 \in I^c$, $2 \neq I^c$, then

$$\begin{aligned}\varphi(\tilde{u}) &= \varphi_{-\infty}^{p_2-2L}(\tilde{u}) + \sum_{i\in I} \varphi_{p_i-2L}^{p_i+2L}(\tilde{u}) + \sum_{i\in I^-\cup I^+} \varphi_{p_i-2L}^{p_i+2L}(\tilde{u}) \\ &\le \varphi(u) + (2\lambda_\varepsilon - \tilde{\Lambda})\,\mathrm{Card}(I) + 2\lambda_\varepsilon\,\mathrm{Card}(I^- \cup I^+).\end{aligned}$$

If $N \in I^c$, $N-1 \neq I^c$, then

$$\begin{aligned}\varphi(\tilde{u}) &= \varphi_{p_{N-1}+2L}^{+\infty}(\tilde{u}) + \sum_{i\in I} \varphi_{p_i-2L}^{p_i+2L}(\tilde{u}) + \sum_{i\in I^-\cup I^+} \varphi_{p_i-2L}^{p_i+2L}(\tilde{u}) \\ &\le \varphi(u) + (2\lambda_\varepsilon - \tilde{\Lambda})\,\mathrm{Card}(I) + 2\lambda_\varepsilon\,\mathrm{Card}(I^- \cup I^+).\end{aligned}$$

In any case, we have

$$\varphi(\tilde{u}) \le \varphi(u) + (2\lambda_\varepsilon - \tilde{\Lambda})\,\mathrm{Card}(I) + 2\lambda_\varepsilon\,\mathrm{Card}(I^- \cup I^+).$$

Since $\mathrm{Card}(I^- \cup I^+) \le 2\mathrm{Card}(I)$, we have that I must be empty, otherwise $\varphi(\tilde{u}) < \varphi(u)$, it's a contradiction because u is a minimizer. □

Theorem 4.1. *Assume that* $(*)$ *and* $(**)$ *hold, then for every odd number* $N \in \mathbb{N}$, $p = (p_1, \dots, p_N) \in \{0, \eta\}^N$, *where* $p_i \neq p_{i-1} \ge 4L$ *and* $\sigma_i \neq \sigma_{i-1}$, *there exists a* $u \in \mathcal{H}_{N,p,\sigma}$ *such that* $\varphi(u) = m_{N,p,\sigma}$. *Moreover, u is classical solution of* (1.1) *with* $\|u\|_{L^\infty(\mathbb{R}^2)} \le R_0$.

Proof. Let $(u_n) \subset \mathcal{H}_{N,p,\sigma}$ be such that $\varphi(u_n) \to m_{N,p,\sigma}$, by Lemma 3.6 there exists a subsequence still noted (u_n) and a $u \in \mathcal{H}$ such that $u_n \to u$ in H^1. By lower semicontinuity of φ, $\varphi(u) \le m_{N,p,\sigma}$. By lower semicontinuity of the H^1 norm, we have

$$D(u(\,\cdot\,, -L), q_0) \le \liminf_{n\to\infty} D(u_n(\,\cdot\,, -L), q_0) \le \delta$$

and

$$D(u(\,\cdot\,, L), q_\eta) \le \liminf_{n\to\infty} D(u_n(\,\cdot\,, L), q_\eta) \le \delta,$$

therefore $\varphi(u) = m_{N,p,\sigma}$.

By Lemmas 4.4 and 4.3, we have $D(u(\,\cdot\,, p_i - L), q_{\sigma_i}) < \delta$ and $D(u(\,\cdot\,, p_i + L), q_{\sigma_{i+1}}) < \delta$ for all $i = 1, \dots, N$. Here we argue as in Lemma 3.5 that

$$D(u(\,\cdot\,, y), q_{\sigma_i}) \le r_0 \quad \text{for} \quad y < p_i - L$$

and

$$D(u(\,\cdot\,, y), q_{\sigma_{i+1}}) \le r_0 \quad \text{for} \quad y > p_i + L.$$

By Lemma 3.2 we have

$$D(u(\,\cdot\,, y), q_0) \to 0 \quad \text{for} \quad y < p_1 - L$$

and

$$D(u(\,\cdot\,, y), q_{\sigma_\eta}) \to 0 \quad \text{for} \quad y > p_N + L.$$

Using standard regularity arguments, we can conclude that u belongs to $C(\mathbb{R}^2)$ and it is a classical solution to (1.1) with $\|u\|_{C^2(\mathbb{R}^2)} \leq C$ (refer to [5] and [7]). $\square$

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