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RELATIONS BETWEEN WIENER AND MARTIN BOUNDARIES

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Introduction

In this paper we shall treat two compactifications—Wiener's and Martin's compactifications. Relations between two ideal boundaries of these compactifications were first remarked by Prof. Y. Kusunoki [5], and Prof. S. Mori [7] discussed this theme by using fine cluster sets. In their book [3], C. Constantinescu and A. Cornea pointed out that Martin space is a quotient space of Wiener space. This fact is fundamental throughout this paper.

Meanwhile, J.L. Doob [4] and L. Naïm [8] studied the behaviour of superharmonic functions at the Martin boundary point by using the notion of the fine limit and succeeded in getting beautiful results.

In this paper we shall discuss the relations between limits of harmonic functions at Wiener boundaries and fine limits at Martin boundaries. From this point of view, the structure of Wiener boundaries will be studied.

In §1, the relations between limits of harmonic functions at Wiener boundaries and fine limits at Martin boundaries are studied by means of the harmonic boundary of Wiener space, and the main result is stated in Theorem 1.3. In §2, the same study as in §1 is made by means of the poles in Wiener space. The notion of poles was first introduced by M. Brelot [2] in the case of metrizable compactifications. We define the poles in Wiener space and get the main result of this section, Theorem 2.4. The set of poles is identical with the fine cluster set defined by S. Mori [7]. Both the set of poles and the harmonic boundary on Wiener space are related to the fine limit, but we shall see that these two sets are not identical. We can consider a new minimum principle and the Dirichlet problem in the set of poles, which are treated in §3. In §4 a short remark on harmonic boundary of Martin space is made. In §5 we study the limits of positive minimal harmonic functions at Wiener boundaries. For this purpose we first define a cluster set $\Omega(s)$, which includes the set of poles but is in general different from the harmonic boundary (Theorem 5.2). In the rest of this section we treat mainly bounded minimal harmonic functions. The results stated there are not always new but they are treated in the light of studies developed in the preceding sections. In the last section we treat the relative Dirichlet problem.

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1. Harmonic boundary

1.1. Let R be a hyperbolic Riemann surface. In the following, we shall consider two sorts of compactifications of R, that is, the Wiener's and the Martin's compactifications which are denoted by R^{*W}, R^{*M} respectively. Their boundaries are denoted by Δ^W, Δ^M respectively. There exists a mapping π from R^{*W} onto R^{*M} which leaves every point of R invariant. If we consider these points to be equivalent which are mapped onto the same point by this mapping π, then the quotient space of R^{*W} obtained by this equivalence relation is homeomorphic to R^{*M}. Therefore π is continuous[1].

Let R^* be an *arbitrary* compactification of R, then R is a subset everywhere dense in R^*. We shall write $\Delta=R^*-R$. For each extended real valued function f on Δ (that is, f may take values $\pm\infty$), we condsider the family $\bar{\mathcal{S}}_f$ of all functions v with the following properties:

a) v is superharmonic on R or $\equiv+\infty$,

b) v is bounded from below,

c) at each point $b\in\Delta$, we have $\liminf_{a\to b} v(a)\geq f(b)$.

The lower envelope of this family

$$\bar{H}_f(a)=\inf\{v(a);v\in\bar{\mathcal{S}}_f\}$$

is harmonic on R or $\equiv+\infty$ or $\equiv-\infty$. Similarly we can define $\underline{H}_f$. If $\bar{H}_f\equiv\underline{H}_f$ holds and this function is harmonic, then it is denoted by H_f and f is said to be *resolutive*. If all bounded continuous functions on Δ are resolutive, R^* is called a *resolutive compactification of* R. Let R^* be a resolutive compactification and let a be a fixed point in R. For a bounded continuous function f on Δ, the correspondence

$$f\to H_f(a)$$

defines a positive mass-distribution on Δ. This mass-distribution is denoted by $d\omega_a$ and is called a *harmonic measure*. In the Wiener space and in the Martin space, harmonic measures are denoted by $d\omega_a^W$ and by $d\omega_a^M$ respectively. Also the notations H_f^W, H_f^M are used.

For a Green potential p on R, we set

1) [3], p. 140.

$$\Gamma_p = \{b \in \Delta;\ \liminf_{a \to b} p(a) = 0\}$$

and

$$\Gamma = \bigcap_p \Gamma_p ,$$

where p ranges over the family of all Green potentials on R. Γ is a non-empty compact subset of Δ and is called a *harmonic boundary*. Notations Γ^W, Γ^M are used to express harmonic boundaries in the Wiener space and in the Martin space.

In Δ^W and Δ^M, harmonic boundaries Γ^W and Γ^M are the carriers of harmonic measures $d\omega^W$ and $d\omega^M$, respectively. Each point of Γ^W is regular for the Dirichlet problem.

1.2. Lemma 1.1. *For a Borel set B in Δ^M, $\pi^{-1}(B)$ is a Borel set in Δ^W.*

Proof. Set

$$\mathfrak{B} = \{A \subset \Delta^M;\ \pi^{-1}(A) \textit{ is a Borel set in } \Delta^W\} .$$

For a compact subset $A \subset \Delta^M$, $\pi^{-1}(A)$ is compact in Δ^W, hence $\mathfrak{B}$ contains all compact sets. Since

$$\pi^{-1}(A) \cap \pi^{-1}(\Delta^M - A) = \phi ,$$
$$\pi^{-1}(A) \cup \pi^{-1}(\Delta^M - A) = \Delta^W ,$$
$$\pi^{-1}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} \pi^{-1}(A_n) ,$$

$A \in \mathfrak{B}$ implies $\Delta^M - A \in \mathfrak{B}$, and $A_n \in \mathfrak{B}$ $(n=1, 2, \cdots)$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{B}$.

$\mathfrak{B}$ is a σ-algebra containing all compact sets, and contains all Borel sets in Δ^M.

Lemma 1.2. *If $A \subset \Delta^M$ is of $d\omega_a^M$-harmonic measure zero, then $\hat{A} = \pi^{-1}(A)$ is of $d\omega_a^W$-harmonic measure zero.*

Proof. Assume that the $d\omega_a^W$-outer harmonic measure of $\hat{A}$ is $\alpha > 0$. We take an arbitrary open set G in Δ^M containing A, and set $\hat{G} = \pi^{-1}(G)$. $\hat{G}$ is open in Δ^W and contains $\hat{A}$. Therefore

$$\alpha \leq \omega_a^W(\hat{G}) = \int_{\Delta^W} \chi_{\hat{G}}\, d\omega_a^W , \tag{1. 1}$$

where $\chi_{\hat{G}}$ is the characteristic function of $\hat{G}$ on Δ^W. In general, we have

$$\int \chi_{\hat{G}}\, d\omega_a^W \leq \bar{H}^W_{\chi_{\hat{G}}}(a)^{2)} .$$

2) [1], p. 34, Satz 16.

Now, at each $b \in G$ there exists an open neighbourhood $U(b)$ of b in R^{*M} such that $U(b) \cap \Delta^M \subset G$. Set

$$U_0 = \bigcup_{b \in G} U(b)$$

and

$$\hat{U}_0 = \pi^{-1}(U_0),$$

then $\hat{U}_0$ is open in R^{*W} and $\hat{G} \subset \hat{U}_0$.

Now let v be a superharmonic function on R, bounded from below satisfying the condition: $\liminf_{a \to b} v(a) \geq \chi_G(b)$ for each $b \in \Delta^M$, where $a \to b$ is considered in the sense of the topology of R^{*M}[3]. Then, we have $v \geq 0$. For sufficiently small positive number ε, there exists an open neighbourhood U_0 of G in R^{*M} such that

$$v \geq 1 - \varepsilon \quad \text{on} \quad U_0 \cap R.$$

Since $\hat{U}_0 \cap R = U_0 \cap R$, we have

$$v \geq 0, \quad \text{and} \quad v/(1-\varepsilon) \geq 1 \quad \text{on} \quad \hat{U}_0 \cap R.$$

Hence at each $\tilde{b} \in \Delta^W$, we have

$$\liminf_{a \to \tilde{b}} v(a)/(1-\varepsilon) \geq \chi_{\hat{G}}(\tilde{b}),$$

where $a \to \tilde{b}$ is considered in the sense of the topology of R^{*W}[3].
Hence we have

$$v/(1-\varepsilon) \geq \bar{H}^W_{\chi_{\hat{G}}}.$$

Since ε and v are arbitrary, we have

$$\bar{H}^W_{\chi_{\hat{G}}} \leq \bar{H}^M_{\chi_G} = H^M_{\chi_G}.$$

Therefore

$$\int_{\Delta^W} \chi_{\hat{G}} \, d\omega_a^W \leq \bar{H}^W_{\chi_{\hat{G}}}(a) \leq H^M_{\chi_G}(a) = \int_{\Delta^M} \chi_G \, d\omega_a^M = \omega_a^M(G). \tag{1.2}$$

From (1.1) and (1.2) we have

$$0 < \alpha \leq \omega_a^M(G).$$

Since G is an arbitrary open set in Δ^M containing A, the $d\omega_a^M$-outer harmonic measure of A is positive, q.e.d.

3) In the following, we shall not mention the topology explicitly, unless confusion would occur.

Theorem 1.1. *π is a measurable mapping.*

Proof. Let $E\subset\Delta^M$ be a $d\omega^M$-measurable set, then there exist three sets B, N_1, N_2 with the following properties:

(i) B is a Borel set in Δ^M,

(ii) N_1, N_2 are of $d\omega^M$-harmonic measure zero,

(iii) $E=(B-N_1)\cup N_2$.

From (iii) we have

$$\pi^{-1}(E)=(\pi^{-1}(B)-\pi^{-1}(N_1))\cup\pi^{-1}(N_2)\,.$$

By Lemmas 1.1 and 1.2, $\pi^{-1}(B)$ is a Borel set in Δ^W, and both $\pi^{-1}(N_1)$ and $\pi^{-1}(N_2)$ are of $d\omega^W$-harmonic measure zero. Thus we see that $\pi^{-1}(E)$ is $d\omega^W$-measurable.

Let $f(b)$ be an extended real valued function defined on Δ^M, we define the function $\hat{f}(\tilde{b})$ on Δ^W by $\hat{f}(\tilde{b})=f[\pi(\tilde{b})]$.

Corollary. *If f is $d\omega_a^M$-measurable, then $\hat{f}$ is $d\omega_a^W$-measurable.*

Theorem 1.2. *If f is resolutive as a function on Δ^M, then $\hat{f}$ is resolutive as a function on Δ^W.*

Proof. Let v be a superharmonic function on R, bounded from below satisfying the condition: $\liminf_{a\to b} v(a)\geq f(b)$ for each $b\in\Delta^M$. For an arbitrary positive number ε and for each point $b\in\Delta^M$, there exists an open neighbourhood $U(b)$ of b in R^{*M} such that

$$v\geq f(b)-\varepsilon \quad \text{on} \quad U(b)\cap R\,.$$

Since $\hat{U}_b=\pi^{-1}(U(b))$ is a neighbourhood of each point of $\pi^{-1}(b)$, we have

$$v+\varepsilon\geq\bar{H}_{\hat{f}}^W\,.$$

Here ε and v are arbitrary and we get

$$\bar{H}_f^M\geq\bar{H}_{\hat{f}}^W\,.$$

Similarly we have

$$\underline{H}_f^M\leq\underline{H}_{\hat{f}}^W\,.$$

Thus we obtain the desired relation

$$H_f^M\leq\underline{H}_{\hat{f}}^W\leq\underline{\int}_{\Delta^W}\hat{f}\,d\omega^W\leq\overline{\int}_{\Delta^W}\hat{f}\,d\omega^W\leq\bar{H}_{\hat{f}}^W\leq H_f^M\,.$$

Corollary. *If f is $d\omega_a^M$-summable, then $\hat{f}$ is $d\omega_a^W$-summable and*

$$\int_{\Delta^M} f d\omega_a^M = \int_{\Delta^W} \hat{f} d\omega_a^W .$$

Theorem 1.3. *If u is a quasi-bounded harmonic function on R, then there exists a set $\hat{E}$ in Δ^W of $d\omega^W$-harmonic measure zero and the following relation holds.*

$$\lim_{a\to\tilde{b}} u(a) = \text{fine} \lim_{a\to\pi(\tilde{b})} u(a), \quad \textit{for each} \quad \tilde{b}\in\Delta^W - \hat{E} .$$

Proof. Let φ be a trace on Δ^W of the continuous extension of u in R^{*W}, that is, the restriction on Δ^W of the continuous extension of u in R^{*W}. Then we have

$$u(a) = \int_{\Delta^W} \varphi(\tilde{b})\, d\omega_a^W(\tilde{b})\text{[4]} . \tag{1.3}$$

On the other hand, there exists a $d\omega_a^M$-measurable function f on Δ^M such that

$$u(a) = \int_{\Delta^M} f(b)\, d\omega_a^M(b)\text{[5]} . \tag{1.4}$$

By the corollary to Theorem 1.2, we have

$$u(a) = \int_{\Delta^W} \hat{f}(\tilde{b})\, d\omega_a^W(\tilde{b}) , \tag{1.5}$$

where $\hat{f}(\tilde{b}) = f[\pi(\tilde{b})]$.

From (1.3), (1.5) we have

$$\int [\varphi(\tilde{b}) - \hat{f}(\tilde{b})]\, d\omega_a^W(\tilde{b}) = 0 .$$

Therefore

$$\int \max\, [\varphi(\tilde{b}) - \hat{f}(\tilde{b}),\, 0]\, d\omega_a^W(\tilde{b}) = 0 \vee 0 = 0 ,$$

$$\int \min\, [\varphi(\tilde{b}) - \hat{f}(\tilde{b}),\, 0]\, d\omega_a^W(\tilde{b}) = 0 \wedge 0 = 0 .$$

From these we conclude

$$\int |\varphi(\tilde{b}) - \hat{f}(\tilde{b})|\, d\omega_a^W(\tilde{b}) = 0 , \tag{1.6}$$

that is, we have

$$\varphi(\tilde{b}) = \hat{f}(\tilde{b}) = f[\pi(\tilde{b})] \tag{1.7}$$

at every point $\tilde{b}$, except at those of a set $\hat{E}_1$ of $d\omega_a^W$-harmonic measure zero.

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4) [6], p. 37, th. 4.4.

5) [2], p. 332.

Let E denote the set of points $b \in \Delta_1^M$[6] at which u has no fine limit or has fine limit not equal to $f(b)$, then E is of $d\omega_a^M$-harmonic measure zero[7]. By Lemma 1.2, we see that

$$\hat{E}_2 = \pi^{-1}(E \cup \Delta_0^M)^{6)}$$

is of $d\omega_a^W$-harmonic measure zero. Then, from (1.7) we have

$$\lim_{a \to \tilde{b}} u(a) = \varphi(\tilde{b}) = \hat{f}(\tilde{b}) = f[\pi(\tilde{b})] = \text{fine} \lim_{a \to \pi(\tilde{b})} u(a)$$

for each $\tilde{b} \in \Delta^W - \hat{E}$, where $\hat{E} = \hat{E}_1 \cup \hat{E}_2$.

Lemma 1.3. *Let u be a positive singular harmonic function, then the continuous extension of u in R^{*W} is zero at each point of Γ^W.*

Proof. Assume that $u(\tilde{b}) = \alpha > 0$ at $\tilde{b} \in \Gamma^W$ (the continuous extension of u in R^{*W} will be denoted by u again). Take α_1 such as $0 < \alpha_1 < \alpha$ and consider

$$p = \inf(u, \alpha_1) .$$

p is a continuous potential. From the assumption, there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that

$$u > \alpha_1 \quad \text{on} \quad \hat{U}(\tilde{b}) \cap R .$$

Therefore

$$p = \alpha_1 \quad \text{on} \quad \hat{U}(\tilde{b}) \cap R .$$

By the definition of Γ^W and by the continuity of p we have

$$0 = \liminf_{a \to \tilde{b}} p(a) = \limsup_{a \to \tilde{b}} p(a) \geq \alpha_1 ,$$

which is a contradiction.

Theorem 1.4. *If u is an HP function (that is, u is the difference of some two non-negative harmonic functions), then we have*

$$\lim_{a \to \tilde{b}} u(a) = \text{fine} \lim_{a \to \pi(\tilde{b})} u(a)$$

at every $\tilde{b}$, except at those of a set of $d\omega^W$-harmonic measure zero.

Proof. Without loss of generality, we can assume $u \geq 0$. u is decomposed into a sum

$$u = u_1 + u_2 ,$$

6) Δ_1^M and Δ_0^M denote the minimal and the non-minimal Martin boundaries, respectively.

7) [4], p. 296, th. 4.1.

where u_1 is quasi-bounded and u_2 is singular. By Theorem 1.3, there exists $\hat{E}$ of $d\omega^W$-harmonic measure zero such that

$$\lim_{a \to \tilde{b}} u_1(a) = \text{fine} \lim_{a \to \pi(\tilde{b})} u_1(a)$$

holds at each $\tilde{b} \in \Delta^W - \hat{E}$.
Since

$$p = \inf(u_2, 1)$$

is a continuous potential on R, there exists a subset E of Δ_1^M of $d\omega^M$-harmonic measure zero such that p has fine limit zero at each point of $\Delta_1^M - E$[8]. Therefore by Lemma 1.3 and by the continuity of p, we have

$$\lim_{a \to \tilde{b}} u_2(a) = 0 ,$$

$$\lim_{a \to \tilde{b}} p(a) = \liminf_{a \to \tilde{b}} p(a) = 0$$

at every point $\tilde{b}$ not in $(\Delta^W - 1^{\prime W}) \cup \pi^{-1}(E)$.
Since $\pi(\tilde{b}) \notin E$,

$$\text{fine} \lim_{a \to \pi(\tilde{b})} p(a) = \text{fine} \lim_{a \to \pi(\tilde{b})} u_2(a) = 0 .$$

This completes the proof.

Corollary. *If v is non-negative and superharmonic on R, then we have*

$$\liminf_{a \to \tilde{b}} v(a) = \text{fine} \lim_{a \to \pi(\tilde{b})} v(a) = \text{fine} \lim_{a \to \pi(\tilde{b})} u(a)$$

at each point $\tilde{b}$, except at those of a set of $d\omega^W$-harmonic measure zero, where u is the greatest harmonic minorant of v. In particular, if v is continuous, we have

$$\lim_{a \to \tilde{b}} v(a) = \text{fine} \lim_{a \to \pi(\tilde{b})} v(a) ,$$

where equality holds almost everywhere in the sense of $d\omega^W$-harmonic measure.

REMARK. As is shown in the follwoing example, the *lim inf* in the above corollary can not be replaced by the *lim*.

EXAMPLE. Let R be a unit disc $\{z; |z| < 1\}$ and let $\{a_n\}$, $a_n \neq 0$, be a sequence of points in R which is dense in R. If G_{a_n} denotes a Green function in R with pole a_n, we can select a sequence of positive numbers $\{c_n\}$ such that

$$\sum_{n=1}^{\infty} c_n G_{a_n}(0) < \infty .$$

8) [8], p. 235, th. 21.

The function

$$v(z)=\sum_{n=1}^{\infty} c_n G_{a_n}(z)$$

has $\limsup_{a\to\tilde{b}} v(a)=+\infty$ at every point $\tilde{b}$ on Δ^W. On the contrary, every potential has fine limit zero at every point of Δ^M, except at those of a set of $d\omega^M$-harmonic measure zero.

2. Poles on the Wiener boundary

2.1. The notion of poles was considered by M. Brelot [2] and by L. Naïm [8] for the case of *metrizable* compactification. In this section we shall extend this notion of poles to an arbitrary compactification R^* of R.

Let s be a minimal Martin boundary point, let K_s be a positive minimal harmonic function on R corresponding to s, and let A be an arbitrary subset of $\Delta=R^*-R$. We define

$$(K_s)_A(a)=\mathit{inf}\, v(a)\,,$$

where v ranges over the family of all non-negative superharmonic functions on R dominating K_s on the intersection of R and some neighbourhood of A in R^*. Clearly we have that

$(K_s)_A$ *is a non-negative harmonic function and* $K_s\geq(K_s)_A$.

Lemma 2.1. $(K_s)_A$ *is either zero or identically equal to* K_s, *and the former case holds if and only if there exists a neighbourhood of* A *in* R^* *such that its trace on* R (*the intersection of* R *and the neighbourhood*) *is thin at* s.

Proof. For a neighbourhood $\hat{U}$ of A in R^*, we define $(K_s)_{\hat{U}\cap R}(a)$ as an infimum of $v(a)$ satisfying the following conditions:

1) v is a non-negative superharmonic function on R,
2) v dominates K_s on $\hat{U}\cap R$.

As is known[9], $(K_s)_{\hat{U}\cap R}$ is equal to a potential or equal to K_s. The former case occurs if and only if $\hat{U}\cap R$ is thin at s. In this case $(K_s)_A=0$ holds, since $(K_s)_A$ is a non-negative harmonic function which is dominated by the potential $(K_s)_{\hat{U}\cap R}$.

Lemma 2.2. *There exists at least one point* z *of* Δ *such that*

$$(K_s)_{\{z\}}\equiv K_s\,.$$

Proof. Assume that there exists no such point. At each point $z\in\Delta$ there

9) [8], p. 204, lemme 1, th. 5; also, p. 233, th. 20.

exists a neighbourhood $\hat{U}(z)$ of z such that $\hat{U}(z)\cap R$ is thin at s. Since Δ is compact, it is covered by a finite number of such neighbourhoods, say $\{\hat{U}(z_i)\}$ $(i=1, 2, \cdots, n)$. $\bigcup_{i=1}^{n} \hat{U}(z_i)\cap R$ is thin at s. On the other hand, $R-\bigcup_{i=1}^{n}(\hat{U}(z_i)\cap R)\cap\Delta=\phi$, that is, $R-\bigcup_{i=1}^{n}(\hat{U}(z_i)\cap R)$ is relatively compact, whence it is also thin at s. But then, R should be thin at s which is a contradiction, q.e.d.

The point z in Lemma 2.2, is called *a pole of s on* Δ.

2.2. In the following, we shall consider the poles in R^{*W}. The set of the poles of s on Δ^W is denoted by $\Phi(s)$, The next theorem gives a new characterization of poles.

Theorem 2.1. *Let* $\mathfrak{F}_s$ *denote the family of subsets E of R such that R—E is thin at s. Then we have*

$$\Phi(s)=\bigcap_{E\in\mathfrak{F}_s}\bar{E},$$

where the closure of E is taken in R^{*W}.

Proof. First we assume $\tilde{b}\in\Phi(s)$, then by Lemma 2.1, for every neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ the set $\hat{U}(\tilde{b})\cap R$ is not thin at s. If for each $E\in\mathfrak{F}_s$ there should exist an $\hat{U}(\tilde{b})$ such that $\hat{U}(\tilde{b})\cap R\cap E=\phi$, then $\hat{U}(\tilde{b})\cap R\subset R-E$, and $\hat{U}(\tilde{b})\cap R$ would be thin at s since $R-E$ is thin at s. This implies that $\hat{U}(\tilde{b})\cap R\cap E\neq\phi$ for an arbitrary neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$, that is, $\tilde{b}\in\bar{E}$.

Next we assume $\tilde{b}\notin\Phi(s)$, then there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that $\hat{U}(\tilde{b})\cap R$ is thin at s. Therefore $R-\hat{U}(\tilde{b})$ belongs to $\mathfrak{F}_s$, and $\tilde{b}\notin R-\hat{U}(\tilde{b})$. This means that $\tilde{b}\notin\bigcap_{E\in\mathfrak{F}_s}\bar{E}$.

Corollary. $\Phi(s)$ *is compact.*

For $s\in\Delta_1^M$, S. Mori [7] defined the fine cluster set:

$$\Psi(s)=\bigcap\{\bar{G};\ G \text{ is open in } R \text{ and } R-G \text{ is thin at } s\},$$

where the closure is taken in R^{*W}. The next theorem gives a new characterization of this Mori's cluster set $\Psi(s)$.

Theorem 2.2. *For* $s\in\Delta_1^M$, *we have* $\Phi(s)=\Psi(s)\subset\pi^{-1}(s)$.

Proof. By Theorem 2.1, it is clear that $\Phi(s)\subset\Psi(s)$. If $\tilde{b}\in\Psi(s)-\Phi(s)$, then, from the compactenss of $\Phi(s)$, there exist an open neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ and an open neighbourhood $\hat{G}$ of $\Phi(s)$ such that $\hat{U}(\tilde{b})\cap\hat{G}=\phi$. We note that $\bar{E}-\hat{G}$ is compact for each $E\in\mathfrak{F}_s$ (the definition of $\mathfrak{F}_s$ is given in Theorem 2.1). Now, if we assume that $\bar{E}\not\subset\hat{G}$ for every $E\in\mathfrak{F}_s$,then we can conclude that for an arbitrary finite number of $E_i\in\mathfrak{F}_s$ $(i=1, 2, \cdots, n)$ holds

$$\bigcap_{i=1}^{n} \bar{E}_i - \hat{G} \supset (\overline{\bigcap_{i=1}^{n} E_i}) - \hat{G} \neq \phi .$$

By the finite intersection property of compactness, we have

$$\Phi(s) - \hat{G} = \bigcap_{E \in \mathfrak{F}_s} \bar{E} - \hat{G} \neq \phi ,$$

which is a contradiction. Therefore we have $\bar{E} \subset \hat{G}$ for some $E \in \mathfrak{F}_s$, and $E \subset \bar{E} \cap R \subset \hat{G} \cap R$. $\underline{\hat{G} \cap R}$ is open in R and $R - (\hat{G} \cap R)$ is thin at s. Since $\tilde{b} \in \Psi(s)$, we get $\tilde{b} \in \overline{\hat{G} \cap R}$, while $\hat{U}(\tilde{b}) \cap \hat{G} = \phi$. This is a contradiction. Thus we have $\Phi(s) = \Psi(s)$.

In order to prove $\Psi(s) \subset \pi^{-1}(s)$, it is sufficient to show that

$$\pi^{-1}(s) = \bigcap \ \{\overline{U \cap R};\ U \text{ is an open neighbourhood of } s \text{ in } R^{*M}\} ,$$

where the closure is taken in R^{*W}. If $\tilde{b} \in \pi^{-1}(s)$, then, for each open neighbourhood U of s in R^{*M} there exists an open neighbourhood $\hat{V}(\tilde{b})$ of $\tilde{b}$ in R^{*W} such that $\pi(\hat{V}(\tilde{b})) \subset U$. Since $\hat{V}(\tilde{b}) \cap R = \pi(\hat{V}(\tilde{b})) \cap R \subset U \cap R$, $\tilde{b} \in \overline{U \cap R}$. On the other hand, if $\tilde{b} \notin \pi^{-1}(s)$, then $\pi(\tilde{b}) = s_1 \neq s$. There exist an open neighbourhood U_1 of s_1 and an open neighbourhood U of s such that $U \cap U_1 = \phi$. We can find a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ in R^{*W} such that $\pi(\hat{U}(\tilde{b})) \subset U_1$. This implies that $\tilde{b} \notin \overline{U \cap R}$, q.e.d.

This theorem shows that *Mori's fine cluster set* $\Psi(s)$ *coincides with the set of poles of s on* Δ^W.

Theorem 2.3. *If we assign an arbitrary neighbourhood* $\hat{V}(\tilde{b})$ *of* $\tilde{b}$ *to each* $\tilde{b} \in \Phi(s)$, *then* $R - (\bigcup_{\tilde{b} \in \Phi(s)} \hat{V}(\tilde{b}) \cap R)$ *is thin at s.*

Proof. As we have seen in the proof of Theorem 2.2,

$$\pi^{-1}(s) = \bigcap \ \overline{U \cap R} ,$$

where U ranges over the family of all open neighbourhoods of s in R^{*M} and the closure is taken in R^{*W}. From this we see that, for an arbitrary open neighbourhood $\hat{G}$ of $\pi^{-1}(s)$, $\hat{G} \cap R$ is the trace on R of some neighbourhood of s in R^{*M}.

We assign an arbitrary open neighbourhood $\hat{V}(\tilde{b})$ to each $\tilde{b} \in \Phi(s)$, and an open neighbourhood $\hat{V}(\tilde{b}')$ such that $\hat{V}(\tilde{b}') \subset R^{*W} - \Phi(s)$ to each $\tilde{b}' \in \pi^{-1}(s) - \Phi(s)$. From the compactness of $\pi^{-1}(s)$ it follows that there exist a finite number of $\tilde{b}_i$ $(i=1, 2, \cdots, n)$ such that $\bigcup_{i=1}^{n} \hat{V}(\tilde{b}_i)$ is an open neighbourhood of $\pi^{-1}(s)$. $R - (\bigcup_{i=1}^{n} \hat{V}(\tilde{b}_i) \cap R)$ is thin at s. Since $\hat{V}(\tilde{b}_i) \cap R$ is thin at s if $\tilde{b}_i \in \pi^{-1}(s) - \Phi(s)$, we can remove such points from $\tilde{b}_i$'s and get the theorem, q.e.d.

For $E\subset\Delta_1^M$ we define $\Phi(E)=\bigcup_{s\in E}\Phi(s)$.

2.3. Theorem 2.4. *If u is a non-negative harmonic function on R, then there exists a polar set $\hat{N}$ in Δ^W such that $\lim_{a\to\tilde{b}} u(a)$ is finite and is equal to the fine limit* $\text{fine}\lim_{a\to\pi(\tilde{b})} u(a)$ *for $\tilde{b}\in\Phi(\Delta_1^M)-\hat{N}$.*

Proof. It is known that there exists a subset N of Δ^M of $d\omega^M$-harmonic measure zero such that u has a finite fine limit at each $s\in\Delta^M-N$[10]. Since N is polar, by Theorem 1.2 $\hat{N}=\pi^{-1}(N)$ is also polar. If $\tilde{b}\in\Phi(\Delta_1^M)-\hat{N}$, then we have

$$\text{fine}\lim_{a\to\pi(\tilde{b})} u(a)=\alpha<+\infty .$$

For an arbitrary positive number ε we set

$$F_\varepsilon=\{a\in R;\ |u(a)-\alpha|\geq\varepsilon\} .$$

F_ε is thin at $\pi(\tilde{b})$. By lemmas 2.1 and 2.2 we see that $\hat{U}(\tilde{b})\cap R$ is not thin at $\pi(\tilde{b})$ for an arbitrary neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$. Hence $\hat{U}(\tilde{b})\cap R\not\subset F_\varepsilon$, that is,

$$\hat{U}(\tilde{b})\cap\{a\in R;\ |u(a)-\alpha|<\varepsilon\}\neq\phi .$$

Therefore we have

$$\inf_{a\in\hat{U}(\tilde{b})\cap R} u(a)\leq\alpha+\varepsilon ,$$

$$\sup_{a\in\hat{U}(\tilde{b})\cap R} u(a)\geq\alpha-\varepsilon .$$

From these

$$\liminf_{a\to\tilde{b}} u(a)\leq\alpha+\varepsilon ,$$

$$\limsup_{a\to\tilde{b}} u(a)\geq\alpha-\varepsilon .$$

Since $\lim_{a\to\tilde{b}} u(a)$ exists, we get

$$\alpha-\varepsilon\leq\lim_{a\to\tilde{b}} u(a)\leq\alpha+\varepsilon .$$

ε being arbitrary, we get the theorem.

REMARK. The set N in the proof of Theorem 2.4 is the union of Δ_0^M and the set of points in Δ_1^M at which u has no finite fine limit. In particular $\pi^{-1}(\Delta_0^M)\subset\hat{N}$.

Corollary 1. *In R^{*W} every singular harmonic function on R has limit zero at each point of $\Phi(\Delta_1^M)$ except at those of some polar set.*

10) [4], p. 297, th. 4.3.

Corollary 2. *In R^{*W} every potential on R has lim inf zero at each point of $\Phi(\Delta_1^M)$ except at those of some polar set.*

REMARK. In general $\Phi(\Delta_1^M) \neq \Gamma^W$. We shall see later (Theorem 5.2 in §5) that there exists even the case where $\Phi(\Delta_1^M) \cap \Gamma^W = \phi$.

3. Minimum principle and Dirichlet problem

3.1. Theorem 3.1. *Let v be a superharmonic function on R, bounded from below. If, at each point $\tilde{b} \in \Phi(\Delta_1^M)$,*

$$\liminf_{a \to \tilde{b}} v(a) \geq \alpha ,$$

then we have $v \geq \alpha$.

Proof. To each $s \in \Delta_1^M$ we assign an arbitrary $\tilde{b}_{(s)} \in \Phi(s)$. Let $\alpha > -\infty$. For every $\alpha' < \alpha$ there exists a neighbourhood $\hat{U}(\tilde{b}_{(s)})$ of $\tilde{b}_{(s)}$ such that

$$v \geq \alpha' \quad \text{on} \quad E_s = \hat{U}(\tilde{b}_{(s)}) \cap R .$$

E_s is not thin at s. Therefore we have $\liminf_{\substack{a \to s \\ a \in E_s}} v(a) \geq \alpha'$ at each $s \in \Delta_1^M$. Hence, $v \geq \alpha'$ [11], and we have the theorem since α' is arbitrary. If $\alpha = -\infty$, the theorem is trivial.

Corollary. *If u is a quasi-bounded harmonic function on R, then we have*

$$\inf_{\tilde{b} \in \Phi(\Delta_1^M)} [\liminf_{a \to \tilde{b}} u(a)] \leq u \leq \sup_{\tilde{b} \in \Phi(\Delta_1^M)} [\limsup_{a \to \tilde{b}} u(a)]$$

at every point of R.

Theorem 3.2. $\underline{H}^W_{\chi_{(\Delta^W - \Phi(\Delta_1^M))}} = 0$, *and* $\bar{H}^W_{\chi_{\Phi(\Delta_1^M)}} = 1$.

Proof. Let u be a subharmonic function on R, bounded from above satisfying the condition:

$$\limsup_{a \to \tilde{b}} u(a) \leq \chi_{(\Delta^W - \Phi(\Delta_1^M))}(\tilde{b})$$

for each point $\tilde{b} \in \Delta^W$. Let ε be an arbitrary positive number. At each point $\tilde{b} \in \Phi(\Delta_1^M)$, there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that

$$u \leq \varepsilon \quad \text{on} \quad \hat{U}(\tilde{b}) \cap R .$$

11) [8], p. 244, th. 22.

If we set $E_s = \bigcup_{\tilde{b} \in \Phi(s)} \hat{U}(\tilde{b}) \cap R$ for each $s \in \Delta_1^M$, then E_s is not thin at s. We have

$$\limsup_{\substack{a \to s \\ a \in E_s}} u(a) \leq \varepsilon .$$

This implies $u \leq \varepsilon$[11]. From this we get

$$\underline{H}^W_{\chi_{(\Delta^W - \Phi(\Delta_1^M))}} = 0 ,$$

and

$$\bar{H}^W_{\chi_{\Phi(\Delta_1^M)}} = 1 - \underline{H}^W_{\chi_{(\Delta^W - \Phi(\Delta_1^M))}} = 1 .$$

Corollary. *$\Phi(\Delta_1^M)$ is not a polar set.*

3.2. Theorem 3.3. *Let f be an extended real valued function defined on Δ^M. We define the function $\hat{f}$ on Δ^W by $\hat{f}(\tilde{b}) = f[\pi(\tilde{b})]$. If an extended real valued function φ defined on Δ^W coincides with $\hat{f}$ on the set $\Phi(\Delta_1^M)$, then*

$$\underline{H}^W_\varphi \leq \underline{H}^M_f \leq \bar{H}^M_f \leq \bar{H}^W_\varphi .$$

Proof. Let $\underline{\mathcal{S}}^W_\varphi$ be the family of all functions u with the following properties:

1) u is subharmonic on R or $\equiv -\infty$,
2) u is bounded from above,
3) at each $\tilde{b} \in \Delta^W$ we have $\limsup_{a \to \tilde{b}} u(a) \leq \varphi(\tilde{b})$.

Corresponding to $\underline{\mathcal{S}}^W_\varphi$ we consider the family $\mathfrak{G}_f$ of all functions u' satisfying the following conditions:

1′) u' is subharmonic on R or $\equiv -\infty$.
2′) u' is bounded from above,
3′) at each $s \in \Delta_1^M$ there exists a subset E_s of R (depending on u' in general) such that

(i) E_s is not thin at s,
(ii) $\limsup_{\substack{a \to s \\ a \in E_s}} u'(a) \leq f(s)$.

Then, it is known that

$$\underline{H}^M_f(a) = \sup \{u'(a);\, u' \in \mathfrak{G}_f\}^{12)} .$$

Let ε be an arbitrary positive number and $u \in \underline{\mathcal{S}}^W_\varphi$. At each $\tilde{b} \in \Delta^W$ there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ in R^{*W} such that

$$a \in \hat{U}(\tilde{b}) \cap R \quad \textit{implies} \quad u(a) < \varphi(\tilde{b}) + \varepsilon .$$

$E_s = \bigcup_{\tilde{b} \in \Phi(s)} \hat{U}(\tilde{b}) \cap R$ is not thin at s and for $\tilde{b} \in \Phi(s)$ we have

12) [8], p. 245, th. 23.

$$\limsup_{\substack{a \to s \\ a \in B_s}} u(a) \leq \varphi(\tilde{b}) + \varepsilon = \hat{f}(\tilde{b}) + \varepsilon = f(s) + \varepsilon .$$

From this $u \in \mathfrak{S}_{f+\varepsilon}$, and therefore

$$u \leq \underline{H}^M_{f+\varepsilon} = \underline{H}^M_f + \varepsilon .$$

Since u and ε are arbitrary, we have

$$\underline{H}^W_\varphi \leq \underline{H}^M_f .$$

As a corollary to this theorem, we get the following result which was obtained by S. Mori[13].

Corollary. $\Gamma^W \subset \overline{\Phi(\Delta_1^M)}$.

Proof. Assume that there exists a point $\tilde{b} \in \Gamma^W$ which does not belong to $\overline{\Phi(\Delta_1^M)}$. Since Γ^W is the carrier of $d\omega^W$-harmonic measure, there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that

(3. 1) $$d\omega^W(\hat{U}(\tilde{b}) \cap \Delta^W) > 0 ,$$

(3. 2) $$\hat{U}(\tilde{b}) \cap \overline{\Phi(\Delta_1^M)} = \phi .$$

By (3.2) we can find a non-negative continuous function ψ defined on Δ^W which is zero on $\overline{\Phi(\Delta_1^M)}$ and is positive on $\hat{U}(\tilde{b}) \cap \Delta^W$. We set

$$u(a) = H^W_\psi(a) = \int_{\Delta^W} \psi(\tilde{b})\, d\omega^W_a(\tilde{b}) .$$

From this and (3.1), u is positive and harmonic. If we take $f \equiv 0$ in Theorem 3.3, since ψ is zero on $\Phi(\Delta_1^M)$, we have

$$u = \underline{H}^W_\psi \leq 0 \leq \bar{H}^W_\psi = u .$$

That is, $u = 0$, which is a contradiction.

4. Remarks on metrizable compactification

In this section we consider the metrizable resolutive compactification. This was discussed by M. Brelot [2] and by L. Naïm [8] from the standpoint related to the Martin's minimal point. Our discussion here is related to the harmonic boundary and to the difference of R^{*M} from R^{*W}.

Let R^* be an arbitrary *metrizable* resolutive compactification. Let $\Delta^* = R^* - R$, and let Γ^* be the harmonic boundary of Δ^*. Let Δ_1' be the set of those points in Δ_1^M that have the unique pole each in Δ^*, and Δ_1^* the set of

13) [7], p. 34, prop. 3.5.

their poles. Then, as is well-known[14])

$$\omega^M(\Delta^M - \Delta_1') = 0\,,$$

$$\omega^*(\Delta^* - \Delta_1^*) = 0\,,$$

where ω^* denotes the harmonic measure on Δ^*.

In particular, if we take R^{*M} as R^*, the symmetric difference of Δ_1^M and Γ^M is of $d\omega^M$-harmonic measure zero and $\Phi(\Delta_1^M) = \Delta_1^M$ holds.

In the next section, we shall see that these situations are quite different in the case of R^{*W}. In fact, there exists even the case where $\Phi(\Delta_1^M) \cap \Gamma^W = \phi$ holds.

REMARK. While Δ_1^M is a G_δ set, Γ^M is a compact set. These two sets are different in general. In the following example $\Delta_1^M \not\subset \Gamma^M$. For $R = \{z;\ 0 < |z| < 1\}$, $z = 0$ is a minimal boundary point in R^{*M} (a minimal positive harmonic function $\log 1/|z|$ corresponds to $z = 0$), but $z = 0$ is not a harmonic boundary point since a Green function on R takes a positive lim inf at $z = 0$.

5. Relations between $\Phi(\Delta_1^M)$ and Γ^W

5.1. Here we discuss some applications of the theory developed in the preceding sections, and get a few results, some are new and some are known. To this end we introduce the following cluster set $\Omega(s)$. Let $s \in \Delta_1^M$ and let K_s be a minimal harmonic function corresponding to s. Set $\bar{\alpha} = \sup_{a \in R} K_s(a)$. For $\alpha < \bar{\alpha}$ set $G_\alpha = \{a \in R;\ K(a) > \alpha\}$. Our cluster set is defined by

$$\Omega(s) = \bigcap_\alpha \bar{G}_\alpha\,,$$

where α ranges over all positive numbers less than $\bar{\alpha}$ and the closure is taken in R^{*W}. Since G_α is open and $R - G_\alpha$ is thin at s, we have the following relation

$$\Phi(s) = \Psi(s) \begin{array}{l} \subset \pi^{-1}(s) \\ \subset \Omega(s) \end{array}$$

Theorem 5.1. *A necessary and sufficient condition for $\tilde{b}$ to be contained in $\Omega(s)$ is that the following equality holds*:

$$\lim_{a \to \tilde{b}} K_s(a) = \sup_{a \in R} K_s(a)\,.$$

Proof. If $\tilde{b} \in \Omega(s)$, then for every neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ and for an arbitrary positive number $\alpha (< \bar{\alpha})$ we have $\hat{U}(\tilde{b}) \cap G_\alpha \neq \phi$, so that $\sup_{a \in \hat{U}(\tilde{b}) \cap R} K_s(a) \geq \alpha$. This means

14) [8], Chap. V, p. 256.

$$\lim_{a\to\tilde{b}} K_s(a)=\lim_{a\to\tilde{b}}\sup K_s(a)\geq\alpha\ .$$

Therefore, $\lim_{a\to\tilde{b}} K_s(a)=K_s(\tilde{b})\geq\alpha$ and $K_s(\tilde{b})\geq\bar{\alpha}=\sup_{a\in R} K_s(a)$. That is, $K_s(\tilde{b})=\bar{\alpha}$. Next, if we assume $\tilde{b}\notin\Omega(s)$, then there exists an α ($<\bar{\alpha}$) such that $\tilde{b}\notin\bar{G}_\alpha$. Hence there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that $\hat{U}(\tilde{b})\cap G_\alpha=\phi$. This means $K_s\leq\alpha$ on $\hat{U}(\tilde{b})\cap R$. Hence

$$\lim_{a\to\tilde{b}} K_s(a)=\lim_{a\to\tilde{b}}\inf K_s(a)\leq\alpha<\bar{\alpha}\ .$$

Corollary. *$\tilde{b}\in\Phi(s)$ implies $\lim_{a\to\tilde{b}} K_s(a)=\sup_{a\in R} K_s(a)$.*

Theorem 5.2. *If $\bar{\alpha}=\sup_{a\in R} K_s(a)=+\infty$, then we have*

$$\Gamma^W\cap\pi^{-1}(s)\cap\Omega(s)=\phi\ ,$$

in particular, $\Gamma^W\cap\Phi(s)=\phi$.

Proof. Since K_s is singular, K_s has limit zero at each point of Γ^W by Lemma 1.3. This gives the theorem.

5.2. In this paragraph we consider only the *bounded* minimal harmonic function K_s.

Theorem 5.3. *There exists only one point $\tilde{b}$ in $\pi^{-1}(s)$ such that*

$$u(a)=\int_{\Delta^W}\chi_{\{\tilde{b}\}}\,d\omega_a^W$$

is a bounded harmonic function on R. Moreover $u(a)=c\cdot K_s(a)$, where $1/c=\bar{\alpha}=\sup_{a\in R} K_s(a)$.

Proof. By Theorem 1.2 we have

$$\int_{\Delta^M}\chi_{\{s\}}\,d\omega_a^M=\int_{\Delta^W}\chi_{\pi^{-1}(s)}\,d\omega_a^W=\int_{\Delta^W}\chi_{(\pi^{-1}(s)\cap\Gamma^W)}\,d\omega_a^W\ .$$

Let this function be denoted by $u_1(a)$,and let λ be the canonical measure of the harmonic function 1. Then we have

$$\int_{\Delta^M}\chi_{\{s\}}\,d\omega_a^M=\int_{\Delta^M}\chi_{\{s\}}(t)K_t(a)\,d\lambda(t)=\lambda(\{s\})\cdot K_s(a)\ ,$$

that is, $u_1(a)=c_1K_s(a)$. Since $\omega^M(\{s\})>0$, we have

$$\operatorname{fine}\lim_{a\to s} u_1(a)=\chi_{\{s\}}(s)=1\ .$$

On the other hand, by Theorem 2.4 we have

$$(5.1) \qquad \lim_{a\to\tilde{b}} u_1(a) = \text{fine} \lim_{a\to s} u_1(a) = 1$$

at $\tilde{b}\in\Phi(s)$, while, by the corollary to Theorem 5.1

$$u_1(\tilde{b}) = \lim_{a\to\tilde{b}} u_1(a) = c_1\cdot K_s(\tilde{b}) = c_1\bar{\alpha}\,.$$

From these, $c_1\cdot\bar{\alpha}=1$, so that $u_1=1/\bar{\alpha}K_s$.

To show that there exists the unique point described in the theorem, we go on as follows. For each $\tilde{b}\in\pi^{-1}(s)$ and for every open neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$, we define

$$u_{\hat{U}}(a) = \int \chi_{(\hat{U}(\tilde{b})\cap\pi^{-1}(s)\cap\Gamma^W)}\, d\omega_a^W\,.$$

Since $0\leq u_{\hat{U}}\leq u_1$, $u_{\hat{U}}$ has the form

$$u_{\hat{U}} = c(\hat{U})\cdot K_s\,,$$

where $0\leq c(\hat{U})\leq 1$. If $\hat{U}_1\supset\hat{U}_2$, then $c(\hat{U}_1)\geq c(\hat{U}_2)$. Clearly we have

$$\int \chi_{\{\tilde{b}\}}\, d\omega_a^W \leq \inf_{\hat{U}\in H_{\tilde{b}}} u_{\hat{U}}(a)\,,$$

where $H_{\tilde{b}}$ denotes the family of all open neighbourhoods of $\tilde{b}$. Then,

$$\int \chi_{\{\tilde{b}\}}\, d\omega_a^W = 0 \quad \text{implies} \quad \inf_{\hat{U}\in H_{\tilde{b}}} u_{\hat{U}} = 0\,.$$

In fact, by the definition of the integral of an upper semi-continuous function $\chi_{\{\tilde{b}\}}$, for every positive number ε there exists a bounded continuous function φ on Δ^W such that

$$\varphi\geq\chi_{\{\tilde{b}\}}\,,$$

$$\int\varphi\, d\omega_a^W < \varepsilon\,.$$

Since $\varphi(\tilde{b})\geq 1$, there exists $\hat{U}\in H_{\tilde{b}}$ such that $\varphi>\frac{1}{2}$ on $\hat{U}\cap\Delta^W$. Therefore we have

$$\varepsilon > \int\varphi\, d\omega_a^W \geq \int \tfrac{1}{2}\chi_{(\hat{U}\cap\pi^{-1}(s)\cap\Gamma^W)}\, d\omega_a^W = \tfrac{1}{2}\cdot u_{\hat{U}}(a)\,.$$

Since ε is arbitrary, we have $\inf_{\hat{U}\in H_{\tilde{b}}} u_{\hat{U}}=0$.

If $u_{\hat{U}}>0$, $u_{\hat{U}}$ is the harmonic measure in the sense of M. Heins, that is,

$$u_{\hat{U}}\wedge(1-u_{\hat{U}})=0\,.$$

Now, there exists a function f defined on Δ^M such that $0 \leq f \leq 1$ and

$$u_{\hat{U}}(a) = \int_{\Delta^M} f(t)\, d\omega_a^M(t)\,.$$

Hence

$$0 = u_{\hat{U}}(a) \wedge (1 - u_{\hat{U}}(a)) = \int_{\Delta^M} \min\,[f(t),\ 1-f(t)]\, d\omega_a^M(t)\,.$$

Therefore we have min $[f(t), 1-f(t)] = 0$ for every point of Δ^M except for those of a set of $d\omega^M$-harmonic measure zero.

$\omega^M(\{s\}) > 0$ implies min $[f(s), 1-f(s)] = 0$, so that $f(s)$ is either equal to zero or to 1. On the other hand, the inequality

$$u_{\hat{U}}(a) = \int_{\Delta^M} f(t)\, d\omega_a^M(t) \leq u_1(a) = \int_{\Delta^M} \chi_{\{s\}}(t)\, d\omega_a^M(t)$$

implies

$$f(t) \leq \chi_{\{s\}}(t)\,,$$

where the inequality holds $d\omega^M$-almost everywhere. Since $u_{\hat{U}} > 0$, it follows that $f(s) = 1$ and $u_{\hat{U}} \equiv u_1$. Therefore we have the alternative:

$u_{\hat{U}}$ is either equal to u_1 or to zero.

If $\int \chi_{\{\tilde{b}\}}\, d\omega^W = 0$, then there exists $\hat{U} \in H_{\tilde{b}}$ such that $u_{\hat{U}} = 0$. Therefore, if we have $\int \chi_{\{\tilde{b}\}}\, d\omega^W = 0$ at each $\tilde{b} \in \pi^{-1}(s)$, then there exists a covering $\{\hat{U}(\tilde{b})\}$ of $\pi^{-1}(s)$ such that $u_{\hat{U}(\tilde{b})} = 0$ holds for each $\hat{U}(\tilde{b})$. From this we get a finite covering of the compact set $\pi^{-1}(s)$, hence

$$\int \chi_{\pi^{-1}(s)}\, d\omega^W = 0\,,$$

which is a contradiction. Hence there exists at least one point $\tilde{b} \in \pi^{-1}(s)$ such that

$$\int \chi_{\{\tilde{b}\}}\, d\omega^W = (1/\alpha) \cdot K_s\,.$$

The uniqueness of such $\tilde{b}$ follows easily from the argument above, q.e.d.

For every $s \in \Delta_1^M$ such that the corresponding minimal harmonic function is bounded, the unique $\tilde{b}$ described in Theorem 5.3 is denoted by $\tilde{b}_s$.

Corollary. $\omega^W(\pi^{-1}(s) - \{\tilde{b}_s\}) = 0$, $\tilde{b}_s \in \Omega(s) \cap \Gamma^W$.

Proof. As in the proof of Theorem 5.3, we have

$$\int \chi_{\{s\}}\, d\omega^M = \int \chi_{\pi^{-1}(s)}\, d\omega^W = \int \chi_{\{\tilde{b}_s\}}\, d\omega^W .$$

This implies $\omega^W(\pi^{-1}(s) - \{\tilde{b}_s\}) = 0$. Also, we have $K_s = \bar{\alpha}. \int \chi_{\{\tilde{b}_s\}}\, d\omega^W$. By Theorem 1.3, we see

$$\lim_{a \to \tilde{b}} K_s(a) = \text{fine} \lim_{a \to s} K_s(a) = \bar{\alpha} .$$

From Theorem 5.1 this means $\tilde{b}_s \in \Omega(s)$, Since $\omega^W(\{\tilde{b}_s\}) > 0$ we have $\tilde{b}_s \in \Gamma^W$.

Theorem 5.4. $\tilde{b}_s \in \Phi(s)$.

Proof. Assume $\tilde{b}_s \notin \Phi(s)$. Since $\Phi(s)$ is compact, there exists a non-negative bounded continuous function ψ defined on Δ^W such that

$$\psi(\tilde{b}_s) = 0 \quad \text{and} \quad \psi = 1 \quad \text{on} \quad \Phi(s) .$$

Set

$$H_\psi^W = \int \psi\, d\omega^W .$$

If we assign to each $\tilde{b} \in \Phi(s)$ an arbitrary open neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ and set

$$E = \bigcup_{\tilde{b} \in \Phi(s)} (\hat{U}(\tilde{b}) \cap R) ,$$

then, by Theorem 2.3, we see that $R - E$ is thin at s.

Let v be non-negative, superharmonic and $\liminf_{a \to \tilde{b}} v(a) \geq \psi(\tilde{b})$ at each $\tilde{b} \in \Delta^W$, and let ε be an arbitrary positive number. For every $\tilde{b} \in \Phi(s)$ there exists an open neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that

$$v \geq 1 - \varepsilon \quad \text{on} \quad \hat{U}(\tilde{b}) \cap R .$$

If we make up above E from these neighbourhoods $\hat{U}(\tilde{b})$, then $v \geq 1 - \varepsilon$ holds on E. Since $v \geq 0$, it follows $v \geq (1-\varepsilon) \bar{H}^M_{\chi_{\{s\}}}$. Therefore

$$H_\psi^W = \bar{H}_\psi^W \geq (1-\varepsilon) \bar{H}^M_{\chi_{\{s\}}} .$$

Since ε is arbitrary, we have

$$H_\psi^W \geq \bar{H}^M_{\chi_{\{s\}}} = H^M_{\chi_{\{s\}}} = (1/\bar{\alpha}) K_s .$$

From the corollary to Theorem 5.3, we can see

$$\lim_{a \to \tilde{b}_s} H_\psi^W(a) \geq 1/\bar{\alpha} \cdot \lim_{a \to \tilde{b}_s} K_s(a) = 1 .$$

On the other hand, from the continuity of ψ and from the fact that $\tilde{b}_s \in \Gamma^W$ is regular for the Dirichlet problem, we have

$$\lim_{a\to\tilde{b}_s} H_\psi^W(a)=\psi(\tilde{b}_s)=0\,,$$

which is a contradiction.

Corollary 1. *For a bounded minimal function K_s we have*

$$\Phi(s)\cap\Gamma^W=\{\tilde{b}_s\}\,.$$

Proof. From the corollary to Theorem 5.3 and from Theorem 5.4, we have

$$\tilde{b}_s\in\Phi(s)\cap\Gamma^W\,.$$

If there exist distinct points $\tilde{b}_1$, $\tilde{b}_2\in\Phi(s)\cap\Gamma^W$, then we can find a bounded continuous function φ on Δ^W such that $\varphi(\tilde{b}_1)\neq\varphi(\tilde{b}_2)$. Set $u=H_\varphi^W$. Since u is quasi-bounded, there exists a function f on Δ^M such that $u=H_f^M$. Since $\omega^M(\{s\})>0$, we have

$$\text{fine}\lim_{a\to s} u(a)=f(s)\,.$$

From the same reason as in the proof of Theorem 5.4, we see

$$\lim_{a\to\tilde{b}_1} u(a)=\varphi(\tilde{b}_1)\quad\text{and}\quad\lim_{a\to\tilde{b}_2} u(a)=\varphi(\tilde{b}_2)\,,$$

but by Theorem 2.4,

$$\varphi(\tilde{b}_1)=f(s)=\varphi(\tilde{b}_2)\,,$$

which is a contradiction.

Corollary 2. *If s and s' are points of Δ_1^M such that K_s and $K_{s'}$ are both bounded minimal harmonic functions, then we have*

$$K_s(\tilde{b}_{s'})=\begin{cases}\sup_{a\in R} K_s(a), & s=s'\\ 0, & s\neq s'\,.\end{cases}$$

Theorem 5.5. *$\Phi(\Delta_1^M)$ is a measurable set.*

Proof. If we define

$$A=\{s\in\Delta_1^M;\ K_s \text{ is bounded minimal}\}\,,$$

then A is a countable set. Hence

$$\tilde{A}=\{\tilde{b}_s;\ s\in A\}$$

is also countable. By the corollary to Theorem 5.3, $\pi^{-1}(A)-\tilde{A}$ is of $d\omega^W$-

harmonic measure zero. Also, by Theorem 5.2, $\Phi(\Delta_1^M - A)$ is of $d\omega^W$-harmonic measure zero. Since $\Phi(\Delta_1^M)$ is the union of a set of $d\omega^W$-harmonic measure zero and a countable set, $\Phi(\Delta_1^M)$ is measurable, q.e.d.

We have known that $\Phi(\Delta_1^M)$ is not polar (Theorem 3.2). In the case $R \notin U$, that is, there exists no bounded minimal positive harmonic function on R, $\Phi(\Delta_1^M)$ is of $d\omega^W$-harmonic measure zero (Theorem 5.2). Further we have

Theorem 5.6. *Let e be an arbitrary polar set in* Δ^M. *If we assign to each* $s \in \Delta_1^M - e$ *an arbitrary point* $\tilde{b}(s) \in \Phi(s)$, *then the set*

$$\hat{E} = \bigcup_{s \in \Delta_1^M - e} \{\tilde{b}(s)\}$$

is not polar.

Proof. Assume that there exists a positive superharmonic function v on R such that $\lim_{a \to \tilde{b}} v(a) = +\infty$ at each point $\tilde{b} \in \hat{E}$, then as in the proof of Theorem 3.1, we can apply the minimum principle and get the conclusion that v is reduced to identically $+\infty$, q.e.d.

Here we mention a theorem obtained by S. Mori[15].

Theorem 5.7. $\tilde{b}_s$ *is an isolated point of* Γ^W.

Proof. Suppose that $\tilde{b}_s$ is not an isolated point of Γ^W. Each neighbourhood of $\tilde{b}_s$ has points of Γ^W except $\tilde{b}_s$. Since Γ^W is the carrier of $d\omega^W$-harmonic measure on Δ^W, an arbitrary neighbourhood of an arbitrary point of Γ^W has a positive $d\omega^W$-harmonic measure. From this and from the corollary to Theorem 5.3, we can conclude that there exists a set $\hat{E}$ in Δ^W of $d\omega^W$-harmonic measure positive such that $s \notin \pi(\hat{E})$. Accordingly $\pi(\hat{E})$ has a positive $d\omega^M$-harmonic measure. We can find above $\hat{E}$ such that $\lim_{a \to \tilde{b}} K_s(a) > 0$ holds at every $\tilde{b} \in \hat{E}$. On the other hand, we have

$$\text{fine} \lim_{a \to t} K_s(a) = 0$$

at each point $t \neq s$ except at those of some subset of Δ^M of $d\omega^M$-harmonic measure zero. From this and from Theorem 1.3, we see, at some point $\tilde{b} \in \hat{E}$,

$$\lim_{a \to \tilde{b}} K_s(a) = 0 ,$$

which is a contradiction.

5.3. In this paragraph we shall investigate the relations between Γ^W and

15) [6], p. 39, th. 5.1., also [3], p. 125, Satz 11.5.

$\Phi(A)$ again. We have already seen in the corollary to Theorem 3.3 that the following relation holds:

$$\Gamma^W \subset \overline{\Phi(\Delta_1^M)}\,.$$

Can we conclude in general $\Gamma^W \cap \pi^{-1}(A) \subset \overline{\Phi(A)}$? This not the case. For instance, let $A=\{s\}$, where $s \in \Delta_1^M$ and K_s is not bounded, then $\overline{\Phi(A)}=\Phi(A)$ and $\Phi(A) \cap (\Gamma^W \cap \pi^{-1}(A))=\phi$. But as is shown in the following theorem, in some situation we have that relation.

Theorem 5.8. *If G is open in Δ^M, then we have*

$$\Gamma^W \cap \pi^{-1}(G) \subset \overline{\Phi(G)}\,.$$

Proof. Suppose that there exists $\tilde{b} \in \Gamma^W \cap \pi^{-1}(G)$ such that $\tilde{b} \notin \overline{\Phi(G)}$. Then there exists an open neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ such that

$$\hat{U}(\tilde{b}) \cap \overline{\Phi(G)}=\phi\,,$$
$$\omega(\hat{U}(\tilde{b}) \cap \Delta^W)>0\,,$$
$$\pi(\hat{U}(\tilde{b}) \cap \Delta^W) \subset G\,.$$

We take a non-negative bounded continuous function ψ defined on Δ^W as follows:

$$\psi \begin{cases} =0 & \text{on} \quad \pi^{-1}(\Delta^M - G) \cup \overline{\Phi(G)} \\ >0 & \text{on} \quad \hat{U}(\tilde{b}) \cap \Delta^W \end{cases}$$

and set

$$H_\psi^W = \int_{\Delta^W} \psi \, d\omega^W\,.$$

We see $H_\psi^W>0$, while in view of Theorem 3.3, we have $H_\psi^W=0$, which is a contradiction

Corollary. *If F is a G_δ set in Δ^M, that is,*

$$F=\bigcap_{n=1}^{\infty} G_n\,,$$

where G_n is an open set in Δ^M, then

$$\Gamma^W \cap \pi^{-1}(F) \subset \bigcap_{n=1}^{\infty} \overline{\Phi(G_n)}\,.$$

In particular, if $\Gamma^W \cap \pi^{-1}(F) \neq \phi$, then we have

$$\overline{\Phi(G_n)} \cap \Gamma^W \neq \phi\,.$$

6. Relative Dirichlet problem

M. Brelot [2] and L. Naïm [8] developed the theory of the relative Dirichlet problem for the case of the *metrizable* compactification of R. In this section we shall consider the relative Dirichlet problem especially for R^{*W}. First we state here the formulation of the relative Dirichlet problem.

Let h be a positive harmonic function on R. h is fixed throughout this section. For an extended real valued function φ on Δ^W, $\overline{\mathfrak{F}}^W_{\varphi,h}$ denotes the family of all functions v satisfying the following conditions:

a) v is superharmonic on R or $\equiv +\infty$,

b) v/h is bounded from below,

c) $\liminf_{a\to\tilde{b}} v(a)/h(a) \geq \varphi(\tilde{b})$ at every point $\tilde{b}\in\Delta^W$.

The lower envelope of this family

$$\overline{\mathfrak{D}}^W_{\varphi,h}(a)=\inf\{v(a);\ v\in\overline{\mathfrak{F}}^W_{\varphi,h}\}$$

is harmonic on R or $\equiv +\infty$ or $\equiv -\infty$. Similarly,we define $\underline{\mathfrak{D}}^W_{\varphi,h}$. In general, we have

$$\underline{\mathfrak{D}}^W_{\varphi,h} \leq \overline{\mathfrak{D}}^W_{\varphi,h}\ .$$

If $\underline{\mathfrak{D}}^W_{\varphi,h} \equiv \overline{\mathfrak{D}}^W_{\varphi,h}$ and this is harmonic, then φ is called *h-resolutive*.

M. Brelot [2] set up the axiom $(\mathfrak{A}_h)$: *all bounded continuous functions on the boundary are h-resolutive,* and obtained many results. But it seems that this axiom is not always effective for other compactifications. We start with the following axiom:

$(\mathfrak{A}^W_h)$ *all bounded continuous functions on* Δ^W *are h-resolutive.*

It is well-known that the axiom $(\mathfrak{A}^W_1)$ is always satisfied, but the following theorem shows that the axiom $(\mathfrak{A}^W_h)$ is not valid in general.

Theorem 6.1. *If the axiom* $(\mathfrak{A}^W_{K_s})$ *is satisfied for* $s\in\Delta^M_1$, *then the pole of* s *on* Δ^W *is unique.*

Proof. Let $\tilde{b}_1$ and $\tilde{b}_2$ be two distinct poles of s on Δ^W. We take a bounded continuous function φ on Δ^W such that

$$\varphi(\tilde{b}_1)=1 \quad\text{and}\quad \varphi(\tilde{b}_2)=0\ .$$

Let $u\in\underline{\mathfrak{F}}^W_{\varphi,h}$. By definition, u is a function on R with the following properties: u is subharmonic on R, u/K_s is bounded from above and $\limsup_{\tilde{b}a\to} u(a)/K_s(a) \leq \varphi(\tilde{b})$ at each $\tilde{b}\in\Delta^W$. Let ε be an arbitrary positive number. There exists a neighbourhood $\hat{U}(\tilde{b}_2)$ of $\tilde{b}_2$ such that

$$u(a)/K_s(a)<\varepsilon \quad \text{for} \quad a\in \hat{U}(\tilde{b}_2)\cap R .$$

Since $\hat{U}(\tilde{b}_2)\cap R$ is not thin at s, we have

$$u<\varepsilon\cdot K_s \quad \text{on} \quad R^{16)} .$$

Therefore we obtain

$$\underline{\mathfrak{D}}^W_{\varphi,K_s}\leq\varepsilon\cdot K_s .$$

and since ε is arbitrary, we have

$$(6.1) \qquad \underline{\mathfrak{D}}^W_{\varphi,K_s}\leq 0 .$$

Next, let $v\in\overline{\mathcal{G}}^W_{\varphi,K_s}$. For $\varepsilon>0$ there exists a neighbourhood $\hat{U}(\tilde{b}_1)$ of $\tilde{b}_1$ such that

$$v(a)\geq(1-\varepsilon)K_s(a) \quad \text{for} \quad a\in\hat{U}(\tilde{b}_1)\cap R .$$

Then, we can see

$$v(a)\geq(1-\varepsilon)(K_s)_{\hat{U}(\tilde{b}_1)\cap R}(a)^{17)} .$$

Since $\hat{U}(\tilde{b}_1)\cap R$ is also not thin at s, we have

$$(K_s)_{\hat{U}(\tilde{b}_1)\cap R}\equiv K_s{}^{18)}$$

whence $v\geq(1-\varepsilon)K_s$ on R.

Therefore we get

$$\overline{\mathfrak{D}}^W_{\varphi,K_s}\geq(1-\varepsilon)\cdot K_s$$

and finally we have

$$(6.2) \qquad \overline{\mathfrak{D}}^W_{\varphi,K_s}\geq K_s .$$

In view of the inequalities (6.1), (6.2), we see that all bounded continuous functions on Δ^W are not always K_s-resolutive.

From the proof of the above theorem we obtain:

Corollary. *Let φ be a bounded continuous function on Δ^W, then we have*

$$\overline{\mathfrak{D}}^W_{\varphi,K_s}\geq(\max_{\tilde{b}\in\Phi(s)}\varphi(\tilde{b}))\cdot K_s ,$$

$$\underline{\mathfrak{D}}^W_{\varphi,K_s}\leq(\min_{\tilde{b}\in\Phi(s)}\varphi(\tilde{b}))\cdot K_s .$$

16) [8], p. 244, th. 22.

17) For $A\subset R$, $(K_s)_A$ denotes the extremization of K_s on A, that is, the lower envelope of all non-negative superharmonic functions on R dominating K_s on A. Cf. §2, 2.1.

18) [8], p. 205, th. 5.

This corollary means that the upper and the lower solutions of the relative Dirichlet problem for K_s are dominated from below and from above by the values of φ on $\Phi(s)$ respectively. Whether the equalities hold or not in the above corollary is unknown, at least to the author. A less precise result is the following:

Theorem 6.2. *Let φ be a bounded continuous function on Δ^W. If we set for every $t\in\Delta^M$*

$$\bar{\varphi}(t)=\max_{\tilde{b}\in\pi^{-1}(t)}\varphi(\tilde{b})$$

and

$$\underline{\varphi}(t)=\min_{\tilde{b}\in\pi^{-1}(t))}\varphi(\tilde{b}),$$

then we have

$$\bar{\mathfrak{D}}^M_{\varphi,K_s}=(\max_{\tilde{b}\in\pi^{-1}(s)}\varphi(\tilde{b}))\cdot K_s\geq\bar{\mathfrak{D}}^W_{\varphi,K_s}$$

and

$$\underline{\mathfrak{D}}^M_{\varphi,K_s}=(\min_{\tilde{b}\in\pi^{-1}(s)}\varphi(\tilde{b}))\cdot K_s\leq\underline{\mathfrak{D}}^W_{\varphi,K_s}.$$

Proof. Suppose that v_1 is a superharmonic function on R such that v_1/K_s is bounded from below and $\liminf_{a\to t} v_1(a)/K_s(a)\geq\bar{\varphi}(t)$ at each $t\in\Delta^M$. Then, there exists a neighbourhood $U(t)$ of t in R^{*M} such that

$$v/K_s\geq\bar{\varphi}(t)-\varepsilon \quad\text{on}\quad U(t)\cap R.$$

For every $\tilde{b}\in\pi^{-1}(t)$, there exists a neighbourhood $\hat{U}(\tilde{b})$ of $\tilde{b}$ in R^{*W} such that $\pi[\hat{U}(\tilde{b})]\subset U(t)$. Hence we have the relation:

$$a\in\hat{U}(\tilde{b})\cap R \text{ implies } v_1(a)/K_s(a)\geq\bar{\varphi}(t)-\varepsilon\geq\varphi(\tilde{b})-\varepsilon$$

This means

$$\liminf_{a\to\tilde{b}} v_1(a)/K_s(a)\geq\varphi(\tilde{b})-\varepsilon.$$

From this we see immediately

$$v_1\geq\bar{\mathfrak{D}}^W_{\varphi,K_s}-\varepsilon\cdot K_s.$$

Varying v_1, we have

$$\bar{\mathfrak{D}}^M_{\varphi,K_s}\geq\bar{\mathfrak{D}}^W_{\varphi,K_s}-\varepsilon\cdot K_s.$$

Since ε is arbitrary, we get the first inequality in the theorem. The second inequality is proved quite similarly, q.e.d.

From the above theorem, we know that for a bounded continuous function

φ the K_s-solutions depend only on the values of φ on $\pi^{-1}(s)$.

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