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<th>A theorem on lattices of a complex solvable Lie group</th>
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1. Introduction

A discrete subgroup $\Gamma$ of a Lie group $G$ is called a lattice of $G$ if the homogeneous space $G/\Gamma$ is of finite volume. It is known that any lattice $\Gamma$ of a solvable Lie group $G$ is uniform, i.e., such that $G/\Gamma$ is compact. In this note we shall prove the following theorem.

**Theorem.** Let $G$ be a connected complex solvable Lie group and $\Gamma$ be a lattice of $G$. Suppose that $\Gamma$ is nilpotent. Then $G$ is nilpotent.

It is known that Theorem is not true in general for real solvable Lie group ([1] Chapter 3, Example 4).

2. Proof of Theorem

First we note that our theorem will be valid in general if it is proved for the case where $G$ is simply connected. In fact, let $\tilde{G}$ be the universal covering group with the projection $\pi: \tilde{G} \rightarrow G$. Then $\Gamma = \pi^{-1}(\Gamma)$ is a lattice in $G$ and it is nilpotent, since the kernel of $\pi$ is contained in the center of $\Gamma$. Thus $\tilde{G}$ is nilpotent by Theorem for the case where the complex solvable Lie group is simply connected, and so is $G$.

From now on assume that $G$ is simply connected. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $I$ the canonical complex structure. We denote by $\mathfrak{n}$ the maximal nilpotent ideal of $\mathfrak{n}$ regarded as real Lie algebra. Since $\mathfrak{n}$ is given by $\{X \in \mathfrak{g} | \text{ad}(X) \text{ is nilpotent}\}$, $\mathfrak{n}$ is invariant by $I$, so that $\mathfrak{n}$ is a complex subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^k$ denote $[\mathfrak{g}, \mathfrak{g}, \ldots, \mathfrak{g}]$ where we put $\mathfrak{g}^0 = \mathfrak{g}$. Then $\{\mathfrak{g}^k\}$ is a descending sequence of ideals. Put $\mathfrak{g}^\infty = \inf \mathfrak{g}^k$. It is obvious that $\mathfrak{g}^\infty$ equals $\mathfrak{g}^m$ for some $m$ and is a complex subalgebra. We thus have a sequence of ideals:

$$\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}^\infty.$$

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Let $g^c$ denote the complexification of $g$. Then $g^c = g^+ + g^-$ (direct sum), where $g^- = \{X \in g^c | IX = \pm \sqrt{-1}X \}$. By Theorem of Lie, we can take a basis $\{X_1, \ldots, X_n\}$ of the complex solvable Lie algebra $g^+$ such that

1) $\{X_1, \ldots, X_s\}$ is a basis of $(g^-)^+$
2) $\{X_{s+1}, \ldots, X_n\}$ is a basis of $[g^+, g^+]$
3) $\{X_{s+1}, \ldots, X_n\}$ is a basis of $n^+$, where $n^+ = \{X \in n^c | IX = \sqrt{-1}X \}$, $n^c$ being the complex subalgebra spanned by $n$.
4) the subspaces $g^p_+ (p=1, \ldots, n)$ spanned by $\{X_p, \ldots, X_n\}$ are ideals of $g^+$.

Put $Y_j = \frac{1}{2} (X_j + \bar{X}_j)$ for $j = 1, \ldots, n$. Then $IY_j = \frac{\sqrt{-1}}{2} (X_j - \bar{X}_j)$ and $\{Y_1, IY_1, \ldots, Y_n, IY_n\}$ is a basis of $g$ (over $R$). Moreover, if $g_{2j-1}$ (resp. $g_{2j}$) denotes the real vector space spanned by $\{Y_j, IY_j, \ldots, Y_n, IY_n\}$ (resp. $\{Y_j, IY_j, \ldots, Y_n, IY_n\}$). Then $g_i$ ($i=1, \ldots, 2n$) are subalgebras of $g$ and $g_{2j+1}$ is contained in $g_i$ as an ideal. Since $G$ is simply connected, it follows that every element $g \in G$ can be written in one and only one way in the form

$$g = (\exp t_1 Y_1)(\exp s_1 IY_1) \cdots (\exp s_n IY_n),$$

where $t_j = t_j(g), s_j = s_j(g) (j = 1, \ldots, n)$ are real numbers (cf. [2]). Since $[IY_j, Y_j] = 0$ for $j = 1, \ldots, n$.

$$g = \exp (t_1 Y_1 + s_1 IY_1) \cdots \exp (t_n Y_n + s_n IY_n).$$

Thus we get a biholomorphic map $\Phi : G \rightarrow C^n$ defined by

$$\Phi(g) = (t_1(g) + \sqrt{-1}s_1(g), \ldots, t_n(g) + \sqrt{-1}s_n(g)).$$

Let $\{2C^{ij}_s\}$ be the structure constants of the Lie algebra $g^+$ with respect to the basis $\{X_1, \ldots, X_n\}$. Then we may regard $\{C^{ij}_s\}$ as the structure constants of the complex Lie algebra $g$ with respect to the basis $\{Y_1, \ldots, Y_n\}$.

Note that, for $i=s+1, \ldots, n$,

$$ad(X_i) = \begin{pmatrix}
\frac{s}{0} & \frac{r-s}{0} & \frac{l-r}{0} & \frac{n-l}{0} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & * & A_i & 0 \\
* & * & * & B_i
\end{pmatrix}_{s}
$$

where $A_i = \begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}$ and $B_i = \begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}$.
and, for \( i=1, \ldots, s \),

\[
\text{ad}(X_i) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & A_i & 0 \\ * & * & B_i & \end{pmatrix}
\]

where

\[
A_i = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}
\quad \text{and} \quad B_i = \begin{pmatrix} 0 \\ \vdots \\ * \\ 2C_{i+1}^j \end{pmatrix}.
\]

In the following, decomposition of a matrix in sixteen blocks is always taken in sizes as indicated above.

We note that \((C_{i,j}, \ldots, C_{i,n}) \neq (0, \ldots, 0)\) for any \( j=l+1, \ldots, n \), by the definition of \( g^a \).

Since \( \text{Ad}(g) = (\exp (\frac{1}{n} \text{ad}(Y_i))) \cdots (\exp \frac{1}{n} \text{ad}(Y_n)) \),

\[
\text{Ad}(g)(Y_i, \ldots, Y_n) = (Y_i, \ldots, Y_n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 B_2 B_3 0 \\ B_4 B_5 B_6 B_7 \end{pmatrix}
\]

where

\[
B_3 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad B_7 = \begin{pmatrix} \exp (\sum_{j=1}^s C_{i+1}^{j+1} z_j(g)) & \cdots & 0 \\ \vdots & \exp (\sum_{j=1}^s C_{i+1}^{j+1} z_j(g)) & \end{pmatrix}.
\]

Consider \( g \) as a real Lie algebra and let \( l(g) \) denote the number of eigenvalues different from 1 of \( \text{Ad}(g) \): \( g \rightarrow g \) for \( g \in G \). Define rank \( G = \sup_{g \in b} l(g) \). An element \( g \in G \) is called regular if \( l(g) = \text{rank } G \). Then it is easy to see that \( g \in G \) is regular if and only if \( \exp (\sum_{j=1}^s C_{i+1}^{j+1} z_j(g)) = 1 \) for all \( k=l+1, \ldots, n \).

**Lemma 1.** Let \( \Gamma \) be a lattice of a simply connected complex solvable Lie group \( G \). Then \( \Gamma \) contains a regular element of \( G \).

**Proof.** If we denote by \( N \) the connected maximal normal nilpotent Lie group of \( G \), \( N \cap \Gamma \) is a lattice of \( N \) by a theorem of Mostow ([3], [4]). Let \( \pi: G \rightarrow G/N \) be the projection. Then \( \pi(\Gamma) \) is a lattice of \( G/N \) and \( (G/N)/\pi(\Gamma) \) is a complex torus. By the definition of \( \Phi: G \rightarrow C^s \), it is obvious that \( G/N \) is biholomorphic to \( C^s \) by \( G/N \ni \pi(g) \rightarrow (z_1(g), \ldots, z_s(g)) \in C^s \). We identify \( G/N \) with \( R^{2s} \) by
\[ \pi(g) = (\text{Re } z_i(g), \text{Im } z_i(g), \cdots, \text{Re } z_i(g), \text{Im } z_i(g)). \]

Consider the real subspaces \( H_k \) of codimension 1 defined by
\[
H_k = \{(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n} | \sum_{j=1}^{n} (\text{Re } (C_j x_j) - \text{Im } (C_j y_j)) = 0 \}
\]
for \( k = l+1, \cdots, n \). Since \( \pi(\Gamma) \) is a lattice of \( \mathbb{R}^{2n} \), there are infinitely many different real subspaces of codimension 1 which are generated by \( 2s-1 \) lattice points of \( \pi(\Gamma) \). Hence, there exists a point \( \gamma \in \Gamma \) such that \( \pi(\gamma) \notin H_k \) for \( k = l+1, \cdots, n \). Then \( |\exp \left( \sum_{j=1}^{n} C_j \gamma_j(\gamma) \right)| \neq 1 \) for all \( k = l+1, \cdots, n \) and \( \gamma \in \Gamma \) is a regular element of \( G \).

**Lemma 2.** (Mostow) Let \( G \) be a simply connected solvable Lie group and \( \Gamma \) a uniform subgroup of \( G \) containing a regular element. Let \( G^\infty \) denote the connected Lie subgroup of \( G \) corresponding to \( g^\infty \). Then \( G^\infty \cap \Gamma \) is uniform in \( G^\infty \).


Proof of Theorem. Suppose that \( G \) is not nilpotent. Then \( G^\infty \neq \{e\} \). Since \( G^\infty \) is a simply connected nilpotent Lie group, \( G^\infty \cap \Gamma \neq \{e\} \) by Lemma 2. Since \( \Gamma \) is nilpotent, \( G^\infty \cap \Gamma \) contains a non-trivial element of the center \( C \) of \( \Gamma \). Choose an element \( \gamma \neq e \) of \( G^\infty \cap \Gamma \cap C \). We can write \( \gamma \) uniquely as
\[ \gamma = (\exp z_{l+1} Y_{l+1}) \cdots (\exp z_n Y_n) \]
where \( (z_{l+1}, \cdots, z_n) \in C^{n-l} \).

Note that \( \text{ad}(Y_j) \) is represented by the basis \( \{Y_1, \cdots, Y_n\} \) as follows:
\[
\text{ad}(Y_j) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
A_j & B_j & C_j & D_j
\end{bmatrix}
\quad \text{for } j = l+1, \cdots, n
\]
where
\[
A_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
C_{j_1}' & \cdots & C_{j_s}'
\end{bmatrix} < j-l,
\]
\[
B_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0 \\
* & \cdots & *
\end{bmatrix} < j-l,
\]
\[
C_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
C_{j_1} & \cdots & C_{j_s}
\end{bmatrix} < j-l,
\]
\[
D_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0 \\
* & \cdots & *
\end{bmatrix} < j-l.
\]
Fix a $j = l + 1, \cdots, n$ and put $\delta_j = (\exp z_j Y_j) \cdots (\exp z_n Y_n)$. Then $Ad(\delta_j) = (\exp z_j ad(Y_j)) \cdots (\exp z_n ad(Y_n))$ is written as follows:

$$\begin{align*}
(2) \quad Ad(\delta_j) &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
P_j & Q_j & R_j & S_j
\end{pmatrix},
\end{align*}$$

where

$$\begin{align*}
P_j &= \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
* & \cdots & * \\
* & \cdots & *
\end{pmatrix} < j-l \\
Q_j &= \begin{pmatrix}
0 & \cdots & 0 \\
* & \cdots & * \\
* & \cdots & *
\end{pmatrix} < j-l,
\end{align*}$$

$$\begin{align*}
R_j &= \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
* & \cdots & *
\end{pmatrix} < j-l \\
S_j &= \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & 0 \\
* & \cdots & * & 1
\end{pmatrix} < j-l.
\end{align*}$$

We claim that if $\gamma_0 \delta_j = \delta_j \gamma_0$ for a regular element $\gamma_0 \in \Gamma$, then $z_j = 0$. Put

$$Ad(\gamma_0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
B_1 & B_2 & B_3 & 0 \\
B_4 & B_5 & B_6 & B_7
\end{pmatrix}. $$

Since $Ad(\gamma_0) Ad(\delta_j) = Ad(\delta_j) Ad(\gamma_0)$, we get

$$B_4 + B_7 P_j = P_j + R_j B_4 + S_j B_4 \in M_{n-l,n}(C).$$
Consider the \((j-l, k)\)-component of both hands of (3), by (1) we get
\[
\exp \left( \sum_{i=1}^{s} C_{j_i}^{i} z_{i}(\gamma_{0}) \right) C_{j_k}^{j} z_{j} = C_{j_k}^{j} z_{j}
\]
for \(k=1, \ldots, s\). Since \(\gamma_{0}\) is a regular element of \(G\), \(\exp \left( \sum C_{j_i}^{i} z_{i}(\gamma_{0}) \right) \neq 1\) and \(C_{j_k}^{j} z_{j} = 0\) for \(k=1, \ldots, s\). Thus \(z_{j} = 0\), since \((C_{j_i}^{i}, \ldots, C_{j_k}^{k}) = (0, \ldots, 0)\).

Now, starting with \(j=l+1\), we get \(z_{j} = 0\) successively for all \(j=l+1, \ldots, n\). This contradicts our assumption \(\gamma \neq e\). Hence, \(G\) is nilpotent, and this proves our Theorem.

**Remark.** The special case of our Theorem has been proved in a stronger form in the section 2 of [5].

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**References**


