COMPLETE MINIMAL HYPERSURFACES IN $S^4(1)$
WITH CONSTANT SCALAR CURVATURE

QING-MING CHENG

1. Introduction

Let M be an n-dimensional closed minimally immersed hypersurface in the unit sphere $S^{n+1}(1)$. If the square S of the length of the second fundamental form h on M satisfies $0 \leqslant S \leqslant n$, then $S \equiv 0$ or $S \equiv n$. In [3], S.S. Chern, M. do Carmo and S. Kobayashi proved that the Clifford tori are the only minimal hypersurfaces with $S=n$. C. K. Peng and C. L. Terng [6] studied the case $S=$ constant and shown, among other things, that if $n=3$ and $S>3$, then $S \geqslant 6$. The condition $S=6$ is also assumed in the examples of Cartan [1] and Hsiang [4]. On the other hand, in Otsuki's examples of minimal hypersurface in $S^{n+1}(1)$ (see [5]), H. D. Hu proved that there exist complete and non-compact minimal hypersurfaces in $S^{n+1}(1)$. Hence, it is interesting to study complete minimal hypersurfaces in $S^{n+1}(1)$. In [2], the author considered a compete minimally immersed hypersurface M in $S^{n+1}(1)$ with $S=$ constant, and proved that if $0 \leqslant S \leqslant n$, then $S=0$ or $S=n$.

In this paper, we generalize the above theorem due to C. K. Peng and C. L. Terng [6] to complete minimal hypersurfaces. That is, we obtain the following.

Theorem. *Let M^3 be a complete minimally immersed hypersuface in $S^4(1)$ with $S=$constant. If $S>3$, then $S \geqslant 6$.*

Corollary. *Let M^3 be a complete minimally immersed hypersurface in $S^4(1)$ with $S=$constant. If $0 \leqslant S \leqslant 6$, then $S=0$, $S=3$ or $S=6$.*

Proof. According to Theorem and the result of the author [2], Corollary is true obviously.

2. Preliminaries

Let M be an n-dimensional immersed hypersurface in the $n+1$-dimensional unit sphere $S^{n+1}(1)$. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+1}$ in $S^{n+1}(1)$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. We use the following convention on the range of indices unless otherwise stated:

$A, B, C, \cdots = 1, 2, \cdots, n+1, i, j, k, \cdots = 1, 2, \cdots, n$. And we agree the repeated indices under a summation sign without indication are summed over the respective range. With respect to the frame field of $S^{n+1}(1)$ chosen above, let $\omega_1, \cdots, \omega_{n+1}$ be the dual frame. Then the structure equations of $S^{n+1}(1)$ are given by

(2.1) $$d\omega_A = -\sum \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

(3.2) $$d\omega_{AB} = -\sum \omega_{AC} \wedge \omega_{CB} + \Omega_{AB},$$

(2.3) $$\Omega_{AB} = \frac{1}{2} \sum K_{ABCD} \, \omega_C \wedge \omega_D.$$

Restricting these forms to M, we have the structure equations of the immersion.

(2.4) $$\omega_{n+1} = 0.$$

(2.5) $$\omega_{n+1,i} = \sum h_{ij} \, \omega_j, \quad h_{ij} = h_{ji},$$

(2.6) $$d\omega_{ij} = -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

(2.7) $$d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{jk} + \frac{1}{2} \sum R_{ijkl} \, \omega_k \wedge \omega_l.$$

The symmetric 2-form

$$h = \sum h_{ij} \, \omega_i \, \omega_j$$

and the scalar

$$H = \frac{1}{3} \sum h_{ii}$$

are called the second fundamental form and the mean curvature of M respectively. If $H=0$, then M is said to be minimal.

Define h_{ijk} by

(2.8) $$\sum h_{ijk} \, \omega_k = dh_{ij} - \sum h_{im} \, \omega_{mj} - \sum h_{mj} \, \omega_{mi},$$

Exterior differentiating (2.5) and using structure equations, we obtain

$$\sum_{k,j} h_{ijk} \, \omega_k \wedge \omega_j = 0.$$

Thus we have

(2.9) $$h_{ijk} = h_{ikj}.$$

Similarly define h_{ijkl} by

(2.10) $$\sum h_{ijkl} \, \omega_l = dh_{ijk} - \sum h_{ijm} \, \omega_{mk} - \sum h_{imk} \, \omega_{mj} - \sum h_{mjk} \, \omega_{mi},$$

then,

(2.11) $$h_{ijkl} - h_{ijlk} = \sum h_{im} R_{mjkl} + \sum h_{mj} R_{mikl}.$$

If the square S of length of h, i.e., $S = \sum h_{ij}^2$, is constant and M is minimal, then the following formulas are well known (see [6]).

For any point $p \in M$, we can choose a frame field $e_1, \cdots, e_n$ so that $h_{ij} = \lambda_i \delta_{ij}$.

(2.12) $$\sum h_{ijk}^2 = S(S-n),$$

(2.13) $$\sum h_{ijkl}^2 = S(S-n)(S-2n-3) + 3(A-2B).$$

where $A = \sum h_{ijk}^2 \lambda_i^2$, $B = \sum h_{ijk}^2 \lambda_i \lambda_j$.

(2.14) $$t_{ij} = h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j).$$

Let $f_m = \sum \lambda_j^m$. Then we have

(2.15) $$\sum t_{ij}^2 = 2[nS - 2S^2 + Sf_4 - f_3^2],$$

(2.16) $$\Delta f_3 = 3[(n-S)f_3 + 2 \sum h_{ijk}^2 \lambda_i].$$

When $n = 3$, we have

Lemma 1 (see [6]).   (1) $f_3 = $ constant if and only if M has constant principal curvature; (2) $-\sqrt{S^3}/6 \leqslant f_3 \leqslant \sqrt{S^3}/6$ and equality is reached if and only if two of the principal curvature are equal.

Lemma 2 (see [7]).   Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function which is bounded from above on M. Then there exists a sequence $\{p_m\}$ such that

(2.17) $$\lim f(p_m) = \sup f, \ \lim \|\nabla f(p_m)\| = 0, \ \lim \sup \Delta f(p_m) \leqslant 0$$

3. Proof of Theorem

At first, we show the following two propositions.

Proposition 1. Let M be a complete minimal hypersurface in $S^4(1)$ with $S = $ constant. If $\inf f_3 \cdot \sup f_3 = 0$, and $S > 3$, then $S \geqslant 6$.

Proof.   Because of $\inf f_3 \cdot \sup f_3 = 0$, we have $\inf f_3 = 0$ or $\sup f_3 = 0$. If $\inf f_3 = \sup f_3 = 0$, namely, f_3 vanishes identically, then it follows from Lemma 1 (1) that M has constant principal curvature. Thus Proposition 1 is true. (cf. [6: Corollary 1])

Next we will only consider the case $f_3 \neq$ constant. Without loss of generality, we can suppose $\sup f_3 = 0$. According to the Gauss' equation and the assumption that S is constnat, we see that the Ricci curvature of M is bounded

from below.   Hence we can apply Lemma 2 to f_3 and we have a sequence $\{p_m\}$ in M such that

$$(3.1) \qquad \lim_{m\to\infty} f_3(p_m) = \sup f_3 = 0, \quad \lim_{m\to\infty} \|\nabla f_3(p_m)\| = 0.$$

$$(3.2) \qquad \limsup_{m\to\infty} \Delta f_3(p_m) \leqslant 0.$$

Since λ_i, h_{ijk} and h_{ijkl} are bounded because of (2.12) and (2.13), we may assume that

$$(3.3) \qquad \lim_{m\to\infty} \lambda_i(p_m) = \lambda_i^\circ$$

$$(3.4) \qquad \lim_{m\to\infty} h_{ijk}(p_m) = h_{ijk}^\circ,$$

$$(3.5) \qquad \lim_{m\to\infty} h_{ijkl}(p_m) = h_{ijkl}^\circ,$$

by taking a subsequence of $\{p_m\}$ if necessary.   Hence

$$(3,6) \qquad \begin{aligned} \lambda_1^\circ + \lambda_2^\circ + \lambda_3^\circ &= 0, \\ \lambda_1^{\circ 2} + \lambda_2^{\circ 2} + \lambda_3^{\circ 2} &= S, \\ \lambda_1^{\circ 3} + \lambda_2^{\circ 3} + \lambda_3^{\circ 3} &= 0, \end{aligned}$$

that is,

$$(3.7) \qquad \lambda_1^\circ = -\sqrt{S}/2, \ \lambda_2^\circ = 0 \quad \text{and} \quad \lambda_3^\circ = \sqrt{S}/2.$$

Here we assume $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$.

By differentiating $\sum h_{ii} = 0$ and $\sum h_{ij}^2 = S = \text{constant}$, we obtain

$$(3.8) \qquad \sum h_{iik} = 0,$$

$$(3.9) \qquad \sum h_{iik} \lambda_i = 0.$$

(3.3) and (3.4) imply

$$(3.10) \qquad \sum h_{iik}^\circ = 0,$$

$$(3.11) \qquad \sum h_{iik}^\circ \lambda_i^\circ = 0.$$

According to (3.1), we have $\lim\limits_{m\to\infty} \|\nabla f_3\|(p_m) = 0$.   Since

$$\|\nabla f_3\| = \left[\sum_k \left(\sum_i h_{iik} \lambda_i^2 \right)^2 \right]^{1/2},$$

we obtain

$$\lim_{m\to\infty} \|\nabla f_3\|(p_m) = \lim_{m\to\infty} \left[\sum_k \left(\sum_i h_{iik} \lambda_i^2 \right)^2 \right]^{1/1}(p_m) = 0.$$

Thus, by (3.3), (3.4) and the above fact, we get

(3.12) $$\sum h_{iik}^{\cdot}\,\lambda_i^{\cdot 2}=0\,, \quad \text{for any } k.$$

Because $\lambda_i^{\cdot}$ are distinct, (3.10), (3.13) and (3.12) yield

(3.13) $$h_{iik}^{\circ} = 0\,, \quad \text{for any } i \text{ and } k.$$

On the other hand,

$$
\begin{aligned}
3(A-2B) &= \sum h_{ijk}^2[\lambda_i^2+\lambda_j^2+\lambda_k^2-2\lambda_i\,\lambda_j-2\lambda_i\,\lambda_k-2\lambda_j\,\lambda_k]\\
&= \sum_{i\neq j\neq k} h_{ijk}^2[2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2]\\
&\quad +3\sum_{i\neq k} h_{iik}^2(\lambda_k^2-4\lambda_i\,\lambda_k)-3\sum h_{iii}^2\,\lambda_i^2\,.
\end{aligned}
$$

Hence

(3.14)
$$
\begin{aligned}
&\lim_{m\to\infty} 3(A-2B)\,(p_m)\\
&= \sum_{\substack{i\neq j\neq k\\ i\neq k}} h_{ijk}^{\circ 2}[2(\lambda_i^{\circ 2}+\lambda_j^{\circ 2}+\lambda_k^{\circ 2})-(\lambda_i^{\circ}+\lambda_j^{\circ}+\lambda_k^{\circ})^2]\\
&\quad +3\sum_{i\neq k} h_{iik}^{\circ 2}(\lambda_k^{\circ 2}-4\lambda_i^{\circ}\,\lambda_k^{\circ})-3\sum h_{iii}^{\circ 2}\,\lambda_i^{\circ 2}\\
&= 2S\sum h_{ijk}^{\circ 2} \quad \text{(by (3.13) and (3.6))}\\
&= 2S^2(S-3) \quad \text{(by (2.12))}\,;
\end{aligned}
$$

(3.15)
$$
\begin{aligned}
\sum h_{ijkl}^2 &\geqslant 3\sum_{i\neq j} h_{ijij}^2+\sum_i h_{iiii}^2\\
&\geqslant 3\sum_{i\neq j}(h_{ijij}-t_{ij}/2)^2+\frac{3}{4}\sum t_{ij}^2\,,
\end{aligned}
$$

where $t_{ij}=h_{ijij}-h_{jiji}=(\lambda_i-\lambda_j)\,(1+\lambda_i\,\lambda_j)$. From (3.2) and (3.5), we get

(3.16) $$\sum h_{ijkl}^{\circ 2} \geqslant 3\sum_{i\neq j}(h_{ijij}^{\circ}-t_{ij}^{\circ}/2)^2+\frac{3}{4}\sum t_{ij}^{\circ 2}\,,$$

where $t_{ij}^{\circ}=h_{ijij}^{\circ}-h_{jiji}^{\circ}=(\lambda_i^{\circ}-\lambda_j^{\circ})\,(1+\lambda_i^{\circ}\,\lambda_j^{\circ})$. (3.7) implies

(3.17) $$\sum t_{ij}^{\circ 2} = S^2-4S^3+6S\,.$$

Accordingly,

(3.18)
$$
\begin{aligned}
&S(S-3)\,(S-9)+2S^2(S-3)\\
&\geqslant 3\sum_{i\neq j}(h_{ijij}^{\circ}-t_{ij}^{\circ}/2)^2+\frac{3}{4}\,(S^3-4S^2+6S)\,.
\end{aligned}
$$

(3.19)
$$
\begin{aligned}
&\sum_{i\neq j}(h_{ijij}^{\circ}-t_{ij}^{\circ}/2)^2\\
&\geqslant 2[(h_{1212}^{\circ}-t_{12}^{\circ}/2)^2+(h_{2323}^{\circ}-t_{23}^{\circ}/2)^2]\\
&= [h_{1212}^{\circ}+h_{323}^{\circ}-\frac{1}{2}\,(t_2^{\circ}-t_3^{\circ})]^2+[h_{1212}^{\circ}-h_{2323}^{\circ}-\frac{1}{2}\,(t_{12}^{\circ}-t_{23}^{\circ})]^2\\
&\geqslant [h_{1212}^{\circ}+h_{2323}^{\circ}]^2\,,
\end{aligned}
$$

here we make use of $t_{12}^\circ=t_{23}^\circ=-\sqrt{S}/2$.   Differentiating $\sum h_{ij}^2=S$, we obtain

$$\sum_{i,j} h_{ijll}\,h_{ij}+\sum_{i,j} h_{ijl}^2 = 0\,, \qquad \text{for}\quad l=1,2,3,$$

which implies

$$\sum_i h_{iill}\,\lambda_i+\sum_{i,j} h_{ijl}^2 = 0\,, \qquad \text{for}\quad l=1,2,3.$$

Hence

(3.20) $$\sum h_{iill}^\circ\,\lambda_i^\circ+\sum h_{ijl}^{\circ 2} = 0\,, \qquad \text{for}\quad l=1,2,3.$$

We substitute (3.7) into (3.20).   Then

$$\sqrt{S}/2\,[h_{1lll}^\circ-h_{33ll}^\circ]=\sum_{i,j} h_{ijl}^{\circ 2}\,, \qquad \text{for}\quad l=1,2,3.$$

In particular, we have

$$\sqrt{S}/2\,[h_{1122}^\circ-h_{3322}^\circ] = \sum h_{ij2}^{\circ 2}\,.$$

Hence

(3.21) $$\begin{aligned} h_{1212}^\circ-h_{2323}^\circ &= h_{1122}^\circ-h_{2233}^\circ \\ &= h_{1122}^\circ-h_{3322}^\circ-t_{23}^\circ = \sqrt{2}/S\,[\sum h_{ij2}^{\circ 2}+S/2] \\ &= \sqrt{2}/S\,[\tfrac{1}{3}\,S(S-3)+S/2]\,, \end{aligned}$$

since we use $\sum h_{ij2}^{\circ 2}=2h_{123}^{\circ 2}=\dfrac{1}{3}\,S(S-3)$ by (3.13).

By means of (3.18), (3.19) and (3.21), we get

$$3S(S-3)^2\geqslant\frac{6}{S}\,[\tfrac{1}{3}\,S(S-3)+\frac{S}{2}]^2+\frac{3}{4}\,S(S^2-4S+6)\,.$$

Namely,

$$S(S-6)\,(19S-42)\geqslant0\,.$$

It is clear that if $S>3$, then $S\geqslant6$. We complete the proof of Proposition 1.

Proposition 2. *Let M be a complete minimal hypersurface in $S^4(1)$ with $S=constant$. If $\inf f_3\cdot\sup f_3\neq0$ and $S>3$, then $S\geqslant6$.*

Proof. If $f_3=$constant, then it follows from Lemma 1 (1) that M has constant principal curvature. Thus Proposition 2 is valid obviously (cf. [6: Corollary 1]). If $f_3\neq$constant, then we can prove that there exists a point $p\in M$ such that $f_3(p)=0$ because of $\inf f_3\cdot\sup f_3\neq0$.

(1)   If $\inf f_3\cdot\sup f_3<0$, then, from the continuation of f_3, we have that there exists a point $p\in M$ such that $f_3(p)=0$.

COMPLETE MINIMAL HYPERSURFACES

(2) If $\inf f_3 \cdot \sup f_3 > 0$, then $\inf f_3$ and $\sup f_3$ have the same sign, we shall prove that this does not occur. In fact, without loss of generality, we can assume $\sup f_3 < 0$. Lemma 1 (2) yields

(3.22)
$$-\sqrt{S^2}/6 < \sup f_3 < 0 \; ;$$

(3.23)
$$\lim_{m\to\infty} f_3(p_m) = \sup f_3 \,, \quad \lim_{m\to\infty} \|\nabla f_3(p_m)\| = 0,$$

(3.24)
$$\limsup_{m\to\infty} \Delta f_3(p_m) \leqslant 0 \,,$$

(3.25)
$$\lim_{m\to\infty} \lambda_i(p_m) = \overset{\circ}{\lambda_i} \,,$$

(3.26)
$$\lim_{m\to\infty} h_{ijk}(p_m) = \overset{\circ}{h}_{ijk} \,,$$

(3.27)
$$\overset{\circ}{\lambda_1} + \overset{\circ}{\lambda_2} + \overset{\circ}{\lambda_3} = 0 \,,$$

(3.28)
$$\overset{\circ}{\lambda_1}^2 + \overset{\circ}{\lambda_2}^2 + \overset{\circ}{\lambda_3}^2 = S \,,$$

(3.29)
$$\overset{\circ}{\lambda_1}^3 + \overset{\circ}{\lambda_2}^3 + \overset{\circ}{\lambda_3}^3 = \sup f_3 \,.$$

(3.22), (3.27), (3.28) and (3.29) imply that $\overset{\circ}{\lambda_1}$, $\overset{\circ}{\lambda_2}$ and $\overset{\circ}{\lambda_3}$ are distinct. By the same proof as in Proposition 1, we can obtain

(3.30)
$$\overset{\circ}{h}_{iik} = 0 \,, \quad \text{for any } i \text{ and } k.$$

On the other hand, from

$$\Delta f_3 = 3\,[(3-S)f_3 + 2 \sum h_{ijk}^2 \lambda_i] \,,$$

(3.23), (3.24), (3.25) and (3.26) yield

(3.31)
$$3\,[(3-S) \sup f_3 + 2 \sum \overset{\circ}{h}_{ijk}^2 \overset{\circ}{\lambda_i}] \leqslant 0 \,.$$

$$3 \sum \overset{\circ}{h}_{ijk}^2 \overset{\circ}{\lambda_i}$$
$$= \sum \overset{\circ}{h}_{ijk}^2 (\overset{\circ}{\lambda_i} + \overset{\circ}{\lambda_j} + \overset{\circ}{\lambda_k})$$
$$= \underset{\substack{i\neq j\neq k \\ i\neq k}}{\sum} \overset{\circ}{h}_{ijk}^2 (\overset{\circ}{\lambda_i} + \overset{\circ}{\lambda_j} + \overset{\circ}{\lambda_k}) \quad \text{(by (3.13))}$$
$$= 0 \quad \text{(by } \overset{\circ}{\lambda_1} + \overset{\circ}{\lambda_3} + \overset{\circ}{\lambda_2} = 0 \text{)} \,.$$

Hence

$$(3-S) \sup f_3 \leqslant 0 \,.$$

Because of $S > 3$ and $\sup f_3 < 0$, we know that this is impossible. Hence there exists a point $p \in M$ such that $f_3(p) = 0$.

Next by the same proof as in [6], we know that Proposition 2 is valid.

Proof of Theorem. From Propositions 1 and 2, Theorem is obvious.

References

[1] E. Cartan: *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espaces spheriques,* Math. Z., **45** (1939), 335–367.

[2] Q.M. Cheng: *A characterization of complete Riemannian manifold minimally immersed in a unit sphere,* to appear in Nagoya Math. J..

[3] S.S. Chern, M. do Carmo and S. Kobayashi: *Minimal submanifolds of a sphere with second fundamental form of constant length,* In functional analysis and related fields, Springer Verlag, Berlin, 1970, 59–75.

[4] W.Y. Hsiang: *Remarks on closed minimal submanifolds in the standard Riemannian m-sphere,* J. Diff. Geom., **1** (1967), 257–267.

[5] W.Y. Hsiang and H.B. Lawson: *Minimal submanifolds of low cohomogeneity,* J. Diff. Geom., **5** (1971), 1–38.

[6] C.K. Peng and C.L. Terng: *Minimal hypersurfaces of spheres with constant scalar curvature,* Seminar on minimal submanifolds, 1983, Princeton Univ. Press, 177–198.

[7] S.T. Yau: *Harmonic functions on complete Riemannian manifolds,* Comm. Pure and Appl. Math., **28** (1975), 201–228.

Department of Mathematics
Northeast University of Technology
Shenyang
China