Title	An analyticity problem and an integration theorem of completely integrable systems with singularities
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Citation	Osaka Journal of Mathematics. 1970, 7(1), p. 225-229
Version Type	VoR
URL	https://doi.org/10.18910/8323
rights	
Note	

Kumano-go, H. and Matsuda, M.
Osaka J. Math.
7 (1970), 225-229

AN ANALYTICITY PROBLEM AND AN INTEGRATION THEOREM OF COMPLETELY INTEGRABLE SYSTEMS WITH SINGULARITIES

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(Received October 31, 1969)

In this note we shall solve an analyticity problem and improve an integration theorem obtained by the second named author [1].

1. Introduction

We shall give a proof to the following

Lemma. *Let $f(x)$ be a real valued C^∞-function on the interval* $(0, 1)$. *Suppose that the radius of convergence of the power series*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is greater than a positive constant r for every x_0 in $(0, 1)$. *Then $f(x)$ is real analytic on the interval* $(0, 1)$.

Applying this lemma we can prove the following

Theorem. *Let M be a C^∞-manifold and L be a Lie subalgebra of the Lie algebra of all C^∞-vector fields on M. For two elements u and v of L, put*

$$g_t(u, v) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}(ad\ v)^k u\,,$$

where $(ad\ v)^k u=[v, (ad\ v)^{k-1}u]$, $k=1, 2, 3, \cdots$. Suppose that for any pair of u and v in L and for any compact subset K in M there exists a positive number $c(u, v; K)$ such that the radius of convergence of $g_t(u, v)$ at x is greater than $c(u, v; K)$ if x is in K. Then through every point x_0 on M there passes a maximal integral manifold $N(x_0)$ of L. Any integral manifold of L containing x_0 is an open submanifold of $N(x_0)$.

This theorem was proved by Matsuda [1] under the additional condition that $g_t(u, v)$ is continuously differentiable with respect to (x, t) term by term.

Acknowledgements. After our preparation of this note, Dr. R. Vaillancourt kindly informed us that our lemma is known as a theorem of Boas (R.P. Boas, Jr.: A theorem on analytic functions of a real variable, Bull. Amer. Math. Soc. 41 (1935), 233–236). Our proof is different from the original one of Boas.

2. Proof of Lemma

The first step is to prove that the set of all points at which $f(x)$ is real analytic is open and dense in (0, 1). Put

$$M(x)=\sup_k \frac{|f^{(k)}(x)|}{k!} r^k .$$

By our assumption $M(x)$ is finite at every x in (0, 1). Take an arbitrary closed interval I_0 in (0, 1). If we put

$$A_n=\{x\in I_0;\ M(x)\leqq n\},$$

then

$$I_0=\bigcup_{n=1}^{\infty} A_n .$$

Since A_n is closed for every n, by Baire's theorem there exist an integer M and an open subinterval I_1 of I_0 such that A_M contains I_1. For two points x and x_0 in I_1, by the mean value theorem we have

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n)}(y)}{n!}(x-x_0)^n ,$$

where $x_0\leqq y\leqq x$ or $x\leqq y\leqq x_0$. If $|x-x_0|=\theta r$ and $0<\theta<1$, then

$$\frac{|f^{(n)}(y)|}{n!}|x-x_0|^n\leqq M\theta^n.$$

Hence $f(x)$ is real analytic at x_0 and on I_1.

The second step is to prove that the set B of all points at which $f(x)$ is not real analytic is empty. To the contrary suppose that B is not empty. Put

$$B_n=\{x\in B;\ M(x)\leqq n\} .$$

Then

$$B=\bigcup_{n=1}^{\infty} B_n .$$

Since B and $B_n (1\leqq n<\infty)$ are closed, by Baire-Hausdorff's theorem there exist an integer N and an open interval I such that B_N contains $I\cap B$ which is not empty. Let us define $N(x)$ by

$$N(x)=\sup_k \frac{|f^{(k)}(x)|}{k!}\left(\frac{r}{2}\right)^k$$

and prove that

$$N(x)\leqq 3N$$

on I, if $|I|<\frac{r}{3}$.

If x is in B, then

$$N(x)\leqq M(x)\leqq N\leqq 3N\,.$$

Suppose that x is not in B. We can take a neighbourhood (a, b) of x in I such that $f(x)$ is real analytic on (a, b) and a or b is a point of B. Fix a point x_0 in (a, b). By the identity

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\,,$$

we have

$$|f^{(n)}(x)|\leqq\sum_{j=0}^{\infty}\frac{|f^{(j+n)}(x_0)|}{j!}\left(\frac{r}{3}\right)^j.$$

Since

$$\frac{(n+j)!}{j!n!}\leqq 2^{n+j}\,,$$

we obtain

$$\begin{aligned}|f^{(n)}(x)|&\leqq 2^n n!\sum_{j=0}^{\infty}\frac{|f^{(j+n)}(x_0)|}{(j+n)!}\left(\frac{2r}{3}\right)^j\\&=\left(\frac{2}{r}\right)^n n!\sum_{j=0}^{\infty}\frac{|f^{(j+n)}(x_0)|}{(j+n)!}r^{j+n}\left(\frac{2}{3}\right)^j\\&\leqq 3M(x_0)\left(\frac{2}{r}\right)^n n!\,.\end{aligned}$$

Hence

$$\frac{|f^{(n)}(x)|}{n!}\left(\frac{r}{2}\right)^n\leqq 3M(x_0)\,.$$

Suppose that a is a point of B. Since $f^{(n)}(x)$ is continuous, we have

$$\frac{|f^{(n)}(a)|}{n!}\left(\frac{r}{2}\right)^n\leqq 3M(x_0)$$

and

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n\,.$$

In the same way as above we obtain

$$N(x) \leqq 3M(a) \leqq 3N .$$

If a is not a point of B and b is a point of B, we can also get this inequality.

Since $N(x)$ is bounded on $I, f(x)$ is real analytic on I. This is a contradiction, because we assumed that $B \cap I$ is not empty.

3. Proof of Theorem

Take an element v of L satisfying $v(x_0) \neq 0$ and any element u of L. Let us show that there exist a neighbourhood U and a positive number c such that we have

$$\phi_t(v)_* u = g_t(u, v)$$

for (x, t) in $U \times (-c, c)$. Here $\phi_t(v)$ is a local one-parameter group of diffeomorphisms generated by v.

This identity is sufficient for our improvement of the theorem, because as shown in [1] the proof of our theorem is reducible to this identity.

Take a cubic neighbourhood

$$V = \{(x^1, \cdots, x^n);\ |x^i - x_0^i| < 2c\}$$

of x_0 such that $c(u, v; \bar{V}) \geqq c$. Here we can assume that $v = \dfrac{\partial}{\partial x^1}$ in V. Then we have

$$(\text{ad } v)^k u = \frac{\partial^k u}{\partial (x^1)^k}, \qquad k = 1, 2, 3, \cdots$$

and

$$\phi_t(v)_* u(x) = u(x-t) ,$$

where

$$x - t = (x^1 - t, x^2, \cdots, x^n) .$$

Hence if we put

$$U = \{(x^1, \cdots, x^n);\ |x^i - x_0^i| < c\} ,$$

then by our lemma we obtain

$$\begin{aligned} \phi_t(v)_* u(x) = u(x-t) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{\partial^k u}{\partial (x^1)^k}(x) \\ &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (\text{ad } v)^k u(x) \end{aligned}$$

for (x, t) in $U \times (-c, c)$.

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[1] M. Matsuda: *An integration theorem for completely integrable systems with singularities,* Osaka J. Math. **5** (1968), 279–283.