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J-GROUPS OF THE SUSPENSIONS OF THE STUNTED LENS SPACES MOD 2p

Dedicated to Professor Michikazu Fujii on his 60th birthday

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1. Introduction

Let $L^n(q)=S^{2n+1}/\boldsymbol{Z}_q$ be the (2n+1)-dimensional standard lens space mod q. As defined in [10], we set

$$(1.1)\qquad \begin{aligned} L_q^{2n+1} &= L^n(q)\,, \\ L_q^{2n} &= \{[z_0, \cdots, z_n]\in L^n(q)\,|\, z_n \text{ is real}\geq 0\}\ . \end{aligned}$$

By the several papers, we determined the KO-groups $\widetilde{KO}(S^j(L_q^m/L_q^n))$ of the suspensions of the stunted lens spaces L_q^m/L_q^n for the cases $j\equiv 1 \pmod 2$ [25], $q=2$ [12], $q=4$ [20] and $q=8$ [21]. Moreover we determined the J-groups $\tilde{J}(S^j(L_q^m/L_q^n))$ for the cases odd primes q [19], $q=2$ [18], $q=4$ [20] and $q=8$ [21]. The purpose of this paper is to determine the KO-groups $\widetilde{KO}(S^j(L_{2p}^m/L_{2p}^n))$ and J-groups $\tilde{J}(S^j(L_{2p}^m/L_{2p}^n))$ for odd primes p.

This paper is organized as follows. In section 2 we state the main theorems: Theorem 2 gives a direct sum decomposition of $\widetilde{KO}(S^j(L_{2^rq}^m/L_{2^rq}^n)$ for $j\equiv 0$ (mod 2), Theorem 3 gives a direct sum decomposition of $\tilde{J}(S^j(L_{2^rq}^m/L_{2^rq}^n))$ for $j\equiv 0$ (mod 2) and $n+j+1\not\equiv 0 \pmod 4$, Theorem 4 gives the structure of $\tilde{J}(S^j(L_{2p}^m/L_{2p}^n))$ for $j\equiv 0 \pmod 2$ and $n+j+1\equiv 0 \pmod 4$ and the necessary conditions for L_{2p}^m/L_{2p}^n and $L_{2p}^{m+t}/L_{2p}^{n+t}$ to be of the same stable homotopy type are given by Theorem 5 which is an application of Theorems 3 and 4. In section 3 we prepare some lemmas and recall known results in [12], [19] and [25]. The proofs of Theorem 2 and Theorem 3 are given in section 4. The proof of Theorem 4 is given in section 5. In the final section we give the proof of Theorem 5.

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2. Statement of results

Let $\nu_p(s)$ denote the exponent of the prime p in the prime power decomposition of s, and $\mathfrak{m}(s)$ the function defined on positive integers as follows (cf.

[3]):

$$\nu_p(\mathfrak{m}(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s \not\equiv 0 \pmod{(p-1)}) \\ 1+\nu_p(s) & (p \neq 2 \text{ and } s \equiv 0 \pmod{(p-1)}) \\ 1 & (p = 2 \text{ and } s \not\equiv 0 \pmod 2) \\ 2+\nu_2(s) & (p = 2 \text{ and } s \equiv 0 \pmod 2) . \end{cases}$$

Let $\boldsymbol{Z}/k$ denote the cyclic group $\boldsymbol{Z}/k\boldsymbol{Z}$ of order k. For the case $j \equiv 1 \pmod 2$, the following result is known.

Theorem 1. *Let q, j, m and n be non-negative integers with $m > n$, $j \equiv 1 \pmod 2$ and $q > 1$.*
(1) ([20] and [25]) *If $q \equiv 0 \pmod 2$, then we have*

$$\widetilde{KO}(S^j(L_q^m/L_q^n)) \cong \widetilde{KO}(S^j(RP(m)/RP(n)))$$

and

$$J(S^j(L_q^m/L_q^n)) \cong J(S^j(RP(m)/RP(n))) .$$

(2) ([19] and [25]) *If $q \equiv 1 \pmod 2$, then we have*

$$\widetilde{KO}(S^j(L_q^m/L_q^n)) \cong B(j, m) \oplus B(j+1, n)$$

and

$$J(S^j(L_q^m/L_q^n)) \cong C(j, m) \oplus C(j+1, n) ,$$

where

$$B(j, m) = \begin{cases} \widetilde{KO}(S^{j+m}) & (m \equiv 1 \pmod 2) \\ 0 & (m \equiv 0 \pmod 2) \end{cases}$$

and

$$C(j, m) = \begin{cases} J(S^{j+m}) & (m \equiv 1 \pmod 2) \\ 0 & (m \equiv 0 \pmod 2) . \end{cases}$$

REMARK. (1) The groups $\widetilde{KO}(S^j(RP(m)/RP(n)))$ and $J(S^j(RP(m)/RP(n)))$ are determined in [12] and [18] respectively.
(2) J-groups of the spheres are well known (cf. [3]).

Theorem 2. *Let j, m, n, q and r be non-negative integers with $m > n$, $j \equiv 0 \pmod 2$, $q \equiv 1 \pmod 2$, $q > 1$ and $r > 0$.*
(1) *If $n+j+1 \not\equiv 0 \pmod 4$, then we have*

$$\widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n)) \cong \widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^n)) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]})) .$$

(2) *If $j \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$ or $j \equiv 2 \pmod 4$ and $n+j+1 \equiv 4 \pmod 8$, then we have*

$$\widetilde{KO}(S^j(L_{2^r q}^m/L_{2^r q}^n)) \cong \boldsymbol{Z} \oplus \widetilde{KO}(S^j(L_{2^r}^m/L_{2^r}^{n+1})) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{n+1})) .$$

(3) *If* $j\equiv 2 \pmod 4$ *and* $n+j+1\equiv 0 \pmod 8$, *then we have*

$$\widetilde{KO}(S^j(L^m_{2^r q}/L^n_{2^r q})) \cong Z\oplus(\widetilde{KO}(S^j(L^m_{2^r}/L^{n+1}_{2^r}))/G)\oplus\widetilde{KO}(S^j(L^{2[m/2]}_q/L^{n+1}_q)),$$

where G denotes the kernel of the homomorphism

$$(p_{m,n})^!\colon \widetilde{KO}(S^j(L^m_{2^r}/L^{n+1}_{2^r})) \to \widetilde{KO}(S^j(L^m_{2^r}/L^n_{2^r}))$$

and ord G *is equal to* 2.

(4) $$\widetilde{KO}(S^j(L^{2[m/2]}_q/L^{2[n/2]}_q)) \cong \oplus_p \widetilde{KO}(S^j(L^{2[m/2]}_{p^{\nu_p(q)}}/L^{2[n/2]}_{p^{\nu_p(q)}})),$$

where p runs over all prime divisors of q.

REMARK. The partial results for the case $j=n=0$ of this theorem have been obtained (cf. [10]).

Theorem 3. *Let j, m, n, q and r be non-negative integers with* $m>n$, $j\equiv 0 \pmod 2$, $q\equiv 1 \pmod 2$, $q>1$ *and* $r>0$.

(1) *If* $n+j+1\not\equiv 0 \pmod 4$, *then we have*

$$\tilde{J}(S^j(L^m_{2^r q}/L^n_{2^r q})) \cong \tilde{J}(S^j(L^m_{2^r}/L^n_{2^r}))\oplus\tilde{J}(S^j(L^{2[m/2]}_q/L^{2[n/2]}_q)).$$

(2) $\tilde{J}(S^j(L^{2[m/2]}_q/L^{2[n/2]}_q))\cong \oplus_p \tilde{J}(S^j(L^{2[m/2]}_{p^{\nu_p(q)}}/L^{2[n/2]}_{p^{\nu_p(q)}}))$,
where p runs over all prime divisors of q.

REMARK. (1) In the cases $r=1$, 2 and 3, the groups $\tilde{J}(S^j(L^m_{2^r}/L^n_{2^r}))$ are determined in [18], [20] and [21] respectively.
(2) For odd primes p, $\tilde{J}(S^j(L^m_p/L^n_p))$ are determined in [19].
(3) The partial results for the case $j=n=0$ of this theorem have been obtained (cf. [10]).

For an integer n, $A(n)$ denotes the group defined by

$$(2.1)\qquad A(n)=\begin{cases} Z/2\oplus Z/2 & (n\equiv 0 \pmod 8)\\ Z/2 & (n\equiv 1 \text{ or } 7 \pmod 8)\\ 0 & (\text{otherwise}).\end{cases}$$

As defined in [1], we denote by $\varphi(m,n)$ the number of integers s with $n<s\leq m$ and $s\equiv 0, 1, 2$ or $4 \pmod 8$. Set

$$(2.2)\qquad \tilde{\varphi}(m,n)=\begin{cases}\varphi(m,n) & (n\not\equiv 3 \pmod 4)\\ \varphi(m,n+1) & (n\equiv 3 \pmod 4).\end{cases}$$

In order to state next theorem, we set $M=\mathfrak{m}((n+j+1)/2)$,

$$(2.3)\qquad a(j,m,n)=\begin{cases}\tilde{\varphi}(m,n) & (j=0)\\ \min\{\nu_2(j)+1,\ \tilde{\varphi}(m+j,n+j)\} & (j>0)\end{cases}$$

and

$$(2.4) \qquad b(j, m, n) = \begin{cases} b_0(m, n) & (j=0) \\ \min\{\nu_p(j)+1,\ b_0(m+j, n+j)\} & (j>0)\,, \end{cases}$$

where $b_0(m, n)=[m/2(p-1)]-[(n+1)/2(p-1)]$.

Theorem 4. *Let j, m, n be non-negative integers with $m>n$, and p be an odd prime.*
(1) *If $j\equiv 0 \pmod 4$ and $n\equiv 3 \pmod 4$, then we have*

$$\tilde{J}(S^j(L^m_{2p}/L^n_{2p})) \cong Z/M\cdot 2^{a(j,m,n)-i_2}\cdot p^{b(j,m,n)-i_p}\oplus Z/p^{i_p}\oplus Z/2^{i_2}\,,$$

where $i_2=\min\{a(j, m, n),\ \nu_2(n+1)\}$ and $i_p=\min\{b(j, m, n),\ \nu_p(n+1),\ \nu_p(M)\}$.
(2) *If $j\equiv 2 \pmod 4$ and $n\equiv 1 \pmod 4$ and $m>n+2$, Then we have*

$$\tilde{J}(S^j(L^m_{2p}/L^n_{2p})) \cong Z/M\cdot p^{b(j,m,n)-i_p}\oplus Z/p^{i_p}\oplus A(m+j-1)\,,$$

where $i_p=\min\ \{b(j, m, n),\ \nu_p(n+1),\ \nu_p(M)\}$ and $A(m+j-1)$ is the group defined by (2.1).

REMARK. In the case $m=n+1$, $S^j(L^{n+1}_q/L^n_q)$ is homeomorphic to the sphere S^{n+j+1}. Moreover in the case $m=n+2$, we have the homotopy equivalences

$$S^j(L^{n+2}_{2p}/L^n_{2p}) \simeq \begin{cases} S^{j+n+2}\vee S^{j+n+1} & (n\colon \text{odd}) \\ S^{j+n}(L^2_{2p}) & (n\colon \text{even})\,. \end{cases}$$

Hence, J-groups $\tilde{J}(S^j(L^m_{2p}/L^n_{2p})$ are determined completely.

Finally we consider the application of the above results. A space X is said to be stably homotopy equivalent to a space Y if there are non-negative integers u and v such that the u-fold suspension S^uX of X is homotopy equivalent to the v-fold suspension S^VY of Y. In order to state next theorem, we set

$$(2.5) \qquad \bar{\varphi}(m, n) = \max\{\tilde{\varphi}(m, n),\ \tilde{\varphi}(-n-2,\ -m-2)\}\,,$$

where $\tilde{\varphi}$ is the function defined by (2.2).

Theorem 5. *Let m, n and t be non-negative integers with $m>n+2$, and p be an odd prime. If L^m_{2p}/L^n_{2p} is stably homotopy equivalent to $L^{m+t}_{2p}/L^{n+t}_{2p}$, then*

$$\nu_2(t)\geq \begin{cases} \varphi(m-n-1, 0)\ (m\leq n+9 \text{ or } \max\{\nu_2(n+1),\ \nu_2(m+1)\}\geq \varphi(m-n-1, 1)) \\ \bar{\varphi}(m, n)-1 \quad (\text{otherwise}) \end{cases}$$

and

$$\nu_p(t)\geq \begin{cases} b(m, n) \quad (n+1\equiv 0 \pmod{2p^{b(m,n)}} \text{ or } m+1\equiv 0 \pmod{2p^{b(m,n)}}) \\ b(m, n+2) \quad (\text{otherwise}), \end{cases}$$

where $b(m, n)=[([m/2]-[(n+1)/2])/(p-1)]$.

REMARK. Theorem 5 shows that the necessary conditions for $q=2p$ coincide with the product of those for $q=2$ and $q=p$ (cf. [8] and [14]).

In order to state the final theorem, we prepare function α defined by

$$(2.6)\qquad \alpha(k, n)=\begin{cases} 1 & (n\equiv 0 \pmod 2 \text{ and } k\equiv 1 \pmod 8 \\ & \text{ or } n\equiv 1 \pmod 2 \text{ and } k=0) \\ 0 & (\text{otherwise}). \end{cases}$$

Theorem 6. *If* $k=m-2[(n+1)/2]\geq 2$ *and* $t\equiv 0$ $(\mathrm{mod}\ 2^{\varphi(k,0)-\alpha(k,n)}p^{[k/2(p-1)]})$, *then* L_{2p}^{m}/L_{2p}^{n} *and* $L_{2p}^{m+t}/L_{2p}^{n+t}$ *are of the same stable homotopy type.*

3. Preliminaries

In this section we recall known results and set up some lemmas needed later.

We begin by setting some notation. Let $\alpha_i(u, v)$ $(1\leq i\leq 8)$ be the integers defined by

$$(3.1)\qquad \begin{cases} (1) & \alpha_1(u, v)=\binom{2u}{u-v}(-1)^{u-v}, \\ (2) & \alpha_2(u, v)=\binom{u+v}{u-v}+\binom{u+v-1}{u-v-1}, \\ (3) & \alpha_3(u, v)=(\binom{2u-1}{u-v}-\binom{2u-1}{u-v-1})(-1)^{u-v}, \\ (4) & \alpha_4(u, v)=\binom{u+v-1}{u-v}, \\ (5) & \alpha_5(u, v)=\binom{v}{u-v}+\binom{v-1}{u-v-1}, \\ (6) & \alpha_6(u, v)=\binom{2u-v-1}{u-v}(-1)^{u-v}, \\ (7) & \alpha_7(u, v)=\binom{v-1}{u-v}, \\ (8) & \alpha_8(u, v)=(\binom{2u-v-2}{u-v}-\binom{2u-v-2}{u-v-1})(-1)^{u-v}. \end{cases}$$

Then we have following lemma.

Lemma 3.2. *We have the following equalities*:

(1) $\alpha_1(u+1, v)=\alpha_1(u, v+1)-2\alpha_1(u, v)+\alpha_1(u, v-1)$,

(2) $\alpha_2(u+1, v)=\alpha_2(u, v-1)+2\alpha_2(u, v)-\alpha_2(u-1, v)$,

(3) $\alpha_3(u+1, v) = \alpha_3(u, v+1) - 2\alpha_3(u, v) + \alpha_3(u, v-1)$,
(4) $\alpha_4(u+1, v) = \alpha_4(u, v-1) + 2\alpha_4(u, v) - \alpha_4(u-1, v)$,
(5) $\alpha_5(u+1, v) = \alpha_5(u, v-1) + \alpha_5(u-1, v-1)$,
(6) $\alpha_6(u+1, v) = \alpha_6(u, v-1) - \alpha_6(u+1, v+1)$,
(7) $\alpha_7(u+1, v) = \alpha_7(u, v-1) + \alpha_7(u-1, v-1)$,
(8) $\alpha_8(u+1, v) = \alpha_8(u, v-1) - \alpha_8(u+1, v+1)$.

Proof. By the definition (3.1), we have

$$\begin{aligned}
\alpha_1(u+1, v) &= \binom{2u+2}{u-v+1}(-1)^{u+1-v} \\
&= \left(\binom{2u}{u-v+1} + 2\binom{2u}{u-v} + \binom{2u}{u-v-1}\right)(-1)^{u+1-v} \\
&= \alpha_1(u, v-1) - 2\alpha_1(u, v) + \alpha_1(u, v+1), \\
\alpha_4(u+1, v) &= \binom{u+v}{u-v+1} \\
&= \binom{u+v-2}{u-v+1} + 2\binom{u+v-2}{u-v} + \binom{u+v-2}{u-v-1} \\
&= \binom{u+v-2}{u-v+1} + 2\binom{u+v-1}{u-v} - \binom{u+v-2}{u-v-1} \\
&= \alpha_4(u, v-1) + 2\alpha_4(u, v) - \alpha_4(u-1, v), \\
\alpha_6(u+1, v) &= \binom{2u-v+1}{u-v+1}(-1)^{u-v+1} \\
&= \left(\binom{2u-v}{u-v+1} + \binom{2u-v}{u-v}\right)(-1)^{u+1-v} \\
&= \alpha_6(u, v-1) - \alpha_6(u+1, v+1), \\
\alpha_7(u+1, v) &= \binom{v-1}{u-v+1} \\
&= \binom{v-2}{u-v+1} + \binom{v-2}{u-v} \\
&= \alpha_7(u, v-1) + \alpha_7(u-1, v-1).
\end{aligned}$$

Thus the equalities (1), (4), (6) and (7) are established.

By making use of the equalities

$$\begin{aligned}
\alpha_2(u, v) &= \alpha_4(u+1, v+1) - \alpha_4(u-1, v+1)\,, \\
\alpha_3(u, v) &= \alpha_1(u-1, v-1) - \alpha_1(u-1, v+1)\,, \\
\alpha_5(u, v) &= \alpha_7(u+1, v+1) + \alpha_7(u-1, v)
\end{aligned}$$

and $\alpha_8(u, v) = \alpha_6(u-1, v-1) + \alpha_6(u, v+1)$, (2), (3), (5) and (8) follows from (4), (1), (7) and (6) respectively. q.e.d.

Lemma 3.3. *In the polynomial ring $\boldsymbol{Z}[x]$, the following equalities hold, where i denotes a positive integer.*

(1) $\sum_{u=1}^{i} \alpha_2(i, u) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k = x^i$.
(2) $\sum_{u=1}^{i} \alpha_4(i, u) \sum_{k=1}^{u} \alpha_3(u, k)\, x^k = x^i$.
(3) $\sum_{u=1}^{i} \alpha_6(i, u) \sum_{k=1}^{u} \alpha_5(u, k)\, x^k = x^i$.
(4) $\sum_{u=1}^{i} \alpha_8(i, u) \sum_{k=1}^{u} \alpha_7(u, k)\, x^k = x^i$.

Proof. (1) Since $\alpha_1(1, 1)=\alpha_2(1, 1)=\alpha_1(2, 2)=\alpha_2(2, 2)=1$ and $-\alpha_1(2, 1) =\alpha_2(2, 1)=4$, the equality holds for $1\leq i\leq 2$. We argue by induction over i; let us assume that $i\geq 2$ and the result is true for i and $i-1$. Using Lemma 3.2 and the inductive hypothesis, we have

$$\begin{aligned}
&\sum_{u=1}^{i+1} \alpha_2(i+1, u) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k \\
&= \sum_{u=1}^{i+1}(\alpha_2(i, u-1)+2\alpha_2(i, u)-\alpha_2(i-1, u)) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k \\
&= \sum_{u=1}^{i+1} \alpha_2(i, u-1) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k+2 \sum_{u=1}^{i} \alpha_2(i, u) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k \\
&\qquad -\sum_{u=1}^{i-1} \alpha_2(i-1, u) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k \\
&= \sum_{u=0}^{i} \alpha_2(i, u) \sum_{k=1}^{u+1} \alpha_1(u+1, k)\, x^k+2x^i-x^{i-1} \\
&= \sum_{u=0}^{i} \alpha_2(i, u) \sum_{k=1}^{u+1}(\alpha_1(u, k+1)-2\alpha_1(u, k)+\alpha_1(u, k-1))\, x^k+2x^i-x^{i-1} \\
&= \sum_{u=0}^{i} \alpha_2(i, u) \sum_{k=1}^{u-1} \alpha_1(u, k+1)\, x^k-2 \sum_{u=1}^{i} \alpha_2(i, u) \sum_{k=1}^{u} \alpha_1(u, k)\, x^k \\
&\qquad +\sum_{u=0}^{i} \alpha_2(i, u) \sum_{k=1}^{u+1} \alpha_1(u, k-1)\, x^k+2x^i-x^{i-1} \\
&= \sum_{u=1}^{i} \alpha_2(i, u) \sum_{k=2}^{u} \alpha_1(u, k)\, x^{k-1}-2x^i+x^{i+1}+\sum_{u=0}^{i} \alpha_2(i, u)\, \alpha_1(u, 0)\, x \\
&\qquad +2x^i-x^{i-1} \\
&= x^{i-1}-\sum_{u=1}^{i} \alpha_2(i, u)\, \alpha_1(u, 1)+x^{i+1}+\sum_{u=0}^{i} \alpha_2(i, u)\, \alpha_1(u, 0)\, x-x^{i-1} \\
&= x^{i+1}+\sum_{u=0}^{i} \alpha_2(i, u)\, (\alpha_1(u+1, 1)-\alpha_1(u, 2)+2\alpha_1(u, 1))\, x \\
&= x^{i+1}+\sum_{u=1}^{i+1} \alpha_2(i, u-1)\, \alpha_1(u, 1)\, x-0^{i-2}\, x \\
&= x^{i+1}+\sum_{u=1}^{i+1}(\alpha_2(i+1, u)+\alpha_2(i-1, u)-2\alpha_2(i, u))\, \alpha_1(u, 1)\, x-0^{i-2}\, x \\
&= x^{i+1}+\sum_{u=1}^{i-1} \alpha_2(i-1, u)\, \alpha_1(u, 1)\, x-0^{i-2}\, x = x^{i+1}.
\end{aligned}$$

This completes the induction.

The proof of (2) is similar to that of (1).

(3) Let $\beta_3(i, k)$ be the integer defined by

$$\beta_3(i, k) = \sum_{u=k}^{i} \alpha_6(i, u)\, \alpha_5(u, k)\,.$$

It suffices to prove

$$(*) \qquad \beta_3(i, k) = \begin{cases} 1 & (i = k\geq 0) \\ 0 & (i > k\geq 0)\,. \end{cases}$$

Since $\alpha_5(i, i)=\alpha_6(i, i)=1$ and $\alpha_5(k+1, k)=-\alpha_6(k+1, k)=k+1$, (*) holds for $k\leq i\leq k+1$. Assume that $i\geq k+2\geq 2$. Then we have

$$\begin{aligned}
\beta_3(i, k) &= \sum_{u=k}^{i} \alpha_6(i, u)\, \alpha_5(u, k) \\
&= \binom{2i-k-1}{i-k}(-1)^{i-k}+\sum_{u=k+1}^{i} \binom{2i-u-1}{i-u}(-1)^{i-u}(u/(u-k))\binom{k-1}{u-k-1} \\
&= \binom{2i-k-1}{i-k}(-1)^{i-k}+\sum_{u=k+1}^{i} \binom{2i-u-1}{i-u}(-1)^{i-u}((2i-u)/(i-k))\binom{k-1}{u-k-1}
\end{aligned}$$

$$+\sum_{u=k+1}^{i}\binom{2i-u-1}{i-u}(-1)^{i-u}((u/(u-k))-((2i-u)/(i-k)))\binom{k-1}{u-k-1}$$
$$=\binom{2i-k-1}{i-k}(-1)^{i-k}-\sum_{u=k}^{i-1}\binom{2i-u-2}{i-u-1}(-1)^{i-u}((2i-u-1)/(i-k))\binom{k-1}{u-k}$$
$$+\sum_{u=k+1}^{i-1}\binom{2i-u-1}{i-u}(-1)^{i-u}((2k-u)(i-u)/(u-k)(i-k))\binom{k-1}{u-k-1}$$
$$=-\sum_{u=k+1}^{i-1}\binom{2i-u-1}{i-u}(-1)^{i-u}((i-u)/(i-k))\binom{k-1}{u-k}$$
$$+\sum_{u=k+1}^{i-1}\binom{2i-u-1}{i-u}(-1)^{i-u}((i-u)/(i-k))\binom{k-1}{u-k}$$
$$=0\,.$$

This completes the proof of (3).

(4) Let $\beta_4(i,k)$ be the integer defined by

$$\beta_4(i,k)=\sum_{u=k}^{i}\alpha_8(i,u)\,\alpha_7(u,k)\,.$$

It suffices to prove

$$(**)\qquad \beta_4(i,k)=\begin{cases}1 & (i=k\geq 1)\\ 0 & (i>k\geq 1)\,.\end{cases}$$

Since $\alpha_7(i,i)=\alpha_8(i,i)=1$ and $\alpha_7(k+1,k)=-\alpha_8(k+1,k)=k-1$, (**) holds for $k\leq i\leq k+1$. Assume that $i\geq k+2\geq 3$. Then we have

$$\beta_4(i,k)=\sum_{u=k}^{i}\alpha_8(i,u)\,\alpha_7(u,k)$$
$$=\sum_{u=k}^{i}\binom{2i-u-2}{i-u}((u-1)/(i-1))(-1)^{i-u}\binom{k-1}{u-k}$$
$$=\sum_{u=k-1}^{i-1}\binom{2i-u-3}{i-u-1}(u/(i-1))(-1)^{i-u-1}\binom{k-1}{u-k+1}$$
$$=\sum_{u=k-1}^{i-1}\binom{2i-u-3}{i-u-1}(-1)^{i-u-1}((k-1+u-k+1)/(i-1))\binom{k-1}{u-k+1}$$
$$=((k-1)/(i-1))\sum_{u=k-1}^{i-1}\binom{2i-u-3}{i-u-1}(-1)^{i-u-1}\left(\binom{k-1}{u-k+1}+\binom{k-2}{u-k}\right)$$
$$=((k-1)/(i-1))\sum_{u=k-1}^{i-1}\alpha_6(i-1,u)\,\alpha_5(u,k-1)$$
$$=((k-1)/((i-1))\;\beta_3(i-1,k-1)=0.$$

This completes the proof of (4). q.e.d.

In the rest of this section j denotes non-negative integer with $j\equiv 0 \pmod 2$. Considering the $(\boldsymbol{Z}/q)$-action on $S^{2n+1}\times\boldsymbol{C}$ given by

$$\exp(2\pi\sqrt{-1}/q)\,(z,u)=(z\cdot\exp(2\pi\sqrt{-1}/q),\,u\cdot\exp(2\pi\sqrt{-1}/q))$$

for $(z,u)\in S^{2n+1}\times\boldsymbol{C}$, we have a complex line bundle

$$\eta_q\colon (S^{2n+1}\times\boldsymbol{C})/(\boldsymbol{Z}/q)\to L_q^{2n+1}\,.$$

Set

$$\sigma_q = \eta_q - 1 \in \tilde{K}(L_q^{2n+1}). \tag{3.4}$$

We also denote by σ_q the restriction of σ_q to L_q^{2n}. Considering the $(\boldsymbol{Z}/2q)$-action on $S^{2n+1}\times\boldsymbol{R}$ given by

$$\exp(2\pi\sqrt{-1}/2q)(z, u) = (z\cdot\exp(2\pi\sqrt{-1}/2q), -u)$$

for $(z, u)\in S^{2n+1}\times\boldsymbol{R}$, we have a real line bundle

$$\nu_{2q}\colon (S^{2n+1}\times\boldsymbol{R})/(\boldsymbol{Z}/2q) \to L_{2q}^{2n+1}.$$

Set

$$\kappa_{2q} = \nu_{2q} - 1 \in \widetilde{KO}(L_{2q}^{2n+1}). \tag{3.5}$$

We also denote by κ_{2q} the restriction of κ_{2q} to L_{2q}^{2n}.

For each integer n with $0\leq n<m$, we denote the inclusion map of L_q^n into L_q^m by i_n^m, and the kernel of homomorphism

$$(i_n^m)^!\colon \widetilde{KO}(S^jL_q^m) \to \widetilde{KO}(S^jL_q^n)$$

by $VO_{m,n}^j(q)$. We set

$$UO_{m,n}^j(q) = \textstyle\sum_k \bigcap_e k^e(\psi^k - 1)\, VO_{m,n}^j(q). \tag{3.6}$$

Let $a_i(q)$, $b_i(q)$ and $c_i(q)$ $(i>0)$ be elements of $\widetilde{KO}(S^jL_q^m)$ defined by

$$\begin{cases} a_i(q) = r(I^{j/2}(\eta_q^i - 1)) \\ b_i(q) = \begin{cases} \sum_{u=1}^i \alpha_1(i, u)\, a_u(q) & (j\equiv 0 \pmod 4) \\ \sum_{u=1}^i \alpha_3(i, u)\, a_u(q) & (j\equiv 2 \pmod 4) \end{cases} \\ c_i(q) = r(I^{j/2}(\sigma_q^i)), \end{cases} \tag{3.7}$$

where $r\colon K\to KO$ denotes the real restriction and $I\colon \tilde{K}(X)\to\tilde{K}(S^2X)$ is the Bott periodicity isomorphism.

We define the function

$$\mu_q\colon \boldsymbol{Z} \to \boldsymbol{Z} \tag{3.8}$$

by setting $\mu_q(k)$ to be the remainder of k divided by q for every $k\in\boldsymbol{Z}$.

Lemma 3.9. *The elements $a_i(q)$, $b_i(q)$ and $c_i(q)$ satisfy following relations.*

(1) $a_1(q) = b_1(q) = c_1(q)$.

(2) $a_i(q) = \begin{cases} \sum_{u=1}^i \alpha_2(i, u)\, b_u(q) & (j\equiv 0 \pmod 4) \\ \sum_{u=1}^i \alpha_4(i, u)\, b_u(q) & (j\equiv 2 \pmod 4). \end{cases}$

(3) $a_i(q) = \sum_{u=1}^i \binom{i}{u} c_u(q)$.

(4) $c_i(q) = \sum_{u=1}^{i} \binom{i}{u} (-1)^{i-u} a_u(q)$.

(5) $c_i(q) = \begin{cases} \sum_{u=1}^{i} \alpha_5(i, u)\, b_u(q) & (j \equiv 0 \pmod 4) \\ \sum_{u=1}^{i} \alpha_7(i, u)\, b_u(q) & (j \equiv 2 \pmod 4)\,. \end{cases}$

(6) $b_i(q) = \begin{cases} \sum_{u=1}^{i} \alpha_6(i, u)\, c_u(q) & (j \equiv 0 \pmod 4) \\ \sum_{u=1}^{i} \alpha_8(i, u)\, c_u(q) & (j \equiv 2 \pmod 4)\,. \end{cases}$

(7) $a_i(q) = a_{\mu_q(i)}(q) = \begin{cases} a_{q-\mu_q(i)}(q) & (j \equiv 0 \pmod 4) \\ -a_{q-\mu_q(i)}(q) & (j \equiv 2 \pmod 4)\,. \end{cases}$

(8) For the Adams operation ψ^k, we have $\psi^k(a_i(q)) = k^{j/2}\, a_{ki}(q)$.

Proof. (1), (3) and (4) are evident from the definition (3.7).

(2) Suppose that $j \equiv 0 \pmod 4$. It follows from the definition (3.7) that wc have

$$\sum_{u=1}^{i} \alpha_2(i, u)\, b_u(q) = \sum_{u=1}^{i} \alpha_2(i, u) \sum_{k=1}^{u} \alpha_1(u, k)\, a_k(q) = a_i(q)$$

by (1) of Lemma 3.3. The proof of the case $j \equiv 2 \pmod 4$ is similar by making use of (2) of Lemma 3.3.

(5) Suppose that $j \equiv 0 \pmod 4$. By (4) and (2) we have

$$\begin{aligned} c_i(q) &= \sum_{k=1}^{i} \binom{i}{k} (-1)^{i-k} a_k(q) \\ &= \sum_{k=1}^{i} \binom{i}{k} (-1)^{i-k} \sum_{u=1}^{k} \alpha_2(k, u)\, b_u(q) \\ &= \sum_{u=1}^{i} \sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \alpha_2(k, u)\, b_u(q). \end{aligned}$$

It suffices to prove

$$(*) \qquad \sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \alpha_2(k, u) = \alpha_5(i, u) \quad (i \geq u \geq 1)\,.$$

Since we have

$$\begin{aligned} &\sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \alpha_2(k, u) = \sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} (2k/(k+u)) \binom{k+u}{k-u} \\ &= \sum_{j=0}^{i-u} \binom{i}{u+j} (-1)^{i-u-j} (2(u+j)/(2u+j)) \binom{2u+j}{j} \\ &= (2(i!)/((i-u)!\,(2u)!)) \sum_{j=0}^{i-u} \binom{i-u}{j} (-1)^{i-u-j} (2u+j-1) \cdots (u+j) \\ &= \begin{cases} 0 & (i > 2u) \\ 2 & (i = 2u) \end{cases} \end{aligned}$$

by [22, Lemma 3.7], (*) holds for $i \geq 2u$. Since we have

$$\sum_{k=u}^{u}\binom{u}{k}(-1)^{u-k}\alpha_2(k,u)=1$$

and

$$\sum_{k=u}^{u+1}\binom{u+1}{k}(-1)^{u-k+1}\alpha_2(k,u)=-(u+1)\,\alpha_2(u,u)+\alpha_2(u+1,u)$$
$$=-(u+1)+2u+1+1=u+1=\alpha_5(u+1,u)\,,$$

$(*)$ holds for $u\leq i\leq u+1$. In particular, $(*)$ holds for $u=1$. We argue by induction over $i-u$ and u; let us assume that $i\geq u+1\geq 3$ and the result is true for $(i,u-1)$ and $(i-1,\,u-1)$. Using Lemma 3.2 and the inductive hypothesis, we have

$$\begin{aligned}
&\sum_{k=u}^{i+1}\binom{i+1}{k}(-1)^{i-k+1}\alpha_2(k,u)\\
&=\sum_{k=u}^{i+1}\left(\binom{i-1}{k}+2\binom{i-1}{k-1}+\binom{i-1}{k-2}\right)(-1)^{i-k+1}\alpha_2(k,u)\\
&=\sum_{k=u}^{i-1}\binom{i-1}{k}(-1)^{i-k+1}\alpha_2(k,u)+2\sum_{k=u}^{i}\binom{i-1}{k-1}(-1)^{i-k+1}\alpha_2(k,u)\\
&\quad+\sum_{k=u}^{i+1}\binom{i-1}{k-2}(-1)^{i-k+1}\alpha_2(k,u)\\
&=\sum_{k=u}^{i-1}\binom{i-1}{k}(-1)^{i-k+1}\alpha_2(k,u)+2\sum_{k=u-1}^{i-1}\binom{i-1}{k}(-1)^{i-k}\alpha_2(k+1,u)\\
&\quad+\sum_{k=u-2}^{i-1}\binom{i-1}{k}(-1)^{i-k-1}\alpha_2(k+2,u)\\
&=\sum_{k=u-2}^{i-1}\binom{i-1}{k}(-1)^{i-k-1}(\alpha_2(k,u)-2\alpha_2(k+1,u)+\alpha_2(k+2,u))\\
&=\sum_{k=u-2}^{i-1}\binom{i-1}{k}(-1)^{i-k-1}\alpha_2(k+1,u-1)\\
&=\sum_{k=u-1}^{i}\binom{i-1}{k-1}(-1)^{i-k}\alpha_2(k,u-1)\\
&=\sum_{k=u-1}^{i}\left(\binom{i}{k}-\binom{i-1}{k}\right)(-1)^{i-k}\alpha_2(k,u-1)\\
&=\sum_{k=u-1}^{i}\binom{i}{k}(-1)^{i-k}\alpha_2(k,u-1)+\sum_{k=u-1}^{i-1}\binom{i-1}{k}(-1)^{i-1-k}\alpha_2(k,u-1)\\
&=\alpha_5(i,u-1)+\alpha_5(i-1,u-1)=\alpha_5(i+1,u)\,.
\end{aligned}$$

This completes the induction.

Suppose that $j\equiv 2 \pmod 4$. By (4) and (2) we have

$$\begin{aligned}
c_i(q)&=\sum_{k=1}^{i}\binom{i}{k}(-1)^{i-k}a_k(q)\\
&=\sum_{k=1}^{i}\binom{i}{k}(-1)^{i-k}\sum_{u=1}^{k}\alpha_4(k,u)\,b_u(q)
\end{aligned}$$

$$= \sum_{u=1}^{i} \sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \alpha_4(k, u)\, b_u(q)\,.$$

It suffices to prove

$$(**) \qquad \sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \alpha_4(k, u) = \alpha_7(i, u) \quad (i \geq u \geq 1)\,.$$

Since we have

$$\sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \alpha_4(k, u) = \sum_{k=u}^{i} \binom{i}{k} (-1)^{i-k} \binom{k+u-1}{k-u}$$
$$= \sum_{j=0}^{i-u} \binom{i}{u+j} (-1)^{i-u-j} \binom{2u+j-1}{j}$$
$$= ((i!)/((i-u)!\,(2u-1)!)) \sum_{j=0}^{i-u} \binom{i-u}{j} (-1)^{i-u-j} (2u+j-1) \cdots (u+j+1)$$
$$= \begin{cases} 0 & (i \geq 2u) \\ 1 & (i = 2u-1) \end{cases}$$

by [22, Lemma 3.7], (**) holds for $i \geq 2u-1$. Since we have

$$\sum_{k=u}^{u} \binom{u}{k} (-1)^{u-k} \alpha_4(k, u) = 1$$

and

$$\sum_{k=u}^{u+1} \binom{u+1}{k} (-1)^{u-k+1} \alpha_4(k, u) = -(u+1)\,\alpha_4(u, u) + \alpha_4(u+1, u)$$
$$= -(u+1) + 2u = u-1 = \alpha_7(u+1, u)\,,$$

(**) holds for $u \leq i \leq u+1$. In particular, (**) holds for $u=1$. The rest of the proof is similar to that for the case $j \equiv 2 \pmod 4$.

(6) Suppose that $j \equiv 0 \pmod 4$. It follows from (5) that we have

$$\sum_{u=1}^{i} \alpha_6(i, u)\, c_u(q) = \sum_{u=1}^{i} \alpha_6(i, u) \sum_{k=1}^{u} \alpha_5(u, k)\, b_k(q) = b_i(q)$$

by (3) of Lemma 3.3. The proof of the case $j \equiv 2 \pmod 4$ is similar by making use of (4) of Lemma 3.3.

(7) is obtained by the properties $\eta_q^q = 1$ and $r \circ t = r$, where $t\colon K \to K$ denotes the complex conjugation.

(8) is immediately obtained by [1] and [4]. q.e.d.

Now we prepare some notations. Set

(3.10)

(1) $A(d, u, i) = \sum_{k=0}^{2u-1} (-1)^{2u-1-k} \binom{2u-1}{k} \alpha_2(d+k-u+1, i)\,.$

(2) $\beta_{u,i} = (-1)^i \binom{2u-1}{i} + 2\sum_{v=0}^{i-1} (-1)^v \binom{2u-1}{v}\,.$

(3) $B(d, u, i) = \sum_{k=0}^{2u-1} \beta_{u,k}\, \alpha_4(d+k-u+1, i)\,.$

Then we have the following lemma.

Lemma 3.11. *Let u be a positive integer. Then we have*

(1) $A(d, u, i) = \alpha_5(2d+1, d+i-u+1)$.

(2) $\beta_{u,i} = 0 \quad (i \geq 2u$ *or* $i < 0)$.

(3) $\beta_{u,2u-1-i} = \beta_{u,i}$.

(4) $\beta_{u,i+1} - \beta_{u,i} = \alpha_3(u, u-i-1)$.

(5) $B(d, u, i) = A(d, u, i)$.

Proof. (1) If $u=1$, then we have

$$\begin{aligned} A(d, u, i) &= \sum_{k=0}^{1}(-1)^{1-k}\binom{1}{k}\alpha_2(d+k, i) \\ &= \alpha_2(d+1, i) - \alpha_2(d, i) \\ &= \binom{d+1+i}{d+1-i} - \binom{d-1+i}{d-1-i} \\ &= \binom{d+i}{d+1-i} + \binom{d+i-1}{d-i} \\ &= \alpha_5(2d+1, d+i)\,. \end{aligned}$$

This implies (1) for the case $u=1$. If $u+1>1$, then we have

$$\begin{aligned} A(d, u+1, i) &= \sum_{k=0}^{2u+1}(-1)^{k+1}\binom{2u+1}{k}\alpha_2(d+k-u, i) \\ &= \sum_{k=0}^{2u+1}(-1)^{k+1}\left(\binom{2u-1}{k} + 2\binom{2u-1}{k-1} + \binom{2u-1}{k-2}\right)\alpha_2(d+k-u, i) \\ &= \sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}\alpha_2(d+k-u, i) \\ &\quad -2\sum_{k=0}^{2u-1}(-1)^{k}\binom{2u-1}{k}\alpha_2(d+k-u+1, i) + \sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k} \\ &\quad \alpha_2(d+k-u+2, i) \\ &= \sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}(\alpha_2(d+k-u, i) - 2\alpha_2(d+k-u+1, i) \\ &\quad +\alpha_2(d+k-u+2, i)) \\ &= \sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}\alpha_2(d+k-u+1, i-1) = A(d, u, i-1)\,. \end{aligned}$$

Thus (1) is proved by the induction with respect to u.

(2) is evident from the definition (3.10) (2).

(3) Suppose $0 \leq i \leq 2u-1$. Then we have

$$\begin{aligned} &\beta_{u,i} - \beta_{u,2u-1-i} \\ &\quad = 2(-1)^{i}\binom{2u-1}{i} + 2\sum_{v=0}^{i-1}(-1)^{v}\binom{2u-1}{v} - 2\sum_{v=0}^{2u-i-2}(-1)^{v}\binom{2u-1}{v} \end{aligned}$$

$$= 2\sum_{v=0}^{i}(-1)^v\binom{2u-1}{v}+2\sum_{v=i+1}^{2u-1}(-1)^v\binom{2u-1}{v}$$
$$= 2\sum_{v=0}^{2u-1}(-1)^v\binom{2u-1}{v}=0\,.$$

(4) Suppose $-1\leq i\leq 2u-1$. Then we have

$$\beta_{u,i+1}-\beta_{u,i}=(-1)^{i+1}\binom{2u-1}{i+1}+2(-1)^i\binom{2u-1}{i}-(-1)^i\binom{2u-1}{i}$$
$$=\alpha_3(u,u-i-1)\,.$$

(5) $$\begin{aligned}B(d,u,i) &= \textstyle\sum_{k=0}^{2u-1}\beta_{u,k}\,\alpha_4(d+k-u+1,i)\\
&= \textstyle\sum_{k=0}^{2u-1}\beta_{u,k}(\alpha_4(d+k-u+2,i+1)-2\alpha_4(d+k-u+1,i+1)\\
&\quad +\alpha_4(d+k-u,i+1))\\
&= \textstyle\sum_{k=0}^{2u-1}\beta_{u,k}(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&\quad -\textstyle\sum_{k=0}^{2u-1}\beta_{u,k}(\alpha_4(d+k-u+1,i+1)-\alpha_4(d+k-u,i+1))\\
&= \textstyle\sum_{k=0}^{2u-1}\beta_{u,k}(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&\quad -\textstyle\sum_{k=-1}^{2u-2}\beta_{u,k+1}(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&= \textstyle\sum_{k=-1}^{2u-1}(\beta_{u,k}-\beta_{u,k+1})(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&= \textstyle\sum_{k=-1}^{2u-1}-\alpha_3(u,u-k-1)(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&= \textstyle\sum_{k=-1}^{2u-1}(-1)^{k+1}\left(\binom{2u-1}{k}-\binom{2u-1}{k+1}\right)(\alpha_4(d+k-u+2,i+1)\\
&\quad -\alpha_4(d+k-u+1,i+1))\\
&= \textstyle\sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&\quad +\textstyle\sum_{k=-1}^{2u-2}(-1)^{k+2}\binom{2u-1}{k+1}(\alpha_4(d+k-u+2,i+1)\\
&\quad -\alpha_4(d+k-u+1,i+1))\\
&= \textstyle\sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u+1,i+1))\\
&\quad +\textstyle\sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}(\alpha_4(d+k-u+1,i+1)-\alpha_4(d+k-u,i+1))\\
&= \textstyle\sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}(\alpha_4(d+k-u+2,i+1)-\alpha_4(d+k-u,i+1))\\
&= \textstyle\sum_{k=0}^{2u-1}(-1)^{k+1}\binom{2u-1}{k}\alpha_2(d+k-u+1,i)\\
&= A(d,u,i).\end{aligned}$$

q.e.d.

Lemma 3.12. *Let $q\geq 3$ be an odd integer and $d=(q-1)/2$. Then we have*

$$b_{d+u}(q)=-\sum_{i=1}^{d}\alpha_5(q,d+i)\,b_{i+u-1}(q)\,,$$

where u is a positive integer.

Proof. Suppose $j\equiv 0 \pmod 4$. Then by Lemma 3.9, we have

$$a_{q-i}(q) = a_i(q) \quad (0 \leq i \leq q).$$

If $0 < u \leq d+1$, then we have

$$\begin{aligned}
0 &= \textstyle\sum_{k=0}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} a_{d+k-u+1}(q) \\
&= \textstyle\sum_{k=0}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} (\sum_{i=1}^{d+k-u+1} \alpha_2(d+k-u+1, i)\, b_i(q)) \\
&= \textstyle\sum_{i=1}^{d+u}(\sum_{k=0}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} \alpha_2(d+k-u+1, i)\, b_i(q)) \\
&= \textstyle\sum_{i=1}^{d+u} A(d, u, i)\, b_i(q) \\
&= \textstyle\sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q).
\end{aligned}$$

If $u > d+1$, then we have

$$\begin{aligned}
0 &= \textstyle\sum_{k=u-d-1}^{u+d}(-1)^{2u-1-k}\binom{2u-1}{k} a_{d+k-u+1}(q) \\
&= -\textstyle\sum_{k=u+d+1}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} a_{-d+k-u}(q) \\
&\quad +\textstyle\sum_{k=u-d}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} a_{d+k-u+1}(q) \\
&= -\textstyle\sum_{k=u+d+1}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} (\sum_{i=1}^{-d+k-u} \alpha_2(k-d-u, i)\, b_i(q)) \\
&\quad +\textstyle\sum_{k=u-d}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} (\sum_{i=1}^{d+k-u+1} \alpha_2(d+k-u+1, i)\, b_i(q)) \\
&= \textstyle\sum_{i=1}^{u-d-1}(\sum_{k=d+u+1}^{2u-1}(-1)^{2u-k}\binom{2u-1}{k} \alpha_2(k-d-u, i))\, b_i(q) \\
&\quad +\textstyle\sum_{i=1}^{d+u}(\sum_{k=u-d}^{2u-1}(-1)^{2u-1-k}\binom{2u-1}{k} \alpha_2(d+k-u+1, i))\, b_i(q) \\
&= \textstyle\sum_{i=1}^{u-d-1} -A(-d-1, u, i)\, b_i(q) + \sum_{i=1}^{d+u} A(d, u, i)\, b_i(q) \\
&= \textstyle\sum_{i=1}^{u-d-1} -\alpha_5(-2d-1, -d+i-u)\, b_i(q) + \sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q) \\
&= \textstyle\sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q).
\end{aligned}$$

Suppose $j \equiv 2 \pmod 4$. Then by Lemma 3.9, we have

$$a_{q-i}(q) = -a_i(q) \quad (0 \leq i \leq q).$$

If $0 < u \leq d+1$, then we have

$$\begin{aligned}
0 &= \textstyle\sum_{k=0}^{2u-1} \beta_{u,k}\, a_{d+k-u+1}(q) \\
&= \textstyle\sum_{k=0}^{2u-1} \beta_{u,k}(\sum_{i=1}^{d+k-u+1} \alpha_4(d+k-u+1, i)\, b_i(q)) \\
&= \textstyle\sum_{i=1}^{d+u}(\sum_{k=0}^{2u-1} \beta_{u,k}\, \alpha_4(d+k-u+1, i))\, b_i(q) \\
&= \textstyle\sum_{i=1}^{d+u} B(d, u, i)\, b_i(q) \\
&= \textstyle\sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q) \\
&= \textstyle\sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q).
\end{aligned}$$

If $u > d+1$, then we have

$$\begin{aligned}
0 &= \textstyle\sum_{k=u-d-1}^{u+d} \beta_{u,k}\, a_{d+k-u+1}(q) \\
&= -\textstyle\sum_{k=d+u+1}^{2u-1} \beta_{u,k}\, a_{-d+k-u}(q) + \sum_{k=u-d}^{2u-1} \beta_{u,k}\, a_{d+k-u+1}(q)
\end{aligned}$$

$$
\begin{aligned}
&= -\textstyle\sum_{k=d+u+1}^{2u-1} \beta_{u,k}(\sum_{i=1}^{-d+k-u} \alpha_4(-d+k-u, i)\, b_i(q)) \\
&\quad +\textstyle\sum_{k=u-d}^{2u-1} \beta_{u,k}(\sum_{i=1}^{d+k-u+1} \alpha_4(d+k-u+1, i)\, b_i(q)) \\
&= -\textstyle\sum_{i=1}^{u-d-1}(\sum_{k=d+u+1}^{2u-1} \beta_{u,k}\, \alpha_4(-d+k-u, i))\, b_i(q) \\
&\quad +\textstyle\sum_{i=1}^{d+u}(\sum_{k=u-d}^{2u-1} \beta_{u,k}\, \alpha_4(d+k-u+1, i))\, b_i(q) \\
&= \textstyle\sum_{i=1}^{u-d-1} -B(-d-1, u, i)\, b_i(q)+\sum_{i=1}^{d+u} B(d, u, i)\, b_i(q) \\
&= \textstyle\sum_{i=1}^{u-d-1} -\alpha_5(-2d-1, -d+i-u)\, b_i(q)+\sum_{i=1}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q) \\
&= \textstyle\sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q).
\end{aligned}
$$

Thus we have

$$0 = \textstyle\sum_{i=u}^{d+u} \alpha_5(2d+1, d+i-u+1)\, b_i(q) = \sum_{i=1}^{d+1} \alpha_5(q, d+i)\, b_{i+u-1}(q)\,.$$

This completes the proof of lemma 3.12. q.e.d.

Lemma 3.13. *Let p be an odd prime, $d=(p-1)/2$, $u=sd+j$ $(1\leq j\leq d)$ and $i\geq 0$. Then we have*

$$b_{u+i}(p)\equiv(-p)^s\, b_{j+1}(p)$$

modulo the subgroup

$$\langle\{p^{s+1}\, b_{1+i}(p), \cdots, p^{s+1}\, b_{j+i}(p), p^s\, b_{j+1+i}(p), \cdots, p^s\, b_{d+i}(p)\}\rangle\,.$$

Proof. We choose inductively integers $B_{u,k}(u\geq 1$ and $1\leq k\leq d)$ such that

$$(*) \qquad b_{u+i}(p) = \textstyle\sum_{k=1}^{d} B_{u,k}\, b_{k+i}(p)$$

with

$$B_{u,k} = \begin{cases} 0 & (\mathrm{mod}\ p^{s+1}) \quad (k<j) \\ (-p)^s & (\mathrm{mod}\ p^{s+1}) \quad (k=j) \\ 0 & (\mathrm{mod}\ p^{s}) \quad (k>j)\,. \end{cases}$$

If $1\leq u\leq d+1$, we put

$$B_{u,k} = \begin{cases} 0 & (1\leq k\leq d \ \text{ and } \ k\neq u\leq d) \\ 1 & (1\leq k=u\leq d) \\ -\alpha_5(p, d+k) & (1\leq k\leq d \ \text{ and } \ u=d+1)\,. \end{cases}$$

It follows from Lemma 3.12 that $B_{u,k}(1\leq k\leq d)$ satisfy (*) $(1\leq u\leq d+1)$. Assume that $u\geq d+1$ and $B_{u,k}(1\leq k\leq d)$ have been chosen to satisfy the condition (*). Put $B_{u+1,1}=B_{u,d}\, B_{d+1,1}$ and $B_{u+1,k}=B_{u,k-1}+B_{u,d}\, B_{d+1,k}(2\leq k\leq d)$. Then we have

$$
\begin{aligned}
b_{u+1+i}(p) &= \textstyle\sum_{k=1}^{d} B_{u,k}\, b_{k+1+i}(p) = \sum_{k=2}^{d+1} B_{u,k-1}\, b_{k+i}(p) \\
&= \textstyle\sum_{k=2}^{d} B_{u,k-1}\, b_{k+i}(p)+B_{u,d}\, b_{d+1+i}(p) \\
&= \textstyle\sum_{k=2}^{d} B_{u,k-1}\, b_{k+i}(p)+B_{u,d} \sum_{k=1}^{d} B_{d+1,k}\, b_{k+i}(p) \\
&= B_{u,d}\, B_{d+1,1}\, b_{1+i}(p)+\textstyle\sum_{k=2}^{d}(B_{u,k-1}+B_{u,d}\, B_{d+1,k})\, b_{k+i}(p)
\end{aligned}
$$

$$= \sum_{k=1}^{d} B_{u+1,k}\, b_{k+i}(p)$$

and

$$B_{u+1,k} \equiv \begin{cases} 0 & (\text{mod } p^{r+1}) \quad (k < l) \\ (-p)^r & (\text{mod } p^{r+1}) \quad (k = l) \\ 0 & (\text{mod } p^r) \quad (k > l)\,, \end{cases}$$

where $u+1=rd+l(1\leq l\leq d)$. The lemma is a direct consequence of the condition (*). q.e.d.

The part (1) of the following proposition is obtained by making use of Lemmas 3.9 and 3.13.

Proposition 3.14. (1) *Let p be an odd prime. Then the group $\widetilde{KO}(S^j(L_p^{2[m/2]}/L_p^{2[n/2]}))$ is isomorphic to $VO^j_{2[m/2],2[n/2]}(p)$, which is isomorphic to the direct sum of cyclic groups of order $p^{b_0(m+j-4i,j)-b_0(n+j-4i,j)}$ generated by $p^{b_0(n+j-4i,j)+1}\, b_i(p)$ $(1\leq i\leq (p-1)/2)$, where b_0 is the function defined in* (2.4).
(2) ([12]) *Assume that* $j\equiv 0$ (mod 4) *and* $n\not\equiv 3$ (mod 4). *Then the group $\widetilde{KO}(S^j(L_2^m/L_2^n))$ is isomorphic to $VO^j_{m,n}(2)$, and*

$$VO^j_{m,n}(2) \cong \begin{cases} \langle 2^{\varphi(n,0)}\, I_R^{j/8}(\kappa_2)\rangle / \langle 2^{\varphi(m,0)}\, I_R^{j/8}(\kappa_2)\rangle & (j\equiv 0 \ (\text{mod } 8)) \\ \langle 2^{\varphi(n+4,4)}\, c_1(2)\rangle / \langle 2^{\varphi(m+4,4)}\, c_1(2)\rangle & (j\equiv 4 \ (\text{mod } 8))\,. \end{cases}$$

REMARK. The partial result for the case $j=n=0$ of Proposition 3.14. (1) has been obtained in [13].

We define the function $h(q, k)$ by setting

$$(3.15) \qquad h(q, k) = \text{ord}\,\langle J(r(\sigma_q))\rangle\,,$$

where $J(r(\sigma_q))$ is the image of $r(\sigma_q)\in\widetilde{KO}(L_q^k)$ by the J-homomorphism $J\colon \widetilde{KO}(L_q^k)\to \tilde{J}(L_q^k)$.

REMARK. The function $h(q, k)$ have been determined completely by K. Fujii (cf. [9], [11] and [10]).

We recall the following lemma from [17] for the proof of Theorems 5 and 6.

Lemma 3.16. *Suppose that* $k=2\,[m/2]+1-2\,[(n+1)/2]\geq 3$, $N\equiv 0$ (mod $2h(q, k)$) *and* $N>m+1$. *Then the S-dual of L_q^m/L_q^n is L_q^{N-n-2}/L_q^{N-m-2}.*

From [6, Propositions (2.6) and (2.9)] and Lemma 3.16, we have

(3.17) (1) *If* $k=m-2[(n+1)/2]\geq 2$ *and* $t\equiv 0$ (mod $2h(q, k)$), *then L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} are of the same stable homotopy type.*
(2) *If* $k=m-2[(n+1)/2]\geq 2$ *and* $n+1\equiv 0$ (mod $2h(q, k)$), *then* $t\equiv 0$ (mod $2h$

$(q, k))$ *if and only if* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*
(3) *If* $l=2[m/2]-n\geq 2$ *and* $t\equiv 0 \pmod{2h(q, l)}$, *then* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *are of the same stable homotopy type.*
(4) *If* $l=2[m/2]-n\geq 2$ *and* $n+1\equiv 0 \pmod{2h(q, l)}$, *then* $t\equiv 0 \pmod{2h(q, l)}$ *if and only if* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*

From [17] we have the following.
(3.18) *Suppose that* $q\equiv 0 \pmod 2$ *and* $m\geq n+2$. *Then* $\nu_2(t)\geq[\log_2 2(m-n-1)]$ *if* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *are of the same stable homotopy type.*

Proposition 3.19 ([18]). *Suppose that* $j\equiv 0 \pmod 4$.
(1) *If* $n\not\equiv 3 \pmod 4$, *then we have*

$$\tilde{J}(S^j(RP(m)/RP(n))) \cong \boldsymbol{Z}/2^{a(j,m,n)},$$

where $a(j, m, n)$ *is the integer defined by* (2.3).
(2) *If* $n\equiv 3 \pmod 4$, *then we have*

$$\tilde{J}(S^j(RP(m)/RP(n))) \cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{a(j,m,n)-i_2}\oplus\boldsymbol{Z}/2^{i_2},$$

where $a(j, m, n)$ *is the integer defined by* (2.3) *and*

$$i_2=\min\{a(j, m, n), \nu_2(n+1)\}.$$

Proposition 3.20 ([19]). *Let* p *be an odd prime, and suppose that* $j\equiv 0 \pmod 2$.
(1) *If* $n\equiv 0 \pmod 2$, *then we have*

$$\tilde{J}(S^j(L_p^{2[m/2]}/L_p^n) \cong \boldsymbol{Z}/p^{b(j,m,n)},$$

where $b(j, m, n)$ *is the integer defined by* (2.4).
(2) *If* $n\equiv 1 \pmod 2$, *then we have*

$$\tilde{J}(S^j(L_p^{2[m/2]}/L_p^n)) \cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot p^{b(j,m,n)-i_p}\oplus\boldsymbol{Z}/p^{i_p},$$

where $b(j, m, n)$ *is the integer defined by* (2.4) *and*

$$i_p=\min\{b(j, m, n), \nu_p(n+1), \mathfrak{m}((n+j+1)/2)\}.$$

4. Proof of Theorems 2 and 3

We denote the projection map of $L_{2^r}^m$ (resp. $L_q^{2[m/2]}$) into $L_{2^rq}^m$ by π_2 (resp. π_q). Then we have

Lemma 4.1. *Let* j *be a positive integer with* $j\equiv 2 \pmod 4$. *Then we have*
(1) *The induced homomorphism* $(\pi_q)^!\colon \widetilde{KO}(S^jL_{2^rq}^m)\to\widetilde{KO}(S^jL_q^{2[m/2]})$ *is an epimorphism.*

(2) *The induced homomorphism* $(\pi_2)^!: \widetilde{KO}(S^j L^m_{2^r q}) \to \widetilde{KO}(S^j L^m_{2^r})$ *is an epimorphism.*

(3) *If* $m+j+1 \not\equiv 0$ (mod 4), *the induced homomorphism*

$$(\pi_2)^!: \widetilde{KO}(S^{j+1}L^m_{2^r q}) \to \widetilde{KO}(S^{j+1}L^m_{2^r})$$

is an isomorphism.

Proof. (1) In the commutative diagram

$$\begin{array}{ccc} \widetilde{KO}(S^j L^m_{2^r q}) & \xrightarrow{(\pi_q)^!} & \widetilde{KO}(S^j L^{2[m/2]}_q) \\ \uparrow r' & & \uparrow r \\ \tilde{K}(S^j L^m_{2^r q}) & \xrightarrow{\pi^j_{q,C}} & \tilde{K}(S^j L^{2[m/2]}_q) \\ \uparrow I^{j/2} & & \uparrow I^{j/2} \\ \tilde{K}(L^m_{2^r q}) & \xrightarrow{\pi_{q,C}} & \tilde{K}(L^{2[m/2]}_q) , \end{array}$$

r is an epimorphism [19, Lemma 3.1] and $I^{j/2}$ is an isomorphism. There exist an element $\sigma_{2^r q} \in \tilde{K}(L^m_{2^r q})$ which maps to a generator $\sigma_q \in \tilde{K}(L^{2[m/2]}_q)$ by $\pi_{q,C}$. This implies that $\pi_{q,C}$ is an epimorphism. Thus, $(\pi_q)^!$ is an epimorphism. This completes the proof of (1).

(3) If $m+j+1 \equiv 5$, 6 or 7 (mod 8), then we have

$$\widetilde{KO}(S^{j+1}L^m_{2^r q}) \cong \widetilde{KO}(S^{j+1}L^m_{2^r}) \cong 0 .$$

If $m+j+1 \equiv 2$ (mod 8), then in the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}(S^{m+j+1}) \oplus \widetilde{KO}(S^{m+j}) & \cong & \widetilde{KO}(S^{j+1}(L^m_{2^r q}/L^{m-2}_{2^r q})) & \xrightarrow{(p_{m-2}{}^m)^!} & \widetilde{KO}(S^{j+1}L^m_{2^r q}) \\ \downarrow g^! \oplus 1 & & \downarrow & & \downarrow (\pi_2)^! \\ \widetilde{KO}(S^{m+j+1}) \oplus \widetilde{KO}(S^{m+j}) & \cong & \widetilde{KO}(S^{j+1}(L^m_{2^r}/L^{m-2}_{2^r})) & \xrightarrow{(p_{m-2}{}^m)^!} & \widetilde{KO}(S^{j+1}L^m_{2^r}) , \end{array}$$

$\deg g = q$ and both $(p_{m-2}{}^m)^!$ are isomorphisms [25, Remark of (3.3)]. Since $q \equiv 1$ (mod 2), $g^!$ is an isomorphism. Hence $(\pi_2)^!$ is an isomorphism.

Next consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}(S^{m+j+1}) & \xrightarrow{(p_{m-1}{}^m)^!} & \widetilde{KO}(S^{j+1}L^m_{2^r q}) & \xrightarrow{(i_{m-1}{}^m)^!} & \widetilde{KO}(S^{j+1}L^{m-1}_{2^r q}) \\ \downarrow g^! & & \downarrow (\pi_2)^! & & \downarrow (\pi'_2)^! \\ \widetilde{KO}(S^{m+j+1}) & \xrightarrow{(p_{m-1}{}^m)^!} & \widetilde{KO}(S^{j+1}L^m_{2^r}) & \xrightarrow{(i_{m-1}{}^m)^!} & \widetilde{KO}(S^{j+1}L^{m-1}_{2^r}) , \end{array}$$

where the rows are exact and

$$\deg g = \begin{cases} 1 & (m \equiv 0 \text{ (mod 2)}) \\ q & (m \equiv 1 \text{ (mod 2)}) . \end{cases}$$

If $m+j+1 \equiv 1$ (mod 8), then we have $\widetilde{KO}(S^{j+1}L^{m-1}_{2^r q}) \cong \widetilde{KO}(S^{j+1}L^{m-1}_{2^r})) \cong \boldsymbol{Z}$ and $\widetilde{KO}(S^{j+1}L^m_{2^r q}) \cong \widetilde{KO}(S^{j+1}L^m_{2^r})) \cong \boldsymbol{Z}/2$. Hence both $(p_{m-1}{}^m)^!$ are epimorphisms. Since $\widetilde{KO}(S^{m+j+1})\cong Z/2$, $g^!$ and both $(p_{m-1}{}^m)^!$ are isomorphisms. Thus $(\pi_2)^!$ is an isomorphism.

If $m+j+1\equiv 3 \pmod 8$, then in the above diagram we have $\widetilde{KO}(S^{m+j+1})\cong 0$. Hence upper $(i_{m-1}{}^m)^!$ is a monomorphism. By the proof in the case $m+j+1\equiv 2 \pmod 8$, $(\pi_2')^!$ is an isomorphism. Hence $(\pi_2)^!$ is a monomorphism. Since ord $\widetilde{KO}(S^{j+1}L^m_{2^rq})=$ ord $\widetilde{KO}(S^{j+1}L^m_{2^r}))=2$, $(\pi_2)^!$ is an isomorphism. This completes the proof of (3).

(2) We consider the commutative diagram

$$\begin{array}{ccccccc}
\tilde{K}(S^{j+2}L^m_{2^rq}) & \xrightarrow{rI^{-1}} & \widetilde{KO}(S^jL^m_{2^rq}) & \xrightarrow{\delta} & \widetilde{KO}(S^{j+1}L^m_{2^rq}) & \xrightarrow{c} & \tilde{K}(S^{j+1}L^m_{2^rq}) \\
\downarrow \pi^{j+2}_{2,C} & & \downarrow (\pi_2)^! & & \downarrow (\pi_2^{j+1})^! & & \downarrow \pi^{j+1}_{2,C} \\
\tilde{K}(S^{j+2}L^m_{2^r}) & \xrightarrow{rI^{-1}} & \widetilde{KO}(S^jL^m_{2^r}) & \xrightarrow{\delta} & \widetilde{KO}(S^{j+1}L^m_{2^r}) & \xrightarrow{c} & \tilde{K}(S^{j+1}L^m_{2^r})
\end{array}$$

in which the rows are exact ([5] and [7, (12.2)]), where c: $\widetilde{KO}(X)\to\tilde{K}(X)$ is the complexification and δ is the homomorphism defined by the exterior product with the generator of $\widetilde{KO}(S^1)$.

If $m+j+1\not\equiv 1$, 2 and 3 (mod 8), then $\widetilde{KO}(S^{j+1}L^m_{2^r}))$ is a free group and ord $\widetilde{KO}(S^jL^m_{2^r})$ is finite. Hence δ is a zero-map. Since lower rI^{-1} and $\pi^{j+2}_{2,C}$ are epimorphisms, $(\pi_2)^!$ is also an epimorphism.

If $m+j+1\equiv 1$, 2 or 3 (mod 8), then $(\pi_2^{j+1})^!$ is an isomorphism by (3), $\pi^{j+2}_{2,C}$ is an epimorphism and $\pi^{j+1}_{2,C}$ is a monomorphism. Thus $(\pi_2)^!$ is an epimorphism from 4-lemma. This completes the proof of (2). q.e.d.

Now we define the homomorphism

$$f_1\colon \widetilde{KO}(S^jL^m_{2^rq}) \to \widetilde{KO}(S^jL^m_{2^r}) \oplus \widetilde{KO}(S^jL_q^{2[m/2]})$$

by $f_1(x)=((\pi_2)^!(x), (\pi_q)^!(x))$ for $x\in\widetilde{KO}(S^jL^m_{2^rq})$.

Lemma 4.2. *Let j be a positive integer with $j\equiv 2$ (mod 4). Then f_1 is an isomorphism.*

Proof. By [25, Theorems 1 and 2]

$$\text{ord}\ \widetilde{KO}(S^jL^m_{2^rq}) = 2^{h(m+j)+1}(2^{r-1}q)^{[(m+2)/4]},$$

(4.3) $$\text{ord}\ \widetilde{KO}(S^jL_q^{2[m/2]}) = q^{[(m+2)/4]}$$

and

(4.4) $$\text{ord}\ \widetilde{KO}(S^jL^m_{2^r}) = 2^{h(m+j)+1}(2^{r-1})^{[(m+2)/4]},$$

where h: $Z\to Z$ is the function defined by

$$h(s)=\begin{cases} 2 & (s\equiv 1 \pmod 8)\\ 1 & (s\equiv 0 \text{ or } 2 \pmod 8)\\ 0 & (\text{otherwise}).\end{cases}$$

Hence, we have

$$\text{ord}\ \widetilde{KO}(S^j L^m_{2^r q}) = \text{ord}\ (\widetilde{KO}(S^j L^m_{2^r}) \oplus \widetilde{KO}(S^j L_q^{2[m/2]})) . \tag{4.5}$$

By Lemma 4.1, $(\pi_2)^!$ and $(\pi_q)^!$ are epimorphisms. For each element $(x, y) \in \widetilde{KO}(S^j L^m_{2^r}) \oplus \widetilde{KO}(S^j L_q^{2[m/2]})$, there exist elements v and $w \in \widetilde{KO}(S^j L^m_{2^r q})$ such that $(\pi_2)^!(v) = x$ and $(\pi_q)^!(w) = y$. Now we put $h(m+j)+1+(r-1)[(m+2)/4] = s$ and $[(m+2)/4] = t$ for the sake of simplicity. Since 2^s is relatively prime to q^t, we can choose integers a and b such that

$$a2^s + bq^t = 1 . \tag{4.6}$$

Set $z = bq^t v + a2^s w$. Then by (4.3), (4.4) and (4.6) we have

$$\begin{aligned} f_1(z) &= bq^t f_1(v) + a2^s f_1(w) \\ &= (bq^t(\pi_2)^!(v) + a2^s(\pi_2)^!(w),\ bq^t(\pi_q)^!(v) + a2^s(\pi_q)^!(w)) \\ &= ((1-a2^s)(\pi_2)^!(v),\ (1-bq^t)(\pi_q)^!(w)) \\ &= ((\pi_2)^!(v),\ (\pi_q)^!(w)) \\ &= (x, y) . \end{aligned}$$

Thus f_1 is an epimorphism. By (4.5), f_1 is an isomorphism. q.e.d.

We have the homomorphisms

$$\begin{aligned} f_2 &: \widetilde{KO}(S^j L^n_{2^r q}) \to \widetilde{KO}(S^j L^n_{2^r}) \oplus \widetilde{KO}(S^j L_q^{2[n/2]}) , \\ f_3 &: \widetilde{KO}(S^{j+1} L^n_{2^r q}) \to \widetilde{KO}(S^{j+1} L^n_{2^r}) \oplus \widetilde{KO}(S^{j+1} L_q^{2[n/2]}) \end{aligned}$$

and

$$f: \widetilde{KO}(S^j (L^m_{2^r q}/L^n_{2^r q})) \to \widetilde{KO}(S^j(L^m_{2^r}/L^n_{2^r})) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]}))$$

defined similarly as f_1. In the following commutative diagram

$$\begin{array}{ccc} \widetilde{KO}(S^{j+1} L^n_{2^r q}) & \xrightarrow{f_3} & \widetilde{KO}(S^{j+1} L^n_{2^r}) \oplus \widetilde{KO}(S^{j+1} L_q^{2[n/2]}) \\ \downarrow & & \downarrow \\ \widetilde{KO}(S^j(L^m_{2^r q}/L^n_{2^r q})) & \xrightarrow{f} & \widetilde{KO}(S^j(L^m_{2^r}/L^n_{2^r})) \oplus \widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]})) \\ \downarrow & & \downarrow \\ \widetilde{KO}(S^j L^m_{2^r q}) & \xrightarrow{f_1} & \widetilde{KO}(S^j L^m_{2^r}) \oplus \widetilde{KO}(S^j L_q^{2[m/2]}) \\ \downarrow & & \downarrow \\ \widetilde{KO}(S^j L^n_{2^r q}) & \xrightarrow{f_2} & \widetilde{KO}(S^j L^n_{2^r}) \oplus \widetilde{KO}(S^j L_q^{2[n/2]}) \end{array}$$

the columns are exact.

If $j \equiv 2 \pmod 4$, $n+j+1 \not\equiv 0 \pmod 4$ and $m \geq n+3$, then f_1, f_2 and f_3 are isomorphisms by Lemmas 4.1 and 4.2. From 4-lemma, f is an epimorphism.

By [25, Theorems 1 and 2]

$$\begin{aligned}&\operatorname{ord}(\widetilde{KO}(S^j(L^m_{2^rq}/L^n_{2^rq})))\\&\quad=\operatorname{ord}(\widetilde{KO}(S^j(L^m_{2^r}/L^n_{2^r}))\oplus\widetilde{KO}(S^j(L_q^{2[m/2]}/L_q^{2[n/2]})))\,.\end{aligned}$$

Thus f is an isomorphism. This completes the proof for the case $j\equiv 2 \pmod 4$ of the part (1) of Theorem 2. The corresponding proof for the case $j\equiv 0 \pmod 4$ is quite similar to that of the above case.

Combining the part (1) and [25, Theorem 2], we obtain the parts (2) and (3) of Theorem 2.

The proof of the part (4) of Theorem 2 is similar to that of the part (1).

Since the isomorphisms of the parts (1) and (4) of Theorem 2 are ψ-maps, Theorem 3 is an easy consequence of Theorem 2.

5. Proof of Theorem 4

Assume that $j\equiv 0 \pmod 4$ and $n\equiv 3 \pmod 4$. It follows from [25] and Theorem 2 that we have the commutative diagram

$$(5.1)\quad \begin{array}{ccccccc} 0\to VO^j_{m,n+1}(2)\oplus VO^j_{2[m/2],n+1}(p) & \xrightarrow{f_1} & \widetilde{KO}(S^j(L^m_{2p}/L^n_{2p})) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n+1})\to 0\\ \| & & \downarrow f_3 & & \\ 0\to VO^j_{m,n+1}(2)\oplus VO^j_{2[m/2],n+1}(p) & \to & \widetilde{KO}(S^jL^m_2)\oplus\widetilde{KO}(S^jL_p^{2[m/2]}) & & \end{array}$$

in which the rows are exact. For each i prime to p (resp. 2), $N_p(i)$ (resp. $N_2(i)$) denote the integer chosen to satisfy the property

$$(5.2)\qquad iN_p(i)\equiv 1 \pmod{p^m}\quad (\text{resp. } iN_2(i)\equiv 1 \pmod{2^m}).$$

As defined in [19], let w be the remainder of $j/2$ divided by $p-1$ and set $v=p-1-w$, $N_v=N(\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N_p(i^{pj/2}))$ and $C_l(p)=c_l(p)-\sum_{i=1}^{l}\binom{l}{i}(-1)^{l-i}N_p(i^{pj/2})\,N_v\,c_v(p)$ $(1\leq l\leq p-1)$. In order to state the next lemma, we set

$$(5.3)\quad \begin{aligned}&(1)\quad u_2=\begin{cases}2^{(n-1)/2}c_1(2) & (n+j+1\equiv 4 \pmod 8)\\ 2^{(n-3)/2}c_1(2) & (n+j+1\equiv 0 \pmod 8)\,.\end{cases}\\ &(2)\quad u_p=\begin{cases}(-p)^s c_v(p) & (n+j+1\equiv 4 \pmod 8)\ \text{and}\ l=v)\\ (-p)^s C_l(p) & (n+j+1\equiv 4 \pmod 8)\ \text{and}\ l\neq v)\\ N_p(2)(-p)^s c_v(p) & (n+j+1\equiv 0 \pmod 8)\ \text{and}\ l=v)\\ N_p(2)(-p)^s C_l(p) & (n+j+1\equiv 0 \pmod 8)\ \text{and}\ l\neq v)\,,\end{cases}\end{aligned}$$

where $s=[n/2(p-1)]$ and $l=(n+1)/2-s(p-1)$.

According to Lemmas 3.9, 3.13 and 3.14, we have the following lemma.

Lemma 5.4. *If $j\equiv 0 \pmod 4$ and $n\equiv 3 \pmod 4$, then $\widetilde{KO}(S^j(L^m_{2p}/L^n_{2p}))$ has* *an element x, which satisfies the following conditions*:

(1) $f_2(x)$ *generates the group* $\widetilde{KO}(S^{j+n+1})$,

(2) $f_3(x)=(u_2, u_p)$.

In the diagram (5.1), since $\widetilde{KO}(S^{j+n+1})$ is isomorphic to $\boldsymbol{Z}$, we have a direct decomposition

$$\widetilde{KO}(S^j(L^m_{2p}/L^n_{2p}) \cong f_1(VO^j_{m,n+1}(2)\oplus VO^j_{2[m/2],n+1}(p))\oplus \boldsymbol{Z}\,\{x\}$$

where $\boldsymbol{Z}\,\{x\}$ means the infinite cyclic group generated by x.

For the Adams operation, we have the following lemma.

Lemma 5.5. *If* $j\equiv 0 \pmod 4$ *and* $n\equiv 3 \pmod 4$, *then the Adams operation* ψ^k *is given as follows.*

$$\psi^k(x) = k^{(n+j+1)/2}\, x+f_1(b_2, b_p)$$

where $b_2\in VO^j_{m,n+1}(2)$, $b_p\in VO^j_{2[m/2],n+1}(p)$,

$$b_p \equiv \begin{cases} 0 \quad (k\not\equiv 0 \pmod p \quad \textit{and} \quad (n+j+1)/2\not\equiv 0 \pmod{(p-1)}) \\ -(((k^{(n+j+1)/2}-1)+(j/2)\,(k^{p-1}-1))/p)\,(pu_p) \\ \qquad (k\not\equiv 0 \pmod p \quad \textit{and} \quad (n+j+1)/2\equiv 0 \pmod{(p-1)}) \end{cases}$$

$(\text{mod } UO^j_{2[m/2],n+1}(p))$ *and*

$$b_2 = \begin{cases} -(k^{(n+j+1)/2}/2)\,(2u_2) & (k\equiv 0 \pmod 2) \\ -((k^{(n+j+1)/2}-k^{j/2})/2)\,(2u_2) & (k\equiv 1 \pmod 2)\,. \end{cases}$$

Proof. We necessarily have

$$\psi^k(x) = \alpha x+f_1(b_2, b_p)$$

for some integer α and an element

$$(b_2, b_p)\in VO^j_{m,n+1}(2)\oplus VO^j_{2[m/2],n+1}(p)\,.$$

By using the ψ-map f_2, we see that $\alpha=k^{(n+j+1)/2}$. Under $f_3, f_1(b_2, b_p)$ maps (b_2, b_p) and x maps into $f_3(x)$, and by above Lemma we see that

$$\psi^k(u_2, u_p) = k^{(n+j+1)/2}(u_2, u_p)+(b_2, b_p)\,.$$

It follows from [18, Lemma 2.3] and [19, Lemma 2.13] that

$$\psi^k(u_2) = \begin{cases} 0 & (k\equiv 0 \pmod 2) \\ k^{j/2}\, u_2 & (k\equiv 1 \pmod 2) \end{cases}$$

and

$$\psi^k(u_p) \equiv \begin{cases} k^{(n+j+1)/2}\, u_p & (n+j+1)/2\not\equiv 0 \pmod{(p-1)} \\ (1+(j/2)\,(1-k^{p-1}))\, u_p & (n+j+1)/2\equiv 0 \pmod{(p-1)} \end{cases}$$

(mod $UO^j_{2[m/2],n+1}$) ($k\not\equiv 0 \pmod p$). Therefore,

$$b_2 = \begin{cases} -k^{(n+j+1)/2}\,u_2 & (k\equiv 0 \pmod 2) \\ (k^{j/2}-k^{(n+j+1)/2})\,u_2 & (k\equiv 1 \pmod 2)\end{cases}$$

and

$$b_p \equiv \begin{cases} 0 & ((n+j+1)/2\not\equiv 0 \pmod{(p-1)}) \\ (1+(j/2)(1-k^{p-1})-k^{(n+j+1)/2})\,u_p & ((n+j+1)/2\equiv 0 \pmod{(p-1)})\end{cases}$$

(mod $UO^j_{2[m/2],n+1}(p)$) ($k\not\equiv 0 \pmod p$). q.e.d.

We now recall some definition in [3], set $Y=\widetilde{KO}(S^j(L^m_{2p}/L^n_{2p}))$ and let f be a function which assigns to each integer k a non-negative integer $f(k)$. Given such a function f, we define Y_f to be the subgroup of Y generated by

$$\{k^{f(k)}(\psi^k-1)(y)\,|\,k\in \boldsymbol{Z}, y\in Y\}\ ;$$

that is

$$Y_f = \langle\{k^{f(k)}(\psi^k-1)(y)\,|\,k\in \boldsymbol{Z}, y\in Y\}\rangle\,.$$

Then the kernel of the homomorphism $J''\colon Y\to J''(Y)$ coincides with $\bigcap_f Y_f$, where the intersection runs over all functions f.

Suppose that f satisfies

(5.6) $f(k)\geq m+\max\{\nu_p(\mathfrak{m}((n+j+1)/2))\,|\ p$ is a prime divisor of $k\}$ for every $k\in\boldsymbol{Z}$.

In the following calculation we put $(n+j+1)/2=u$ and

$$U_{n+1} = VO^j_{m,n+1}(2)\oplus VO^j_{2[m/2],n+1}(p)$$

for the sake of simplicity.

Now we consider the case $(n+j+1)/2\equiv 0 \pmod{(p-1)}$. From Lemma 5.5, we have

$$\begin{aligned}
&k^{f(k)}(\psi^k-1)(x)\\
&\quad\equiv k^{f(k)}(k^u-1)x-k^{f(k)}(((k^u-1)+(j/2)(k^{p-1}-1))/p)f_1(pu_p)\\
&\qquad -k^{f(k)}((k^u-k^{j/2})/2)f_1(2u_2) \qquad\qquad (\mathrm{mod}\ f_1(U_{n+1}))\\
&\quad\equiv k^{f(k)}(k^u-1)x\\
&\qquad -k^{f(k)}N_p(u/p^{\nu_p(u)})((u(k^u-1)-(j/2)(k^u-1))/p^{\nu_p(u)+1})f_1(pu_p)\\
&\qquad -k^{f(k)}N_2(u/2^{\nu_2(u)})((u(k^u-1)-(j/2)(k^u-1))/2^{\nu_2(u)+1})f_1(2u_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{mod}\ f_1(U_{n+1}))\\
&\quad= (k^{f(k)}(k^u-1)/p^{\nu_p(u)+1}2^{\nu_2(u)+1})(2^{\nu_2(u)+1}p^{\nu_p(u)+1}x\\
&\qquad -N_p(u/p^{\nu_p(u)})((n+1)/2)2^{\nu_2(u)+1}f_1(pu_p)\\
&\qquad -N_2(u/2^{\nu_2(u)})((n+1)/2)p^{\nu_p(u)+1}f_1(2u_2))\,.
\end{aligned}$$

By virtue of [3, Theorem (2.7) and Lemma (2.12), we have

$$\begin{aligned}&\langle f_1(U_{n+1})\cup\{k^{f(k)}(\psi^k-1)(x)\,|\,k\in\boldsymbol{Z}\}\rangle\\&=\langle f_1(U_{n+1})\cup\{(\mathfrak{m}(u)/p^{\nu_p(u)+1}2^{\nu_2(u)+1})(2^{\nu_2(u)+1}p^{\nu_p(u)+1}x\\&\quad-N_p(u/p^{\nu_p(u)})((n+1)/2)\,2^{\nu_2(u)+1}f_1(pu_p)\\&\quad-N_2(u/2^{\nu_2(u)})((n+1)/2)\,p^{\nu_p(u)+1}f_1(2u_2))\}\rangle\,.\end{aligned}$$

Therefore,

$$Y_f=\langle f_1(U_{n+1})\cup\{\mathfrak{m}(u)\,x-M_pf_1(pu_p)-M_2f_1(2u_2)\}\rangle\,,$$

where $u=(n+j+1)/2$,

$$\begin{aligned}M_p&=(\mathfrak{m}(u)/p^{\nu_p(u)+1})\,N_p(u/p^{\nu_p(u)})\,((n+1)/2)\,,\\M_2&=(\mathfrak{m}(u)/2^{\nu_2(u)+2})\,N_2(u/2^{\nu_2(u)})\,(n+1)\,.\end{aligned}$$

Since this is true for every function f which satisfies (5.6), we have

$$J''(Y)\cong Y/\langle f_1(U_{n+1})\cup\{\mathfrak{m}(u)\,x-M_pf_1(pu_p)-M_2f_1(2u_2)\}\rangle\,.$$

Therefore,

$$J''(Y)\cong\langle\{x,u_2,u_p\}\rangle/\langle\{X_1,X_2,X_3\}\rangle$$

where $M_0=\mathfrak{m}((n+j+1)/2)$, $X_1=M_0\,x-M_2\,u_2-M_p\,u_p$, $X_2=2^{a(j,m,n)}\,u_2$ and $X_3=p^{b(j,m,n)}\,u_p$.
we set

$$i_2=\min\{a(j,m,n),\nu_2(n+1)\}$$

and

$$i_p=\min\{b(j,m,n),\nu_p(n+1),\nu_p(\mathfrak{m}((n+j+1)/2))\}\,.$$

Since $\nu_2(M_2)=\nu_2(n+1)$ and $\nu_p(M_p)=\nu_p(n+1)$, the greatest common divisor of $2^{a(j,m,n)}$ and $M_2\,p^{b(j,m,n)-i_p}$ is equal to 2^{i_2}, and the greatest common divisor of $p^{b(j,m,n)}$ and $M_p\,2^{a(j,m,n)-i_2}$ is equal to p^{i_p}. Choose integers e_1, e_2, e_3 and e_4 with

$$e_1\,2^{a(j,m,n)}+e_2\,M_2\,p^{b(j,m,n)-i_p}=2^{i_2}$$

and

$$e_3\,p^{b(j,m,n)}+e_4\,M_p\,2^{a(j,m,n)-i_2}=p^{i_p}\,.$$

For the sake of simplicity, we put $a=a(j,m,n)$ and $b=b(j,m,n)$ in the following calculation. Set

$$A=\begin{pmatrix}2^{a-i_2}p^{b-i_p}&p^{b-i_p}M_2/2^{i_2}&2^{a-i_2}M_p/p^{i_p}\\e_2\,p^{b-i_p}&-e_1&e_2\,M_p/p^{i_p}\\e_4\,2^{a-i_2}&e_4\,M_2/2^{i_2}&-e_3\end{pmatrix},$$

then we have

$$A\begin{pmatrix} M_0\,x-M_2\,u_2-M_p\,u_p \\ 2^a\,u_2 \\ p^b\,u_p \end{pmatrix}=\begin{pmatrix} 2^{a-i_2}p^{b-i_p}M_0\,x \\ 2^{i_2}(e_2\,p^{b-i_p}M_0\,x/2^{i_2}-u_2) \\ p^{i_p}(e_4\,2^{a-i_2}M_0\,x/p^{i_p}-u_p) \end{pmatrix}$$

and det $A=1$. This implies that

$$J''(Y)\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{a(j,m,n)-i_2}\cdot p^{b(j,m,n)-i_p}\oplus \boldsymbol{Z}/p^{i_p}\oplus \boldsymbol{Z}/2^{i_2}\,.$$

Thus the proof of (1) for the case $(n+j+1)/2\equiv 0 \pmod{(p-1)}$ is completed by [24].

We now turn to the case $u=(n+j+1)/2\not\equiv 0 \pmod{(p-1)}$. Then we have

$$\begin{aligned} &k^{f(k)}(\psi^k-1)(x) \\ &\quad\equiv (k^{f(k)}(k^u-1)/2^{\nu_2(u)+1})\,(2^{\nu_2(u)+1}x-N_2(u/2^{\nu_2(u)})\,((n+1)/2)f_1(2u_2)) \end{aligned}$$

$(\bmod f_1(U_{n+1}))$. Hence

$$J''(Y)\cong Y/\langle f_1(U_{n+1})\cup\{\mathfrak{m}(u)\,x-M_2f_1(2u_2)\}\rangle\,.$$

Therefore,

$$J''(Y)\cong\langle\{x,\,u_2,\,u_p\}\rangle/\langle\{X_1,\,X_2,\,X_3\}\rangle$$

where $M_0=\mathfrak{m}((n+j+1)/2)$, $X_1=M_0\,x-M_2\,u_2$, $X_2=2^{a(j,m,n)}\,u_2$ and $X_3=p^{b(j,m,n)}\,u_p$. We set

$$i_2=\min\,\{a(j,m,n),\,\nu_2(n+1)\}\,.$$

Since $\nu_2(M_2)=\nu_2(n+1)$ the greatest common divisor of $2^{a(j,m,n)}$ and M_2 is equal to 2^{i_2}. Choose integers e_1 and e_2 with

$$e_1\,2^{a(j,m,n)}+e_2=2^{i_2}\,.$$

Set

$$B=\begin{pmatrix} 2^{a-i_2} & M_2/2^{i_2} & 0 \\ e_2 & -e_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then we have

$$B\begin{pmatrix} M_0\,x-M_2\,u_2 \\ 2^a\,u_2 \\ p^b\,u_p \end{pmatrix}=\begin{pmatrix} 2^{a-i_2}M_0\,x \\ 2^{i_2}((e_2\,M_0/2^{i_2})\,x-u_2) \\ p^b\,u_p \end{pmatrix}$$

and det $B=-1$. This implies that

$$\begin{aligned} J''(Y)&\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{a(j,m,n)-i_2}\oplus \boldsymbol{Z}/p^{b(j,m,n)}\oplus \boldsymbol{Z}/2^{i_2} \\ &\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{a(j,m,n)-i_2}\cdot p^{b(j,m,n)}\oplus \boldsymbol{Z}/2^{i_2}\,. \end{aligned}$$

Thus the proof of (1) is completed by [24].

Now we turn to the case $j\equiv 2 \pmod 4$ and $n\equiv 1 \pmod 4$. In the corresponding diagram of (5.1), $\widetilde{KO}(S^j(L^m_{2p}/L^n_{2p}))$ has an element x, which satisfies the following conditions:

i) $f_2(x)$ generates the group $\widetilde{KO}(S^{j+n+1})$,
ii) the 2-component of $f_3(x)$ is equal to 0.

Since the Adams operations are given by $\psi^k=k-2\,[k/2]$ on the 2-component of $\widetilde{KO}(S^j(L^m_{2p}/L^n_{2p}))$, the rest of the proof of (2) is similar to that of (1).

6. Proofs of Theorems 5 and 6

In this section we state proofs of Theorems 5 and 6.

Proof of Theorem 5. Suppose that the spaces L^m_{2p}/L^n_{2p} and $L^{m+t}_{2p}/L^{n+t}_{2p}$ are of the same stable homotopy type with $m>n+2$. Then there exists a homotopy equivalence

$$f\colon S^j(L^m_{2p}/L^n_{2p}) \to S^{j-t}(L^{m+t}_{2p}/L^{n+t}_{2p})\,,$$

which induces an isomorphism

$$J(f^!)\colon \tilde{J}(S^{j-t}(L^{m+t}_{2p}/L^{n+t}_{2p})) \to \tilde{J}(S^j(L^m_{2p}/L^n_{2p}))\,. \tag{6.1}$$

We can assume that $\nu_2(j)\geq \max\{3,\ \bar{\varphi}(m,n)\}$. By (3.18), $t\equiv 0 \pmod 4$. It follows from Proposition 3.19, Theorem 3 and Theorem 4, that we have

$$\min\{\nu_2(j)+1,\ \bar{\varphi}(m,n)\} = \min\{\nu_2(j-t)+1,\ \bar{\varphi}(m,n)\}\,.$$

Thus we have

$$\nu_2(t)\geq \bar{\varphi}(m,n)-1\,. \tag{6.2}$$

If $m\geq n+9$ and $\nu_2(n+1)\geq \varphi(m-n-1,0)-1$, then we have the following from Theorem 4:

$$\begin{aligned}&\tilde{J}(S^{j-t}(L^{m+t}_{2p}/L^{n+t}_{2p}))\\ &\quad\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{\varphi(m,n+1)-k_2}\cdot p^{b(j-t,m+t,n+t)-k_p}\oplus \boldsymbol{Z}/p^{k_p}\oplus \boldsymbol{Z}/2^{k_2}\end{aligned}$$

and

$$\begin{aligned}&\tilde{J}(S^j(L^m_{2p}/L^n_{2p}))\\ &\quad\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{\varphi(m,n+1)-i_2}\cdot p^{b(j,m,n)-i_p}\oplus \boldsymbol{Z}/p^{i_p}\oplus \boldsymbol{Z}/2^{i_2}\,,\end{aligned}$$

where

$$\begin{aligned}k_2 &= \min\{\varphi(m,n+1),\ \nu_2(n+t+1)\}\,,\\ k_p &= \min\{b(j-t,m+t,n+t),\ \nu_p(n+t+1),\ \nu_p(\mathfrak{m}((n+j+1)/2))\}\,,\\ i_2 &= \min\{\varphi(m,n+1),\ \nu_2(n+1)\}\end{aligned}$$

and

$$i_p = \min \{b(j, m, n), \nu_p(n+1), \nu_p(\mathrm{m}((n+j+1)/2))\} .$$

By the isomorphism (6.1), we have $k_2 = i_2$. This implies that we have $\nu_2(n+t+1) \geq \varphi(m, n+1)$ if $\nu_2(n+1) \geq \varphi(m, n+1)$ and $\nu_2(n+t+1) = \varphi(m, n+1)-1$ if $\nu_2(n+1) = \varphi(m, n+1)-1$. Since $\nu_2(n+1)+1 \geq \varphi(m-n-1, 0) = \varphi(m, n+1)$, we have

(6.3) *If* $m \geq n+9$ *and* $\nu_2(n+1) \geq \varphi(m-n-1, 0)-1$, *then we have*

$$\nu_2(t) \geq \varphi(m-n-1, 0) .$$

On the other hand, we can assume that

$$j \equiv 0 \pmod{2p^{[([m/2]-[(n+3)/2])/(p-1)]}}$$

and $j/2 \equiv p-2-[(n+1)/2] \pmod{(p-1)}$. It follows from Proposition 3.20, Theorem 3 and Theorem 4, that we have

$$\begin{aligned} &\min \{\nu_p(j-t)+1, [(m+j)/2\,(p-1)]-[(n+j+1)/2\,(p-1)]\} \\ &\quad = \min \{\nu_p(j)+1, [(m+j)/2\,(p-1)]-[(n+j+1)/2\,(p-1)]\} \\ &\quad = [(m+j)/2\,(p-1)]-[(n+j+1)/2\,(p-1)] \\ &\quad = [[(m+j)/2]/(p-1)]-([(n+1)/2]-p+2+(j/2))/(p-1) \\ &\quad = [([m/2]-[(n+3)/2])/(p-1)]+1 . \end{aligned}$$

This implies

(6.4) $$\nu_p(t) \geq [([m/2]-[(n+3)/2])/(p-1)] .$$

In the case $n+1 \equiv 0 \pmod{2p^{[([m/2]-[(n+1)/2])/(p-1)]}}$, we assume that

$$j \equiv 0 \pmod{2p^{[([m/2]-[(n+1)/2])/(p-1)]}}$$

and $n+j+1 \equiv 0 \pmod{2\,(p-1)}$. It follows from Theorem 4 that we have

$$\begin{aligned} &\min \{\nu_p(n+t+1), [(m+j)/2\,(p-1)]-[(n+j+1)/2\,(p-1)]\} \\ &\quad = \min \{\nu_p(n+1), [(m+j)/2\,(p-1)]-[(n+j+1)/2\,(p-1)]\} \\ &\quad = \min \{\nu_p(n+1), [([m/2]-[(n+1)/2])/(p-1)]\} \\ &\quad = [([m/2]-[(n+1)/2])/(p-1)] . \end{aligned}$$

This implies

(6.5) *If* $n+1 \equiv 0 \pmod{2p^{[([m/2]-[(n+1)/2])/(p-1)]}}$, *we have*

$$\nu_p(t) \geq [([m/2]-[(n+1)/2])/(p-1)] .$$

Combining (6.2), (6.3), (6.4), (6.5), Lemma 3.16 and (3.18), we obtain Theorem 5.

Proof of Theorem 6. According to [10], we have

$$h(2p, k) = 2^{\varphi(k,0)-1} p^{[k/2(p-1)]}.$$

Then Theorem 6 follows from (3.17).

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