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AN ANALYTICITY PROBLEM AND AN INTEGRATION THEOREM OF COMPLETELY INTEGRABLE SYSTEMS WITH SINGULARITIES

HITOSHI KUMANO-GO AND MICHIHIKO MATSUDA

(Received October 31, 1969)

In this note we shall solve an analyticity problem and improve an integration theorem obtained by the second named author [1].

1. Introduction

We shall give a proof to the following

Lemma. Let \( f(x) \) be a real valued \( C^\infty \)-function on the interval \((0, 1)\). Suppose that the radius of convergence of the power series

\[
\sum_{k=0}^\infty \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k
\]

is greater than a positive constant \( r \) for every \( x_0 \) in \((0, 1)\). Then \( f(x) \) is real analytic on the interval \((0, 1)\).

Applying this lemma we can prove the following

Theorem. Let \( M \) be a \( C^\infty \)-manifold and \( L \) be a Lie subalgebra of the Lie algebra of all \( C^\infty \)-vector fields on \( M \). For two elements \( u \) and \( v \) of \( L \) put

\[
g_t(u, v) = \sum_{k=0}^\infty \frac{(-t)^k}{k!}(ad v)^k u,
\]

where \( (ad v)^k u = [v, (ad v)^{k-1} u] \), \( k = 1, 2, 3, \ldots \). Suppose that for any pair of \( u \) and \( v \) in \( L \) and for any compact subset \( K \) in \( M \) there exists a positive number \( c(u, v; K) \) such that the radius of convergence of \( g_t(u, v) \) at \( x \) is greater than \( c(u, v; K) \) if \( x \) is in \( K \). Then through every point \( x_0 \) on \( M \) there passes a maximal integral manifold \( N(x_0) \) of \( L \). Any integral manifold of \( L \) containing \( x_0 \) is an open submanifold of \( N(x_0) \).

This theorem was proved by Matsuda [1] under the additional condition that \( g_t(u, v) \) is continuously differentiable with respect to \((x, t)\) term by term.
Acknowledgements. After our preparation of this note, Dr. R. Vaillancourt kindly informed us that our lemma is known as a theorem of Boas (R.P. Boas, Jr.: A theorem on analytic functions of a real variable, Bull. Amer. Math. Soc. 41 (1935), 233–236). Our proof is different from the original one of Boas.

2. Proof of Lemma

The first step is to prove that the set of all points at which $f(x)$ is real analytic is open and dense in $(0, 1)$. Put

$$M(x) = \sup_{k} \frac{|f^{(k)}(x)|}{k!}.$$ 

By our assumption $M(x)$ is finite at every $x$ in $(0, 1)$. Take an arbitrary closed interval $I_0$ in $(0, 1)$. If we put

$$A_n = \{x \in I_0; M(x) \leq n\},$$

then

$$I_0 = \bigcup_{n=1}^{\infty} A_n.$$ 

Since $A_n$ is closed for every $n$, by Baire's theorem there exist an integer $M$ and an open subinterval $I_1$ of $I_0$ such that $A_M$ contains $I_1$. For two points $x$ and $x_0$ in $I_1$, by the mean value theorem we have

$$f(x) = \sum_{k=0}^{M} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{f^{(M)}(y)}{M!} (x-x_0)^M,$$

where $x_0 \leq y \leq x$ or $x \leq y \leq x_0$. If $|x-x_0| = \theta r$ and $0<\theta<1$, then

$$\frac{|f^{(M)}(y)|}{M!} |x-x_0|^M \leq M\theta^n.$$ 

Hence $f(x)$ is real analytic at $x_0$ and on $I_1$.

The second step is to prove that the set $B$ of all points at which $f(x)$ is not real analytic is empty. To the contrary suppose that $B$ is not empty. Put

$$B_n = \{x \in B; M(x) \leq n\}.$$ 

Then

$$B = \bigcup_{n=1}^{\infty} B_n.$$ 

Since $B$ and $B_n(1 \leq n < \infty)$ are closed, by Baire-Hausdorff's theorem there exist an integer $N$ and an open interval $I$ such that $B_N$ contains $I \cap B$ which is not empty. Let us define $N(x)$ by
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\[ N(x) = \sup_{k} \left| \frac{f^{(k)}(x)}{k!} \right| \left( \frac{r}{2} \right)^k \]

and prove that

\[ N(x) \leq 3N \]

on \( I \), if \(|I| < \frac{r}{3}\).

If \( x \) is in \( B \), then

\[ N(x) \leq M(x) \leq N \leq 3N. \]

Suppose that \( x \) is not in \( B \). We can take a neighbourhood \((a, b)\) of \( x \) in \( I \) such that \( f(x) \) is real analytic on \((a, b)\) and \( a \) or \( b \) is a point of \( B \). Fix a point \( x_0 \) in \((a, b)\). By the identity

\[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, \]

we have

\[ |f^{(n)}(x)| \leq \sum_{j=0}^{\infty} \frac{|f^{(j+n)}(x_0)|}{j!} \left( \frac{r}{3} \right)^j. \]

Since

\[ \frac{(n+j)!}{j!n!} \leq 2^{n+j}, \]

we obtain

\[ |f^{(n)}(x)| \leq 2^n n! \sum_{j=0}^{\infty} \frac{|f^{(j+n)}(x_0)|}{(j+n)!} \left( \frac{2r}{3} \right)^j \]

\[ = \left( \frac{2}{r} \right)^n n! \sum_{j=0}^{\infty} \frac{|f^{(j+n)}(x_0)|}{(j+n)!} \left( \frac{2r}{3} \right)^j \]

\[ \leq 3M(x_0) \left( \frac{2}{r} \right)^n n!. \]

Hence

\[ \left| \frac{f^{(n)}(x)}{n!} \right| \left( \frac{r}{2} \right)^n \leq 3M(x_0). \]

Suppose that \( a \) is a point of \( B \). Since \( f^{(n)}(x) \) is continuous, we have

\[ \left| \frac{f^{(n)}(a)}{n!} \right| \left( \frac{r}{2} \right)^n \leq 3M(x_0) \]

and

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n. \]

In the same way as above we obtain
If $a$ is not a point of $B$ and $b$ is a point of $B$, we can also get this inequality.

Since $N(x)$ is bounded on $I$, $f(x)$ is real analytic on $I$. This is a contradiction, because we assumed that $B \cap I$ is not empty.

3. Proof of Theorem

Take an element $v$ of $L$ satisfying $v(x_0) \neq 0$ and any element $u$ of $L$. Let us show that there exist a neighbourhood $U$ and a positive number $c$ such that we have

$$
\phi_t(v)_*u = g_t(u, v)
$$

for $(x, t)$ in $U \times (-c, c)$. Here $\phi_t(v)$ is a local one-parameter group of diffeomorphisms generated by $v$.

This identity is sufficient for our improvement of the theorem, because as shown in [1] the proof of our theorem is reducible to this identity.

Take a cubic neighbourhood $V = \{(x', \cdots, x^n); |x' - x_0'| < 2c\}$ of $x_0$ such that $c(u, v; V) \leq c$. Here we can assume that $v = \frac{\partial}{\partial x^i}$ in $V$. Then we have

$$
(ad v)^k u = \frac{\partial^k u}{\partial (x^i)^k}, \quad k = 1, 2, 3, \ldots
$$

and

$$
\phi_t(v)_*u(x) = u(x-t),
$$

where

$$
x-t = (x'_1-t, x'_2, \cdots, x'_n).
$$

Hence if we put

$$
U = \{(x', \cdots, x^n); |x' - x_0'| < c\},
$$

then by our lemma we obtain

$$
\phi_t(v)_*u(x) = u(x-t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{\partial^k u}{\partial (x^i)^k}(x)
$$

$$
= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (ad v)^k u(x)
$$

for $(x, t)$ in $U \times (-c, c)$. 

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Bibliography
