

Title	Complete minimal hypersurfaces in $S^4(1)$ with constant scalar curvature
Author(s)	Cheng, Qing-ming
Citation	Osaka Journal of Mathematics. 1990, 27(4), p. 885-892
Version Type	VoR
URL	https://doi.org/10.18910/8317
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

COMPLETE MINIMAL HYPERSURFACES IN $S^4(1)$ WITH CONSTANT SCALAR CURVATURE

QING-MING CHENG

(Received October 31, 1989)

1. Introduction

Let M be an n -dimensional closed minimally immersed hypersurface in the unit sphere $S^{n+1}(1)$. If the square S of the length of the second fundamental form h on M satisfies $0 \leq S \leq n$, then $S \equiv 0$ or $S \equiv n$. In [3], S.S. Chern, M. do Carmo and S. Kobayashi proved that the Clifford tori are the only minimal hypersurfaces with $S=n$. C. K. Peng and C. L. Terng [6] studied the case $S=\text{constant}$ and shown, among other things, that if $n=3$ and $S>3$, then $S \geq 6$. The condition $S=6$ is also assumed in the examples of Cartan [1] and Hsiang [4]. On the other hand, in Otsuki's examples of minimal hypersurface in $S^{n+1}(1)$ (see [5]), H. D. Hu proved that there exist complete and non-compact minimal hypersurfaces in $S^{n+1}(1)$. Hence, it is interesting to study complete minimal hypersurfaces in $S^{n+1}(1)$. In [2], the author considered a complete minimally immersed hypersurface M in $S^{n+1}(1)$ with $S=\text{constant}$, and proved that if $0 \leq S \leq n$, then $S=0$ or $S=n$.

In this paper, we generalize the above theorem due to C. K. Peng and C. L. Terng [6] to complete minimal hypersurfaces. That is, we obtain the following.

Theorem. *Let M^3 be a complete minimally immersed hypersurface in $S^4(1)$ with $S=\text{constant}$. If $S>3$, then $S \geq 6$.*

Corollary. *Let M^3 be a complete minimally immersed hypersurface in $S^4(1)$ with $S=\text{constant}$. If $0 \leq S \leq 6$, then $S=0$, $S=3$ or $S=6$.*

Proof. According to Theorem and the result of the author [2], Corollary is true obviously.

2. Preliminaries

Let M be an n -dimensional immersed hypersurface in the $n+1$ -dimensional unit sphere $S^{n+1}(1)$. We choose a local field of orthonormal frames e_1, \dots, e_{n+1} in $S^{n+1}(1)$ such that, restricted to M , the vectors e_1, \dots, e_n are tangent to M . We use the following convention on the range of indices unless otherwise stated:

$A, B, C, \dots = 1, 2, \dots, n+1, i, j, k, \dots = 1, 2, \dots, n$. And we agree the repeated indices under a summation sign without indication are summed over the respective range. With respect to the frame field of $S^{n+1}(1)$ chosen above, let $\omega_1, \dots, \omega_{n+1}$ be the dual frame. Then the structure equations of $S^{n+1}(1)$ are given by

$$(2.1) \quad d\omega_A = -\sum \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(3.2) \quad d\omega_{AB} = -\sum \omega_{AC} \wedge \omega_{CB} + \Omega_{AB},$$

$$(2.3) \quad \Omega_{AB} = \frac{1}{2} \sum K_{ABCD} \omega_C \wedge \omega_D.$$

Restricting these forms to M , we have the structure equations of the immersion.

$$(2.4) \quad \omega_{n+1} = 0.$$

$$(2.5) \quad \omega_{n+1,i} = \sum h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

$$(2.6) \quad d\omega_{ij} = -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(2.7) \quad d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{jk} + \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l.$$

The symmetric 2-form

$$h = \sum h_{ij} \omega_i \omega_j$$

and the scalar

$$H = \frac{1}{3} \sum h_{ii}$$

are called the second fundamental form and the mean curvature of M respectively. If $H=0$, then M is said to be minimal.

Define h_{ijk} by

$$(2.8) \quad \sum h_{ijk} \omega_k = dh_{ij} - \sum h_{im} \omega_{mj} - \sum h_{mj} \omega_{mi},$$

Exterior differentiating (2.5) and using structure equations, we obtain

$$\sum_{k,j} h_{ijk} \omega_k \wedge \omega_j = 0.$$

Thus we have

$$(2.9) \quad h_{ijk} = h_{ikj}.$$

Similarly define h_{ijkl} by

$$(2.10) \quad \sum h_{ijkl} \omega_l = dh_{ijk} - \sum h_{ijm} \omega_{mk} - \sum h_{imk} \omega_{mj} - \sum h_{mjk} \omega_{mi},$$

then,

$$(2.11) \quad h_{ijkl} - h_{ijlk} = \sum h_{im} R_{mjkl} + \sum h_{mj} R_{mikl}.$$

If the square S of length of h , i.e., $S = \sum h_{ij}^2$, is constant and M is minimal, then the following formulas are well known (see [6]).

For any point $p \in M$, we can choose a frame field e_1, \dots, e_n so that $h_{ij} = \lambda_i \delta_{ij}$.

$$(2.12) \quad \sum h_{ijk}^2 = S(S-n),$$

$$(2.13) \quad \sum h_{ijkl}^2 = S(S-n)(S-2n-3) + 3(A-2B),$$

where $A = \sum h_{ijk}^2 \lambda_i^2$, $B = \sum h_{ijk}^2 \lambda_i \lambda_j$.

$$(2.14) \quad t_{ij} = h_{ijij} - h_{jiij} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j).$$

Let $f_m = \sum \lambda_j^m$. Then we have

$$(2.15) \quad \sum t_{ij}^2 = 2[nS - 2S^2 + Sf_4 - f_3^2],$$

$$(2.16) \quad \Delta f_3 = 3[(n-S)f_3 + 2 \sum h_{ijk}^2 \lambda_i].$$

When $n=3$, we have

Lemma 1 (see [6]). (1) $f_3 = \text{constant}$ if and only if M has constant principal curvature; (2) $-\sqrt{S^3}/6 \leq f_3 \leq \sqrt{S^3}/6$ and equality is reached if and only if two of the principal curvature are equal.

Lemma 2 (see [7]). Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2 -function which is bounded from above on M . Then there exists a sequence $\{p_m\}$ such that

$$(2.17) \quad \lim f(p_m) = \sup f, \quad \lim \|\nabla f(p_m)\| = 0, \quad \lim \sup \Delta f(p_m) \leq 0$$

3. Proof of Theorem

At first, we show the following two propositions.

Proposition 1. Let M be a complete minimal hypersurface in $S^4(1)$ with $S = \text{constant}$. If $\inf f_3 \cdot \sup f_3 = 0$, and $S > 3$, then $S \geq 6$.

Proof. Because of $\inf f_3 \cdot \sup f_3 = 0$, we have $\inf f_3 = 0$ or $\sup f_3 = 0$. If $\inf f_3 = \sup f_3 = 0$, namely, f_3 vanishes identically, then it follows from Lemma 1 (1) that M has constant principal curvature. Thus Proposition 1 is true. (cf. [6: Corollary 1])

Next we will only consider the case $f_3 \neq \text{constant}$. Without loss of generality, we can suppose $\sup f_3 = 0$. According to the Gauss' equation and the assumption that S is constant, we see that the Ricci curvature of M is bounded

from below. Hence we can apply Lemma 2 to f_3 and we have a sequence $\{p_m\}$ in M such that

$$(3.1) \quad \lim_{m \rightarrow \infty} f_3(p_m) = \sup f_3 = 0, \quad \lim_{m \rightarrow \infty} \|\nabla f_3(p_m)\| = 0.$$

$$(3.2) \quad \limsup_{m \rightarrow \infty} \Delta f_3(p_m) \leq 0.$$

Since λ_i, h_{ijk} and h_{ijkl} are bounded because of (2.12) and (2.13), we may assume that

$$(3.3) \quad \lim_{m \rightarrow \infty} \lambda_i(p_m) = \lambda_i^\circ$$

$$(3.4) \quad \lim_{m \rightarrow \infty} h_{ijk}(p_m) = h_{ijk}^\circ,$$

$$(3.5) \quad \lim_{m \rightarrow \infty} h_{ijkl}(p_m) = h_{ijkl}^\circ,$$

by taking a subsequence of $\{p_m\}$ if necessary. Hence

$$(3.6) \quad \begin{aligned} \lambda_1^\circ + \lambda_2^\circ + \lambda_3^\circ &= 0, \\ \lambda_1^{\circ 2} + \lambda_2^{\circ 2} + \lambda_3^{\circ 2} &= S, \\ \lambda_1^{\circ 3} + \lambda_2^{\circ 3} + \lambda_3^{\circ 3} &= 0, \end{aligned}$$

that is,

$$(3.7) \quad \lambda_1^\circ = -\sqrt{S}/2, \lambda_2^\circ = 0 \quad \text{and} \quad \lambda_3^\circ = \sqrt{S}/2.$$

Here we assume $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

By differentiating $\sum h_{ii} = 0$ and $\sum h_{ij}^2 = S = \text{constant}$, we obtain

$$(3.8) \quad \sum h_{iik} = 0,$$

$$(3.9) \quad \sum h_{iik} \lambda_i = 0.$$

(3.3) and (3.4) imply

$$(3.10) \quad \sum h_{iik}^\circ = 0,$$

$$(3.11) \quad \sum h_{iik}^\circ \lambda_i^\circ = 0.$$

According to (3.1), we have $\lim_{m \rightarrow \infty} \|\nabla f_3\|(p_m) = 0$. Since

$$\|\nabla f_3\| = [\sum_k (\sum_i h_{iik} \lambda_i^2)^2]^{1/2},$$

we obtain

$$\lim_{m \rightarrow \infty} \|\nabla f_3\|(p_m) = \lim_{m \rightarrow \infty} [\sum_k (\sum_i h_{iik} \lambda_i^2)^2]^{1/2}(p_m) = 0.$$

Thus, by (3.3), (3.4) and the above fact, we get

$$(3.12) \quad \sum h_{iik}^{\circ} \lambda_i^2 = 0, \quad \text{for any } k.$$

Because λ_i are distinct, (3.10), (3.13) and (3.12) yield

$$(3.13) \quad h_{iik}^{\circ} = 0, \quad \text{for any } i \text{ and } k.$$

On the other hand,

$$\begin{aligned} 3(A-2B) &= \sum h_{ijk}^2 [\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i \lambda_j - 2\lambda_i \lambda_k - 2\lambda_j \lambda_k] \\ &= \sum_{\substack{i \neq j \neq k \\ i \neq k}} h_{ijk}^2 [2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2] \\ &\quad + 3 \sum_{i \neq k} h_{iik}^2 (\lambda_k^2 - 4\lambda_i \lambda_k) - 3 \sum h_{iii}^2 \lambda_i^2. \end{aligned}$$

Hence

$$\begin{aligned} (3.14) \quad \lim_{m \rightarrow \infty} 3(A-2B)(p_m) &= \sum_{\substack{i \neq j \neq k \\ i \neq k}} h_{ijk}^{\circ 2} [2(\lambda_i^{\circ 2} + \lambda_j^{\circ 2} + \lambda_k^{\circ 2}) - (\lambda_i^{\circ} + \lambda_j^{\circ} + \lambda_k^{\circ})^2] \\ &\quad + 3 \sum_{i \neq k} h_{iik}^{\circ 2} (\lambda_k^{\circ 2} - 4\lambda_i^{\circ} \lambda_k^{\circ}) - 3 \sum h_{iii}^{\circ 2} \lambda_i^{\circ 2} \\ &= 2S \sum h_{ijk}^{\circ 2} \quad (\text{by (3.13) and (3.6)}) \\ &= 2S^2(S-3) \quad (\text{by (2.12)}); \end{aligned}$$

$$\begin{aligned} (3.15) \quad \sum h_{ijk}^2 &\geq 3 \sum_{i \neq j} h_{ijij}^2 + \sum_i h_{iii}^2 \\ &\geq 3 \sum_{i \neq j} (h_{ijij} - t_{ij}/2)^2 + \frac{3}{4} \sum t_{ij}^2, \end{aligned}$$

where $t_{ij} = h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j)$. From (3.2) and (3.5), we get

$$(3.16) \quad \sum h_{ijk}^2 \geq 3 \sum_{i \neq j} (h_{ijij} - t_{ij}/2)^2 + \frac{3}{4} \sum t_{ij}^2,$$

where $t_{ij}^{\circ} = h_{ijij}^{\circ} - h_{jiji}^{\circ} = (\lambda_i^{\circ} - \lambda_j^{\circ})(1 + \lambda_i^{\circ} \lambda_j^{\circ})$. (3.7) implies

$$(3.17) \quad \sum t_{ij}^{\circ 2} = S^2 - 4S^3 + 6S.$$

Accordingly,

$$\begin{aligned} (3.18) \quad S(S-3)(S-9) + 2S^2(S-3) \\ \geq 3 \sum_{i \neq j} (h_{ijij}^{\circ} - t_{ij}^{\circ}/2)^2 + \frac{3}{4} (S^3 - 4S^2 + 6S). \end{aligned}$$

$$\begin{aligned} (3.19) \quad \sum_{i \neq j} (h_{ijij}^{\circ} - t_{ij}^{\circ}/2)^2 \\ \geq 2[(h_{1212}^{\circ} - t_{12}^{\circ}/2)^2 + (h_{2323}^{\circ} - t_{23}^{\circ}/2)^2] \\ = [h_{1212}^{\circ} + h_{323}^{\circ} - \frac{1}{2}(t_2^{\circ} - t_3^{\circ})]^2 + [h_{1212}^{\circ} - h_{2323}^{\circ} - \frac{1}{2}(t_{12}^{\circ} - t_{23}^{\circ})]^2 \\ \geq [h_{1212}^{\circ} + h_{2323}^{\circ}]^2, \end{aligned}$$

here we make use of $t_{12}^\circ = t_{23}^\circ = -\sqrt{S}/2$. Differentiating $\sum h_{ij}^2 = S$, we obtain

$$\sum_{i,j} h_{ijl} h_{ij} + \sum_{i,j} h_{ijl}^2 = 0, \quad \text{for } l = 1, 2, 3,$$

which implies

$$\sum_i h_{iil} \lambda_i + \sum_{i,j} h_{ijl}^2 = 0, \quad \text{for } l = 1, 2, 3.$$

Hence

$$(3.20) \quad \sum h_{iil}^\circ \lambda_i^\circ + \sum h_{ijl}^{\circ 2} = 0, \quad \text{for } l = 1, 2, 3.$$

We substitute (3.7) into (3.20). Then

$$\sqrt{S}/2 [h_{iil}^\circ - h_{33il}^\circ] = \sum_{i,j} h_{ijl}^{\circ 2}, \quad \text{for } l = 1, 2, 3.$$

In particular, we have

$$\sqrt{S}/2 [h_{1122}^\circ - h_{3322}^\circ] = \sum h_{ij2}^{\circ 2}.$$

Hence

$$(3.21) \quad \begin{aligned} h_{1212}^\circ - h_{2323}^\circ &= h_{1122}^\circ - h_{2233}^\circ \\ &= h_{1122}^\circ - h_{3322}^\circ - t_{23}^\circ = \sqrt{2}/S [\sum h_{ij2}^{\circ 2} + S/2] \\ &= \sqrt{2}/S [\frac{1}{3} S(S-3) + S/2], \end{aligned}$$

since we use $\sum h_{ij2}^{\circ 2} = 2h_{123}^{\circ 2} = \frac{1}{3} S(S-3)$ by (3.13).

By means of (3.18), (3.19) and (3.21), we get

$$3S(S-3)^2 \geq \frac{6}{S} [\frac{1}{3} S(S-3) + \frac{S}{2}]^2 + \frac{3}{4} S(S^2 - 4S + 6).$$

Namely,

$$S(S-6)(19S-42) \geq 0.$$

It is clear that if $S > 3$, then $S \geq 6$. We complete the proof of Proposition 1.

Proposition 2. *Let M be a complete minimal hypersurface in $S^4(1)$ with $S = \text{constant}$. If $\inf f_3 \cdot \sup f_3 \neq 0$ and $S > 3$, then $S \geq 6$.*

Proof. If $f_3 = \text{constant}$, then it follows from Lemma 1 (1) that M has constant principal curvature. Thus Proposition 2 is valid obviously (cf. [6: Corollary 1]). If $f_3 \neq \text{constant}$, then we can prove that there exists a point $p \in M$ such that $f_3(p) = 0$ because of $\inf f_3 \cdot \sup f_3 \neq 0$.

(1) If $\inf f_3 \cdot \sup f_3 < 0$, then, from the continuation of f_3 , we have that there exists a point $p \in M$ such that $f_3(p) = 0$.

(2) If $\inf f_3 \cdot \sup f_3 > 0$, then $\inf f_3$ and $\sup f_3$ have the same sign, we shall prove that this does not occur. In fact, without loss of generality, we can assume $\sup f_3 < 0$. Lemma 1 (2) yields

$$(3.22) \quad -\sqrt{S^2/6} < \sup f_3 < 0 ;$$

$$(3.23) \quad \lim_{m \rightarrow \infty} f_3(p_m) = \sup f_3, \quad \lim_{m \rightarrow \infty} \|\nabla f_3(p_m)\| = 0,$$

$$(3.24) \quad \limsup_{m \rightarrow \infty} \Delta f_3(p_m) \leq 0,$$

$$(3.25) \quad \lim_{m \rightarrow \infty} \lambda_i(p_m) = \lambda_i^\circ,$$

$$(3.26) \quad \lim_{m \rightarrow \infty} h_{ijk}(p_m) = h_{ijk}^\circ,$$

$$(3.27) \quad \lambda_1^\circ + \lambda_2^\circ + \lambda_3^\circ = 0,$$

$$(3.28) \quad \lambda_1^{\circ 2} + \lambda_2^{\circ 2} + \lambda_3^{\circ 2} = S,$$

$$(3.29) \quad \lambda_1^{\circ 3} + \lambda_2^{\circ 3} + \lambda_3^{\circ 3} = \sup f_3.$$

(3.22), (3.27), (3.28) and (3.29) imply that λ_1° , λ_2° and λ_3° are distinct. By the same proof as in Proposition 1, we can obtain

$$(3.30) \quad h_{iik}^\circ = 0, \quad \text{for any } i \text{ and } k.$$

On the other hand, from

$$\Delta f_3 = 3[(3-S)f_3 + 2 \sum h_{ijk}^2 \lambda_i],$$

(3.23), (3.24), (3.25) and (3.26) yield

$$(3.31) \quad 3[(3-S) \sup f_3 + 2 \sum h_{ijk}^2 \lambda_i] \leq 0.$$

$$\begin{aligned} & 3 \sum h_{ijk}^2 \lambda_i \\ &= \sum h_{ijk}^2 (\lambda_i^\circ + \lambda_j^\circ + \lambda_k^\circ) \\ &= \sum_{\substack{i \neq j \neq k \\ i \neq k}} h_{ijk}^2 (\lambda_i^\circ + \lambda_j^\circ + \lambda_k^\circ) \quad (\text{by (3.13)}) \\ &= 0 \quad (\text{by } \lambda_1^\circ + \lambda_3^\circ + \lambda_2^\circ = 0). \end{aligned}$$

Hence

$$(3-S) \sup f_3 \leq 0.$$

Because of $S > 3$ and $\sup f_3 < 0$, we know that this is impossible. Hence there exists a point $p \in M$ such that $f_3(p) = 0$.

Next by the same proof as in [6], we know that Proposition 2 is valid.

Proof of Theorem. From Propositions 1 and 2, Theorem is obvious.

References

- [1] E. Cartan: *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espaces spheriques*, Math. Z., **45** (1939), 335–367.
- [2] Q.M. Cheng: *A characterization of complete Riemannian manifold minimally immersed in a unit sphere*, to appear in Nagoya Math. J..
- [3] S.S. Chern, M. do Carmo and S. Kobayashi: *Minimal submanifolds of a sphere with second fundamental form of constant length*, In functional analysis and related fields, Springer Verlag, Berlin, 1970, 59–75.
- [4] W.Y. Hsiang: *Remarks on closed minimal submanifolds in the standard Riemannian m -sphere*, J. Diff. Geom., **1** (1967), 257–267.
- [5] W.Y. Hsiang and H.B. Lawson: *Minimal submanifolds of low cohomogeneity*, J. Diff. Geom., **5** (1971), 1–38.
- [6] C.K. Peng and C.L. Terng: *Minimal hypersurfaces of spheres with constant scalar curvature*, Seminar on minimal submanifolds, 1983, Princeton Univ. Press, 177–198.
- [7] S.T. Yau: *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math., **28** (1975), 201–228.

Department of Mathematics
Northeast University of Technology
Shenyang
China