

Title	Pseudo diagrams of knots, links and spatial graphs
Author(s)	Hanaki, Ryo
Citation	Osaka Journal of Mathematics. 2010, 47(3), p. 863-883
Version Type	VoR
URL	<a href="https://doi.org/10.18910/8303">https://doi.org/10.18910/8303</a>
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## PSEUDO DIAGRAMS OF KNOTS, LINKS AND SPATIAL GRAPHS

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(Received February 9, 2009, revised May 14, 2009)

### Abstract

A pseudo diagram of a spatial graph is a spatial graph projection on the 2-sphere with over/under information at some of the double points. We introduce the trivializing (resp. knotting) number of a spatial graph projection by using its pseudo diagrams as the minimum number of the crossings whose over/under information lead the triviality (resp. nontriviality) of the spatial graph. We determine the set of non-negative integers which can be realized by the trivializing (resp. knotting) numbers of knot and link projections, and characterize the projections which have a specific value of the trivializing (resp. knotting) number.

### 1. Introduction

Throughout this paper we work in the piecewise linear category. Let  $G$  be a finite graph which does not have degree zero or one vertices. We consider  $G$  as a topological space in the usual way. Let  $f$  be an embedding of  $G$  into the 3-sphere  $\mathbf{S}^3$ . Then  $f$  is called a *spatial embedding* of  $G$  and the image  $\mathcal{G} = f(G)$  is called a *spatial graph*. In particular,  $f(G)$  is called a *knot* if  $G$  is homeomorphic to a circle and an  *$r$ -component link* if  $G$  is homeomorphic to disjoint  $r$  circles. In this paper, we say that two spatial graphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are said to be *ambient isotopic* if there exists an orientation-preserving self-homeomorphism  $h$  on  $\mathbf{S}^3$  such that  $h(\mathcal{G}_1) = \mathcal{G}_2$ . A graph  $G$  is said to be *planar* if there exists an embedding of  $G$  into the 2-sphere  $\mathbf{S}^2$ . A spatial graph  $\mathcal{G}$  is said to be *trivial* (or *unknotted*) if  $\mathcal{G}$  is ambient isotopic to a graph in  $\mathbf{S}^2$  where we consider  $\mathbf{S}^2$  as a subspace of  $\mathbf{S}^3$ . Thus only planar graphs have trivial spatial graphs. We consider only planar graphs from now on. It is known in [11] that a trivial spatial graph of  $G$  is unique up to ambient isotopy in  $\mathbf{S}^3$ .

A continuous map  $\varphi: G \rightarrow \mathbf{S}^2$  is called a *regular projection*, or simply a *projection*, of  $G$  if the multiple points of  $\varphi$  are only finitely many transversal double points away from the vertices. Then  $P = \varphi(G)$  is also called a *projection*. A *diagram*  $D$  is a projection  $P$  with over/under information at the every double point. Then we say that  $D$  is *obtained from*  $P$  and  $P$  is a *projection of*  $D$ . A diagram  $D$  uniquely represents a spatial graph up to ambient isotopy. Let  $\mathcal{G}$  be a spatial graph represented by  $D$  and  $\mathcal{G}'$  a spatial graph ambient isotopic to  $\mathcal{G}$ . Then we also say that  $P$  is a *pro-*



Fig. 1.1. Projection and diagrams obtained from it.

jection of  $\mathcal{G}$ . A double point with over/under information and a double point without over/under information are called a *crossing* and a *pre-crossing*, respectively. Thus a diagram has crossings and has no pre-crossings, and a projection has pre-crossings and has no crossings.

A projection  $P$  is said to be *trivial* if any diagram obtained from  $P$  represents a trivial spatial graph. On the other hand, a projection  $P$  is said to be *knotted* [22] if any diagram obtained from  $P$  represents a nontrivial spatial graph. Moreover, the following definitions for a projection  $P$  are known. A projection  $P$  is said to be *identifiable* [9] if every diagram obtained from  $P$  yields a unique labeled spatial graph, and *completely distinguishable* [14] if any two different diagrams obtained from  $P$  represent different labeled spatial graphs. Nikkuni showed in [13, Theorem 1.2] that a projection  $P$  is identifiable if and only if  $P$  is trivial.

Let  $\mathcal{G}$  be a spatial graph and  $P$  a projection of  $\mathcal{G}$ . Then we ask the following question.

QUESTION 1.1. Can we determine from  $P$  whether the original spatial graph  $\mathcal{G}$  is trivial or knotted?

If  $P$  is neither trivial nor knotted, then the (non)triviality of  $\mathcal{G}$  cannot be determined from  $P$ . For example, let  $P$  be a projection of a circle with 3 pre-crossings as illustrated in Fig. 1.1. Then we have  $2^3$  diagrams obtained from  $P$ . Two diagrams represent a nontrivial knot and six diagrams represent a trivial knot.

It is well known in knot theory that for any projection  $P$  of disjoint circles there exists a diagram  $D$  obtained from  $P$  such that  $D$  represents a trivial link. Namely  $P$  never admits a knotted projection. However it is known in [22] that there exists a knotted projection of a planar graph. For example, let  $\mathcal{G}$  be a spatial graph of the octahedron graph and  $P$  a projection of  $\mathcal{G}$  as illustrated in Fig. 1.2. Then we can see that any diagram obtained from  $P$  contains a diagram of a Hopf link. Namely  $P$  is knotted. However there exists a projection of  $\mathcal{G}$  which is neither trivial nor knotted. In general, we have the following proposition.

**Proposition 1.2.** *For any spatial graph  $\mathcal{G}$  of a graph  $G$ , there exists a projection  $P$  of  $\mathcal{G}$  such that  $P$  is neither trivial nor knotted.*

We give a proof of Proposition 1.2 in Section 2. Then it is natural to ask the following question.

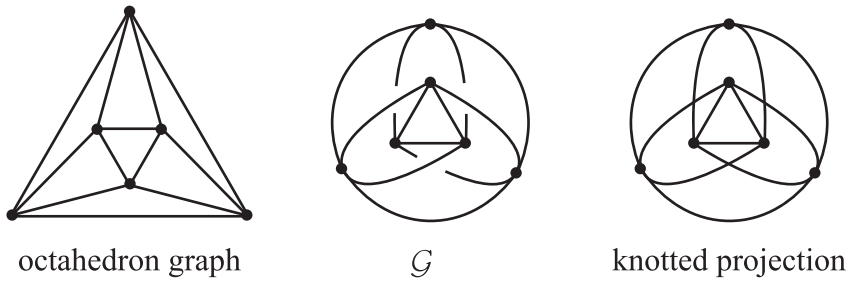


Fig. 1.2. Octahedron graph and a knotted projection of it.

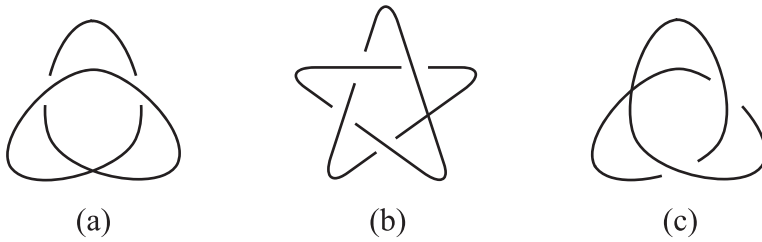


Fig. 1.3. Pseudo diagrams.

QUESTION 1.3. Let  $\mathcal{G}$  be a spatial graph and  $P$  a projection of  $\mathcal{G}$ . Which pre-crossings of  $P$  and the over/under information lead the (non)triviality of  $\mathcal{G}$ ?

Now we introduce the notion of a pseudo diagram as a generalization of a projection and a diagram. Let  $P$  be a projection of a graph  $G$ . A *pseudo diagram*  $Q$  of  $G$  is a projection  $P$  with over/under information at some of the pre-crossings. Then we say that  $Q$  is obtained from  $P$  and  $P$  is a projection of  $Q$ . Thus a pseudo diagram  $Q$  has crossings and pre-crossings. Here we allow the possibility that a pseudo diagram has no crossings or has no pre-crossings, that is, a pseudo diagram is possibly a projection or a diagram. We denote the number of crossings and pre-crossings of  $Q$  by  $c(Q)$  and  $p(Q)$ , respectively. For a pseudo diagram  $Q$ , by giving over/under information to some of the pre-crossings, we can get another (possibly same) pseudo diagram  $Q'$ . Then we say that  $Q'$  is obtained from  $Q$ .

We say that a pseudo diagram  $Q$  is *trivial* if for any diagram obtained from  $Q$  represents a trivial spatial graph. On the other hand, we say that  $Q$  is *knotted* if any diagram obtained from  $Q$  represents a nontrivial spatial graph. For example, in Fig. 1.3, a pseudo diagram (a) is trivial, (b) is knotted, and (c) is neither trivial nor knotted.

Let  $P$  be a projection of a graph  $G$ . Then we define the *trivializing number* (resp. *knotted number*) of  $P$  by the minimum of  $c(Q)$ , where  $Q$  varies over all trivial (resp. knotted) pseudo diagrams obtained from  $P$ , and denote it by  $tr(P)$  (resp.  $kn(P)$ ). Note that there does not exist a knotted (resp. trivial) pseudo diagram obtained from  $P$  if and only

if  $tr(P) = 0$  (resp.  $kn(P) = 0$ ), namely  $P$  is trivial (resp. knotted). In this case we define that  $kn(P) = \infty$  (resp.  $tr(P) = \infty$ ). Note that for any graph  $G$  there exists a projection  $P$  of  $G$  with  $kn(P) = \infty$ . For example,  $P$  is an image of a planar embedding of  $G$ . We also note that for a certain graph  $G$  there exists a projection  $P$  of  $G$  with  $tr(P) = \infty$  as in Fig. 1.2.

We remark here that the observation of DNA knots was an opportunity of this research, namely we cannot determine over/under information at some of the crossings in some photos of DNA knots. DNA knots barely become visual objects by examining the protein-coated one by electromicroscope. However there are still cases in which it is hard to confirm the over/under information of some of the crossings. If we can know the (non-)triviality of a knot without checking every over/under information of crossings, then it may give a reasonable way to detect the (non-)triviality of a DNA knot. In addition, it is known that there exists an enzyme, called topoisomerase, which plays a role of crossing change. The research of pseudo diagrams may provide an effective method to change a given DNA knot to a trivial (nontrivial) one. See [7, 4, 12] on DNA knots.

We start from two questions on the trivializing number and the knotting number of projections of a circle.

**QUESTION 1.4.** For any non-negative integer  $n$ , does there exist a projection  $P$  of a circle with  $tr(P) = n$ ?

**QUESTION 1.5.** For any non-negative integer  $n$ , does there exist a projection  $P$  of a circle with  $kn(P) = n$ ?

We have the following theorem and propositions as answers to Questions 1.4 and 1.5.

**Theorem 1.6.** *For any projection  $P$  of a circle, the trivializing number of  $P$  is even.*

**Proposition 1.7.** *For any non-negative even number  $n$ , there exists a projection  $P$  of a circle with  $tr(P) = n$ .*

**Proposition 1.8.** *There does not exist a projection of a circle whose knotting number is less than three. For any positive integer  $n \geq 3$ , there exists a projection  $P$  of a circle with  $kn(P) = n$ .*

We give proofs of Theorem 1.6 and Proposition 1.7 in Section 3 and a proof of Proposition 1.8 in Section 4. Moreover we see from the following proposition that there are no relations between trivializing number and knotting number in general.

**Proposition 1.9.** *For any non-negative even number  $n$  and any positive integer  $l \geq 3$ , there exists a projection  $P$  of a circle with  $tr(P) = n$  and  $kn(P) = l$ .*

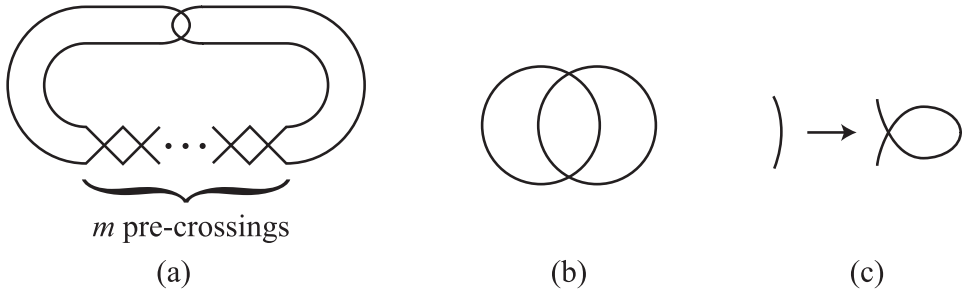


Fig. 1.4.

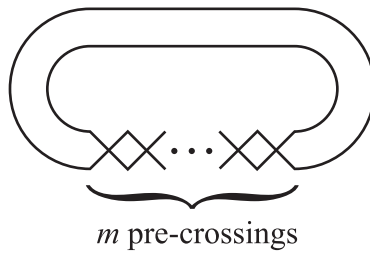


Fig. 1.5.

We give a proof of Proposition 1.9 in Section 5. In addition, we have the following theorems.

**Theorem 1.10.** *Let  $P$  be a projection of disjoint circles. Then  $tr(P) = 2$  if and only if  $P$  is obtained from one of the projections as illustrated in Fig. 1.4 (a) and (b) where  $m$  is a positive integer by possibly adding trivial circles and by a series of replacing a sub-arc of  $P$  as illustrated in Fig. 1.4 (c) where a trivial circle means an embedding of a circle into  $S^2$  which does not intersect any other component of the projection.*

We see that for any projection  $P$  of disjoint circles,  $tr(P) \leq p(P)$  by the definitions. We also see that for any projection  $P$  with  $kn(P) \neq \infty$ ,  $kn(P) \leq p(P)$  by the definitions. Then we have the following theorems.

**Theorem 1.11.** *Let  $P$  be a projection of a circle with at least one pre-crossing. Then it holds that  $tr(P) \leq p(P) - 1$ . The equality holds if and only if  $P$  is one of the projections as illustrated in Fig. 1.5 where  $m$  is a positive odd integer.*

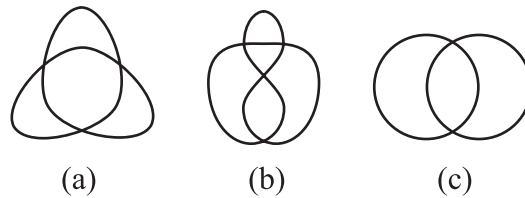


Fig. 1.6. Projections  $P$  of a circle with  $kn(P) = p(P)$ .

**Theorem 1.12.** *Let  $P$  be a projection of  $n$  disjoint circles. Let  $C_1, C_2, \dots, C_n$  be the image of the circles of  $P$ . Then  $tr(P) = p(P)$  if and only if each of  $C_1, C_2, \dots, C_n$  has no self-pre-crossings where a self-pre-crossing is a pre-crossing whose preimage is contained in a circle.*

**Theorem 1.13.** *Let  $P$  be a projection of disjoint circles. Then  $kn(P) = p(P)$  if and only if  $P$  is obtained from one of the projections as illustrated in Fig. 1.6 by possibly adding trivial circles.*

We give proofs of Theorems 1.10, 1.11 and 1.12 in Section 3 and a proof of Theorem 1.13 in Section 4.

Let  $Q$  be a pseudo diagram of a circle. By giving an orientation to the circle, we can regard  $Q$  as a *singular knot*, namely an immersion of a circle into  $\mathbf{S}^3$  whose multiple points are only finitely many transversal double points of arcs spanning a sufficiently small flat plane. We consider a singular knot up to ambient isotopy preserving the flatness at each double point. A singular knot  $K$  is said to be *trivial* if  $K$  is deformed by ambient isotopy preserving the flatness at each double point to a singular knot in  $\mathbf{S}^2$ . See [17] for details. We can also regard a singular knot as a spatial 4-valent graph up to *rigid vertex isotopy*, see [10, 28]. Then we have the following.

**Theorem 1.14.** *Let  $Q$  be a trivial pseudo diagram of a circle. Let  $K_Q$  be a singular knot obtained from  $Q$  by giving an orientation to the circle. Then  $K_Q$  is trivial.*

We give a proof of Theorem 1.14 in Section 3. In Section 6 we give an application of the trivializing number and the knotting number.

## 2. Fundamental property

First of all, we prove Proposition 1.2.

**Proof of Proposition 1.2.** First we show that  $\mathcal{G}$  has a projection which is not knotted. For any spatial graph  $\mathcal{G}$  we can transform  $\mathcal{G}$  into a trivial spatial graph by crossing changes and ambient isotopies. Thus any spatial graph can be expressed as a band sum

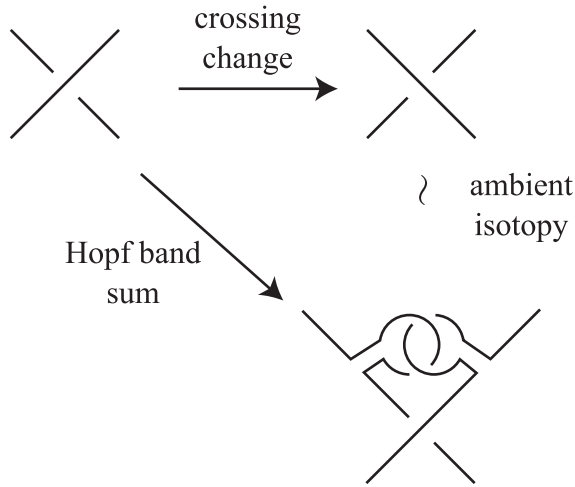


Fig. 2.1.

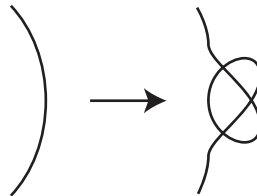


Fig. 2.2.

of a trivial spatial graph and Hopf links, see Fig. 2.1. See [19, 29, 24] for details. Then we can get a diagram  $D$  of  $\mathcal{G}$  which is identical with a planar embedding of  $G$  except the Hopf bands. Let  $P$  be the projection of  $D$ . Then  $P$  is also a projection of a band sum of a trivial spatial graph and trivial 2-component links which is itself a trivial spatial graph. Therefore  $P$  is not knotted.

If  $P$  is not trivial then  $P$  is neither trivial nor knotted. Suppose that  $P$  is trivial. Let  $l$  be a simple arc in  $P$  which belongs to the image of a cycle of  $P$ . Let  $P'$  be a projection obtained from  $P$  by applying the local deformation to  $l$  as illustrated in Fig. 2.2. Then  $P'$  is also a projection of  $\mathcal{G}$  which is neither trivial nor knotted.  $\square$

In the rest of this section, we show fundamental properties of the trivializing number and the knotting number which are needed later. Let  $P$  be a projection of a circle. We say that a simple closed curve  $S$  in  $\mathbf{S}^2$  is a *decomposing circle* of  $P$  if the intersection of  $P$  and  $S$  is the set of just two transversal double points. See Fig. 2.3.



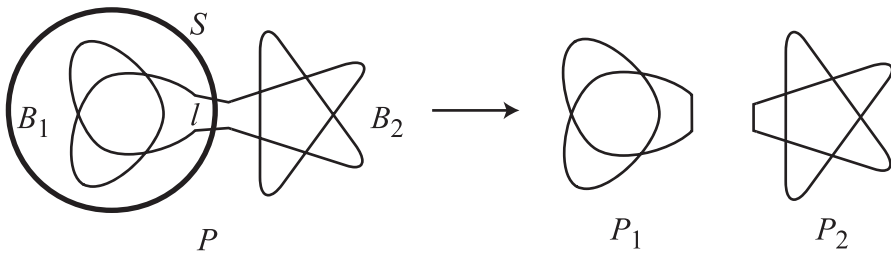


Fig. 2.3. Decomposing circle.

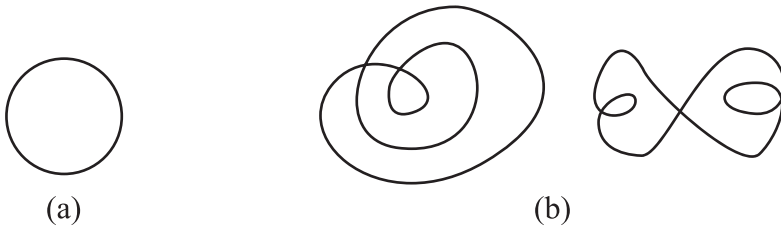


Fig. 2.4. Projections  $P$  of a circle with  $tr(P) = 0$ .

**Proposition 2.1.** *Let  $P$  be a projection of a circle and  $S$  a decomposing circle of  $P$ . Let  $\{q_1, q_2\} = P \cap S$ . Let  $B_1$  and  $B_2$  be the disks such that  $B_1 \cup B_2 = \mathbf{S}^2$  and  $B_1 \cap B_2 = S$ . Let  $l$  be one of the two arcs on  $S$  joining  $q_1$  and  $q_2$ . Let  $P_1 = (P \cap B_1) \cup l$  and  $P_2 = (P \cap B_2) \cup l$ . Then  $tr(P) = tr(P_1) + tr(P_2)$  and  $kn(P) = \min\{kn(P_1), kn(P_2)\}$ .*

*Proof.* Let  $Q$  be a pseudo diagram obtained from  $P$ . Let  $Q_1$  (resp.  $Q_2$ ) be the pseudo diagram obtained from  $P_1$  (resp.  $P_2$ ) corresponding to  $Q$ . Then  $Q$  is trivial if and only if both  $Q_1$  and  $Q_2$  are trivial. This implies that  $tr(P) = tr(P_1) + tr(P_2)$ . We also see that  $Q$  is knotted if and only if either  $Q_1$  or  $Q_2$  is knotted. This implies that  $kn(P) = \min\{kn(P_1), kn(P_2)\}$ . □

The following proposition is shown in [5, 15, 20, 21] as a characterization of trivializing number zero projections of disjoint circles.

**Proposition 2.2** ([5, 15, 20, 21]). *Let  $P$  be a projection of disjoint circles. Then  $tr(P) = 0$  if and only if  $P$  is obtained from the projection in Fig. 2.4 (a) by possibly adding trivial circles and by a series of replacing a sub-arc of  $P$  as illustrated in Fig. 1.4 (c).*

As an example we illustrate a projection of two circles whose trivializing number equals to zero in Fig. 2.4 (b).

Let  $P$  be a projection of disjoint circles. A pre-crossing  $p$  of a projection  $P$  is said to be *nugatory* if the number of connected components of  $P - p$  is greater than that of  $P$ . A crossing  $c$  of a diagram  $D$  obtained from a projection  $P$  is also said to be *nugatory* if the pre-crossing corresponding to  $c$  is nugatory in  $P$ . Then we can rephrase that  $P$  is a projection of disjoint circles with  $tr(P) = 0$  if and only if all pre-crossings of  $P$  are nugatory. A projection  $P$  (resp. a diagram  $D$ ) is said to be *reduced* if  $P$  (resp.  $D$ ) has no nugatory pre-crossings (resp. no nugatory crossings). Then the following propositions hold.

**Proposition 2.3.** *Let  $P$  be a projection of disjoint circles with nugatory pre-crossings and  $tr(P) = k$ . Let  $p$  be a nugatory pre-crossing of  $P$ . Let  $Q$  be a trivial pseudo diagram obtained from  $P$  with  $k$  crossings. Then  $p$  is a pre-crossing of  $Q$ .*

*Proof.* Suppose that  $p$  is a crossing in  $Q$ . By forgetting the over/under information of  $p$ , we can get another trivial pseudo diagram. Then we have  $tr(P) < k$ . This is a contradiction.  $\square$

Similarly we have the following proposition.

**Proposition 2.4.** *Let  $P$  be a projection of disjoint circles with nugatory pre-crossings and  $kn(P) = k$ . Let  $p$  be a nugatory pre-crossing of  $P$ . Let  $Q$  be a knotted pseudo diagram obtained from  $P$  with  $k$  crossings. Then  $p$  is a pre-crossing of  $Q$ .*

### 3. Trivializing number

In this section, we study trivializing number. First we prove Theorem 1.6 and Proposition 1.7.

For a pseudo diagram of a circle, we recall a chord diagram of pre-crossings to prove Theorem 1.6. Let  $Q$  be a pseudo diagram of a circle with  $n$  pre-crossings. A *chord diagram* of  $Q$  is a circle with  $n$  chords marked on it by dashed line segment, where the preimage of each pre-crossing is connected by a chord. We denote it by  $CD_Q$ . For example, let  $Q$  be a pseudo diagram (a) in Fig. 3.1. Then a chord diagram (b) in Fig. 3.1 is  $CD_Q$ . Note that for each chord of a chord diagram of a projection, each of the two arcs in the circle bounded by the end points of the chord contains even number of end points of the other chords. Moreover, a realization problem of a chord diagram by a projection is known in [8].

To prove Theorem 1.6, we regard a pseudo diagram of a circle as a singular knot by giving an orientation to the circle and consider the Vassiliev invariant. Let  $v$  be a knot invariant which takes values in an additive group. We can extend  $v$  to singular knots by the Vassiliev skein relation:

$$v(K_{\times}) = v(K_{+}) - v(K_{-})$$

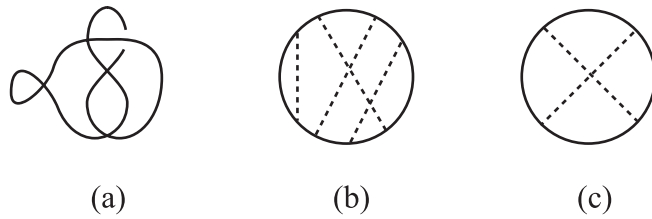


Fig. 3.1. Chord diagram.

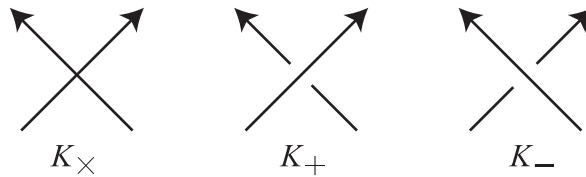


Fig. 3.2.

where  $K_\times$ ,  $K_+$  and  $K_-$  are singular knots which are identical except inside the depicted regions as illustrated in Fig. 3.2. Then  $v$  is called a *Vassiliev invariant of order  $k$*  if  $v(K) = 0$  for any singular knot  $K$  with more than  $k$  double points and there exists a singular knot  $J$  with exactly  $k$  double points such that  $v(J) \neq 0$ . See [27, 2, 3, 17] for Vassiliev invariants. Then the following lemmas hold.

**Lemma 3.1.** *Let  $Q$  be a trivial pseudo diagram of a circle with  $p(Q) > 0$ . Let  $K_Q$  be a singular knot obtained from  $Q$  by giving an orientation to the circle. Then  $v(K_Q) = 0$  where  $v$  is a Vassiliev invariant of oriented knots.*

Proof. It is clear from the definitions of Vassiliev invariants. □

**Lemma 3.2.** *Let  $Q$  be a pseudo diagram of a circle with two pre-crossings such that  $CD_Q$  is (c) in Fig. 3.1. Then  $Q$  is not trivial.*

Proof. Let  $K_Q$  be a singular knot obtained from  $Q$ . Let  $a_2$  be the second coefficient of the Conway polynomial which is extended to singular knots as above. It is well known that  $a_2(K_Q) = 1$ . Thus  $Q$  is not trivial by Lemma 3.1. □

We have the following lemma by applying Lemma 3.2.

**Lemma 3.3.** *Let  $Q$  be a trivial pseudo diagram of a circle. Then  $CD_Q$  contains no sub-chord diagrams as in Fig. 3.1 (c).*

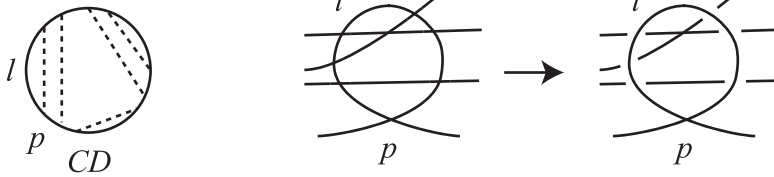


Fig. 3.3.

Proof. Suppose that  $Q$  contains sub-chord diagrams as in Fig. 3.1 (c). Let  $Q'$  be a pseudo diagram obtained from  $Q$  such that  $CD_{Q'}$  is (c) in Fig. 3.1. By Lemma 3.2, a diagram representing nontrivial knot is obtained from  $Q'$ , hence from  $Q$ . This implies that  $Q$  is not trivial. This completes the proof.  $\square$

Proof of Theorem 1.6. Let  $CD$  be a sub-chord diagram of  $CD_P$  with the maximum number of chords over all sub-chord diagrams of  $CD_P$  which do not contain (c) in Fig. 3.1. We show that a trivial pseudo diagram whose chord diagram is  $CD$  is obtained from  $P$ . Let  $p_1$  be a pre-crossing of  $P$  which corresponds to an outer most chord  $c_1$  in  $CD$  and  $l_1$  the sub-arc on  $P$  which corresponds to the outer most arc. By giving over/under information to each pre-crossing on  $l_1$  so that  $l_1$  goes over the others as in Fig. 3.3, we obtain a pseudo diagram  $Q_1$  from  $P$ . Next, let  $p_2$  be a pre-crossing of  $Q_1$  which corresponds to an outer most chord  $c_2$  under forgetting  $c_1$  in  $CD$ , and  $l_2$  the sub-arc on  $Q_1$  which corresponds to the outer most arc. By giving over/under information to each pre-crossing on  $l_2$  so that  $l_2$  goes over the others except  $l_1$ , we obtain a pseudo diagram  $Q_2$  from  $Q_1$ . By repeating this procedure until all of the chords are forgotten, we obtain a pseudo diagram  $Q$  from  $P$ . For any diagram  $D$  obtained from  $Q$ , first we can vanish the crossings on  $l_1$  and the crossing corresponding to  $p_1$ , next we can vanish the crossings on  $l_2$  and the crossing corresponding to  $p_2$ , similarly we can vanish all crossings of  $D$ . Therefore, we see that  $Q$  is trivial. Moreover  $c(Q)$  is even because each  $l_i$  has no self-crossings by the maximality of chords in  $CD$ . Since  $tr(P) = c(Q)$  by Lemma 3.3,  $tr(P)$  is even.  $\square$

Proof of Proposition 1.7. The projection of Fig. 1.5 where  $m = n + 1$  has trivializing number  $n$ .  $\square$

Then we have the following corollary of Theorem 1.6 for projections of  $n$  disjoint circles.

**Corollary 3.4.** *Let  $P$  be a projection of  $n$  disjoint circles. Let  $C_1, C_2, \dots, C_n$  be the images of the circles of  $P$ . Then the following formula holds.*

$$tr(P) = \sum_{1 \leq i < j \leq n} \#(C_i \cap C_j) + \sum_{k=1}^n tr(C_k)$$

where  $\#A$  is the cardinality of a set  $A$ . Therefore,  $tr(P)$  is even.

**Proof.** First we show that

$$tr(P) \geq \sum_{1 \leq i < j \leq n} \#(C_i \cap C_j) + \sum_{k=1}^n tr(C_k).$$

Let  $Q$  be a trivial pseudo diagram obtained from  $P$ . Suppose that there exists a pre-crossing in  $C_i \cap C_j$  ( $i \neq j$ ) such that it is also a pre-crossing of  $Q$ . Then a diagram whose sub-diagram represents a 2-component link with nonzero linking number is obtained from  $Q$ , namely  $Q$  is not trivial. Thus each of the pre-crossings in  $C_i \cap C_j$  is a crossing of  $Q$ . Note that  $\#(C_i \cap C_j)$  is even. Moreover each  $C_k$  ( $1 \leq k \leq n$ ) has to be a trivial pseudo diagram in  $Q$ . This implies that the above inequality holds.

Next we construct a trivial pseudo diagram obtained from  $P$  with  $\sum_{1 \leq i < j \leq n} \#(C_i \cap C_j) + \sum_{k=1}^n tr(C_k)$  crossings. We give over/under information to the pre-crossings in  $C_i \cap C_j$  so that  $C_i$  goes over  $C_j$  for  $i > j$  and some pre-crossings of  $C_k$  so that a pseudo diagram obtained from  $C_k$  is trivial and has  $tr(C_k)$  crossings. Then it is easy to see that the pseudo diagram obtained from  $P$  by the above way is trivial. This completes the proof.  $\square$

In general, we have the following proposition.

**Proposition 3.5.** *Let  $P$  a projection of a graph. Then  $tr(P) \neq 1$ .*

**Proof.** Suppose that there exists a projection  $P$  with  $tr(P) = 1$ . Let  $Q$  be a trivial pseudo diagram obtained from  $P$  with only one crossing  $c$ . Let  $Q'$  be the pseudo diagram obtained from  $Q$  by changing the over/under information of  $c$ . We show that  $Q'$  is trivial. Let  $D$  be a diagram obtained from  $Q'$ . The mirror image diagram of  $D$  is obtained from  $Q$ . Since the mirror image of a trivial spatial graph is also trivial,  $D$  represents a trivial spatial graph. Hence  $Q'$  is trivial. Thus this implies that  $tr(P) = 0$ . This is a contradiction.  $\square$

However, for a certain graph  $G$  there exists a projection  $P$  of  $G$  with  $tr(P) = 3$ . For example, let  $G$  be a graph which is homeomorphic to the disjoint union of a circle and a  $\theta$ -curve as illustrated in the left side of Fig. 3.4. Then there exists a projection  $P$  of  $G$  with  $tr(P) = 3$ , see the right side of Fig. 3.4. Moreover for each  $n \geq 2$  there exists a projection  $P_n$  of  $G$  with  $tr(P_n) = n$ , see Fig. 3.5.

Next we prove Theorem 1.10 that characterizes trivializing number two projections of disjoint circles.

**Proof of Theorem 1.10.** The ‘if’ part is obvious. Let  $P$  be a projection of  $n$  disjoint circles with  $tr(P) = 2$ . Let  $C_1, C_2, \dots, C_n$  be the image of the circles in  $P$ . Suppose that there exist pre-crossings in  $C_i \cap C_j$  ( $i \neq j$ ). In this case, such pre-crossings

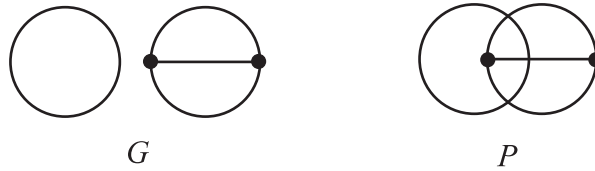


Fig. 3.4.

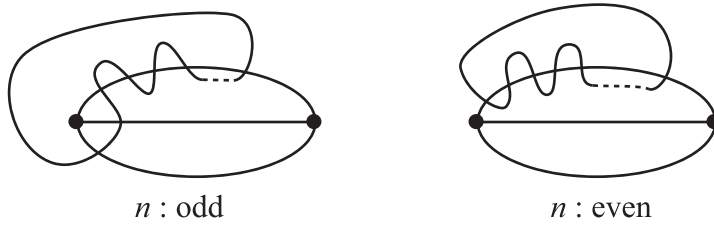


Fig. 3.5.

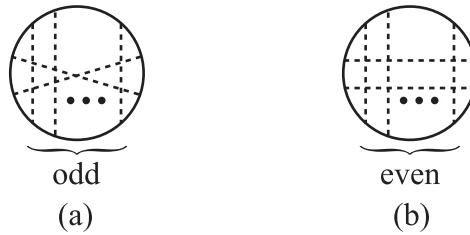


Fig. 3.6.

must be crossings in a trivial pseudo diagram by the same reason as we said in the proof of Corollary 3.4. Since  $tr(P) = 2$ , such pre-crossings belong to the intersection of only one pair of  $C_i$  and  $C_j$  and each  $C_i$  is a trivial projection by Corollary 3.4. Thus  $P$  is a projection obtained from (b) in Fig. 1.4 by adding trivial circles and by a series of replacing a sub-arc of  $P$  as illustrated in Fig. 1.4 (c).

Suppose that  $C_i \cap C_j = \emptyset$  ( $i \neq j$ ). Since  $tr(P) = 2$ , by Theorem 1.6 and Corollary 3.4, only one of  $C_1, C_2, \dots, C_n$  is not a trivial projection. Then by the proof of Theorem 1.6 we see that  $CD_P$  is obtained from one of the chord diagrams (a) or (b) in Fig. 3.6 by adding chords which do not cross the other chords. These chord diagrams (a) or (b) in Fig. 3.6 are realized by the projections (a) in Fig. 1.4. It follows from [8, Theorem 1] that the realizations of these chord diagrams are unique up to mirror image and ambient isotopy. Adding chords which do not cross the other chords corresponds to a series of replacing a sub-arc as illustrated in Fig. 1.4 (c). This completes the proof.  $\square$

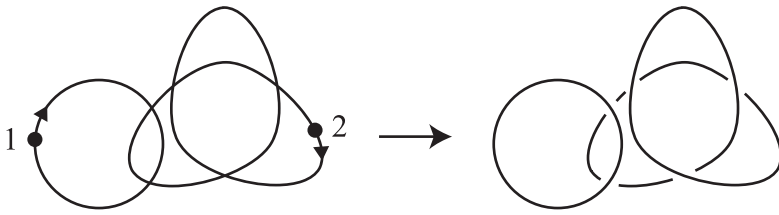


Fig. 3.7. A descending procedure.

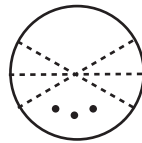


Fig. 3.8.

We use the following procedure which is called a *descending procedure* to prove Theorem 1.11 and Proposition 1.8. Let  $P$  be a projection of  $n$  disjoint circles. Let  $C_1, C_2, \dots, C_n$  be the image of the circles in  $P$ . We give an arbitrary orientation and an arbitrary base point which is not a pre-crossing to each  $C_i$ . We trace  $C_1, C_2, \dots, C_n$  in order and from their base points along their orientation. We give the over/under information to each pre-crossing of  $P$  so that every crossing may be first traced as an over-crossing as illustrated in Fig. 3.7. Then the diagram obtained from  $P$  by the procedure as above represents a trivial link.

Proof of Theorem 1.11. First we show that  $tr(P) \leq p(P) - 1$ . Let  $P$  be a projection of a circle. We give an orientation to the circle. Let  $b_1$  be a base point on  $P$  which is not a pre-crossing. Let  $p$  be the pre-crossing of  $P$  which first appears when we trace  $P$  from  $b_1$  along the orientation. Let  $b_2$  be a base point which is slightly before it than  $p$  with respect to the orientation.

Let  $D_1$  (resp.  $D_2$ ) be the diagram obtained from  $P$  by the descending procedure from a base point  $b_1$  (resp.  $b_2$ ) along the orientation. Here each of  $D_1$  and  $D_2$  represents a trivial knot. The difference of  $D_1$  and  $D_2$  is only the over/under information of  $p$ . Let  $Q$  be the pseudo diagram obtained from  $D_1$  (or  $D_2$ ) by forgetting the over/under information of  $p$ . Then  $Q$  is trivial. This implies that  $tr(P) \leq p(P) - 1$ .

Next we show that the equality holds if and only if  $P$  is one of the projections as illustrated in Fig. 1.5. The ‘if’ part is obvious. Let  $P$  be a projection of a circle with  $tr(P) = p(P) - 1$ . Then  $CD_P$  is a chord diagram in Fig. 3.8 since there exists no pair of parallel chords by the proof of Theorem 1.6. Note that  $CD_P$  has odd chords. These chord diagrams are realized by the projections as illustrated in Fig. 1.5 where  $m$  is a positive odd integer. It follows from [8, Theorem 1] that the realizations of these

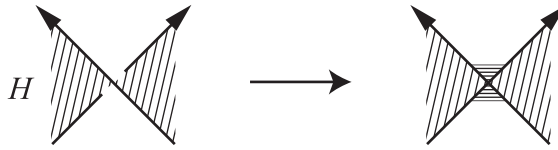


Fig. 3.9.

chord diagrams are unique up to mirror image and ambient isotopy. This completes the proof. □

Proof of Theorem 1.12. This is an immediate consequence of Theorem 1.11 and Corollary 3.4. □

Note that similar results on the *unknotting number* for knot diagrams and link diagrams as Theorem 1.11 and Theorem 1.12 are known in [25, Theorem 1.4, Theorem 1.5].

In the rest of this section, we prove Theorem 1.14. To accomplish this, we use the following Theorem 3.6. Let  $D$  be a diagram of a circle and  $K$  a knot represented by  $D$ . Then a disk  $E$  in  $\mathbf{S}^3$  is called a *crossing disk* for a crossing of  $D$  if  $E$  intersects  $K$  only in its interior exactly twice with zero algebraic intersection number and these two intersections correspond the crossing.

**Theorem 3.6** ([1]). *Let  $K$  be a trivial knot and  $D$  a diagram of  $K$ . Let  $c_1, c_2, \dots, c_n$  be crossings of  $D$  and  $E_1, E_2, \dots, E_n$  crossing disks corresponding to  $c_1, c_2, \dots, c_n$  respectively. Suppose that for any nonempty subset  $C \subset \{c_1, c_2, \dots, c_n\}$  the diagram obtained from  $D$  by crossing changes at  $C$  represents a trivial knot. Then  $K$  bounds an embedded disk in the complement of  $\partial E_1 \cup \partial E_2, \cup \dots \cup \partial E_n$ .*

Proof of Theorem 1.14. Let  $p_1, p_2, \dots, p_n$  be all of the pre-crossings of  $Q$ . Let  $D$  be a diagram representing a trivial knot  $K$  obtained from  $Q$ . Let  $c_1, c_2, \dots, c_n$  be the crossings of  $D$  corresponding to  $p_1, p_2, \dots, p_n$  respectively. Let  $E_1, E_2, \dots, E_n$  be crossing disks corresponding to  $c_1, c_2, \dots, c_n$  respectively. For any nonempty subset  $C$  of  $\{c_1, c_2, \dots, c_n\}$ , a diagram obtained from  $D$  by crossing changes at  $C$  represents a trivial knot by the definition of a trivial pseudo diagram. By Theorem 3.6, there exists an embedded disk  $H$  whose boundary is  $K$  in the complement of  $\partial E_1 \cup \partial E_2, \cup \dots \cup \partial E_n$ . By taking sufficiently small sub-disk of  $E_i$  if necessary, we may assume that each  $H \cap E_i$  ( $i = 1, 2, \dots, n$ ) is a simple arc. By contracting each simple arc to a point, we obtain a singular disk bounding  $K_Q$ . Here, we stick two disks at each double point of  $K_Q$  as illustrated in Fig. 3.9. Then we have a disk containing  $K_Q$ . Therefore,  $K_Q$  is trivial. □



#### 4. Knotting number

In this section, we study knotting number and give proofs of Proposition 1.8 and Theorem 1.13.

Proof of Proposition 1.8. First we show that there does not exist a projection of a circle whose knotting number is less than three. Suppose that there exists a projection  $P$  of a circle with  $kn(P) = 2$ . Let  $Q$  be a knotted pseudo diagram obtained from  $P$  with two crossings  $c_1$  and  $c_2$ . Let  $p_1$  and  $p_2$  be the pre-crossings of  $P$  which correspond to  $c_1$  and  $c_2$  respectively.

Without loss of generality, we may assume that the position of  $p_1$  and  $p_2$  (resp.  $c_1$  and  $c_2$ ) on  $P$  (resp.  $Q$ ) is (a) or (b) (resp. (c) or (d)) as in Fig. 4.1. We give an orientation and a base point to the image of the circle as illustrated in Fig. 4.1. In case (a) (resp. (b)), let  $D_1$  (resp.  $D_2$ ) be the diagram obtained from  $P$  by the descending procedure from a base point  $b$ . Here under any of the over/under information of  $c_1$  and  $c_2$ , each of  $D_1$  and  $D_2$  represents a trivial knot. This is a contradiction. In case (c) (resp. (d)), let  $D_3$  (resp.  $D_4$ ) be the diagram obtained from  $Q$  by the descending procedure from a base point  $b_1$  (resp.  $b_2$ ). Then each of  $D_3$  and  $D_4$  represents a trivial knot. This is a contradiction.

Similarly we can show that there do not exist projections of a circle whose knotting number is less than two.

For  $n \geq 3$ , the projection of Fig. 1.5 where  $m = 2n - 3$  has knotting number  $n$ . This completes the proof.  $\square$

Note that there exists a projection  $P$  of two circles with  $kn(P) = 2$  as (c) in Fig. 1.6. In general, we have the following proposition which is similar to Proposition 3.5.

**Proposition 4.1.** *Let  $P$  be a projection of a graph  $G$ . Then  $kn(P) \neq 1$ .*

Proof. Since the mirror image of a nontrivial spatial graph is also nontrivial, we can prove it in the same way as the proof of Proposition 3.5.  $\square$

We prepare some known theorems to prove Theorem 1.13. Let  $D$  be a diagram of disjoint circles. We give an orientation to the image of each circle in  $D$ . Then each crossing has a sign as illustrated in Fig. 4.2. A diagram  $D$  is said to be *positive* if all crossings of  $D$  are positive. Then the following is known.

**Theorem 4.2** ([5, 26, 15, 6]). *Let  $D$  be a positive diagram of disjoint circles with a crossing which is not nugatory. Then  $D$  represents a nontrivial link.*

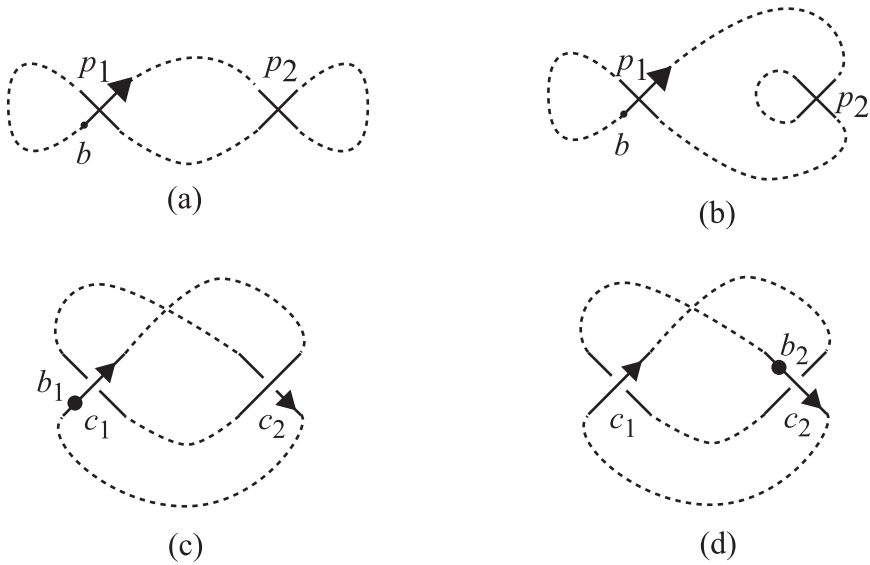


Fig. 4.1.

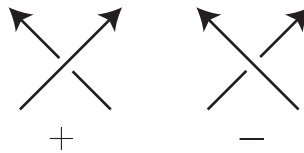


Fig. 4.2.

A diagram  $D$  is said to be *almost positive* if all crossings except one crossing of  $D$  are positive. The following theorem is shown in [18, 16] for knots and in [16] for links.

**Theorem 4.3** ([18, 16]). *Let  $D$  be an almost positive diagram representing a trivial link. Then  $D$  can be obtained from one of the diagrams (a), (b), (c) in Fig. 4.3 by possibly adding trivial circles and by a series of replacing a sub-arc by a part as illustrated in Fig. 4.3 (d).*

Proof of Theorem 1.13. The ‘if’ part is obvious. Let  $P$  be a projection with  $tr(P) \neq 0$  which is not obtained from any of the projections as illustrated in Fig. 1.6 by possibly adding trivial circles. We show that there exists a knotted pseudo diagram with at least one pre-crossing obtained from  $P$ , that is,  $kn(P) < p(P)$ .

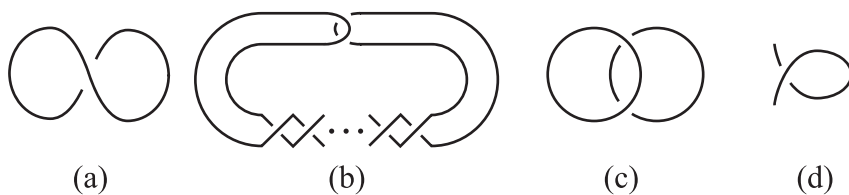


Fig. 4.3.

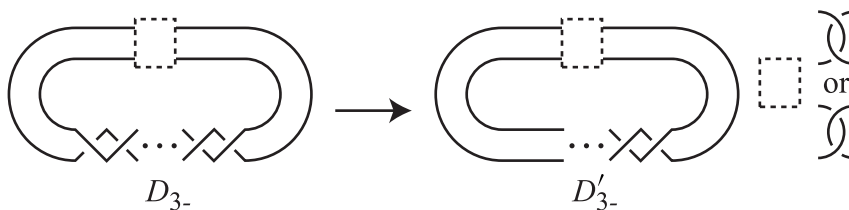


Fig. 4.4.

First we suppose that  $P$  has a nugatory pre-crossing  $p_1$ . By Proposition 2.4 there exists a knotted pseudo diagram obtained from  $P$  with a pre-crossing  $p_1$ . This implies that  $kn(P) < p(P)$ .

Next we suppose that  $P$  has no nugatory pre-crossings. Suppose that  $P$  is not a projection as (a) or (b) in Fig. 1.4. Let  $p_2$  be a pre-crossing of  $P$  and  $Q_2$  the pseudo diagram obtained from  $P$  by giving over/under information to all pre-crossings except  $p_2$  to be positive. We show that  $Q_2$  is knotted. Let  $D_{2+}$  be the diagram obtained from  $Q_2$  by giving the over/under information to  $p_2$  to be positive. Since  $D_{2+}$  is a positive diagram,  $D_{2+}$  represents a nontrivial link by Theorem 4.2. Let  $D_{2-}$  be the diagram obtained from  $Q_2$  by giving the over/under information to  $p_2$  to be negative. Since  $D_{2-}$  is an almost positive diagram,  $D_{2-}$  represents a nontrivial link by Theorem 4.3. Thus  $Q_2$  is knotted.

Suppose that  $P$  is a projection (a) in Fig. 1.4. Note that  $m > 2$  since  $P$  is not obtained from one of the projections as illustrated in Fig. 1.6. Let  $p_3$  be one of  $m$  pre-crossings in a row. Let  $Q_3$  be the pseudo diagram obtained from  $P$  by giving over/under information to all crossings except  $p_3$  to be positive. We show that  $Q_3$  is knotted. Let  $D_{3+}$  be the diagram obtained from  $Q_3$  by giving the over/under information to  $p_3$  to be positive. Since  $D_{3+}$  is a positive diagram,  $D_{3+}$  represents a nontrivial link by Theorem 4.2. Let  $D_{3-}$  be the diagram obtained from  $Q_3$  by giving the over/under information to  $p_3$  to be negative. We deform  $D_{3-}$  into  $D'_{3-}$  as illustrated in Fig. 4.4. Since  $D'_{3-}$  is a positive diagram with crossings which are not nugatory,  $D'_{3-}$  represents a nontrivial link by Theorem 4.2. Thus  $Q_3$  is knotted.  $\square$

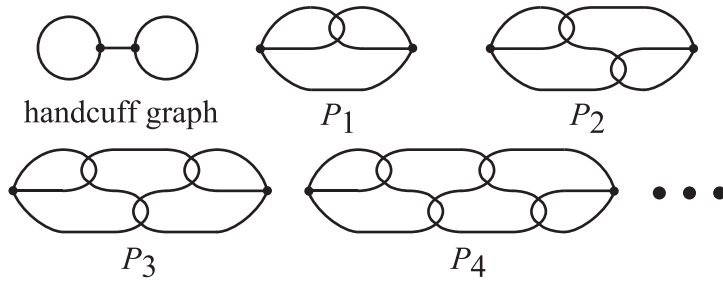


Fig. 4.5.

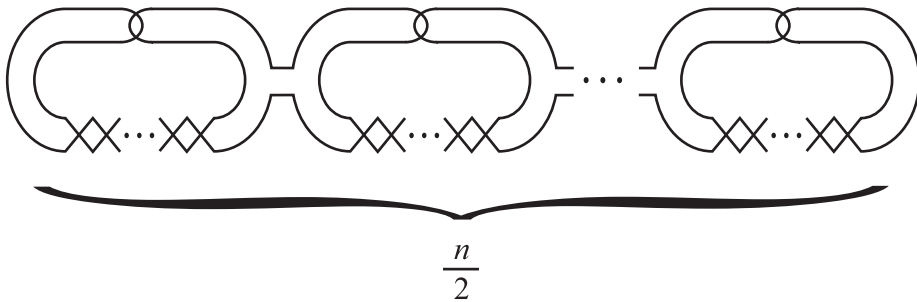


Fig. 5.1.

Note that for a certain graph  $G$  there exist infinitely many projections  $P$  of  $G$  with  $kn(P) = p(P)$ . For example, let  $G$  be a handcuff graph and  $\{P_i\}_{i=1,2,\dots}$  is the family of the projections as illustrated in Fig. 4.5. It is known in [23] that a diagram representing a nontrivial spatial graph is obtained from  $P_i$  ( $i = 1, 2, 3, \dots$ ). Then it is easy to check  $kn(P_i) = p(P_i)$ .

**5. Relations between trivializing number and knotting number**

In this section, we study relations between the trivializing number and the knotting number. We give a proof of Proposition 1.9.

Proof of Proposition 1.9. Let  $P_1$  be a projection of a circle as illustrated in Fig. 1.4 where  $l = 2m - 5$ . Then we have  $tr(P_1) = 2$  and  $kn(P_1) = l$ . Let  $P$  be the projection which is the composition of  $n/2$  copies of  $P_1$  as illustrated in Fig. 5.1. Thus  $tr(P) = n$  and  $kn(P) = l$  by Proposition 2.1. □

**6. An application of trivializing number and knotting number**

We ask the following question. For a projection  $P$  of a graph, how many diagrams obtained from  $P$  which represent trivial spatial graphs (resp. nontrivial spatial

graphs)? We denote the number of diagrams obtained from  $P$  which represent trivial spatial graphs (resp. nontrivial spatial graphs) by  $n_{\text{tri}}(P)$  (resp.  $n_{\text{nontri}}(P)$ ). Then we have the following inequality between  $n_{\text{tri}}(P)$  (resp.  $n_{\text{nontri}}(P)$ ) and  $\text{tr}(P)$  (resp.  $\text{kn}(P)$ ) for any graphs.

**Proposition 6.1.** *Let  $P$  be a projection of a graph. If  $P$  is neither trivial nor knotted, then  $n_{\text{tri}}(P) \geq 2^{p(P)-\text{tr}(P)+1}$  and  $n_{\text{nontri}}(P) \geq 2^{p(P)-\text{kn}(P)+1}$ .*

*Proof.* We show that  $n_{\text{tri}}(P) \geq 2^{p(P)-\text{tr}(P)+1}$ . Let  $Q$  be a trivial pseudo diagram obtained from  $P$  with  $\text{tr}(P)$  crossings. Then  $2^{p(P)-\text{tr}(P)}$  diagrams which represent trivial spatial graphs are obtained from  $Q$ . Let  $Q'$  be the pseudo diagram obtained from  $Q$  by changing over/under information at all crossings of  $Q$ . Then  $Q'$  is trivial in the same way as the proof of Proposition 3.5. Then  $2^{p(P)-\text{tr}(P)}$  diagrams which represent spatial graphs are obtained from  $Q'$ . Thus  $n_{\text{tri}}(P) \geq 2^{p(P)-\text{tr}(P)+1}$ . Similarly we can show that  $n_{\text{nontri}}(P) \geq 2^{p(P)-\text{kn}(P)+1}$ .  $\square$

**ACKNOWLEDGMENT.** The author expresses his gratitude to Professors Shin'ichi Suzuki, Kouki Taniyama and Ryo Nikkuni for their encouragement and advices. He is also grateful to Professors Koya Shimokawa and Andrzej Stasiak for their suggestions about DNA knots. He is also grateful to Professors Yoshiyuki Ohyama and Atsushi Ishii for their suggestions about a proof of Theorem 1.6. He is also grateful to Professor Makoto Ozawa for his suggestion about a proof of Theorem 1.14. He is also grateful to the referee for his/her comments.

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