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A THEOREM ON LATTICES OF A COMPLEX SOLVABLE LIE GROUP

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1. Introduction

A discrete subgroup Γ of a Lie group G is called a lattice of G if the homogeneous space G/Γ is of finite volume. It is known that any lattice Γ of a solvable Lie group G is uniform, i.e., such that G/Γ is compact. In this note we shall prove the following theorem.

Theorem. *Let G be a connected complex solvable Lie group and Γ be a lattice of G. Suppose that Γ is nilpotent. Then G is nilpotent.*

It is known that Theorem is not true in general for real solvable Lie group ([1] Chapter 3, Example 4).

2. Proof of Theorem

First we note that our theorem will be valid in general if it is proved for the case where G is simply connected. In fact, let $\tilde{G}$ be the universal covering group with the projection π: $\tilde{G}\to G$. Then $\tilde{\Gamma}=\pi^{-1}(\Gamma)$ is a lattice in $\tilde{G}$ and it is nilpotent, since the kernel of π is contained in the center of $\tilde{\Gamma}$. Thus $\tilde{G}$ is nilpotent by Theorem for the case where the complex solvable Lie group is simply connected, and so is G.

From now on assume that G is simply connected. Let $\mathfrak{g}$ be the Lie algebra of G and I the canonical complex structure. We denote by $\mathfrak{n}$ the maximal nilpotent ideal of $\mathfrak{n}$ regarded as real Lie algebra. Since $\mathfrak{n}$ is given by $\{X\in\mathfrak{g}\,|\,ad(X)$ is nilpotent$\}$, $\mathfrak{n}$ is invariant by I, so that $\mathfrak{n}$ is a complex subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^k$ denote $[\mathfrak{g}, \mathfrak{g}^{k-1}]$ where we put $\mathfrak{g}^0=\mathfrak{g}$. Then $\{\mathfrak{g}^k\}$ is a descending sequence of ideals. Put $\mathfrak{g}^\infty=\inf_k \mathfrak{g}^k$. It is obvious that $\mathfrak{g}^\infty$ equals $\mathfrak{g}^m$ for some m and is a complex subalgebra. We thus have a sequence of ideals:

$$\mathfrak{g}\supset\mathfrak{n}\supset[\mathfrak{g}, \mathfrak{g}]\supset\mathfrak{g}^\infty .$$

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Let $\mathfrak{g}^C$ denote the complexification of $\mathfrak{g}$. Then $\mathfrak{g}^C=\mathfrak{g}^+ +\mathfrak{g}^-$ (direct sum), where $\mathfrak{g}^{\pm}=\{X\in\mathfrak{g}^C \mid IX=\pm\sqrt{-1}X\}$. By Theorem of Lie, we can take a basis $\{X_1, \cdots, X_n\}$ of the complex solvable Lie algebra $\mathfrak{g}^+$ such that

1) $\{X_{l+1}, \cdots, X_n\}$ is a basis of $(\mathfrak{g}^\infty)^+$

2) $\{X_{r+1}, \cdots, X_n\}$ is a basis of $[\mathfrak{g}^+, \mathfrak{g}^+]$

3) $\{X_{s+1}, \cdots, X_n\}$ is a basis of $\mathfrak{n}^+$, where $\mathfrak{n}^+=\{X\in\mathfrak{n}^C \mid IX=\sqrt{-1}X\}$, $\mathfrak{n}^C$ being the complex subalgebra spanned by $\mathfrak{n}$.

4) the subspaces $\mathfrak{g}_p^+$ $(p=1, \cdots, n)$ spanned by $\{X_p, \cdots, X_n\}$

are ideals of $\mathfrak{g}^+$.

Put $Y_j=\frac{1}{2}(X_j+\bar{X}_j)$ for $j=1, \cdots, n$. Then $IY_j=\frac{\sqrt{-1}}{2}(X_j-\bar{X}_j)$ and $\{Y_1, IY_1, \cdots, Y_n, IY_n\}$ is a basis of $\mathfrak{g}$ (over $\boldsymbol{R}$). Moreover, if $\mathfrak{g}_{2j-1}$ (resp. $\mathfrak{g}_{2j}$) denotes the real vector space spanned by $\{Y_j, IY_j, \cdots, Y_n, IY_n\}$ (resp. $\{IY_j, Y_{j+1}, IY_{j+1}, \cdots, Y_n, IY_n\}$). Then $\mathfrak{g}_i$ $(i=1, \cdots, 2n)$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_{i+1}$ is contained in $\mathfrak{g}_i$ as an ideal. Since G is simply connected, it follows that every element $g\in G$ can be written in one and only one way in the form

$$g = (\exp t_1 Y_1)(\exp s_1 I Y_1)\cdots(\exp t_n Y_n)(\exp s_n I Y_n)\,,$$

where $t_j=t_j(g)$, $s_j=s_j(g)$ $(j=1, \cdots, n)$ are real numbers (cf. [2]). Since $[IY_j, Y_j]=0$ for $j=1, \cdots, n$.

$$g = \exp(t_1 Y_1+s_1 I Y_1)\cdots\exp(t_n Y+s_n I Y_n)\,.$$

Thus we get a biholomorphic map Φ: $G\to\boldsymbol{C}^n$ defined by

$$\Phi(g) = (t_1(g)+\sqrt{-1}s_1(g), \cdots, t_n(g)+\sqrt{-1}s_n(g))\,.$$

Let $\{2C_{ij}^k\}$ be the structure constants of the Lie algebra $\mathfrak{g}^+$ with respect to the basis $\{X_1, \cdots, X_n\}$. Then we may regard $\{C_{ij}^k\}$ as the structure constants of the complex Lie algebra $\mathfrak{g}$ with respect to the basis $\{Y_1, \cdots, Y_n\}$.

Note that, for $i=s+1. \cdots, n$,

$$ad(X_i) = \begin{array}{c} \begin{array}{cccc} \overbrace{\quad}^{s} & \overbrace{\quad}^{r-s} & \overbrace{\quad}^{l-r} & \overbrace{\quad}^{n-l} \end{array} \\ \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & A_i & 0 \\ * & * & * & B_i \end{array}\right) \end{array} \begin{array}{l} \\ \}\, s \\ \}\, r-s \\ \}\, l-r \\ \}\, n-l \end{array}$$

where $A_i = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix}$ and $B_i = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix}$

and, for $i=1, \cdots, s$,

$$ad(X_i) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & A_i & 0 \\ * & * & * & B_i \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} 2C_{i\,l+1}^{l+1} & & 0 \\ & \ddots & \\ * & & 2C_{in}^{n} \end{pmatrix}.$$

In the following, decomposition of a matrix in sixteen blocks is always taken in sizes as indicated above.

We note that $(C_{1j}^{j}, \cdots, C_{sj}^{j}) \neq (0, \cdots, 0)$ for any $j=l+1, \cdots, n$, by the definition of $\mathfrak{g}^\infty$.

Since $Ad(g)=(\exp z_1(g)\, ad(Y_1))\cdots(\exp z_n(g)\, ad(Y_n))$,

$$Ad(g)(Y_i, \cdots, Y_n) = (Y_1, \cdots, Y_n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix} \tag{1}$$

where

$$B_3 = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix}, \quad B_7 = \begin{pmatrix} \exp\left(\sum_{j=1}^{s} C_{j\,l+1}^{l+1} z_j(g)\right) & & 0 \\ & \ddots & \\ * & & \exp\left(\sum_{j=1}^{s} C_{jn}^{n} z_j(g)\right) \end{pmatrix}.$$

Consider $\mathfrak{g}$ as a real Lie algebra and let $l(g)$ denote the number of eigenvalues different from 1 of $Ad(g)\colon \mathfrak{g}\to\mathfrak{g}$ for $g\in G$. Define rank $G=\sup_{g\in G} l(g)$. An element $g\in G$ is called regular if $l(g)=\text{rank } G$. Then it is easy to see that $g\in G$ is regular if and only if $\exp\left(\sum_{j=1}^{s} C_{jk}^{k} z_j(g)\right)\neq 1$ for all $k=l+1, \cdots, n$.

Lemma 1. *Let Γ be a lattice of a simply connected complex solvable Lie group G. Then Γ contains a regular element of G.*

Proof. If we denote by N the connected maximal normal nilpotent Lie group of G, $N\cap\Gamma$ is a lattice of N by a theorem of Mostow ([3], [4]). Let $\pi\colon G\to G/N$ be the projection. Then $\pi(\Gamma)$ is a lattice of G/N and $(G/N)/\pi(\Gamma)$ is a complex torus. By the definition of $\Phi\colon G\to \boldsymbol{C}^n$, it is obvious that G/N is biholomorphic to $\boldsymbol{C}^s$ by $G/N\ni\pi(g)\to(z_1(g), \cdots, z_s(g))\in\boldsymbol{C}^s$. We identify G/N with $\boldsymbol{R}^{2s}$ by

$$\pi(g) = (\mathrm{Re}\, z_1(g),\ \mathrm{Im}\, z_1(g),\ \cdots,\ \mathrm{Re}\, z_s(g),\ \mathrm{Im}\, z_s(g))\,.$$

Consider the real subspaces H_k of codimension 1 defined by

$$H_k = \{(x_1, y_1, \cdots, x_s, y_s) \in \boldsymbol{R}^{2s} \mid \sum_{j=1}^{s} (\mathrm{Re}\,(C^k_{jk}) x_j - \mathrm{Im}\,(C^k_{jk}) y_j) = 0\}$$

for $k=l+1, \cdots, n$. Since $\pi(\Gamma)$ is a lattice of $\boldsymbol{R}^{2s}$, there are infinitely many different real subspaces of codimension 1 which are generated by $2s-1$ lattice points of $\pi(\Gamma)$. Hence, there exists a point $\gamma \in \Gamma$ such that $\pi(\gamma) \notin H_k$ for $k=l+1, \cdots, n$. Then $|\exp(\sum_{j=1}^{s} C^k_{jk} z_j(\gamma))| \neq 1$ for all $k=l+1, \cdots, n$ and $\gamma \in \Gamma$ is a regular element of G. q.e.d.

Lemma 2. (Mostow) *Let G be a simply connected solvable Lie group and Γ a uniform subgroup of G containing a regular element. Let G^∞ denote the connected Lie subgrovp of G corresponding to $\mathfrak{g}^\infty$. Then $G^\infty \cap \Gamma$ is uniform in G^∞.*

Proof. See [3] Lemma 5.

Proof of Theorem. Suppose that G is not nilpotent. Then $G^\infty \neq \{e\}$. Since G^∞ is a simply connected nilpotent Lie group, $G^\infty \cap \Gamma \neq \{e\}$ by Lemma 2. Since Γ is nilpotent, $G^\infty \cap \Gamma$ contains a non-trivial element of the center C of Γ. Choose an element $\gamma \neq e$ of $G^\infty \cap \Gamma \cap C$. We can write γ uniquely as

$$\gamma = (\exp z_{l+1} Y_{l+1}) \cdots (\exp z_n Y_n)$$

where $(z_{l+1}, \cdots, z_n) \in \boldsymbol{C}^{n-l}$.

Note that $ad(Y_j)$ is represented by the basis $\{Y_1, \cdots, Y_n\}$ as follows:

$$ad(Y_j) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_j & B_j & C_j & D_j \end{pmatrix} \qquad \text{for} \quad j = l+1, \cdots, n$$

where

$$A_j = \begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ C^j_{j1} & \cdots\cdots & C^j_{js} \\ \vdots & & \vdots \\ C^n_{j1} & \cdots\cdots & C^n_{js} \end{pmatrix} < j-l, \qquad B_j = \begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \\ * & \cdots\cdots & * \\ * & \cdots\cdots & * \end{pmatrix} < j-l,$$

$$
C_j=\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \\ * & \cdots\cdots & * \\ \vdots & & \vdots \\ * & \cdots\cdots & * \end{pmatrix}\begin{matrix} \\ \\ \\ <j-l \\ \\ \\ \\ \end{matrix} \quad \text{and} \quad D_j=\begin{array}{c}
\begin{matrix} & j-l & \\ & \vee & \end{matrix} \\
\begin{pmatrix} 0 & \cdots\cdots & 0 & \cdots\cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots\cdots & 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 & \cdots\cdots & 0 \\ * & \cdots\cdots & * & 0 \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ * & \cdots\cdots & \cdots & * & 0 \end{pmatrix}
\end{array}\begin{matrix} \\ \\ \\ \\ \\ <j-l \\ \\ \\ \\ \end{matrix}.
$$

Fix a $j=l+1, \cdots, n$ and put $\delta_j=(\exp z_j Y_j)\cdots(\exp z_n Y_n)$. Then $Ad(\delta_j)=(\exp z_j ad(Y_j))\cdots(\exp z_n ad(Y_n))$ is written as follows:

$$
(2) \qquad Ad(\delta_j)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ P_j & Q_j & R_j & S_j \end{pmatrix}
$$

where

$$
P_j=\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ C^j_{j1} z_j & \cdots\cdots & C^j_{js} z_j \\ * & \cdots\cdots & * \\ \vdots & & \vdots \\ * & \cdots\cdots & * \end{pmatrix}\begin{matrix} \\ \\ \\ <j-l, \\ \\ \\ \\ \end{matrix} \qquad Q_j=\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \\ * & \cdots\cdots & * \\ \vdots & & \vdots \\ * & \cdots\cdots & * \end{pmatrix}\begin{matrix} \\ \\ \\ <j-l, \\ \\ \\ \\ \end{matrix}
$$

$$
R_j=\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \\ * & \cdots\cdots & * \\ \vdots & & \vdots \\ * & \cdots\cdots & * \end{pmatrix}\begin{matrix} \\ \\ \\ <j-l \\ \\ \\ \\ \end{matrix} \quad \text{and} \quad S_j=\begin{pmatrix} 1 & 0 & \cdots\cdots & 0 & \cdots\cdots & 0 \\ 0 & 1 & 0 & \vdots & \cdots\cdots & \vdots \\ \vdots & & \ddots & \vdots & & \vdots \\ \vdots & & & \ddots \vdots & & \vdots \\ 0 & \cdots\cdots & 1 & 0 & 0 & 0 \\ * & \cdots\cdots & * & 1 & 0 \cdots\cdots & 0 \\ \vdots & & & * & \ddots & \vdots \\ \vdots & & & \vdots & \ddots & 0 \\ * & \cdots\cdots & \cdots & \cdots & * & 1 \end{pmatrix}\begin{matrix} \\ \\ \\ \\ <j-l. \\ \\ \\ \\ \\ \end{matrix}
$$

We claim that if $\gamma_0\delta_j=\delta_j\gamma_0$ for a regular element $\gamma_0\in\Gamma$, then $z_j=0$. Put

$$
Ad(\gamma_0)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix}.
$$

Since $Ad(\gamma_0)Ad(\delta_j)=Ad(\delta_j)Ad(\gamma_0)$, we get

$$
(3) \qquad B_4+B_7P_j=P_j+R_jB_1+S_jB_4\in M_{n-l,s}(C)\,.
$$

Consider the $(j-l, k)$-component of both hands of (3), by (1) we get

$$\exp\left(\sum_{i=1}^{s} C_{ij}^{j} z_i(\gamma_0)\right) C_{jk}^{j} z_j = C_{jk}^{j} z_j$$

for $k=1, \cdots, s$. Since γ_0 is a regular element of G, $\exp(\sum C_{ij}^{j} z_j(\gamma_0)) \neq 1$ and $C_{jk}^{j} z_j = 0$ for $k=1, \cdots, s$. Thus $z_j=0$, since $(C_{1j}^{j}, \cdots. C_{sj}^{j}) \neq (0, \cdots, 0)$.

Now, starting with $j=l+1$, we get $z_j=0$ successively for all $j=l+1, \cdots, n$. This contradicts our assumption $\gamma \neq e$. Hence, G is nilpotent, and this proves our Theorem.

REMARK. The special case of our Theorem has been proved in a stronger form in the section 2 of [5].

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