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AN ANALYTICITY PROBLEM AND AN INTEGRATION THEOREM OF COMPLETELY INTEGRABLE SYSTEMS WITH SINGULARITIES

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In this note we shall solve an analyticity problem and improve an integration theorem obtained by the second named author [1].

1. Introduction

We shall give a proof to the following

Lemma. *Let $f(x)$ be a real valued C^∞-function on the interval* $(0, 1)$. *Suppose that the radius of convergence of the power series*

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is greater than a positive constant r for every x_0 in $(0, 1)$. *Then $f(x)$ is real analytic on the interval* $(0, 1)$.

Applying this lemma we can prove the following

Theorem. *Let M be a C^∞-manifold and L be a Lie subalgebra of the Lie algebra of all C^∞-vector fields on M. For two elements u and v of L, put*

$$g_t(u, v) = \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}(ad\ v)^k u\ ,$$

where $(ad\ v)^k u=[v, (ad\ v)^{k-1}u]$, $k=1, 2, 3, \cdots$. Suppose that for any pair of u and v in L and for any compact subset K in M there exists a positive number $c(u, v; K)$ such that the radius of convergence of $g_t(u, v)$ at x is greater than $c(u, v; K)$ if x is in K. Then through every point x_0 on M there passes a maximal integral manifold $N(x_0)$ of L. Any integral manifold of L containing x_0 is an open submanifold of $N(x_0)$.

This theorem was proved by Matsuda [1] under the additional condition that $g_t(u, v)$ is continuously differentiable with respect to (x, t) term by term.

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2. Proof of Lemma

The first step is to prove that the set of all points at which $f(x)$ is real analytic is open and dense in (0, 1). Put

$$M(x)=\sup_k \frac{|f^{(k)}(x)|}{k!} r^k .$$

By our assumption $M(x)$ is finite at every x in (0, 1). Take an arbitrary closed interval I_0 in (0, 1). If we put

$$A_n=\{x\in I_0;\ M(x)\leqq n\} ,$$

then

$$I_0=\bigcup_{n=1}^{\infty} A_n .$$

Since A_n is closed for every n, by Baire's theorem there exist an integer M and an open subinterval I_1 of I_0 such that A_M contains I_1. For two points x and x_0 in I_1, by the mean value theorem we have

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n)}(y)}{n!}(x-x_0)^n ,$$

where $x_0\leqq y\leqq x$ or $x\leqq y\leqq x_0$. If $|x-x_0|=\theta r$ and $0<\theta<1$, then

$$\frac{|f^{(n)}(y)|}{n!}|x-x_0|^n\leqq M\theta^n.$$

Hence $f(x)$ is real analytic at x_0 and on I_1.

The second step is to prove that the set B of all points at which $f(x)$ is not real analytic is empty. To the contrary suppose that B is not empty. Put

$$B_n=\{x\in B;\ M(x)\leqq n\} .$$

Then

$$B=\bigcup_{n=1}^{\infty} B_n .$$

Since B and $B_n (1\leqq n<\infty)$ are closed, by Baire-Hausdorff's theorem there exist an integer N and an open interval I such that B_N contains $I\cap B$ which is not empty. Let us define $N(x)$ by

$$N(x)=\sup_k \frac{|f^{(k)}(x)|}{k!}\left(\frac{r}{2}\right)^k$$

and prove that

$$N(x)\leqq 3N$$

on I, if $|I|<\frac{r}{3}$.

If x is in B, then

$$N(x)\leqq M(x)\leqq N\leqq 3N\,.$$

Suppose that x is not in B. We can take a neighbourhood (a, b) of x in I such that $f(x)$ is real analytic on (a, b) and a or b is a point of B. Fix a point x_0 in (a, b). By the identity

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\,,$$

we have

$$|f^{(n)}(x)|\leqq\sum_{j=0}^{\infty}\frac{|f^{(j+n)}(x_0)|}{j!}\left(\frac{r}{3}\right)^j.$$

Since

$$\frac{(n+j)!}{j!n!}\leqq 2^{n+j}\,,$$

we obtain

$$\begin{aligned}|f^{(n)}(x)| &\leqq 2^n n!\sum_{j=0}^{\infty}\frac{|f^{(j+n)}(x_0)|}{(j+n)!}\left(\frac{2r}{3}\right)^j\\ &=\left(\frac{2}{r}\right)^n n!\sum_{j=0}^{\infty}\frac{|f^{(j+n)}(x_0)|}{(j+n)!}r^{j+n}\left(\frac{2}{3}\right)^j\\ &\leqq 3M(x_0)\left(\frac{2}{r}\right)^n n!\,.\end{aligned}$$

Hence

$$\frac{|f^{(n)}(x)|}{n!}\left(\frac{r}{2}\right)^n\leqq 3M(x_0)\,.$$

Suppose that a is a point of B. Since $f^{(n)}(x)$ is continuous, we have

$$\frac{|f^{(n)}(a)|}{n!}\left(\frac{r}{2}\right)^n\leqq 3M(x_0)$$

and

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n\,.$$

In the same way as above we obtain

$$N(x) \leqq 3M(a) \leqq 3N\,.$$

If a is not a point of B and b is a point of B, we can also get this inequality.

Since $N(x)$ is bounded on I, $f(x)$ is real analytic on I. This is a contradiction, because we assumed that $B \cap I$ is not empty.

3. Proof of Theorem

Take an element v of L satisfying $v(x_0) \neq 0$ and any element u of L. Let us show that there exist a neighbourhood U and a positive number c such that we have

$$\phi_t(v)_* u = g_t(u, v)$$

for (x, t) in $U \times (-c, c)$. Here $\phi_t(v)$ is a local one-parameter group of diffeomorphisms generated by v.

This identity is sufficient for our improvement of the theorem, because as shown in [1] the proof of our theorem is reducible to this identity.

Take a cubic neighbourhood

$$V = \{(x^1, \cdots, x^n);\ |x^i - x_0^i| < 2c\}$$

of x_0 such that $c(u, v; \bar{V}) \geqq c$. Here we can assume that $v = \frac{\partial}{\partial x^1}$ in V. Then we have

$$(\mathrm{ad}\ v)^k u = \frac{\partial^k u}{\partial (x^1)^k}, \qquad k = 1, 2, 3, \cdots$$

and

$$\phi_t(v)_* u(x) = u(x-t)\,,$$

where

$$x - t = (x^1 - t, x^2, \cdots, x^n)\,.$$

Hence if we put

$$U = \{(x^1, \cdots, x^n);\ |x^i - x_0^i| < c\}\,,$$

then by our lemma we obtain

$$\begin{aligned} \phi_t(v)_* u(x) = u(x-t) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{\partial^k u}{\partial (x^1)^k}(x) \\ &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (\mathrm{ad}\ v)^k u(x) \end{aligned}$$

for (x, t) in $U \times (-c, c)$.

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Bibliography

[1] M. Matsuda: *An integration theorem for completely integrable systems with singularities*, Osaka J. Math. **5** (1968), 279–283.