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A THEOREM ON LATTICES OF A COMPLEX
SOLVABLE LIE GROUP

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1. Introduction

A discrete subgroup \( \Gamma \) of a Lie group \( G \) is called a lattice of \( G \) if the homogeneous space \( G/\Gamma \) is of finite volume. It is known that any lattice \( \Gamma \) of a solvable Lie group \( G \) is uniform, i.e., such that \( G/\Gamma \) is compact. In this note we shall prove the following theorem.

Theorem. Let \( G \) be a connected complex solvable Lie group and \( \Gamma \) be a lattice of \( G \). Suppose that \( \Gamma \) is nilpotent. Then \( G \) is nilpotent.

It is known that Theorem is not true in general for real solvable Lie group ([1] Chapter 3, Example 4).

2. Proof of Theorem

First we note that our theorem will be valid in general if it is proved for the case where \( G \) is simply connected. In fact, let \( \tilde{G} \) be the universal covering group with the projection \( \pi: \tilde{G} \rightarrow G \). Then \( \Gamma = \pi^{-1}(\Gamma) \) is a lattice in \( \tilde{G} \) and it is nilpotent, since the kernel of \( \pi \) is contained in the center of \( \Gamma \). Thus \( \tilde{G} \) is nilpotent by Theorem for the case where the complex solvable Lie group is simply connected, and so is \( G \).

From now on assume that \( G \) is simply connected. Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( I \) the canonical complex structure. We denote by \( \mathfrak{n} \) the maximal nilpotent ideal of \( \mathfrak{g} \) regarded as real Lie algebra. Since \( \mathfrak{n} \) is given by \( \{X \in \mathfrak{g} \mid \text{ad}(X) \text{ is nilpotent}\} \), \( \mathfrak{n} \) is invariant by \( I \), so that \( \mathfrak{n} \) is a complex subalgebra of \( \mathfrak{g} \). Let \( \mathfrak{g}^k \) denote \( \mathfrak{g} \cap \mathfrak{g}^k \) where we put \( \mathfrak{g}^1 = \mathfrak{g} \). Then \( \{\mathfrak{g}^k\} \) is a descending sequence of ideals. Put \( \mathfrak{g}^\infty = \inf \mathfrak{g}^k \). It is obvious that \( \mathfrak{g}^\infty \) equals \( \mathfrak{g}^m \) for some \( m \) and is a complex subalgebra. We thus have a sequence of ideals:

\[ \mathfrak{g} \supset \mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{g}^\infty. \]

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Let \( g^c \) denote the complexification of \( g \). Then \( g^c = g^+ + g^- \) (direct sum), where \( g^- = \{ X \in g^c | IX = \pm \sqrt{-1}X \} \). By Theorem of Lie, we can take a basis \( \{ X_1, \ldots, X_n \} \) of the complex solvable Lie algebra \( g^+ \) such that

1) \( \{ X_{r+1}, \ldots, X_n \} \) is a basis of \( (g^-)^+ \)

2) \( \{ X_{r+1}, \ldots, X_n \} \) is a basis of \( [g^+, g^+] \)

3) \( \{ X_{s+1}, \ldots, X_n \} \) is a basis of \( n^+ \), where \( n^+ = \{ X \in n^c | IX = \sqrt{-1}X \} \), \( n^c \)

being the complex subalgebra spanned by \( n \).

4) the subspaces \( g_\beta^+ (\beta = 1, \ldots, n) \) spanned by \( \{ X_\beta, \ldots, X_n \} \)

are ideals of \( g^c \).

Put \( Y_j = \frac{1}{2} (X_j + \bar{X}_j) \) for \( j = 1, \ldots, n \). Then \( IY_j = \frac{-1}{2} (X_j - \bar{X}_j) \) and \( \{ Y_1, IY_1, \ldots, Y_n, IY_n \} \) is a basis of \( g \) (over \( R \)). Moreover, if \( g_{2j-1} \) (resp. \( g_{2j} \)) denotes the real vector space spanned by \( \{ Y_j, IY_j, \ldots, Y_n, IY_n \} \) (resp. \( \{ IY_j, Y_{j+1}, IY_{j+1}, \ldots, Y_n, IY_n \} \)). Then \( g_\beta \) \( (\beta = 1, \ldots, 2n) \) are subalgebras of \( g \) and \( g_{2r+1} \) is contained in \( g_\beta \) as an ideal. Since \( G \) is simply connected, it follows that every element \( g \in G \) can be written in one and only one way in the form

\[
g = (\exp t_1 Y_1) (\exp t_2 Y_2) \cdots (\exp s_1 IY_1) \cdots (\exp s_n IY_n),
\]

where \( t_1 = t_1(g), s_j = s_j(g) \) \( (j = 1, \ldots, n) \) are real numbers (cf. [2]). Since \( [IY_j, Y_j] = 0 \) for \( j = 1, \ldots, n \).

\[
g = \exp (t_1 Y_1 + s_1 IY_1) \cdots \exp (t_n Y_n + s_n IY_n).
\]

Thus we get a biholomorphic map \( \Phi: G \to \mathbb{C}^n \) defined by

\[
\Phi(g) = (t_1(s_1(g)), \ldots, t_n(s_n(g))).
\]

Let \( \{ 2C^\pm_{ij} \} \) be the structure constants of the Lie algebra \( g^+ \) with respect to the basis \( \{ X_1, \ldots, X_n \} \). Then we may regard \( \{ C^\pm_{ij} \} \) as the structure constants of the complex Lie algebra \( g \) with respect to the basis \( \{ Y_1, \ldots, Y_n \} \).

Note that, for \( i = \text{s} + 1, \ldots, n \),

\[
ad(X_i) = \begin{pmatrix}
& s & r-s & l-r & n-l \\
0 & 0 & 0 & 0 & \end{pmatrix}
s
\]

where \( A_i = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \) and \( B_i = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \)
and, for \(i=1, \ldots, s\),

\[
ad(X_i) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & A_i & 0 \\ * & * & * & B_i \end{pmatrix}
\]

where

\[
A_i = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \quad \text{and} \quad B_i = \begin{pmatrix} 2C_{i+1}^1 & \cdots & 0 \\ * & 2C_{i+n}^1 \end{pmatrix}.
\]

In the following, decomposition of a matrix in sixteen blocks is always taken in sizes as indicated above.

We note that \((C_{ij}^1, \ldots, C_{ij}^n) = (0, \ldots, 0)\) for any \(j=l+1, \ldots, n\), by the definition of \(g^m\).

Since \(\text{Ad}(g) = (\exp(\gamma(g)\text{ad}(Y)) \cdots (\exp(z_s(g)\text{ad}(Y))\),

\[
\text{Ad}(g)(Y_1, \ldots, Y_n) = (Y_1, \ldots, Y_n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix}
\]

where

\[
B_3 = \begin{pmatrix} 0 & 0 \\ \star & 0 \end{pmatrix}, \quad B_7 = \begin{pmatrix} \exp\left(\sum_{j=1}^s C_{ij}^1 z_j(g)\right) & \cdots & 0 \\ \star & \exp\left(\sum_{j=1}^s C_{ij}^n z_j(g)\right) \end{pmatrix}.
\]

Consider \(g\) as a real Lie algebra and let \(l(g)\) denote the number of eigenvalues different from 1 of \(\text{Ad}(g): g \rightarrow g\) for \(g \in G\). Define \(\text{rank } G = \sup_{g \in b} l(g)\). An element \(g \in G\) is called regular if \(l(g) = \text{rank } G\). Then it is easy to see that \(g \in G\) is regular if and only if \(\exp\left(\sum_{j=1}^s C_{ij}^k z_j(g)\right) \neq 1\) for all \(k=l+1, \ldots, n\).

**Lemma 1.** Let \(\Gamma\) be a lattice of a simply connected complex solvable Lie group \(G\). Then \(\Gamma\) contains a regular element of \(G\).

**Proof.** If we denote by \(N\) the connected maximal normal nilpotent Lie group of \(G\), \(N \cap \Gamma\) is a lattice of \(N\) by a theorem of Mostow ([3], [4]). Let \(\pi: G \rightarrow G/N\) be the projection. Then \(\pi(\Gamma)\) is a lattice of \(G/N\) and \((G/N)/\pi(\Gamma)\) is a complex torus. By the definition of \(\Phi: G \rightarrow \mathbb{C}^s\), it is obvious that \(G/N\) is biholomorphic to \(\mathbb{C}^s\) by \(\pi(g) \mapsto (z_1(g), \ldots, z_s(g)) \in \mathbb{C}^s\). We identify \(G/N\) with \(\mathbb{R}^{2s}\) by
\( \pi(g) = (\Re z_i(g), \Im z_i(g), \cdots, \Re z_s(g), \Im z_s(g)) \).

Consider the real subspaces \( H_k \) of codimension 1 defined by

\[
H_k = \left\{ (x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n} \mid \sum_{j=1}^s (\Re (C^*_j) x_j - \Im (C^*_j) y_j) = 0 \right\}
\]

for \( k=l+1, \cdots, n \). Since \( \pi(\Gamma) \) is a lattice of \( \mathbb{R}^{2n} \), there are infinitely many different real subspaces of codimension 1 which are generated by \( 2s-1 \) lattice points of \( \pi(\Gamma) \). Hence, there exists a point \( \gamma \in \Gamma \) such that \( \pi(\gamma) \in H_k \) for \( k=l+1, \cdots, n \). Then \( |\exp(\sum_{j=1}^s C_j^* z_j(\gamma))| \neq 1 \) for all \( k=l+1, \cdots, n \) and \( \gamma \in \Gamma \) is a regular element of \( G \).

**Lemma 2.** (Mostow) Let \( G \) be a simply connected solvable Lie group and \( \Gamma \) a uniform subgroup of \( G \) containing a regular element. Let \( G^0 \) denote the connected Lie subgroup of \( G \) corresponding to \( \Gamma^0 \). Then \( G^0 \cap \Gamma \) is uniform in \( G^0 \).

**Proof.** See [3] Lemma 5.

**Proof of Theorem.** Suppose that \( G \) is not nilpotent. Then \( G^0 \neq \{e\} \).

Since \( G^0 \) is a simply connected nilpotent Lie group, \( G^0 \cap \Gamma = \{e\} \) by Lemma 2. Since \( \Gamma \) is nilpotent, \( G^0 \cap \Gamma \) contains a non-trivial element of the center \( C \) of \( \Gamma \). Choose an element \( \gamma \neq e \) of \( G^0 \cap \Gamma \cap C \). We can write \( \gamma \) uniquely as

\[
\gamma = (\exp z_{l+1} Y_{l+1}) \cdots (\exp z_n Y_n)
\]

where \( (z_{l+1}, \cdots, z_n) \in \mathbb{C}^{n-l} \).

Note that \( \text{ad}(Y_j) \) is represented by the basis \( \{Y_1, \cdots, Y_n\} \) as follows:

\[
\text{ad}(Y_j) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
A_j & B_j & C_j & D_j
\end{bmatrix}
\]

for \( j=l+1, \cdots, n \),

where

\[
A_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
C_{j1} & \cdots & C_{js} \\
\vdots & \cdots & \vdots \\
C_{js} & \cdots & C_{js}
\end{bmatrix} < j-l,
\]

\[
B_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & * \\
\cdots & \cdots & *
\end{bmatrix} < j-l,
\]

\[
C_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
C_{j1} & \cdots & C_{js} \\
\vdots & \cdots & \vdots \\
C_{js} & \cdots & C_{js}
\end{bmatrix} < j-l,
\]

\[
D_j = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & * \\
\cdots & \cdots & *
\end{bmatrix} < j-l.
\]
Fix a $j=l+1, \cdots, n$ and put $\delta_j=(\exp x_j Y_j)\cdots(\exp x_n Y_n)$. Then $Ad(\delta_j)=(\exp x_1 ad(Y_j))\cdots(\exp x_n ad(Y_n))$ is written as follows:

\[ Ad(\delta_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ P_j & Q_j & R_j & S_j \end{pmatrix} \]

where

\[ P_j = C_{j1} x_j \cdots C_{jn} x_j <j-l, \quad Q_j = 0 \cdots 0 <j-l, \]

\[ R_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ * & \cdots & * \end{pmatrix} <j-l \quad \text{and} \quad S_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & * & 0 \end{pmatrix} <j-l. \]

We claim that if $y_0 \delta_j = \delta_j y_0$ for a regular element $y_0 \in \Gamma$, then $x_j = 0$. Put

\[ Ad(y_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 \\ B_4 & B_5 & B_6 & B_7 \end{pmatrix}. \]

Since $Ad(y_0) Ad(\delta_j) = Ad(\delta_j) Ad(y_0)$, we get

\[ B_4 + B_7 P_j = P_j + R_j B_1 + S_j B_4 \in M_{n-l,d}(C). \]
Consider the \((j-l, k)\)-component of both hands of (3), by (1) we get
\[
\exp \left( \sum_{i=1}^s C_{ij} z_i(\gamma_0) \right) C_{jk} z_j = C_{jk} z_j
\]
for \(k=1, \cdots, s\). Since \(\gamma_0\) is a regular element of \(G\), \(\exp (\sum C_{ij} z_j(\gamma_0)) \neq 1\) and \(C_{jk} z_j = 0\) for \(k=1, \cdots, s\). Thus \(z_j = 0\), since \((C_{ij}, \cdots, C_{ij}) \neq (0, \cdots, 0)\).

Now, starting with \(j=l+1\), we get \(z_j = 0\) successively for all \(j=l+1, \cdots, n\). This contradicts our assumption \(\gamma \neq e\). Hence, \(G\) is nilpotent, and this proves our Theorem.

**Remark.** The special case of our Theorem has been proved in a stronger form in the section 2 of [5].

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**References**


