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A THEOREM ON LATTICES OF A COMPLEX SOLVABLE LIE GROUP

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1. Introduction

A discrete subgroup Γ of a Lie group G is called a lattice of G if the homogeneous space G/Γ is of finite volume. It is known that any lattice Γ of a solvable Lie group G is uniform, i.e., such that G/Γ is compact. In this note we shall prove the following theorem.

Theorem. *Let G be a connected complex solvable Lie group and Γ be a lattice of G . Suppose that Γ is nilpotent. Then G is nilpotent.*

It is known that Theorem is not true in general for real solvable Lie group ([1] Chapter 3, Example 4).

2. Proof of Theorem

First we note that our theorem will be valid in general if it is proved for the case where G is simply connected. In fact, let \tilde{G} be the universal covering group with the projection $\pi: \tilde{G} \rightarrow G$. Then $\tilde{\Gamma} = \pi^{-1}(\Gamma)$ is a lattice in \tilde{G} and it is nilpotent, since the kernel of π is contained in the center of $\tilde{\Gamma}$. Thus \tilde{G} is nilpotent by Theorem for the case where the complex solvable Lie group is simply connected, and so is G .

From now on assume that G is simply connected. Let \mathfrak{g} be the Lie algebra of G and I the canonical complex structure. We denote by \mathfrak{n} the maximal nilpotent ideal of \mathfrak{g} regarded as real Lie algebra. Since \mathfrak{n} is given by $\{X \in \mathfrak{g} \mid \text{ad}(X) \text{ is nilpotent}\}$, \mathfrak{n} is invariant by I , so that \mathfrak{n} is a complex subalgebra of \mathfrak{g} . Let \mathfrak{g}^k denote $[\mathfrak{g}, \mathfrak{g}^{k-1}]$ where we put $\mathfrak{g}^0 = \mathfrak{g}$. Then $\{\mathfrak{g}^k\}$ is a descending sequence of ideals. Put $\mathfrak{g}^\infty = \inf_k \mathfrak{g}^k$. It is obvious that \mathfrak{g}^∞ equals \mathfrak{g}^m for some m and is a complex subalgebra. We thus have a sequence of ideals:

$$\mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}^\infty.$$

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Let \mathfrak{g}^c denote the complexification of \mathfrak{g} . Then $\mathfrak{g}^c = \mathfrak{g}^+ + \mathfrak{g}^-$ (direct sum), where $\mathfrak{g}^+ = \{X \in \mathfrak{g}^c \mid IX = \pm \sqrt{-1}X\}$. By Theorem of Lie, we can take a basis $\{X_1, \dots, X_n\}$ of the complex solvable Lie algebra \mathfrak{g}^+ such that

- 1) $\{X_{l+1}, \dots, X_n\}$ is a basis of $(\mathfrak{g}^\infty)^+$
- 2) $\{X_{r+1}, \dots, X_n\}$ is a basis of $[\mathfrak{g}^+, \mathfrak{g}^+]$
- 3) $\{X_{s+1}, \dots, X_n\}$ is a basis of \mathfrak{n}^+ , where $\mathfrak{n}^+ = \{X \in \mathfrak{n}^c \mid IX = \sqrt{-1}X\}$, \mathfrak{n}^c being the complex subalgebra spanned by \mathfrak{n} .
- 4) the subspaces \mathfrak{g}_p^+ ($p=1, \dots, n$) spanned by $\{X_p, \dots, X_n\}$

are ideals of \mathfrak{g}^+ .

Put $Y_j = \frac{1}{2}(X_j + \bar{X}_j)$ for $j=1, \dots, n$. Then $iy_j = \frac{\sqrt{-1}}{2}(X_j - \bar{X}_j)$ and $\{Y_1, iy_1, \dots, Y_n, iy_n\}$ is a basis of \mathfrak{g} (over \mathbf{R}). Moreover, if \mathfrak{g}_{2j-1} (resp. \mathfrak{g}_{2j}) denotes the real vector space spanned by $\{Y_j, iy_j, \dots, Y_n, iy_n\}$ (resp. $\{iy_j, Y_{j+1}, iy_{j+1}, \dots, Y_n, iy_n\}$). Then \mathfrak{g}_i ($i=1, \dots, 2n$) are subalgebras of \mathfrak{g} and \mathfrak{g}_{i+1} is contained in \mathfrak{g}_i as an ideal. Since G is simply connected, it follows that every element $g \in G$ can be written in one and only one way in the form

$$g = (\exp t_1 Y_1)(\exp s_1 iy_1) \cdots (\exp t_n Y_n)(\exp s_n iy_n),$$

where $t_j = t_j(g), s_j = s_j(g)$ ($j=1, \dots, n$) are real numbers (cf. [2]). Since $[iy_j, Y_j] = 0$ for $j=1, \dots, n$.

$$g = \exp(t_1 Y_1 + s_1 iy_1) \cdots \exp(t_n Y_n + s_n iy_n).$$

Thus we get a biholomorphic map $\Phi: G \rightarrow \mathbf{C}^n$ defined by

$$\Phi(g) = (t_1(g) + \sqrt{-1}s_1(g), \dots, t_n(g) + \sqrt{-1}s_n(g)).$$

Let $\{2C_{ij}^k\}$ be the structure constants of the Lie algebra \mathfrak{g}^+ with respect to the basis $\{X_1, \dots, X_n\}$. Then we may regard $\{C_{ij}^k\}$ as the structure constants of the complex Lie algebra \mathfrak{g} with respect to the basis $\{Y_1, \dots, Y_n\}$.

Note that, for $i=s+1, \dots, n$,

$$ad(X_i) = \begin{pmatrix} \overbrace{0}^s & \overbrace{0}^{r-s} & \overbrace{0}^{l-r} & \overbrace{0}^{n-l} \\ 0 & 0 & 0 & 0 \\ * & * & A_i & 0 \\ * & * & * & B_i \end{pmatrix} \begin{matrix} s \\ r-s \\ l-r \\ n-l \end{matrix}$$

where $A_i = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$ and $B_i = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$

and, for $i=1, \dots, s,$

$$ad(X_i) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & A_i & 0 \\ * & * & * & B_i \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} 0 & 0 \\ \ddots & \\ * & 0 \end{pmatrix} \text{ and } B_i = \begin{pmatrix} 2C_{i'l+1}^{l+1} & & 0 \\ & \ddots & \\ * & & 2C_{i'n}^n \end{pmatrix}.$$

In the following, decomposition of a matrix in sixteen blocks is always taken in sizes as indicated above.

We note that $(C_{1j}^j, \dots, C_{sj}^j) \neq (0, \dots, 0)$ for any $j=l+1, \dots, n,$ by the definition of $\mathfrak{g}^\infty.$

Since $Ad(g) = (\exp z_1(g) ad(Y_1)) \cdots (\exp z_n(g) ad(Y_n)),$

$$(1) \quad Ad(g)(Y_1, \dots, Y_n) = (Y_1, \dots, Y_n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix}$$

where

$$B_3 = \begin{pmatrix} 0 & 0 \\ \ddots & \\ * & 0 \end{pmatrix}, \quad B_7 = \begin{pmatrix} \exp \left(\sum_{j=1}^s C_{j'l+1}^{l+1} z_j(g) \right) & & 0 \\ & \ddots & \\ * & & \exp \left(\sum_{j=1}^s C_{j'n}^n z_j(g) \right) \end{pmatrix}.$$

Consider \mathfrak{g} as a real Lie algebra and let $l(g)$ denote the number of eigenvalues different from 1 of $Ad(g): \mathfrak{g} \rightarrow \mathfrak{g}$ for $g \in G.$ Define $\text{rank } G = \sup_{g \in G} l(g).$ An element $g \in G$ is called regular if $l(g) = \text{rank } G.$ Then it is easy to see that $g \in G$ is regular if and only if $\exp \left(\sum_{j=1}^s C_{jk}^k z_j(g) \right) \neq 1$ for all $k=l+1, \dots, n.$

Lemma 1. *Let Γ be a lattice of a simply connected complex solvable Lie group $G.$ Then Γ contains a regular element of $G.$*

Proof. If we denote by N the connected maximal normal nilpotent Lie group of $G, N \cap \Gamma$ is a lattice of N by a theorem of Mostow ([3], [4]). Let $\pi: G \rightarrow G/N$ be the projection. Then $\pi(\Gamma)$ is a lattice of G/N and $(G/N)/\pi(\Gamma)$ is a complex torus. By the definition of $\Phi: G \rightarrow \mathbb{C}^n,$ it is obvious that G/N is biholomorphic to \mathbb{C}^s by $G/N \ni \pi(g) \rightarrow (z_1(g), \dots, z_s(g)) \in \mathbb{C}^s.$ We identify G/N with \mathbb{R}^{2s} by

$$\pi(g) = (\operatorname{Re} z_1(g), \operatorname{Im} z_1(g), \dots, \operatorname{Re} z_s(g), \operatorname{Im} z_s(g)).$$

Consider the real subspaces H_k of codimension 1 defined by

$$H_k = \{(x_1, y_1, \dots, x_s, y_s) \in \mathbf{R}^{2s} \mid \sum_{j=1}^s (\operatorname{Re}(C_{jk}^k)x_j - \operatorname{Im}(C_{jk}^k)y_j) = 0\}$$

for $k=l+1, \dots, n$. Since $\pi(\Gamma)$ is a lattice of \mathbf{R}^{2s} , there are infinitely many different real subspaces of codimension 1 which are generated by $2s-1$ lattice points of $\pi(\Gamma)$. Hence, there exists a point $\gamma \in \Gamma$ such that $\pi(\gamma) \notin H_k$ for $k=l+1, \dots, n$. Then $|\exp(\sum_{j=1}^s C_{jk}^k z_j(\gamma))| \neq 1$ for all $k=l+1, \dots, n$ and $\gamma \in \Gamma$ is a regular element of G . q.e.d.

Lemma 2. (Mostow) *Let G be a simply connected solvable Lie group and Γ a uniform subgroup of G containing a regular element. Let G^∞ denote the connected Lie subgroup of G corresponding to \mathfrak{g}^∞ . Then $G^\infty \cap \Gamma$ is uniform in G^∞ .*

Proof. See [3] Lemma 5.

Proof of Theorem. Suppose that G is not nilpotent. Then $G^\infty \neq \{e\}$. Since G^∞ is a simply connected nilpotent Lie group, $G^\infty \cap \Gamma \neq \{e\}$ by Lemma 2. Since Γ is nilpotent, $G^\infty \cap \Gamma$ contains a non-trivial element of the center C of Γ . Choose an element $\gamma \neq e$ of $G^\infty \cap \Gamma \cap C$. We can write γ uniquely as

$$\gamma = (\exp z_{l+1} Y_{l+1}) \cdots (\exp z_n Y_n)$$

where $(z_{l+1}, \dots, z_n) \in \mathbf{C}^{n-l}$.

Note that $ad(Y_j)$ is represented by the basis $\{Y_1, \dots, Y_n\}$ as follows:

$$ad(Y_j) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_j & B_j & C_j & D_j \end{pmatrix} \quad \text{for } j = l+1, \dots, n$$

where

$$A_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ C_{j1}^j & \cdots & C_{js}^j \\ \vdots & & \vdots \\ C_{j1}^n & \cdots & C_{js}^n \end{pmatrix} < j-l, \quad B_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix} < j-l,$$

$$C_j = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{pmatrix} \left\langle j-l \right. \text{ and } D_j = \begin{matrix} j-l \\ \downarrow \\ \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \\ * & \dots & * & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ * & \dots & * & 0 & \dots & 0 \end{pmatrix} \left\langle j-l \right.$$

Fix a $j=l+1, \dots, n$ and put $\delta_j=(\exp z_j Y_j)\cdots(\exp z_n Y_n)$. Then $Ad(\delta_j)=(\exp z_j ad(Y_j))\cdots(\exp z_n ad(Y_n))$ is written as follows:

$$(2) \quad Ad(\delta_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ P_j & Q_j & R_j & S_j \end{pmatrix}$$

where

$$P_j = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ C_{j,1}^j z_j & \dots & C_{j,s}^j z_j \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{pmatrix} \left\langle j-l, \quad Q_j = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{pmatrix} \left\langle j-l,$$

$$R_j = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{pmatrix} \left\langle j-l \text{ and } S_j = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & & & \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 \\ * & \dots & * & 1 & 0 & \dots & 0 \\ \vdots & & & * & & \ddots & \vdots \\ * & \dots & \dots & \dots & * & & 1 \end{pmatrix} \left\langle j-l.$$

We claim that if $\gamma_0 \delta_j = \delta_j \gamma_0$ for a regular element $\gamma_0 \in \Gamma$, then $z_j=0$. Put

$$Ad(\gamma_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix}.$$

Since $Ad(\gamma_0)Ad(\delta_j)=Ad(\delta_j)Ad(\gamma_0)$, we get

$$(3) \quad B_4+B_7P_j = P_j+R_jB_1+S_jB_4 \in M_{n-l,s}(C).$$

Consider the $(j-l, k)$ -component of both hands of (3), by (1) we get

$$\exp\left(\sum_{i=1}^s C_{i_j, z_i}^j(\gamma_0)\right) C_{j_k, z_j}^j = C_{j_k, z_j}^j$$

for $k=1, \dots, s$. Since γ_0 is a regular element of G , $\exp\left(\sum C_{i_j, z_j}^j(\gamma_0)\right) \neq 1$ and $C_{j_k, z_j}^j = 0$ for $k=1, \dots, s$. Thus $z_j = 0$, since $(C_{1_j}^j, \dots, C_{s_j}^j) \neq (0, \dots, 0)$.

Now, starting with $j=l+1$, we get $z_j = 0$ successively for all $j=l+1, \dots, n$. This contradicts our assumption $\gamma \neq e$. Hence, G is nilpotent, and this proves our Theorem.

REMARK. The special case of our Theorem has been proved in a stronger form in the section 2 of [5].

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