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Title	On null-equivalent knots
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Citation	Osaka Mathematical Journal. 1959, 11(2), p. 95-113
Version Type	VoR
URL	https://doi.org/10.18910/8288
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Osaka Math. J.
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On Null-Equivalent Knots

To Professor Zyoiti SUETUNA on his 60th birthday

By Hidetaka TERASAKA

Let $C_1, C_2, \cdots, C_n$ be a finite number of, not only one another but also as a whole, non linking trivial knots (=circles). Connect a pair of small arcs γ_1 and γ_2 of C_1 and C_2, γ_2' and γ_3 of C_2 and C_3, etc. respectively by narrow stripes of disjoint bands $B_1, B_2, \cdots, B_{n-1}$, so that the join of the circles $C_1, C_2, \cdots, C_n$ and the boundaries of the bands $B_1, B_2, \cdots, B_{n-1}$ minus the arcs $\gamma_1, \gamma_2, \gamma_2', \gamma_3, \cdots, \gamma_{n-1}', \gamma_n$ forms a knot κ. Then κ is the so-called *null equivalent knot* according to R. H. Fox and J. W. Milnor. Our symmetric (and skew symmetric) union of a knot [4] happens to be a special kind of null-equivalent knots [3]. It is the main result of Fox-Milnor [2] that the Alexander polynomial $\Delta_\kappa(x)$ of κ is of the form $\Delta_\kappa(x) = \pm x^m f(x)\cdot f(x^{-1})$, where $f(1) = \pm 1$. The purpose of the present paper is to show conversely that given a polynomial $f(x)$ with $f(1) = \pm 1$ there can be found a null equivalent knot connecting two circles C_1 and C_2 whose Alexander polynomial is of the form $\pm x^m f(x) f(x^{-1})$.

After some preliminary remarks (§1) the Alexander polynomial of a null equivalent knot is established (§2) and an alternative proof of the Fox-Milnor theorem is given (§3). The converse of the Fox-Milnor theorem (§4) is then almost immediate. §1 and the first half of §2 contain nothing new, and especially what concerns the band it is in another form fully developed in G. Torres [6].

§1. Preliminary remarks.

Since we are concerned principally with the study of Alexander polynomials of knots, every knot is considered as represented by its regular projection on a fixed plane—the ground plane—, and the knot group K is represented in the Wirtinger fashion [5].

Let κ be a knot obtained by connecting two given knots A and C by a band B. Moving along the band B in the direction from A to C, which we call the *positive direction* of the band, let its boundary on the right be named the *positive side* of the band, and the boundary on the left, the *negative side*.

Now consider a location of the knot where a part of the band B crosses over another, thus making a quadruplet of crossing points.

1) Let c and $\bar{c}$ be the generating elements of the knot group K of κ corresponding to the boundary of the band B which crosses over, c being that corresponding to the positive side, $\bar{c}$ to the negative side, of the band. Likewise let $a, \bar{a}, d, \bar{d}, b$ and $\bar{b}$ be respectively the generating elements of K corresponding to the boundary of the band which crosses below. Then we have either the relations

$$(1.1)\quad \begin{array}{l} c\ d\ c^{-1}a^{-1}=1\,,\\ \bar{c}^{-1}b\ \bar{c}\ d^{-1}=1\,,\\ c\ \bar{d}^{-1}c^{-1}\ \bar{a}=1\,,\\ \bar{c}^{-1}\ \bar{b}^{-1}\ \bar{c}\ \ \bar{d}=1\,, \end{array} \qquad \text{or} \qquad (1.1)'\quad \begin{array}{l} \bar{c}\ d\ \bar{c}^{-1}a^{-1}=1\,,\\ c^{-1}b\ c\ d^{-1}=1\,,\\ \bar{c}\ \bar{d}^{-1}\bar{c}^{-1}\ \bar{a}=1\,,\\ c^{-1}\ \bar{b}^{-1}\ c\ \ \bar{d}=1\,, \end{array}$$

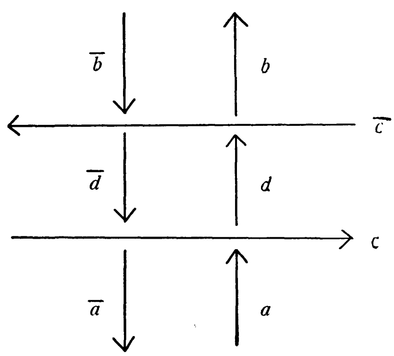

Fig. 1.

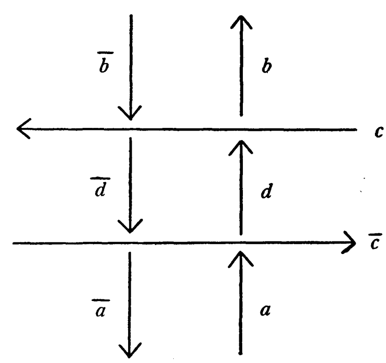

Fig. 2.

according as the *index* of crossing is $+1$ or -1, i.e., according as the band below crosses the band above from the positive side of the latter or from the negative side. Note that the product of group elements is written from left to right. We shall henceforth call a relation of the form (1.1) a *Wirtinger relation.*

d and $\bar{d}$ can therefore be eliminated from the generating elements of K, and we have from the first two relations a new relation

$$(1.2)\qquad c\bar{c}^{-1}b\bar{c}c^{-1}a^{-1}=1\,,$$

which, if we set

$$(1.3)\qquad c\bar{c}^{-1}=C\,,$$

becomes

$$(1.4)\qquad CbC^{-1}a^{-1}=1\,.$$

From (1.1)′ we have similarly

$$(1.4)'\qquad C^{-1}bCa^{-1}=1\,.$$

It will be convenient to write the relations (1.4) and (1.4)′ in one form

(1.5) $$C^{\varepsilon}bC^{-\varepsilon}a^{-1}=1\,,$$

where $\varepsilon=1$ or $=-1$ according as the index of crossing is positive or negative.

2) From the last two equations of (1.1) we have likewise

$$C\bar{b}^{-1}C^{-1}\bar{a}=1\,.$$

Combining this with (1.4) we have, if we set

$$a\bar{a}^{-1}=A\,,\qquad b\bar{b}^{-1}=B\,,$$

the relation

(1.6) $$CBC^{-1}A^{-1}=1\,.$$

Since the elements $A=a\bar{a}^{-1}$, $B=b\bar{b}^{-1}$ and $C=c\bar{c}^{-1}$ correspond each to a closed path whose linking number with κ is equal to zero, they are made commutative when computing the Alexander polynomial, and we have then from (1.6) the relation

(1.7) $$A=B\,.$$

3) Let

(1.8) $$cac^{-1}b^{-1}=1$$

be a Wirtinger relation of a knot. To obtain the *Alexander equation*[1] corresponding to (1.8), we proceed as follows:

Take any one of the Wirtinger generators of K fixed and let it be denoted by x. Then

(1.9) $$ax^{-1}=a^{*}\,,\qquad bx^{-1}=b^{*}\,,\qquad cx^{-1}=c^{*}$$

are elements of K which should be considered as commutative one another. Putting these in (1.8) we have

$$c^{*}x\cdot a^{*}x\cdot x^{-1}c^{*-1}\cdot x^{-1}b^{*-1}=1$$

or

(1.10) $$c^{*}\cdot xa^{*}x^{-1}\cdot xc^{*-1}x^{-1}\cdot b^{*-1}=1\,,$$

where c^{*}, $xa^{*}x^{-1}$, $xc^{*-1}x^{-1}$ and b^{*-1} commute one another. Making use of the abridged notation ([1], p. 291), (1.10) can be written in additional form

1) The equation of the diagram according to Alexander ([1], p. 278).

$$c^* + xa^* - xc^* - b^* = 0\,.$$

Dropping * in this equation we have finally the Alexander equation

(1. 11) $$(1-x)c + xa - b = 0\,.$$

If (1. 8) is of the form

$$c^{-1}acb^{-1} = 1$$

we obtain likewise

$$(x-1)c + a - xb = 0\,.$$

Letting $\varepsilon = 1$ or $= -1$ we have thus

(1. 12) *To the Wirtinger relation*

$$c^{\varepsilon}ac^{-\varepsilon}b^{-1} = 1$$

corresponds the Alexander equation

$$(1-x^{\varepsilon})c + x^{\varepsilon}a - b = 0\,.$$

Finally, since

$$A = a\bar{a}^{-1} = a^*x\cdot(\bar{a}^*x)^{-1} = a^*\cdot\bar{a}^{*-1}\,,$$

we have

(1. 13) *When an element of K such as A appears in a Wirtinger relation, it should be transferred as it is to the corresponding Alexander equation.*

It should be remarked however that for example

(1. 14) *To the Wirtinger relation*

$$cAd^{-1}B^{-1} = 1$$

corresponds the Alexander equation

$$c - d + xA - B = 0\,,$$

and not

$$c - xd + xA - B = 0\,,$$

as would be expected from (1. 11), for $cAd^{-1}B^{-1} = 0$ becomes by the above changing of generators

$$c^*x\cdot A\cdot x^{-1}d^{*-1}\cdot B^{-1} = 0\,,$$

whence

$$c^* + xA - d^* - B = 0\,.$$

§ 2. Alexander polynomials of null equivalent knots.

After the foregoing preliminary remarks, we are now going to give a detailed exposition of computing the Alexander polynomials of null equivalent knots. Let us begin our discussion as Fox and Milnor did [2], with a knot κ of the following type.

Let A be a knot represented on the ground plane by the usual regular projection, and let C be a trivial knot represented similarly by a circle on the ground plane, and which is disjoint from A. Connect a small arc α of A to a small arc γ of C by a band B represented on the ground plane by a pair of parellel curves, thus forming together with A and C minus α and γ a knot κ of a regular projection on the ground plane.

Let $a_{1,0}$, $a_{1,1}$, $\cdots$, $a_{1,n(1)}$, $a_{2,0}$, $\cdots$, $a_{N,n(N)}$ be the elements of the knot group corresponding to the arcs of A with the same letter in this order, beginning at the end point of the arc α, and where $a_{i,n(i)}$ and $a_{i+1,0}$ are the consecutive arcs crossed over by the band B, and let $c_1, c_2, \cdots, c_M$ be those of C beginning at the end point of γ. Starting from the arc A move along the band B in the positive direction as far as one is first crossed over either by A or by C, crossing under several parts of B on the way. Let this part of the band B be denoted by B_1 and let $b_{1,0}$,

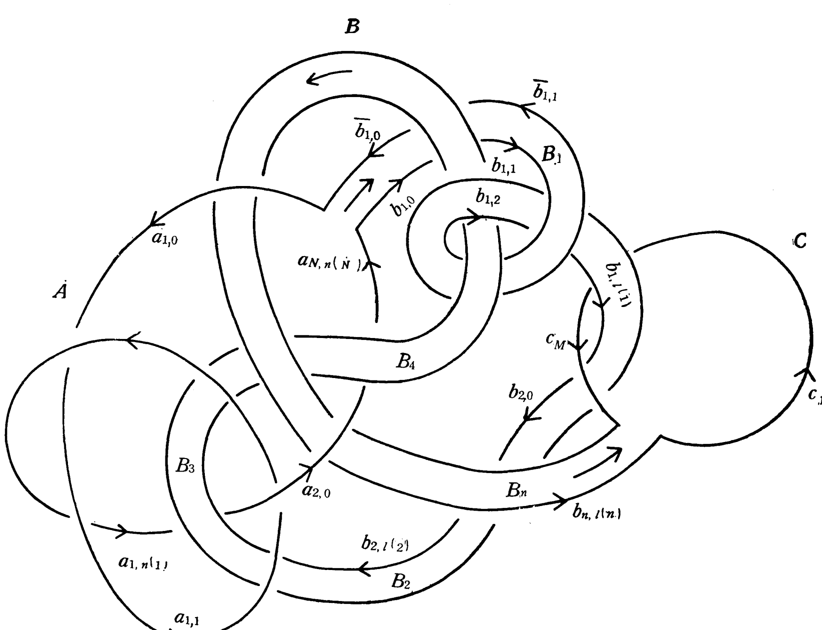

Fig. 3.

$\bar{b}_{1,0}$, $b_{1,1}$, $\bar{b}_{1,1}$, $\cdots$, $b_{1,l(1)}$, $\bar{b}_{1,l(1)}$ be the generating elements of K corresponding to the boundary of B_1. Move further along the band as far as one crosses next under A or C, denote this part of B by B_2, and let $b_{2,0}$, $\bar{b}_{2,0}$, $\cdots$, $b_{2,l(2)}$, $\bar{b}_{2,l(2)}$ be the corresponding elements of K, etc. Let $b_{n,l(n)}$, $\bar{b}_{n,l(n)}$ be the generating elements of K corresponding to the last part B_n of the band B that reaches C.

1) According to the formula (1.7) we can set first of all

$$b_{i,0}\bar{b}_{i,0}^{-1} = b_{i,1}\bar{b}_{i,1}^{-1} = \cdots = b_{i,l(i)}\bar{b}_{i,l(i)}^{-1} = B_i\,. \tag{2.1}$$

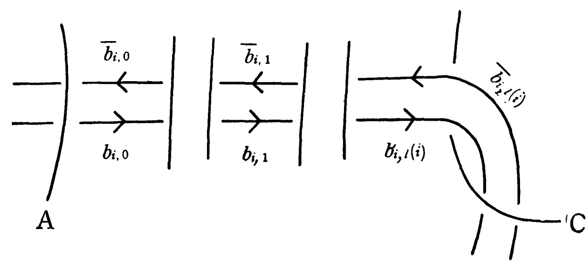

Fig. 4.

2) Next consider the place where the band crosses under an arc $a_{i,j}$ of A or an arc c_i of C, which we indiscriminatively denote by ξ. Then we have the relations

$$\begin{aligned} &\xi^{\delta_i}b_{i+1,0}\xi^{-\delta_i}b_{i,l(i)}^{-1} = 1\,, \\ &\xi^{\delta_i}\bar{b}_{i+1,0}^{-1}\xi^{-\delta_i}\bar{b}_{i,l(i)} = 1\,, \end{aligned} \tag{2.2}$$

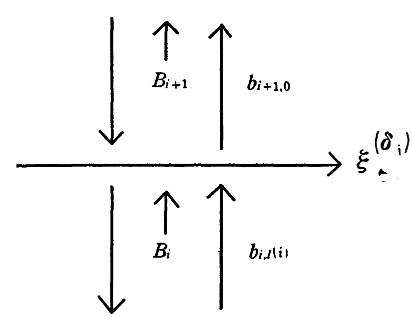

Fig. 5.

where δ_i indicates the index of crossing of ξ and B. These relations yield at once

$$\xi^{\delta_i}B_{i+1}\xi^{-\delta_i}B_i^{-1} = 1 \tag{2.3}$$

if we put

$$b_{i,l(i)}\bar{b}_{i,l(i)}^{-1} = B_i\,, \qquad b_{i+1,0}\bar{b}_{i+1,0}^{-1} = B_{i+1}\,. \tag{2.4}$$

Calculating the corresponding Alexander equation by the procedure of §1, 3) (cf. (1.14)), we have immediately

$$x^{\delta_i}B_{i+1} - B_i = 0\,. \tag{2.5}$$

If we consider instead of the band its positive side alone, we have from the first relation of (2.2) the corresponding Alexander equation

$$(1-x^{\delta_i})\xi + x^{\delta_i}b_{i+1,0} - b_{i,l(i)} = 0\,. \tag{2.6}$$

3) Next consider the place where the part of the band B crosses over A or C or the positive side of B, thus giving rise to pairs of arcs $a_{i,n(i)}$, $a_{i+1,0}$ or c_i, c_{i+1} or $b_{j,i}$, $b_{j,i+1}$, which we indiscriminatively denote again by ξ_i, ξ_{i+1}. Then we have the Wirtinger relation (cf. (1. 5))

$$B^{\varepsilon_i}\xi_{i+1}B^{-\varepsilon_i}\xi_i^{-1} = 1\,, \tag{2. 7}$$

whence the corresponding Alexander equation

$$\varepsilon_i(1-x)B+\xi_{i+1}-\xi_i = 0\,. \tag{2. 8}$$

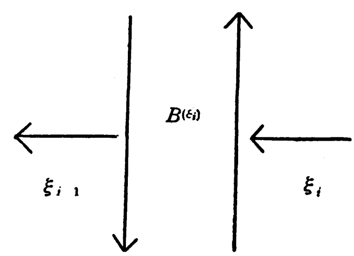

Fig. 6.

4) Let the part B_i of the band B be crossed over in succession by the parts $B^{\eta_1}_{(1)}, B^{\eta_2}_{(2)}, \cdots, B^{\eta_{l(i)}}_{(l(i))}$ of B, where each (k) denotes one of the suffixes $1, 2, 3, \cdots, l(i)$ and η_i denotes the index of crossing. Then we have first by (2. 7)

$$B^{\eta_1}_{(1)}b_{i,1}B^{-\eta_1}_{(1)}b^{-1}_{i,0} = 1$$

or

$$b_{i,0}B^{\eta_1}_{(1)}b^{-1}_{i,1} = B^{\eta_1}_{(1)}\,.$$

Similarly

$$b_{i,1}B^{\eta_2}_{(2)}b^{-1}_{i,2} = B^{\eta_2}_{(2)}$$

$$\vdots$$

$$b_{i,l(i-1)}B^{\eta_{l(i)}}_{(l(i))}b^{-1}_{i,l(i)} = B^{\eta_{l(i)}}_{(l(i))}\,.$$

Multiplying side by side

$$b_{i,0}B^{\eta_1}_{(1)}B^{\eta_2}_{(2)}\cdots B^{\eta_{l(i)}}_{(l(i))}b^{-1}_{i,l(i)} = B^{\eta_1}_{(1)}B^{\eta_2}_{(2)}\cdots B^{\eta_{l(i)}}_{(l(i))}\,.$$

As the corresponding Alexander equation we obtain by means of the procedure of § 1, 3)

$$b_{i,0}+\eta_1 xB_{(1)}+\eta_2 xB_{(2)}+\cdots+\eta_{l(i)}xB_{(l(i))}-b_{i,l(i)} =$$
$$\eta_1 B_{(1)}+\eta_2 B_{(2)}+\cdots+\eta_{l(i)}B_{(l(i))}\,,$$

whence, arranging with respect to $B_1, \cdots, B_n$,

$$\begin{aligned}\eta_{i,1}(x-1)B_1+\eta_{i,2}(x-1)B_2+\cdots\\ +\eta_{i,n}(x-1)B_n+b_{i,0}-b_{i,l(i)} = 0\,.\end{aligned} \tag{2. 9}$$

Subtracting (2. 6) from this, we have

$$\begin{aligned}\eta_{i,1}(x-1)B_1+\eta_{i,2}(x-1)B_2+\cdots\\ +\eta_{i,n}(x-1)B_n+b_{i,0}-x^{\delta_i}b_{i+1,0}-(1-x^{\delta_i})\xi = 0\,.\end{aligned} \tag{2. 10}$$

5) Finally the following relations should be added. Since

$$a_{1,0} = \bar{b}_{1,0}\,,$$
$$a_{N,n(N)} = b_{1,0}\,,$$
$$c_1 = b_{n,l(n)}\,,$$
$$c_M = \bar{b}_{n,l(n)}\,,$$

applying the same procedure already known (§1, 3)), we obtain the following Alexander equations:

$$a_{1,0}-b_{1,0}+B_1 = 0\,, \tag{2.11}$$
$$a_{N,n(N)}-b_{1,0} = 0\,, \tag{2.12}$$
$$c_1-b_{n,l(n)} = 0\,, \tag{2.13}$$
$$c_M-b_{n,l(n)}+B_n = 0\,. \tag{2.14}$$

Since we can take as the generators of the knot group K the elements $B_1, B_2, \cdots, B_n, b_{1,0}, \cdots, a_{1,0}, \cdots, c_1, \cdots, c_M$ we can write down immediately the Alexander matrix of κ from the Alexander equations above obtained. But to do this the following synopsis of the Alexander equation will be of use.

(2.5)′ means for example

$$x^{-\delta_i}B_i-B_{i+1} = 0\,,$$

which is the same as (2.5).

(2.5)′

B_i	B_{i+1}
$x^{-\delta_i}$	-1

(2.9)′

B_1	$\cdots$	B_n	$b_{i,0}$	$b_{i,l(i)}$
$\eta_{i,1}(x-1)$		$\eta_{i,n}(x-1)$	1	-1

(2.10)′

B_1	$\cdots$	B_n	$b_{i,0}$	$b_{i+1,0}$	$\xi^{(\delta_i)}$
$\eta_{i,1}(x-1)$		$\eta_{i,n}(x-1)$	1	$-x^{\delta_i}$	$x^{\delta_i}-1$

(2.8)′

$B^{(\varepsilon_i)}$	ξ_i	ξ_{i+1}
$\varepsilon_i(x-1)$	1	-1

(2.11)′

B_1	$b_{1,0}$	$a_{1,0}$
-1	1	-1

(2.14)′

B_n	$b_{n,l(n)}$	c_M
1	-1	1

(2.12)′

$b_{1,0}$	$a_{N,n(N)}$
-1	1

(2.13)′

$b_{n,l(n)}$	c_1
1	-1

Then the Alexander matrix [1] of κ runs as follows.

To make the explanation easy the rows of the matrix are divided into groups as indicated by the letters A', $B=B'$ & B'', $C=C'$ & C'' & C''' and $E=E'$ & E''. The groups of the columns will also be named as B-column and a-columns, etc. The meaning of the denomination Minor [a-columns, B-rows] will then be self-explanatory.

A'-rows correspond to the formula (2.5)′ of the synopsis: band contra band.

B'-rows correspond to (2.10)′ and B''-row corresponds to (2.9)′ with $i=n$. The part $B=B'$ & B'' concerns with the positive sides of the band B as crossed over by A or by C. The term $x^{\delta_i}-1$ appears either in the a-columns or in the c-columns, according as the ith part B_i ends at A or at C. In Minor [c-columns, B'-rows] we inserted $x^{\delta_i^c}-1$ for later use, where $\delta_i^c=\delta_i$ if B ends at C and $\delta_i^c=0$, that is, $x^{\delta_i^c}-1=0$ if B ends at A.

C-rows and E-rows represent the relations between the band B and the arcs of A or C as these are crossed over respectively by B. Especially, C' correspond to the formula (2.8)′ with $\xi_i=a_{j,i}$, $\varepsilon_i=\varepsilon_{a,i}$, C'' to (2.11)′ and (2.12)′. C'''-rows correspond to the Alexander equations derived from the ordinary Wirtinger relations between the arcs of A, and hence closely related to the Alexander matrix of the knot A. E' correspond again to the formula (2.8)′ with $\xi_i=c_i$ and $\varepsilon_i=\varepsilon_{c,i}$, and E'' to (2.14)′ and (2.13)′.

The Alexander polynomial $\Delta_\kappa(x)$ of our knot κ is obtained as follows:

	$B_1\ B_2 \cdots B_{n-1}\ B_n$	$b_{1,0}$	$b_{2,0} \cdots b_{n,0}\ b_{n,l(n)}$	$a_{1,0}\ a_{2,0} \cdots a_{N,0}$	$a_{1,1} \cdots a_{1,n(1)}\ a_{2,1} \cdots a_{N,n(N)}$	$c_1\ c_2 \cdots c_M$	
	$x^{-\delta_1}\ -1$ $x^{-\delta_2}$ $\ddots$ $x^{-\delta_{n-1}}\ -1$		0		0	0	A'
	$\eta_{i,j}(x-1)$	1	$-x^{\delta_1}$ $1\ -x^{\delta_2}$ $\ddots$ $1\ -x^{\delta_{n-1}}$	$x^{\delta_i}-1$		$x^{\delta_1^c}-1$ $x^{\delta_{n-1}^c}-1$	B'
			$1\ \ -1$			0	B''
	$\varepsilon_{a,1}(x-1)$ $\varepsilon_{a,N-1}(x-1)$		0	-1 -1 $\ddots$ -1	1 1 $\ddots$ 1	0	C'
**	-1	1 -1		-1	1		C''
*	0		0	$[\Delta_A(x)]$		0	C'''
	$\varepsilon_{c,1}(x-1)$ $\vdots$ $\varepsilon_{c,M-1}(x-1)$		0	0		$1\ \ -1$ 1 -1 $1\ \ -1$	E'
	1		-1 1			1 -1	E''

If the old $b_{1,0}$ of the band B introduced at the beginning of §2 is taken as x, the new $b_{1,0}$, which is by our convention (cf. §1, 3)) nothing other than $b^*_{1,0}=b_{1,0}x^{-1}$, is equal to unity. And since one of the Alexander equations is as well known superfluous, let it be one of the rows of C''', say the row designated by $*$. Then the Alexander polynomial of κ is the determinant obtained from the above matrix by deleting the column under $b_{1,0}$ and the row $*$.

To compute this determinant $\Delta_\kappa(x)$ first add all c-columns to the column under $b_{n,l(n)}$.

Next add all rows of E to the lowermost row. Then all elements of the latter becomes 0 except those under B's. Transferring this new row in between the $n-1$th and the nth rows, we have a square matrix of order n on the top left side of the determinant, while in the adjoining matrix of n rows on the top of the determinant stands zeros alone.

$$g(x)=\begin{vmatrix} x^{-\delta_1} & -1 & & 0 & \\ & x^{-\delta_2} & -1 & & \\ & & \ddots & & \\ & 0 & & x^{-\delta_{n-1}} & -1 \\ \varepsilon_{c,1}(x-1), & \cdots, & & \varepsilon_{c,M-1}(x-1) & 1 \end{vmatrix}$$

$$f(x)=\begin{vmatrix} -x^{\delta_1} & & & & x^{\delta_1^c}-1 \\ 1 & -x^{\delta_2} & 0 & & x^{\delta_2^c}-1 \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ 0 & & 1 & -x^{\delta_{n-1}} & x^{\delta_{n-1}^c}-1 \\ & & & 1 & -1 \end{vmatrix}.$$

Thus the determinant $\Delta_\kappa(x)$ is split into the product of determinants $g(x)$, $f(x)$ and the rest $h(x)$, where $h(x)$ is nothing other than the Alexander polynomial $\Delta_A(x)$ of the knot A up to a factor $\pm x^m$ [2]. To see this latter, add the columns under $a_{1,0}, a_{2,0}, \cdots, a_{N,0}$ respectively to the columns under $a_{N,n(N)}, a_{1,n(1)}, \cdots, a_{N-1,n(N-1)}$. Since the adding together of $a_{i+1,0}$ and $a_{i,n(i)}$ means the identification of these arcs, or which is the same, the elimination of the band B that crosses over A, Minor [columns under $a_{1,1}, \cdots, a_{N,n(N)}$, C'''-rows] becomes the Alexander matrix of A. But since the row $*$ in C'''-rows is already deleted in the determinant $\Delta_\kappa(x)$ and since the column under $a_{N,n(N)}$ can also be dropped in computing the determinant, because there stands only one unity on the row $**$, the minor under consideration yields the Alexander polynomial $\Delta_A(x)$ of A, and our assertion follows immediately.

§ 3. Proof that $g(x^{-1}) = \pm f(x)$

We are now going to prove that $g(x^{-1})$ is equal to $f(x)$ up to a factor ± 1 [2].

Since $\Delta_\kappa(x)$ is as we have seen the product of $f(x)$, $g(x)$ and $\Delta_A(x)$ up to a factor $\pm x^m$, the elements of Minor [B-columns, B'-rows] in no way contribute to the evaluation of $\Delta_\kappa(x)$. This shows geometrically that the interchanging of over crossing and under crossing of the band B at any quadruplet of crossing in no way affects the value of $\Delta_\kappa(x)$. Therefore we can rearrange if necessary the order of the crossings of the band as they appear over or under C such that they occur successively in the positive sense, all the while the value of $\Delta_\kappa(x)$ remaining unaltered. If further there take place two successive crossings over C or two successive crossings under C, these two crossings can be substituted by

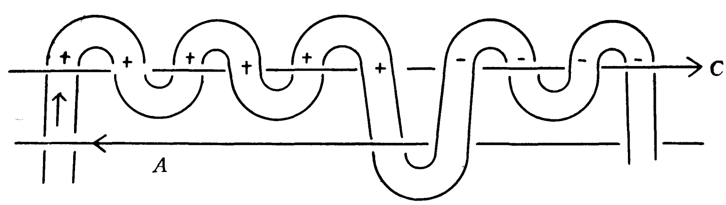

Fig. 7.

no crossing whatever, so that in the final stage of rearrangement a crossing over C is followed by the crossing under C and vice versa, when one moves along the band B in the positive sense. If the crossings of the band on C are arranged as above, the crossings and the projection of the knot itself will be called *irreducible*. In the following we shall consider only the irreducible projection.

There are two cases to be considered.

First case: *positive crossing*. B_i reaches C from outside and B_{i+1} goes out of it by crossing over it. In this case we have

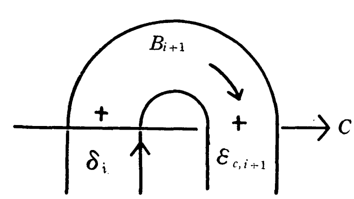

Fig. 8.

$$(3.1) \qquad \delta_i^c = \delta_i = 1\,, \qquad \varepsilon_{c,i+1} = 1\,.$$

Second case: *negative crossing*. B_i enters inside C by crossing over it and B_{i+1} goes out of it. Then

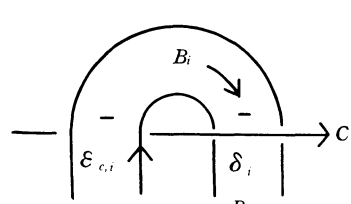

Fig. 9.

$$(3.2) \qquad \delta_i^c = \delta_i = -1\,, \qquad \varepsilon_{c,i} = -1\,.$$

It should be noticed that since the crossings are assumed to be irreducible, the positive and the negative crossing (and vice versa) can not occur for consecutive B_i and B_{i+1}. Either the positive crossings alone or the negative crossings alone occur successively on C, separated by some crossings on A, when one moves along the band B in the positive direction. To this corresponds in the matrix $f(x)$ the phenomenon that on the last column of the matrix there appear in groups successions of $x-1$ alone or successions of $x^{-1}-1$ alone, separated only by a zero or zeros.

Our proof that $g(x^{-1})=\pm f(x)$ will best be illustrated by the following example (see Fig. 7):

$$f(x)=$$

$$\begin{array}{c|cccccccc|cl}
1 & -x &&&&&&& x-1 & & \delta_1=\delta_1^c=1\\
x & 1 & -x &&& 0 &&& x-1 & & \delta_2=\delta_2^c=1\\
x^2 & & 1 & -x &&&&& x-1 & P & \delta_3=\delta_3^c=1\\
\cline{2-9}
x^3 &&& 1 & -x^{-1} &&&& 0 & & \delta_4=1,\ \delta_4^c=0\\
\cline{2-9}
x^2 &&&& 1 & -x^{-1} &&& x^{-1}-1 & Q & \delta_5=\delta_5^c=-1\\
x && 0 &&& 1 & -x^{-1} && x^{-1}-1 & & \delta_6=\delta_6^c=-1\\
\cline{2-9}
1 &&&&&& 1 & -x & 0 & & \delta_7=1,\ \delta_7^c=0\\
\cline{2-9}
x &&&&&&& 1 & -1 & &
\end{array}$$

$$=-x[(x-1)(1+x+x^2)+(x^{-1}-1)(x^2+x)-x]$$

$$g'(x^{-1})=$$

$$\begin{array}{c|cccccccc|cl}
1 & -x &&&&&&& 0 & & \\
x & 1 & -x &&& 0 &&& 1-x^{-1} & & \varepsilon_{c,2}=1\\
x^2 & & 1 & -x &&&&& 1-x^{-1} & P' & \varepsilon_{c,3}=1\\
\cline{2-9}
x^3 &&& 1 & -x^{-1} &&&& 1-x^{-1} & & \varepsilon_{c,4}=1\\
\cline{2-9}
x^2 &&&& 1 & -x^{-1} &&& -(1-x^{-1}) & Q' & \varepsilon_{c,5}=-1\\
x && 0 &&& 1 & -x^{-1} && -(1-x^{-1}) & & \varepsilon_{c,6}=-1\\
\cline{2-9}
1 &&&&&& 1 & -x & 0 & & \\
\cline{2-9}
x &&&&&&& 1 & -1 & &
\end{array}$$

$$=-x[(1-x^{-1})(x+x^2+x^3)-(1-x^{-1})(x^2+x)-x]$$

Multiply the rows of $f(x)$ downward by 1, x, x^2, x^3, x^2, x, 1 and x respectively and add together to the lowermost row. Then all elements of this row but the right corner become zeros and we have

$$f(x) = -x[(x-1)(1+x+x^2)+(x^{-1}-1)(x^2+x)-x]. \tag{3.3}$$

In the determinant $g'(x^{-1})$, which is the transposed of $g(x^{-1})$, the groups of $(1-x^{-1})$ are lower by one as compared with the corresponding groups of $(x-1)$ in $f(x)$, and we obtain by just the same procedure as the precedent

$$g'(x^{-1}) = -x[(1-x^{-1})(x+x^2+x^3)-(1-x^{-1})(x^2+x)-x] \tag{3.4}$$

and it is evidently

$$f(x) = g'(x^{-1}) = g(x^{-1}).$$

The general proof will be done quite similarly:

Divide the matrix of $f(x)$ into groups of rows and first let P be a "maximal" group of rows such that the elements on the last column are all $x-1$ except for the lowermost row where 0 stands, and such that the row of $f(x)$ directly over P contains no $x-1$. If P' is the corresponding group of rows of the determinant of $g'(x^{-1})$, which is the transposed to $g(x^{-1})$, then, since each row of $1-x^{-1}$ is by (3.1) lower than that of $x-1$ by one row, the elements on the last column are all $1-x^{-1}$ except for the upper corner where 0 stands.

$$\begin{array}{l} x^{p} \\ x^{p+1} \\ \vdots \\ x^{p+i-2} \\ \\ x^{p+i-1} \end{array}
\left.\left|\begin{array}{ccccccc}
1 & -x & & & & & x-1 \\
 & 1 & -x & & & & x-1 \\
 & & \ddots & & & & \vdots \\
 & & & 1 & -x & & x-1 \\
\hline
 & & & & 1 & -x^{\delta} & 0
\end{array}\right|\right\} P$$

$$\begin{array}{l} x^{p} \\ x^{p+1} \\ \vdots \\ \vdots \\ \\ x^{p+i-1} \end{array}
\left.\left|\begin{array}{ccccccc}
1 & -x & & & & & 0 \\
 & 1 & -x & & & & 1-x^{-1} \\
 & & \ddots & & & & \vdots \\
 & & & 1 & -x & & 1-x^{-1} \\
\hline
 & & & & 1 & -x^{\delta} & 1-x^{-1}
\end{array}\right|\right\} P'$$

$$
\begin{array}{l|ccccccc|l}
x^q & 1 & -x^{-1} & & & & x^{-1}-1 & \\
x^{q-1} & & 1 & -x^{-1} & & & x^{-1}-1 & \\
\vdots & & & \ddots & & & \vdots & Q \\
x^{q-i+1} & & & & 1 & -x^{-1} & x^{-1}-1 &
\end{array}
$$

$$
\begin{array}{l|ccccccc|l}
x^q & 1 & -x^{-1} & & & & -(1-x^{-1}) & \\
x^{q-1} & & 1 & -x^{-1} & & & -(1-x^{-1}) & \\
\vdots & & & \ddots & & & \vdots & Q' \\
x^{q-i+1} & & & & 1 & -x^{-1} & -(1-x^{-1}) &
\end{array}
$$

If the rows of P and P' are multiplied downward successively by

$$x^p,\ x^{p+1},\ \cdots,\ x^{p+i-1},$$

where i is the number of rows of P and P', and added together to the lowermost row of P and P' respectively, then we have at the right corners of P and P'

$$(x-1)(x^p+\ \cdots\ +x^{p+i-1})$$

and

$$(1-x^{-1})(x^{p+1}+\ \cdots\ +x^{p+i})$$

respectively, which are evidently equal to each other.

Secondly, let Q be a maximal group of rows with $1-x^{-1}$ on the last column and let Q' be the corresponding group of rows of the transposed determinant $g'(x^{-1})$ of $g(x^{-1})$. Then, since this time each row of $x^{-1}-1$ is on the same level as that of $-(1-x^{-1})$, if we multiply the rows of Q and Q' downward by

$$x^q,\ x^{q-1},\ \cdots,\ x^{q-i+1}$$

and add together to the lowermost row respectively, then we have at the right corners of Q and Q'

$$(x^{-1}-1)(x^q+x^{q-1}+\ \cdots\ +x^{q-i+1})$$

and

$$-(1-x^{-1})(x^q+x^{q-1}+\ \cdots\ +x^{q-i+1}),$$

which are clearly equal.

Since the determinant $f(x)$ and the transposed determinant $g'(x^{-1})$ of $g(x^{-1})$ can be divided into the groups of rows of the above types and those having zero elements on the last column, if each row of them is multiplied suitably by x^p and added together to the lowermost row respectively such that all elements but the last of this row become 0, then the elements on the right corners of the determinants become identical for both $f(x)$ and the transposed $g'(x^{-1})$, and hence

$$f(x) = g'(x^{-1}) = g(x^{-1}),$$

which was to be proved, and hence the Fox-Milnor's Theorem (see [2]).

§4. The converse of the Fox-Milnor theorem.

We are now in a position to prove the following theorem:

Theorem. *If $f(x)$ is a polynomial in x with $f(1)=\pm 1$, then there is a null equivalent knot connecting two non-linking circles by a band, with the Alexander polynomial of the form $\pm x^m f(x)\cdot f(x^{-1})$.*

Proof. We can assume without loss of generality that $f(1)=-1$. Then there is a polynomial $F(x)$ such that

$$f(x) = (x-1)F(x)-1.$$

Splitting $F(x)$ into the difference of two polynomials with positive coefficients, we have

$$\begin{aligned}(4.1)\quad f(x) &= (x-1)[(a_0+a_1x+\cdots+a_nx^n)-(b_0+b_1x+\cdots+b_mx^m)]-1\\ &= (x-1)(a_0+a_1x+\cdots+a_nx^n)\\ &\quad +(x^{-1}-1)(b_0x+b_1x^2+\cdots+b_mx^{m+1})-1,\\ &\quad \text{where}\quad a_i\geqq 0,\quad b_i\geqq 0.\end{aligned}$$

Before entering into the proper proof, let us again illustrate the idea of the proof by an example. Given

$$f(x) = (x-1)(3+x)+(x^{-1}+1)x^3-1,$$

construct the following determinant.

$$\begin{array}{c|ccccccccccc|}
1 & -x & & & & & & & & & & x-1 \\
x & 1 & -x^{-1} & & & & & & & & & 0 \\
1 & & 1 & -x & & & & & 0 & & & x-1 \\
x & & & 1 & -x^{-1} & & & & & & & 0 \\
1 & & & & 1 & -x & & & & & & x-1 \\
x & & & & & 1 & -x & & & & & x-1 \\
x^2 & & & & & & 1 & -x & & & & 0 \\
x^3 & & & & & & & 1 & -x^{-1} & & & x^{-1}-1 \\
x^2 & & & 0 & & & & & 1 & -x^{-1} & & 0 \\
x & & & & & & & & & 1 & -x^{-1} & 0 \\
1 & & & & & & & & & & 1 & -1
\end{array}$$

If we multiply the rows of this determinant downward in succession by 1, x, 1, x, 1, x, x^2, x^3, x^2, x and 1 and add together to the last row, we obtain as the value of the determinant $-f(x)$. Now, compared with the general form of $f(x)$ in § 2, we have the following table of values of δ_i and δ_i^c.

δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}
1	-1	1	-1	1	1	1	-1	-1	-1
δ_1^c	δ_2^c	δ_3^c	δ_4^c	δ_5^c	δ_6^c	δ_7^c	δ_8^c	δ_9^c	δ_{10}^c
1	0	1	0	1	1	0	-1	0	0

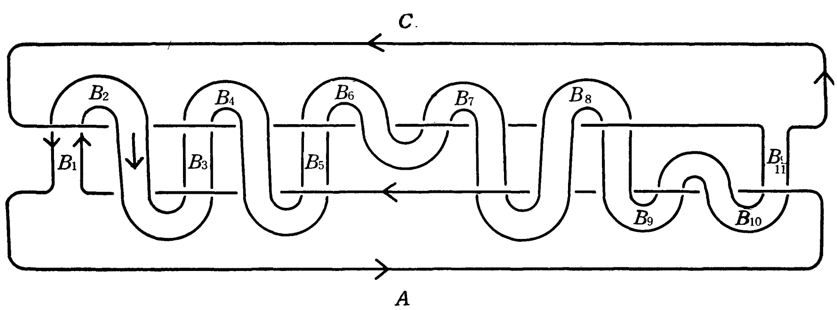

Fig. 10.

Fig. 10 represents then the desired knot with Alexander polynomial $f(x)\cdot f(x^{-1})$: The band starting from A first crosses under $C(\delta_1=\delta_1^c=1)$, then crossing over C comes out of C and enter A crossing over it and comes out of A again crossing under it ($\delta_2=-1$, $\delta_2^c=0$), etc.

Now to the proof of our theorem. If we sacrify the compactness of the determinant but allow possibly a fantastic bulk, the following construction may be theoretically simple.

Let us introduce two kinds of matrices, positive blocks and negative blocks, which look like B_n and B_n' below. All empty spaces are occupied by zeros. *A positive block of nth order* B_n consists of $2(n+1)$ rows, on the nth row of the last column standing $x-1$. If we multiply the rows downward by 1, x, x^2, $\cdots$, x^n, x^{n+1}, x^n, $\cdots$, x in succession and add together to the lowermost row, then this row obtains the form

$$\cdots\cdots 1 \cdots\cdots -1 \cdots\cdots x^n(x-1),$$

the empty spaces being all occupied by zeros. *A negative block of nth order* B_n' consists of $2n$ rows and on the nth row of the last column stands $x^{-1}-1$. If we multiply the rows downward by 1, x, x^2, $\cdots$, x^n, x^{n-1}, $\cdots$, x and add together to the lowermost row, then we have likewise

$$\cdots\cdots 1 \cdots\cdots -1 \cdots\cdots x^n(x^{-1}-1)$$

$$
\begin{array}{c|cccccccccc|}
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & -x & & & & & & & & 0 \\
x & & 1 & -x & & & & & 0 & & 0 \\
\vdots & & & & \ddots & & & & & & \vdots \\
x^{n} & & & & 1 & -x & & & & & x-1 \\
x^{n+1} & & & & & 1 & -x^{-1} & & & & 0 \\
x^{n} & 0 & & & & & 1 & -x^{-1} & & & 0 \\
\vdots & & & & & & & \ddots & & & \vdots \\
x & & & & & & & & 1 & -x^{-1} & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

B_n

$$
\begin{array}{c|cccccccccc|}
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & -x & & & & & & & & 0 \\
x & & 1 & -x & & & & & 0 & & 0 \\
\vdots & & & & \ddots & & & & & & \vdots \\
x^{n-1} & & & & 1 & -x & & & & & 0 \\
x^{n} & & & & & 1 & -x^{-1} & & & & x^{-1}-1 \\
x^{n-1} & 0 & & & & & 1 & -x^{-1} & & & 0 \\
\vdots & & & & & & & \ddots & & & \vdots \\
x & & & & & & & & 1 & -x^{-1} & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

B'_n

Now if $f(x)$ is given in the form (4.1), construct the determinant $D(x)$ from positive and negative blocks as follows:

The elements on the principal diagonal should be all 1. The top of $D(x)$ consists of a_0 positive blocks of 0th order. Then come a_1 positive blocks of 1st order, etc., and finally a_n positive blocks of nth order. Next come b_0 negative blocks of 1st order, etc. and ends with b_m negative blocks of $m+1$th order. Last of all, to make the determinant complete, two rows of the form

$$
\begin{array}{cccc}
0 \cdots & 0 & 1 & -1 \\
0 \cdots & 0 & 0 & 1
\end{array}
$$

should be added. If we delete from this determinant the first column and the last row, we have, as our repeated computations show, the desired determinant with the value $\pm x^{m} f(x) \cdot f(x^{-1})$.

The construction of the corresponding knot will also be clear from what we have explained above.

(Received July 15, 1959)

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