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2-KNOTS IN $S^2 \times S^2$, AND HOMOLOGY 4-SPHERES

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1. Introduction

Let K be a 2-knot in S^4 with exterior X. Since any self-homeomorphism of $S^2 \times S^1$ extends across $D^3 \times S^1$, the closed 4-manifold $M(K) = X \cup_f D^3 \times S^1$ obtained from S^4 by surgery on K is independent of the choice of a gluing homeomorphism f and well defined. One may study 2-knots K through the corresponding closed oriented 4-manifolds $M(K)$ [5]. The core $\{0\} \times S^1$ of $D^3 \times S^1$ $\subset M(K)$ is well defined up to isotopy by the conjugacy class of a meridian in $\pi_1(S^4 - K) \cong \pi_1 M(K)$, and two 2-knots K and K' have homeomorphic exteriors if and only if there is a homeomorphism from $M(K)$ onto $M(K')$ carrying the conjugacy class of m to the conjugacy class of $(m')^{\pm 1}$. Here $m \in \pi_1 M(K)$ and $m' \in \pi_1 M(K')$ are elements corresponding to meridians of K and K' via natural isomorphisms $\pi_1(S^4 - K) \cong \pi_1 M(K)$ and $\pi_1(S^4 - K') \cong \pi_1 M(K')$ respectively. There are, however, 2-knots in S^4 which have homeomorphic exteriors but are inequivalent [3, 11]. Therefore, in general, 2-knots K in S^4 are not determined by $M(K)$ together with the conjugacy class of a meridian in $\pi_1(S^4 - K)$.

In this paper, for 2-knots in $S^2 \times S^2$ we consider the analogue of 2-knots in S^4. Let S be a 2-knot in $S^2 \times S^2$ homologous to $S^2 \times \{*\}$ and let E be the exterior of S. Then $\Sigma(S) = E \cup_{\partial E} D^3 \times S^1$ is a homology 4-sphere such that $\pi_1 \Sigma(S)$ is isomorphic to the knot group of S, $\pi_1(S^2 \times S^2 - S)$. We shall study 2-knots S in $S^2 \times S^2$ homologous to $S^2 \times \{*\}$ through the corresponding homology 4-spheres $\Sigma(S)$. In contrast with the case of 2-knots in S^4, since it is proved in §1 that each 2-knot in $S^2 \times S^2$ homologous to $S^2 \times \{*\}$ is uniquely determined by its exterior, it is also uniquely determined by its homology 4-sphere together with the conjugacy class of its meridian. Thus we classify such 2-knots in $S^2 \times S^2$ in terms of homology 4-spheres Σ together with the conjugacy class of weight elements for $\pi_1 \Sigma$ (Theorem 3.1 in §3). Moreover, in §§3 and 4, we exhibit a family of 2-knots in $S^2 \times S^2$ which have one of the following properties:
(1). Their knot groups are isomorphic mutually.
(2). The homology 4-spheres obtained from $S^2 \times S^2$ by surgery on 2-knots are homeomorphic (or diffeomorphic) to a spun homology 3-sphere.
So in the case where knot groups are infinite, we can get an example of infinitely many 2-knots in $S^2 \times S^2$ with the same knot group. However note that in the

case of finite knot groups, there are only finitely many 2-knots in $S^2\times S^2$ with the same finite knot group (Corollary 3.1 and Theorem 3.2).

2. Exteriors of 2-knots in $S^2\times S^2$

Let ζ and η be natural generators of $H_2(S^2\times S^2; \boldsymbol{Z})$ with $\zeta\cdot\zeta=\eta\cdot\eta=0$ and $\zeta\cdot\eta=\eta\cdot\zeta=1$. Let $D(n)$ be the D^2-bundle over S^2 with Euler number n, and with zero serction $\nu: S^2\to D(n)$.

DEFINITION 2.1. Let $f: D(n)\to S^2\times S^2$ be a topological embedding. Putting $S=f\circ\nu(S^2)$, S is a submanifold of $S^2\times S^2$ homeomorphic to S^2. We call S *a 2-knot in* $S^2\times S^2$. Note that, if S represents $p\zeta+q\eta\in H_2(S^?\times S^2; \boldsymbol{Z})$, then $n=2pq$. In particular, a 2-knot S in $S^2\times S^2$ is said to be *smooth*, provided that $f: D(n)\to S^2\times S^2$ is smooth. Moreover, we call $\pi_1(S^2\times S^2-S)$ *the kont group of* S.

DEFINITION 2.2. Let S and S' be 2-knots in $S^2\times S^2$. Then we say that S and S' are equivalent, if there is a self-homeomorphism of $S^2\times S^2$ taking S onto S'. We denote the equivalence class of S by $\langle S\rangle$.

Note that, if S and S' are 2-knots in $S^2\times S^2$ representing the same homology class with the trivial knot group, then S and S' are equivalent [12, 13]. Let S denote a 2-knot in $S^2\times S^2$ representing ζ with exterior E. Then one can recover a 2-knot S from E as follows:

$$(S^2\times S^2, S) = (E\cup_g S^2\times D^2, S^2\times \{0\}),$$

where $g: S^2\times\partial D^2\to\partial E=S^2\times S^1$ is some gluing homeomorphism. By the structure of the homeotopy group of $S^2\times S^1$ [2], we may assume that g is the identity or the twisting map τ. The twisting map τ is defined by $\tau(x,\theta)=(\rho(\theta),\theta)$, where $\rho(\theta)$ rotates S^2 about its polar S^0 through angle θ and corresponds to the generator of $\pi_1(SO(3))\cong\boldsymbol{Z}/2\boldsymbol{Z}$. Hence there are at most two knot types $\langle S\rangle$, $\langle S^*\rangle$ of 2-knots with a given exterior E. Here S and S^* are 2-knots characterized by the identity and the twisting map τ respectively. Putting $M=E\cup_\tau S^2\times D^2$, S^* is a 2-knot in M. Since τ corresponds to the generator of $\pi_1(SO(3))$, the second Stiefel-Whitney class $w_2(M)$ of M is nontrivial. In fact, it follows that M is a closed 1-connected 4-manifold with the intersection form $(1)\oplus(-1)$. (See [14].) Therefore M is not homeomorphic to $S^2\times S^2$. Thus we have the following:

Theorem 2.1 ([14]). *There exists, up to homeomorphism, exactly one 2-knot in $S^2\times S^2$ representing ζ with a given exterior.*

3. Classification of 2-knots in $S^2\times S^2$

We begin with some definitions.

DEFINITION 3.1. Let k be $\boldsymbol{Z}$ or $\boldsymbol{Q}$. A topological 4-manifold Σ is said to be *a k-homology 4-sphere*, if Σ is a closed 4-manifold with the k-homology of S^4. In particular, $\boldsymbol{Z}$-homology 4-sphere is simply called *a homology 4-sphere.*

DEFINITION 3.2. A group G is said to *have weight one*, if there is an element α of G such that the normal closure of α in G, *i.e.* the smallest normal subgroup of G which contains α, is equal to G. We call such an element α *a weight element for G.*

The knot group π of a 2-knot S in $S^2 \times S^2$ has weight one. In fact, if we let E be the exterior of S and m a simple closed curve on ∂E representing a meridian of S, then $E \cup_m D^2$ is a deformation retract of $S^2 \times S^2 - \{*\}$, which is simply connected. A meridian of S is a weight element for the knot group π. See [7].

Lemma 3.1. *Let π be a group isomorphic to the fundamental group $\pi_1\Sigma$ of a homology 4-sphere Σ. If π has weight one, then there exists a 2-knot in $S^2 \times S^2$ representing ζ with knot group isomorphic to π and whose meridian is a weight element for π.*

Proof: We give Σ an orientation. Suppose that α is a weight element for $\pi_1\Sigma$. We take a simple closed curve C in Σ representing α. If we let $N(C)$ be a closed regular neighborhood of C in Σ, homeomorphic to $D^3 \times S^1$, and if we put $E = \Sigma - intN(C)$, then $\pi_1 E$ is isomorphic to $\pi_1\Sigma$. Putting $X = E \cup S^2 \times D^2$ with the trivial framing, by van Kampen's theorem, X is a closed 1-connected 4-manifold. Moreover we give X the orientation inherited from that of Σ. By the Alexander duality, E looks like $S^2 \times D^2$ homologically, and so it follows that $H_2(X; \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ and the intersection form $(H_2(E; \boldsymbol{Z}), \cdot)$ is isomorphic to $(\boldsymbol{Z}, (0))$. Thus it is easily seen that the intersection form on $H_2(X; \boldsymbol{Z})$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to a basis x, y for $H_2(X; \boldsymbol{Z})$. Here y is the class represented by a core $S^2 \times \{0\}$, S, of $S^2 \times D^2$ in $X = E \cup S^2 \times D^2$. It follows from Freedman's theorem [1] that there is a homeomorphism $h: X \to S^2 \times S^2$. Then $h(S)$ is a 2-knot in $S^2 \times S^2$ representing $h_*(y)$ whose knot group $\pi_1(S^2 \times S^2 - h(S))$ is isomorphic to π. By composing an orientation reversing self-homeomorphism of $S^2 \times S^2$ if necessary, we may assume that h is orientation preserving. Hence the induced isomorphism $h_*: H_2(X; \boldsymbol{Z}) \to H_2(S^2 \times S^2; \boldsymbol{Z})$ is an automorphism of $(\boldsymbol{Z} \oplus \boldsymbol{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. Since $Aut(\boldsymbol{Z} \oplus \boldsymbol{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$, $h(S)$ represents $\pm\zeta$ or $\pm\eta$. After changing the orientation of $S^2 \times S^2$ and/or the orientation of ζ and η if necessary, $h(S)$ may represent ζ. Thus we get a required 2-knot in $S^2 \times S^2$. ■

REMARK. For any homology 3-sphere M obtained by Dehn surgery on a knot in S^3, there is a smooth 2-knot in $S^2\times S^2$ representing ζ whose knot group is isomorphic to $\pi_1 M$ [12]. Note that the fundamental group of a homology 3-sphere obtained by Dehn surgery on a knot in S^3 has weight one. Considering spins of homology 3-spheres (Definition 3.5), however, it follows from Lemma 3.1 that for any homology 3-sphere M whose fundamental group has weight one, there is a 2-knot in $S^2\times S^2$ representing ζ whose knot group is isomorphic to $\pi_1 M$.

Let S be a 2-knot in $S^2\times S^2$ representing ζ with exterior E and with knot group π. Set $\Sigma(S)=E\cup_f D^3\times S^1$, where f is a self-homeomorphism of $S^2\times S^1$. However, since every self-homeomorphism of $S^2\times S^1$ extends to a self-homeomorphism of $D^3\times S^1$, $\Sigma(S)$ does not depend on the choice of a gluing map f. The fact that S represents ζ implies that $\Sigma(S)$ is a homology 4-sphere whose fundamental group is isomorphic to π. We call $\Sigma(S)$ *a homology 4-sphere obtained by surgery on S*. When S is a smooth 2-knot, we sometimes use a self-diffeomorphism of $S^2\times S^1$ instead of a self-homeomorphism of $S^2\times S^1$ and $\Sigma(S)$ is a smooth homology 4-sphere. Then, it follows from the proof of Lemma 3.1 that, for any homology 4-sphere Σ whose fundamental group has weight one, there is a 2-knot S in $S^2\times S^2$ representing ζ such that $\Sigma(S)$ is homeomorphic to Σ. Moreover, by Lemma 3.1, we have:

Proposition 3.1. *The set of isomorphism classes of knot groups of all 2-knots in $S^2\times S^2$ representing ζ is equal to the set of isomorphism classes of all fundamental groups of homology 4-spheres of weight one.*

DEFINITION 3.3. Let Σ be a homology 4-sphere. Let α and β be elements of $\pi_1\Sigma=\pi_1(\Sigma, *)$. Then we say that α is geometrically equivalent to β, if the image of a circle representing $\alpha^{\pm 1}$ via a self-homeomorphism of Σ is free homotopic to a circle representing β.

REMARK. Two elements α, β of $\pi_1\Sigma$ are geometrically equivalent if and only if there is a self-homeomorphism f of Σ such β is conjugate to $w_* f_\sharp(\alpha^{\pm 1})$, where $f_\sharp$ is the induced isomorphism $\pi_1(\Sigma, *)\to\pi_1(\Sigma, f(*))$ from f and $w_*\colon \pi_1(\Sigma, f(*))\to\pi_1(\Sigma, *)$ is the isomorphism induced from a path w with $w(0)=f(*)$ and $w(1)=*$.

DEFINITION 3.4. We give the relation $\sim$ on the set of all pairs $(\Sigma, [\alpha]_\Sigma)$ as follows: $(\Sigma, [\alpha]_\Sigma)\sim(\Sigma', [\beta]_{\Sigma'})$ if and only if there is a homeomorphism $h\colon \Sigma\to\Sigma'$ such that $w_* h_\sharp(\alpha)$ is geometrically equivalent to $\beta\in\pi_1(\Sigma', *')$, where $w_*\colon \pi_1(\Sigma', h(*))\to\pi_1(\Sigma', *')$ is the isomorphism induced from a path w with $w(0)=h(*)$ and $w(1)=*'$. This relation $\sim$ is clearly an equivalence relation and we denote the equivalence class of $(\Sigma, [\alpha]_\Sigma)$ by $[\Sigma, [\alpha]_\Sigma]$.

Then we have the following theorem.

Theorem 3.1. *Let π be a group isomorphic to the fundamental group of a homology 4-sphere. Suppose π has weight one. Then there exists a bijection* $\{\langle S\rangle$; *S is a 2-knot in $S^2\times S^2$ representing ζ whose knot group is isomorphic to $\pi\}\leftrightarrow\{[\Sigma, [\alpha]_\Sigma]$; Σ is a homology 4-sphere whose fundamental group $\pi_1\Sigma$ is isomorphic to π and α is a weight element for $\pi_1\Sigma\}$.*

Proof: Let S be a 2-knot in $S^2\times S^2$ representing ζ whose knot group is isomorphic to π. Let $\Sigma=\Sigma(S)$ denote the homology 4-sphere obtained by surgery on S. Then we define a map ψ from the first set to the second set by $\psi(\langle S\rangle)=[\Sigma, [\alpha]_\Sigma]$, where α is an element induced from a meridian of S. This map ψ is clearly well defined, *i.e.* independent of the choice of S. By Lemma 3.1, ψ is surjective. To verify that ψ is injective, let us suppose $\psi(\langle S\rangle)=\psi(\langle S'\rangle)$, *i.e.* $[\Sigma, [\alpha]_\Sigma]=[\Sigma', [\alpha']_{\Sigma'}]$. If we let E and E' be exteriors of S and S' respectively, then $\Sigma=E\cup D^3\times S^1$ and $\Sigma'=E'\cup D^3\times S^1$. The elements α and α' are represented by the cores C and C' of the $D^3\times S^1$-parts in Σ and Σ' respectively. Our assumption implies that there are homeomorphisms $h\colon\Sigma'\to\Sigma$ and $f\colon\Sigma\to\Sigma$ such that $f\circ h(C')$ is free homotopic to C. Note that $\Sigma=f\circ h(E')\cup f\circ h(D^3\times S^1)$. Since C and $f\circ h(C')$ are free homotopic in Σ and the dimension of Σ is four, there is an isotopy of Σ taking $f\circ h(C')$ to C. Hence there is a homeomorphism $g\colon f\circ h(E')\to E$, and so we have a homeomorphism $g\circ f\circ h\colon E'\to E$. Since the equivalence class of 2-knots in $S^2\times S^2$ representing ζ is uniquely determined by the knot exterior by Theorem 2.1, $\langle S\rangle=\langle S'\rangle$, that is, ψ is injective. This completes the proof. ■

REMARK. If π is trivial, then there is exactly one equivalence class of 2-knots representing ζ in $S^2\times S^2$ whose knot group is trivial. Moreover there is exactly one equivalence class of such 2-knots via topological ambient isotopies $S^2\times S^2$. See [12].

Let M be a smooth homology 3-sphere and $M^\circ=M-intB^3$ be a punctured copy of M. Then a 4-manifold obtained by gluing $S^2\times D^2$ to $M^\circ\times S^1$ along $S^2\times S^1$ is a smooth homology 4-sphere, which is called *a spin of M*. Note that for a fixed M, there are, up to homeomorphism, at most two spins of M. One is a homology 4-sphere obtained by using the trivial framing, and the other is a homology 4-sphere obtained by using the nontrivial framing τ. In particular, we call the former the untwisted spin of M or the untwisted spun homology 3-sphere M, which is denoted by *spunM*. We call the latter the twisted spin of M or the twisted spun homology 3-sphere M, which is denoted by $\widetilde{spun}M$. Since $\tau|S^2\vee S^1$ is the identity and τ induces the identity automorphism of $\pi_1(S^2\times S^1)$, note that there is a nutural isomorphism between the fundamental group of a spin of M and that of M.

EXAMPLE 3.1. For an aspherical homology 3-sphere M, $\widetilde{spun}M$ and *spunM* are distinct homology 4-spheres with isomorphic π. In fact, these have the same 3-skelton but distinct equivariant intersection forms [11]. Let M be an aspherical homology 3-sphere whose fundamental group has weight one, for example the Brieskorn homology 3-sphere $\Sigma(p, q, r)$ except $\Sigma(2, 3, 5)$. By Lemma 3.1, we can construct two 2-knots S_α and S'_β in $S^2\times S^2$ representing ζ from *spunM* with a weight element α and $\widetilde{spunM}$ with a weight element β respectively. Since *spunM* and $\widetilde{spunM}$ are not homeomorphic, two 2-knots S_α and S'_β are not equivalent for any weight element $\alpha\in\pi_1 spunM$ and any weight element $\beta\in\pi_1 \widetilde{spunM}$. Therefore, in particular, there exist distinct 2-knots S and S' in $S^2\times S^2$ representing ζ with the same knot groups and with the same meridians through a natural isomorphism between their knot groups.

Corollary 3.1. *There exist infinitely many distinct 2-knots in $S^2\times S^2$ representing ζ with the same knot group.*

Proof: Let p,q and r be pairwise coprime integers ≥ 2 except $\{p, q, r\}=\{2, 3, 5\}$. Let Σ denote a spin of the Brieskorn homology 3-sphere $\Sigma(p, q, r)$. By [10], there are infinitely many algebraically distinct weight elements for $\pi_1\Sigma(p,q,r)$. Here we say that two elements $\alpha,\beta\in\pi_1\Sigma(p,q,r)$ are algebraically distinct, if there is no automorphism $\varphi\colon \pi_1\Sigma(p,q,r)\to\pi_1\Sigma(p,q,r)$ such that $\beta=\varphi(\alpha^{\pm1})$. Hence there are infinitely many geometrically distinct weight elements for $\pi_1\Sigma\cong\pi_1\Sigma(p,q,r)$. Thus Theorem 3.1 implies the required result. ■

In contrast with Corollary 3.1, we have the following in the case of finite knot groups.

Theorem 3.2. *Let p be a nonzero integer. Let π be a finite group isomorphic to the knot group of a 2-knot representing $p\zeta$. Then the set $\{\langle S\rangle$; S is a 2-knot in $S^2\times S^2$ representing $p\zeta$ whose knot group is isomorphic to $\pi\}$ is finite.*

Proof: Let S be a 2-knot in $S^2\times S^2$ representing $p\zeta$ whose knot group is isomorphic to π. Then since the exterior E of S looks like $S^2\times D^2$ $\boldsymbol{Q}$-homologically, the manifold V obtained by gluing $D^3\times S^1$ to E along $S^2\times S^1$ is a $\boldsymbol{Q}$-homology 4-sphere with $\pi_1V\cong\pi_1E\cong\pi$. Then by the finiteness of π, there are only finitely many geometric equivalence classes of weight elements for π_1V. Moreover, by [4] there are only finitely many homeomorphism types of $\boldsymbol{Q}$-homology 4-spheres whose fundamental group are isomorphic to π. Hence it follows from the proof of Theorem 3.1 that there are only finitely many homeomorphism types of exteriors of 2-knots whose knot group are isomorphic to π. Therefore, noting that there are at most two equivalence classes of 2-knots with a given exterior [2], we get the required result. ■

We can construct a smooth 2-knot in $S^2\times S^2$ from a smooth 2-knot in S^4. Let K be a smooth 2-knot in S^4 and C a smoothly embedded circle in S^4 disjoint from K. Then we may assume that C is standardly embedded in S^4 up to ambient isotopy. So surgery on C in S^4 gives a smooth 2-knot S in $S^2\times S^2$ and $\pi_1(S^2\times S^2-S)$ is isomorphic to $\pi_1(S^4-K)/H$, where H is the normal closure generated by the element represented by C in $\pi_1(S^4-K)$. If C is homologous in S^4-K to the p-th power of a meridian of K, then S represents $p\zeta$. See [12].

EXAMPLE 3.2. Let O be the trivial 2-knot in S^4 and C a smooth circle in S^4-O representing the p-th power of a meridian of O. Then the smooth 2-knot S in $S^2\times S^2$ obtained from O and C represents $p\zeta$ and has the knot group $\pi_1(S^2\times S^2-S)\cong \boldsymbol{Z}/|p|\boldsymbol{Z}$.

EXAMPLE 3.3. Let G be the binary dodecahedral group, which is of order 120. From the 5-twist spun 2-knot of the trefoil, we can construct a smooth 2-knot in $S^2\times S^2$ representing $p\zeta$ whose knot group is isomorphic to $G\times \boldsymbol{Z}/|p|\boldsymbol{Z}$ for each p. In particular, when $p=1$, there are at least two distinct smooth 2-2-knot in $S^2\times S^2$ representing ζ whose knot group are isomorphic to G. See [13].

We conclude this section with the following proposition.

Proposition 3.2. *Let K be a smooth 2-knot in S^4 and C a smooth circle in S^4 disjoint from K. Let H be the normal closure generated by the element represented by C in $\pi_1(S^4-K)$. If C is homologous in S^4-K to a meridian of K, then there exists a smooth homology 4-sphere Σ with $\pi_1\Sigma\cong\pi_1(S^4-K)/H$.*

Proof: There is a smooth 2-knot S in $S^2\times S^2$ representing ζ with $\pi_1(S^2\times S^2-S)\cong\pi_1(S^4-K)/H$. Since $\pi_1\Sigma(S)\cong\pi_1(S^2\times S^2-S)$, the resultant manifold $\Sigma(S)$ obtained by gurgery on S is the required homology 4-sphere. ■

4. Spun homology 3-spheres obtained by surgery on 2-knots in $S^2\times S^2$

Let Σ be a homology 4-sphere such that $\pi_1\Sigma$ has weight one. If there are algebraically (or geometrically) distinct weight elements for $\pi_1\Sigma$, then by Theorem 3.1, there are distinct 2-knots in $S^2\times S^2$ representing ζ such that the homology 4-sphere obtained by surgery on these 2-knots are homeomorphic to the original homology 4-sphere Σ. In this section, we concern ourself with untwisted spun homology 3-spheres as homology 4-spheres.

Let M be a smooth homology 3-sphere whose fundamental group $\pi_1 M$ has weight one and let α be a weight element for $\pi_1 M$. (From now on we may not distinguish notationally between an element of π_1 and a circle representing it).) Then surgery on αt in $M\times S^1$ gives a 2-knot K_α in S^4 with exterior X_α equal to $M\times S^1$ minus an open tubular neighborhood of αt and $\alpha t\in\pi_1 X_\alpha\cong\pi_1(M\times S^1)$ is a meridian of K_α. Here t is the element of $\pi_1(M\times S^1)\cong\pi_1 M\times \boldsymbol{Z}$ represented by $\{*\}\times S^1$. (See [10]. We do not know whether K_α is smoothable or not, for the 4-dimensional Poincaré conjecture has not been solved in the DIFF category yet.)

Now since $t\in\pi_1X_\alpha$ is homologous in X_α to a meridian αt of K_α, surgery on t in S^4 gives a 2-knot S_α in $S^2\times S^2$ representing ζ with knot group isomorphic to π_1M and with meridian α. In particular, if K_α is smooth, then S_α is also smooth. Let E_α denote the exterior of S_α. Then E_α is homeomorphic to $(X_\alpha-intN(t))\cup_{\partial N(t)} S^2\times D^2$, where $N(t)$ is a closed tubular neighborhood of t. Hence the homology 4-sphere $\Sigma(S_\alpha)$ obtained by surgery on S_α is homeomorphic to

$$\begin{aligned}
&D^3\times S^1\cup_{\partial X}(X_\alpha-intN(t))\cup_{\partial N(t)} S^2\times D^2\\
&=D^3\times S^1\cup_{\partial N(\alpha t)}(M\times S^1-(intN(\alpha t)\amalg intN(t)))\cup_{\partial N(t)} S^2\times D^2\\
&=(M\times S^1-intN(t))\cup_{\partial N(t)} S^2\times D^2\\
&=M^\circ\times S^1\cup_{\partial M^\circ\times S^1} S^2\times D^2\\
&=spunM.
\end{aligned}$$

Thus we have the following:

Theorem 4.1. *Let M be a homology 3-sphere suth that π_1M has weight one and let α be a weight element for π_1M. Then there exisis a 2-knot S in $S^2\times S^2$ representing ζ with meridian α such thai $\Sigma(S)$ is homeomorphic to spunM.*

REMARK. This theorem follows also from Lemma 3.1, but the proof of this theorem in the manner as above is useful for proving Propositions 4.1 and 4.2.

Moreover, we have the following propositions.

Proposition 4.1. *There exist arbitrarily many distinct smooth 2-knots in $S^2\times S^2$ representing ζ with isomorphic knot group.*

Proof: Let $M=\Sigma(p,q,r)$ and let α be a weight element for π_1M. Then the exterior X_α of a 2-knot K_α in S^4 is fibered over S^1 with fiber M°. If some power of α lies in the center $Z(\pi_1M)$ of π_1M, then the monodromy of X_α fibered overe S^1 has finite order, and so by [9], K_α is a smooth 2-knot in S^4.

By picking p,q and r large, we get arbitrarily many algebraically distinct weight elements $\{\alpha_i\}_i$ such that some power of α_i lies in the center $Z(\pi_1M)$ for each i. Hence there are arbitrarily many distinct smooth 2-knots K_{α_i} in S^4 with knot group isomorphic to $\pi_1M\times \boldsymbol{Z}$ and with meridian $\alpha_i t$, and so by performing surgery on t, there are arbitrarily many distinct smooth 2-knots S_{α_i} in $S^2\times S^2$ representing ζ with knot group isomorphic to π_1M. ■

Considering a fixed $M=\Sigma(p,q,r)$ in the proof of Proposition 4.1, we have the following.

Proposition 4.2. *For any pairwise coprime integers p, q and r, there exist many smooth 2-knots S in $S^2\times S^2$ representing ζ such that $\Sigma(S)$ is diffeomorphic to spun $\Sigma(p, q, r)$.*

Proof: Let α be a weight element for $\pi_1\Sigma(p, q, r)$ such that some power of α lies in $Z(\pi_1\Sigma(p, q, r))$. Then since K_α is a smooth 2-knot, the exterior E_α of a smooth 2-knot S_α in $S^2\times S^2$ is diffeomorphic to $(X_\alpha - intN(t))\cup_{\partial N(t)} S^2\times D^2$. Hence $\Sigma(S_\alpha)$ is diffeomorphic to $spun\Sigma(p, q, r)$. ■

Proposition 4.3. *Exclude $\{p, q, r\}=\{2, 3, 5\}$. For a given $\{p, q, r\}$, there exist infinitely many 2-knots S in $S^2\times S^2$ representing ζ such that $\Sigma(S)$ is homeomorphic to $spun\Sigma(p, q, r)$.*

Proof: There are infinitely many algebraically distinct weight elements for $\pi\Sigma_1(p, q, r)$ [10]. By Theorem 4.1, we get the required result. ■

REMARK. (1). Since $\pi_1(S^2\times S^2-S)$ is isomorphic to $\pi_1\Sigma(S)$, Corollary 3.1 follows also from Proposition 4.3.
(2). Exclude $\{p, q, r\}=\{2, 3, 5\}$. By making use of Lemma 3.1, it also follows that there exist infinitely many 2-knots S in $S^2\times S^2$ representing ζ such that $\Sigma(S)$ is homeomorphic to $\widetilde{spun}\Sigma(p, q, r)$.

5. Concluding remark

We have the following:

Proposition 5.1. *For any smooth 2-knot S in $S^2\times S^2$ representing ζ, there is a smooth action of $Z/2Z$ on S^5 whose fixed point set is diffeomorphic to the smooth homology 4-sphere $\Sigma(S)$.*

For a smooth 2-knot S in $S^2\times S^2=S^2\times\partial D^3$ representing ζ, let W_S denote the smooth 5-manifold obtained by attaching the 3-handle $H^{(3)}=D^3\times D^2$ to $S^2\times D^3$ along S: $W_S=S^2\times D^3\cup_f H^{(3)}$. Then the boundary ∂W_S is the smooth smooth homology 4-sphere $\Sigma(S)$. We can show the following lemma in the same manner as [8].

Lemma 5.1. *$W_S\times I$ is diffeomorphic to the 6-ball D^6.*

Proof: Let S_0 be the attaching sphere of the 3-handle $H^{(3)}$, $\partial D^3\times\{0\}$. Let $N(S)$ and $N(S_0)$ be closed tubular neighborhoods of S and S_0 in $S^2\times\partial D^3$ and $\partial H^{(3)}$ respectively. We consider $N(S)\times I\subset(S^2\times D^3)\times I\subset W_S\times I$ and $N(S_0)\times I\subset H^{(3)}\times I\subset W_S\times I$. Then $N(S)\times I$ and $N(S_0)\times I$ can be regarded as closed tubular neighborhoods of S and S_0 in $\partial((S^2\times D^3)\times I)$ and $\partial(H^{(3)}\times I)$ respectively. Defining a diffeomorphism $F:N(S_0)\times I\to N(S)\times I$ by $F(x, t)=(f(x), t)$ for any $(x, t)\in N(S_0)\times I$,

$$W_S\times I\cong(S^2\times D^3)\times I\cup_F H^{(3)}\times I.$$

Meanwhile, $(S^2\times D^3)\times I\cong S^2\times D^4$ and $H^{(3)}\times I\cong D^6$, so that $S\subset\partial((S^2\times D^3)\times I)\cong S^2\times S^3$ and $S_0\subset\partial(H^{(3)}\times I)\cong S^5$. Hence there is a diffeomorphism g:

$\partial((S^2\times D^3)\times I)\to S^2\times S^3$ taking S onto the standard 2-sphere $S^2\times\{*\}$ in $S^2\times S^3$.

By modifying F via g in necessary, $W_S\times I$ is diffeomorphic to

$$\begin{aligned}&S^2\times D^4\cup_F D^3\times D^3\\&\quad\cong S^2\times D^4\cup_G D^3\times D^3\,,\end{aligned}$$

where $G\colon \partial D^3\times D^3\to S^2\times\partial D^4$ is a smooth embedding which takes the standard 2-sphere $\partial D^3\times\{0\}\subset\partial(D^3\times D^3)\cong S^5$ onto the standard 2-sphere $S^2\times\{*\}\subset S^2\times\partial D^4\cong S^2\times S^3$. Therefore $W_S\times I$ is diffeomorphic to D^6. ■

REMARK. By Lemma 5.1, W_S is contractible, and so the smooth homology 4-sphere $\Sigma(S)$ bounds a contractible smooth 5-manifold. In general, for any smooth homology 4-sphere Σ, one can construct a contractible smooth 5-manifold that Σ bounds by the Pontrjagin-Thom construction and surgeries [6].

Proof of Proposition 5.1: By Lemma 5.1, the double $2W_S$ of W_S is diffeomorphic to $\partial(W_S\times I)\cong\partial D^6\cong S^5$. Let φ: $S^5\to S^5$ be the mapping which switches copies of W_S. Then φ is a diffeomorphism of period 2 and keeps $\partial W_S=\Sigma(S)$ fixed. ■

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