

Title	Polynomial representation of strongly-invertible knots and strongly-negative-amphicheiral knots
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Citation	Osaka Journal of Mathematics. 43(3) P.625-P.639
Issue Date	2006-09
Text Version	publisher
URL	https://doi.org/10.18910/8342
DOI	10.18910/8342
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POLYNOMIAL REPRESENTATION OF STRONGLY-INVERTIBLE KNOTS AND STRONGLY-NEGATIVE-AMPHICHEIRAL KNOTS

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(Received August 24, 2005, revised October 26, 2005)

Abstract

It is shown that the symmetric behaviour of certain class of knots can be realized by their polynomial representations. We prove that every *strongly invertible* knot (open) can be represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ of \mathbb{R} in \mathbb{R}^3 where among the polynomials $f(t)$, $g(t)$ and $h(t)$ two of them are odd polynomials and one is an even polynomial. We also prove that a subclass of *strongly negative amphicheiral* knots can be represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ of \mathbb{R} in \mathbb{R}^3 where all three polynomials $f(t)$, $g(t)$ and $h(t)$ are odd polynomials.

1. Introduction

It has been proved [10] that every smooth knot in S^3 is isotopy equivalent to the closure of the image of an embedding $\phi: \mathbb{R} \mapsto \mathbb{R}^3$ defined by $\phi(t) = (f(t), g(t), h(t))$ where $f(t)$, $g(t)$ and $h(t)$ are polynomials over the field of real numbers \mathbb{R} . In fact any two such polynomial embeddings representing the same knot-type can be joined by a polynomial isotopy [12]. Thus the set of polynomially isotopic classes of polynomial knots is in one to one correspondence with the set of ambient isotopy classes of all classical knots. In [10], Shastri constructed polynomial embeddings for the trefoil knot and for the figure eight knot respectively. Later, we could find a general procedure ([8], [5]) to construct a polynomial embedding representing any torus knot of type (p, q) . Let us look at Shastri's polynomial embeddings for these two simple knots: For the trefoil knot it was given by $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ and for the figure eight knot it was $t \mapsto (t^3 - 3t, t^5 - 5t^3 + 4t, t^7 - 42t)$. A 3D plot of these knots using *Mathematica* is shown in Fig. 1 and Fig. 2 respectively at the end of the paper.

One can easily see that the involution map $(x, y, z) \mapsto (-x, y, -z)$ from \mathbb{R}^3 to \mathbb{R}^3 , which is orientation preserving, preserves this representation of the trefoil knot set wise but reverses the orientation; similarly for the figure eight representation, the orientation reversing involution $(x, y, z) \mapsto (-x, -y, -z)$ brings the same knot with

2000 Mathematics Subject Classification. 57M25.

This work was supported by The 21st Century COE Program, "Constitution of wide-angle mathematical basis focused on knots;" leader Akio Kawauchi, Osaka City University.

opposite orientation. Thus, in a way, this provides us a proof that the trefoil knot is *strongly invertible* ([9], [3]) and the figure eight knot is *strongly-negative-amphicheiral* ([4], [1]). With this in mind Kawauchi made the following two conjectures:

(1) *Every strongly invertible knot can be represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ where among $f(t)$, $g(t)$ and $h(t)$, two of them must be odd polynomials and one must be an even polynomial.*

(2) *Every strongly negative amphicheiral knot can be represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ where all three polynomials $f(t)$, $g(t)$ and $h(t)$ must be odd polynomials.*

In this paper we prove the first conjecture and give a partial proof of the second conjecture. We also construct some examples of such embeddings for few such knots.

2. Definitions and Remarks

DEFINITION 2.1. A knot K in S^3 is said to be *strongly invertible* if there exists an orientation preserving involution $h: S^3 \rightarrow S^3$ such that $h(K) = K$ setwise but with the reverse orientation.

REMARK 2.2. By the well known Smith theory [7] the fixed point set $Fix(h)$ will be an unknotted S^1 intersecting the knot K at exactly two points. Then it has been proved that h will be equivalent (up to conjugate) to a π -rotation ([13], [6], [11]).

REMARK 2.3. Regarding S^1 and S^3 as one point compactifications of \mathbb{R} and \mathbb{R}^3 respectively, we can define a long knot (\mathbb{R} in \mathbb{R}^3) to be strongly invertible if there exists an orientation preserving involution $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which preserves the knot setwise but reverses its orientation. By the previous remark $Fix(h)$ will be an unknotted line intersecting the knot at exactly one point and then such an involution will be equivalent to the π -rotation along this line as axis. By linear change of coordinates we can assume that this axis is the Y axis and the involution $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is given by the standard map $(x, y, z) \mapsto (-x, y, -z)$.

DEFINITION 2.4. A knot K in S^3 is said to be *strongly-negative-amphicheiral* if there exists an orientation reversing involution $h: S^3 \rightarrow S^3$ such that $h(K) = K$ setwise but with the reverse orientation. Regarding $S^1 = \mathbb{R} \cup \{\infty\}$ and $S^3 = \mathbb{R}^3 \cup \{\infty\}$ we can modify the definition of *strongly negative amphicheiral long knots*.

REMARK 2.5. It is well known by the Smith theory that in this case the set $Fix(h)$ will be homeomorphic to either S^2 or to S^0 and in either case $Fix(h)$ intersects the knot at exactly two points.

If the set $Fix(h)$ is homeomorphic to S^2 then we can switch it to another involution say h_1 of S^3 satisfying the same condition of preserving K setwise but reversing

the orientation and $\text{Fix}(h_1)$ is homeomorphic to S^0 . In this situation we have a famous *open conjecture* due to *Folklore*:

Conjecture 2.6. *Every involution h on $S^3 = \mathbb{R}^3 \cup \{\infty\}$ with $\text{Fix}(h) = S^0$ is conjugate to the standard involution $(x, y, z) \mapsto (-x, -y, -z)$ on \mathbb{R}^3 and $\infty \mapsto \infty$. Proving this conjecture is equivalent to proving the following:*

Conjecture 2.7. *Every irreducible homotopy $P^2 \times I$ is homeomorphic to $P^2 \times I$ [2].*

Keeping this conjecture in mind we modify our definition as:

DEFINITION 2.8. A long knot K in \mathbb{R}^3 is said to be *faithfully strongly negative amphicheiral* if there exists an involution $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ conjugate to the standard map $(x, y, z) \mapsto (-x, -y, -z)$ from \mathbb{R}^3 to \mathbb{R}^3 such that $h(K) = K$ setwise but with the reverse orientation.

REMARK 2.9. Thus every faithfully strongly negative amphicheiral knot is strongly amphicheiral and if the above conjecture is true then both the notions, the faithfully strongly negative amphicheiral and the strongly negative amphicheiral, are equivalent.

REMARK 2.10. By Remark 2.3 it follows that every strongly invertible long knot can be represented by a smooth embedding $\phi: \mathbb{R} \rightarrow \mathbb{R}^3$ defined as $\phi(t) = (\alpha(t), \beta(t), \gamma(t))$ where α and γ are smooth odd functions and β is a smooth even function.

REMARK 2.11. Also by definition it is clear that every faithfully strongly negative amphicheiral knot can be represented by a smooth embedding $t \mapsto (\alpha(t), \beta(t), \gamma(t))$ where α, β and γ are smooth odd functions.

3. Main Results

Theorem 3.1. *Every strongly invertible (open) knot can be represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ from \mathbb{R} to \mathbb{R}^3 where $f(t)$ and $h(t)$ are odd polynomials and $g(t)$ is an even polynomial.*

Proof. Let K be a strongly invertible knot. By Remark 2.10, K can be represented by a smooth embedding $\phi: \mathbb{R} \rightarrow \mathbb{R}^3$ defined as $\phi(t) = (\alpha(t), \beta(t), \gamma(t))$ where α and γ are smooth odd functions and β is a smooth even function. Let us assume that the map $\tilde{\phi}: \mathbb{R} \rightarrow \mathbb{R}^2$ defined as $\tilde{\phi}(t) = (\alpha(t), \beta(t))$ is a generic immersion, i.e., defines a regular projection of K . Since, there are only finitely many double points in the image of $\tilde{\phi}$ and the embedding K has an asymptotic behaviour at infinity, we can choose an interval $[-A, A]$ such that $\tilde{\phi}([-A, A])$ contains all the double points

and the derivatives α' , γ' are positive everywhere outside the interval $[-A, A]$ and the derivative β' is negative in $(-\infty, -A]$ and positive in $[A, \infty)$. Let $[-M, M]$ be an interval containing the interval $[-A, A]$ be such that $\phi([-M, M])$ is contained in a ball of radius R with $\|\phi(-M)\| = \|\phi(M)\| = R$ and $\alpha(M)$, $\beta(M)$ and $\gamma(A)$ are positive. Let $[-N, N]$ be an interval such that $\phi([-N, N])$ is contained in a ball of radius $2R$ with $\|\phi(-N)\| = \|\phi(N)\| = 2R$. By scaling we can assume that the intervals $[-M, M]$ and $[-N, N]$ are sufficiently small, $M < N = 1$. So we have $\|\phi\|$ is increasing outside $[-M, M]$ with respect to $|t|$ and in fact we can assume that $\alpha' > 1$, $\gamma' > 1$ and $\beta' < -1$ in $(-\infty, -M]$ and > 1 in $[M, \infty)$. Consider the restriction of ϕ to the interval $[-N, N]$ i.e.,

$$\phi|_{[-N, N]}: [-N, N] \rightarrow \mathbb{R}^3.$$

Since the set of embeddings from a compact, Hausdorff manifold to any manifold forms an open set in the set of all smooth maps with the C^1 -topology, there exists $\epsilon_0 > 0$ such that $\psi \in N(\phi, \epsilon_0) \implies \psi: [-N, N] \rightarrow \mathbb{R}^3$ is an embedding, where

$$(I) \quad N(\phi, \epsilon_0) = \left\{ \psi: \sup_{t \in [-N, N]} \{ \|\psi(t) - \phi(t)\|, \|\psi'(t) - \phi'(t)\| \} < \epsilon_0 \right\}.$$

Let $\epsilon < \min\{\alpha(M), R/2, \epsilon_0\}$. For this ϵ , let $\psi_1 = (f_1, g_1, h_1)$ be an $(\epsilon/2)$ -Taylor approximation of $\phi = (\alpha, \beta, \gamma)$ inside $[-N, N]$. Clearly f_1 and h_1 are odd polynomials and g_1 is an even polynomial. Also as we are in C^1 topology $f'_1 > 1 - \epsilon/2$, $h'_1 > 1 - \epsilon/2$ inside $[-N, -M] \cup [M, N]$ and $g'_1 < -1 + \epsilon/2$ in $[-N, -M]$ and $> 1 - \epsilon/2$ in $[M, N]$. Now, for any $\delta \in (0, \epsilon/2)$, we can choose a positive integer n large enough so that $\psi(t) = (f_1(t) + (\delta/(2n+1))t^{2n+1}, g_1(t) + (\delta/(2n))t^{2n}, h_1(t) + (\delta/(2n+1))t^{2n+1}) = (f(t), g(t), h(t))$ (say) is an ϵ - C^1 approximation of ϕ inside $[-N, N]$ and $f' > 0$, $h' > 0$ outside $[-N, N]$ and $g' < 0$ in $(-\infty, -N]$ and > 0 in $[N, \infty)$ and f and h are odd polynomials and g is an even polynomial. In fact $\|\psi\|$ is increasing outside $[-M, M]$ with respect to $|t|$. Now we have

Claim 1. $\psi: \mathbb{R} \rightarrow \mathbb{R}^3$ is an embedding.

Claim 2. The knot type determined by ψ is ambient isotopic to K .

Proof of Claim 1. As the restriction of ψ to $[-N, N]$ belongs to $N(\phi, \epsilon)$ by (I), ψ is an embedding inside $[-N, N]$. Thus $\psi: \mathbb{R} \rightarrow \mathbb{R}^3$ is an immersion. It remains to show that ψ is injective, i.e., $\psi(t_1) \neq \psi(t_2) \forall t_1 \neq t_2 \in \mathbb{R}$. ψ is an embedding inside $[-N, N]$ so is injective also there. Now, if both $t_1, t_2 \in (-\infty, -M)$ or both $\in (M, \infty)$, $\psi(t_1) \neq \psi(t_2)$ as f' , g' and h' are never zero in these intervals. Also, if $t_1 \in (-\infty, M)$ and $t_2 \in (M, \infty)$ then $\alpha(t_1) < -\alpha(M)$ and $\alpha(t_2) > \alpha(M)$. Also we have $f(t_1) < 0$ and $f(t_2) > 0$ and $\epsilon < \alpha(M)$ by using $\psi|_{[-N, N]} \in N(\phi, \epsilon)$ we obtain $\psi(t_1) \neq \psi(t_2)$.

Let $t_1 \in (-\infty, -N)$ and $t_2 \in [-M, M]$, then

$$\begin{aligned} \|\psi(t_1) - \psi(t_2)\| &\geq \|\psi(t_1)\| - \|\psi(t_2)\| \\ &\geq (2R - \epsilon) - (R + \epsilon) \\ &= R - 2\epsilon. \end{aligned}$$

Hence $\psi(t_1) \neq \psi(t_2)$. Similarly we can show that if $t_1 \in (N, \infty)$ and $t_2 \in [-M, M]$ then also $\psi(t_1) \neq \psi(t_2)$. This completes the proof of Claim 1. \square

Proof of Claim 2. We define $F: \mathbb{R} \times I \rightarrow \mathbb{R}^3$ as

$$F(s, t) = (1 - t)\phi(s) + t\psi(s).$$

Clearly $F(s, 0) = \phi(s)$ and $F(s, 1) = \psi(s)$. We must show that for each $t \in (0, 1)$ the map $F_t(s)$ defined as $F(s, t)$ as above is an embedding. Now, inside $[-N, N]$ we have

$$\begin{aligned} \|(1 - t)\phi + t\psi - \phi\| &= \|-t(\phi - \psi)\| \\ &= t\|\phi - \psi\| \\ &< \epsilon. \end{aligned}$$

Similarly we have $\|(1 - t)\phi' + t\psi' - \phi'\| < \epsilon$ inside $[-N, N]$. Thus $(1 - t)\phi + t\psi \in N(\phi, \epsilon)$ and hence is an embedding inside $[-N, N]$. Also for each $t \in (0, 1)$ $F'_t = (1 - t)\phi' + t\psi'$ is never zero outside $[-M, M]$. By a similar argument used in the proof of Claim 1 we can show that each F_t is an embedding. Thus the above map $F: \mathbb{R} \times I \rightarrow \mathbb{R}^3$ defines an isotopy between the embeddings ϕ and ψ and hence the knot K is ambient isotopic to the knot type determined by ψ . This completes the proof of Claim 2. \square

Thus, we have shown that the knot K , which is strongly invertible, can be represented by a polynomial embedding $\psi(t) = (f(t), g(t), h(t))$ where f and h are odd polynomials and g is an even polynomial. This completes the proof of the theorem. \square

Theorem 3.2. *Every faithfully strongly negative amphicheiral (open) knot can be represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ from \mathbb{R} to \mathbb{R}^3 where $f(t)$, $g(t)$ and $h(t)$ are odd polynomials.*

Proof. Let K be a faithfully strongly negative amphicheiral knot. By Remark 2.11, K can be represented by a smooth embedding $\phi: \mathbb{R} \rightarrow \mathbb{R}^3$ defined as $\phi(t) = (\alpha(t), \beta(t), \gamma(t))$ where α , β and γ are smooth odd functions. Let us assume that the map $\tilde{\phi}: \mathbb{R} \rightarrow \mathbb{R}^2$ defined as $\tilde{\phi}(t) = (\alpha(t), \beta(t))$ is a generic immersion, i.e., defines a regular projection of K . Since, there are only finitely many double points in the image of $\tilde{\phi}$ and the embedding K has an asymptotic behaviour at infinity, we can

choose an interval $[-A, A]$ such that $\tilde{\phi}([-A, A])$ contains all the double points and the derivatives α', β' and γ' are positive everywhere outside the interval $[-A, A]$. Let $[-M, M]$ be an interval containing the interval $[-A, A]$ such that $\phi([-M, M])$ is contained in a ball of radius R with $\|\phi(-M)\| = \|\phi(M)\| = R$ and $\alpha(M), \beta(M), \gamma(M) > 0$. Let $[-N, N]$ be an interval such that $\phi([-N, N])$ is contained in a ball of radius $2R$ with $\|\phi(-N)\| = \|\phi(N)\| = 2R$. By scaling we can assume that the intervals $[-M, M]$ and $[-N, N]$ are sufficiently small, $M < N = 1$. So we have $\|\phi\|$ is increasing outside $[-M, M]$ with respect to $|t|$ and in fact we can assume that $\alpha' > 1, \beta' > 1$ and $\gamma' > 1$ outside $[-M, M]$. Consider the restriction of ϕ to the interval $[-N, N]$ i.e.,

$$\phi|_{[-N, N]}: [-N, N] \rightarrow \mathbb{R}^3.$$

Since the set of embeddings from a compact, Hausdorff manifold to any manifold forms an open set in the set of all smooth maps with the C^1 -topology, there exists $\epsilon_0 > 0$ such that $\psi \in N(\phi, \epsilon_0) \implies \psi: [-N, N] \rightarrow \mathbb{R}^3$ is an embedding, where

$$(II) \quad N(\phi, \epsilon_0) = \left\{ \psi: \sup_{t \in [-N, N]} \{ \|\psi(t) - \phi(t)\|, \|\psi'(t) - \phi'(t)\| \} < \epsilon_0 \right\}.$$

Let $\epsilon < \min\{\alpha(M), R/2, \epsilon_0\}$. For this ϵ , let $\psi_1 = (f_1, g_1, h_1)$ be an $(\epsilon/2)$ -Taylor approximation of $\phi = (\alpha, \beta, \gamma)$ inside $[-N, N]$. Clearly f_1, g_1 and h_1 are odd polynomials. Also as we are in C^1 topology $f'_1 > 1 - \epsilon/2, g'_1 > 1 - \epsilon/2, h'_1 > 1 - \epsilon/2$, inside $[-N, -M] \cup [M, N]$. Now, for any $\delta \in (0, \epsilon/2)$, we can choose a positive integer n large enough so that $\psi(t) = (f_1(t) + (\delta/(2n + 1))t^{2n+1}, g_1(t) + (\delta/(2n + 1))t^{2n+1}, h_1(t) + (\delta/(2n + 1))t^{2n+1}) = (f(t), g(t), h(t))$ (say) is an ϵ - C^1 approximation of ϕ inside $[-N, N]$ and $f' > 0, g' > 0$ and $h' > 0$ outside $[-N, N]$ and f, g and h are odd polynomials. Infact $\|\psi\|$ is increasing outside $[-M, M]$ with respect to $|t|$. Now using a similar argument as in Theorem 3.1 we can check the following:

- (1) $\psi: \mathbb{R} \rightarrow \mathbb{R}^3$ is an embedding.
- (2) The knot type determined by ψ is ambient isotopic to K .

This completes the proof of the theorem. □

4. Examples

(1) In Fig. 2 we had seen a polynomial representation of figure eight knot by three odd polynomials demonstrating that this knot is faithfully strongly negative amphicheiral. We have another representation of figure eight knot by two odd and one even polynomials, given by $t \mapsto (t(t^2 - 4)(t^2 - 2), (t^2 - 0.2)(t^2 - 4)(t^2 - 8), t(t^2 - (2.05)^2)(t^2 - (1.5)^2)(t^2 - 0.25))$ which verifies that this knot is strongly invertible also. A 3D plot of this knot, using Mathematica, is shown in Fig. 3, at the end.

(2) The knot 5_1 which is a torus knot of type $(2, 5)$ is represented by $t \mapsto (t(t^2 -$

15), $(t^2 - 4)(t^2 - 10)(t^2 - 36)(t^2 - 15)$, $t(t^2 - 16)(t^2 - 9)(t^2 - (2.5)^2)(t^2 - (4.4)^2)$) demonstrating that it is strongly invertible and its 3D plot is shown in Fig. 4.

(3) The knot 6_3 is represented by an embedding given by three odd polynomials as $t \mapsto (t(t^2 - 45), -t(t^2 - 8)(t^2 - 35)(t^2 - 53), t(t^2 - 1)(t^2 - 9)(t^2 - 36)(t^2 - 49)(t^2 - (7.4)^2))$ demonstrating that it is faithfully strongly negative amphicheiral. Its 3D plot is shown in Fig. 5.

(4) The torus knot of type $(3, 4)$ is represented by $t \mapsto (t(t^2 - 45)(t^2 - 20), t(t^2 - 5)(t^2 - 36)(t^2 - 49), (t^2 - 1)(t^2 - 9)(t^2 - (7.1)^2)(t^2 - (6.5)^2))$ showing that this knot is strongly invertible, shown in Fig. 6.

(5) The knot 8_{17} shown in Fig. 7 is represented by $t \mapsto (t(t^2 - 45)(t^2 - 24), -t(t^2 - 8)(t^2 - 35)(t^2 - 53), t(t^2 - 1)(t^2 - 9)(t^2 - (7.2)^2)(t^2 - 36)(t^2 - 49)(t^2 - (5.4)^2)(t^2 - (0.80)^2))$ verifying that it is faithfully strongly negative amphicheiral.

5. Conclusion

To find a nice method for proving that a given knot is non invertible has been a problem of interest for a long time. The simplest non invertible knot known is 8_{17} . If we can produce an argument, by elementary algebra, that it is impossible to represent this knot by a polynomial embeddings using two odd polynomials and one even polynomial. Then by our theorem it will prove that it is not strongly invertible. Since this knot is hyperbolic, it also proves that it is not invertible. We may then be able to generalize our method for detecting invertibility for *hyperbolic knots* at least.

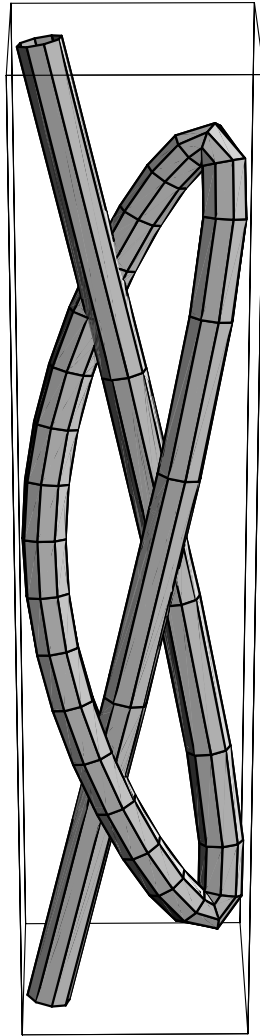


Fig. 1. Shastri's trefoil

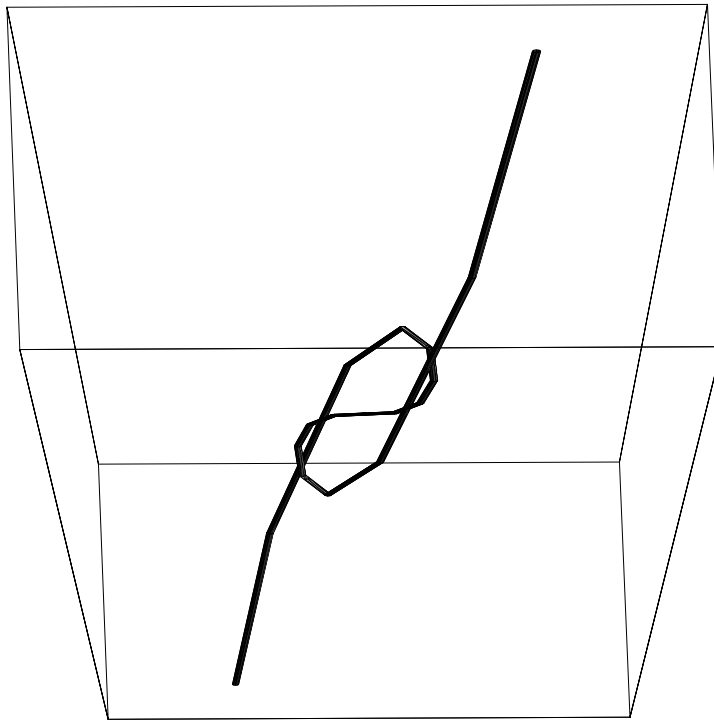


Fig. 2. Shastri's figure eight knot

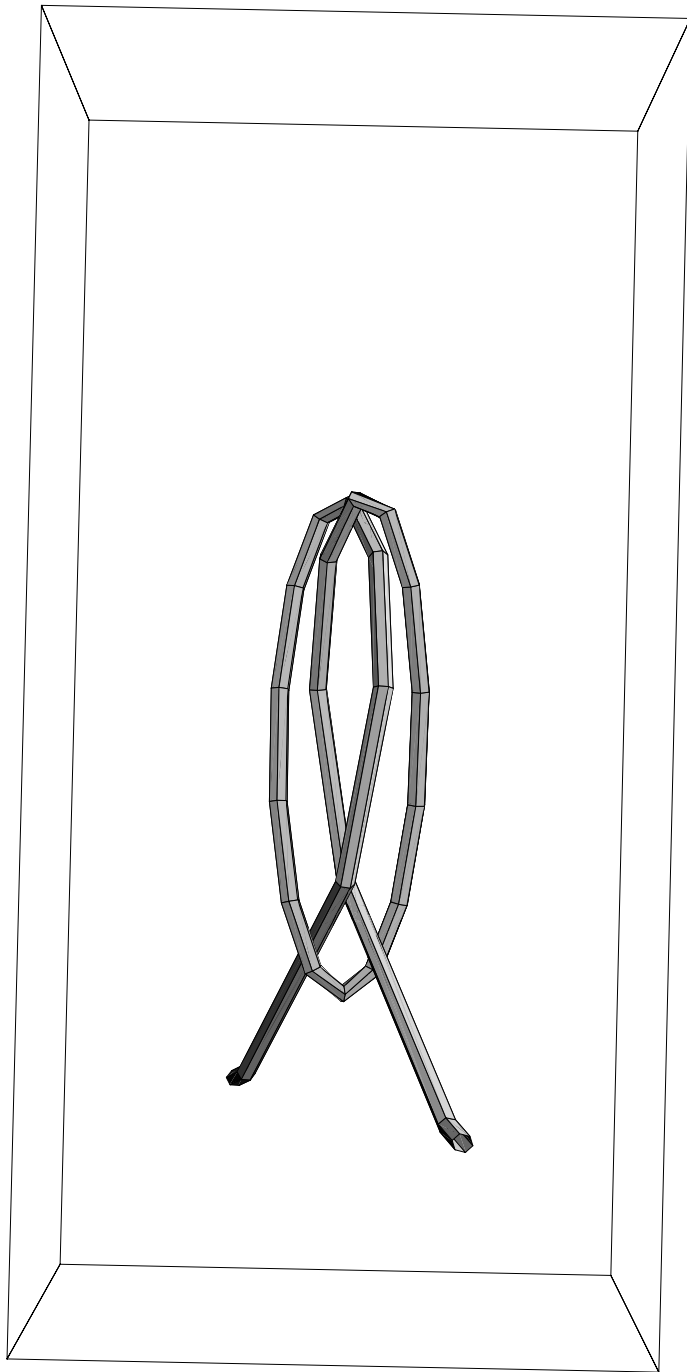


Fig. 3. Strongly invertible representation of figure eight knot

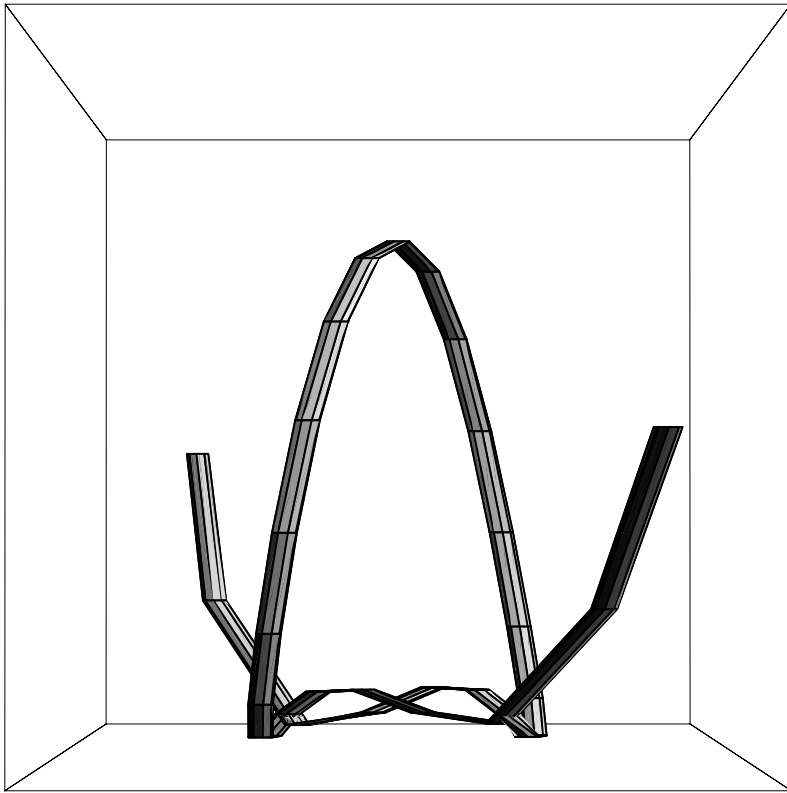


Fig. 4. Strongly invertible representation of 5_1

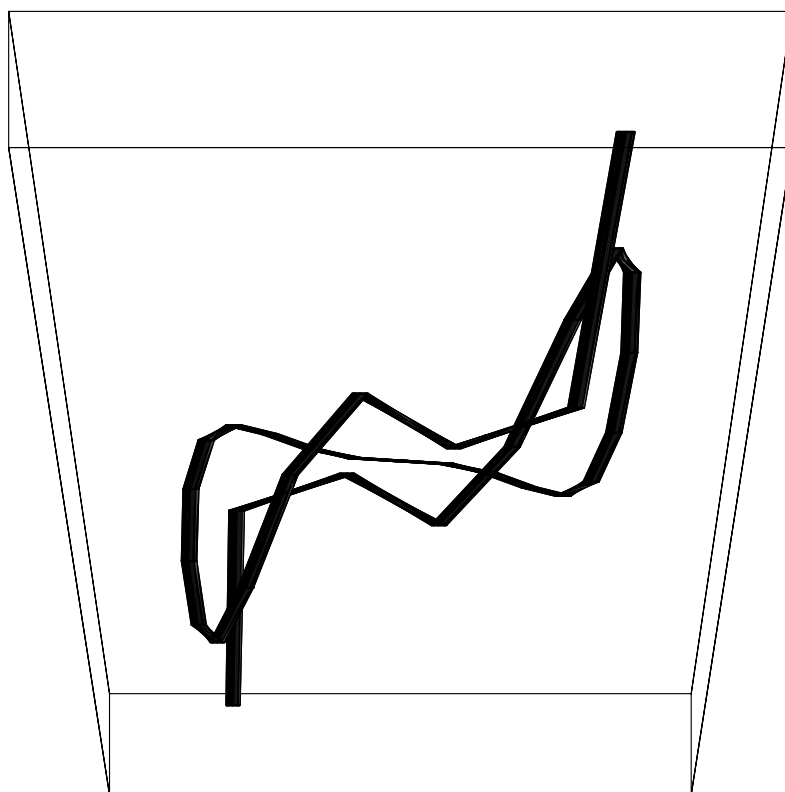


Fig. 5. Faithfully strongly negative amphicheiral representation of 6_3

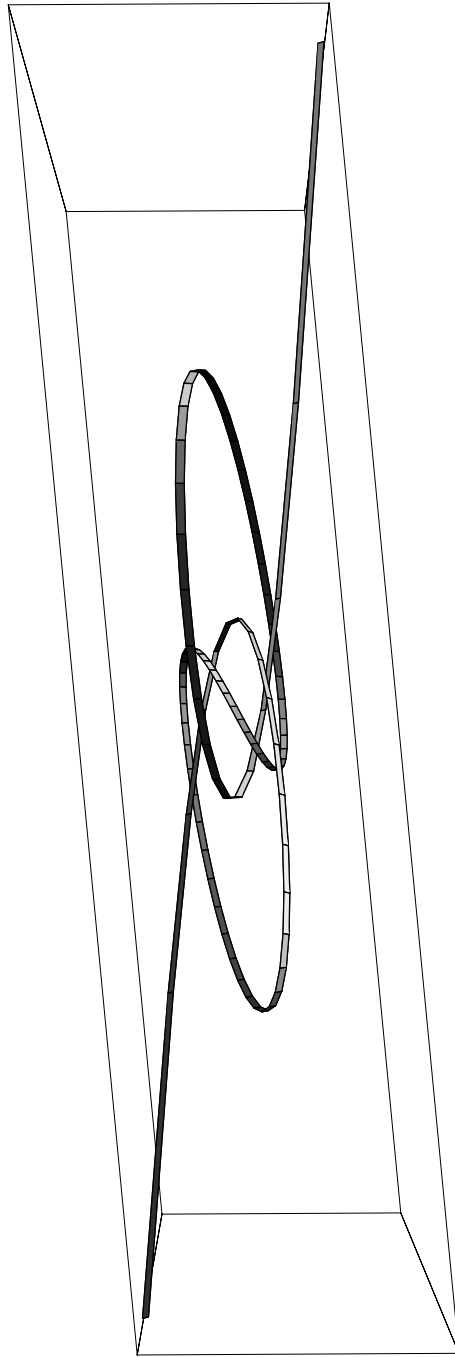


Fig. 6. Strongly invertible representation of $(3, 4)$ torus knot

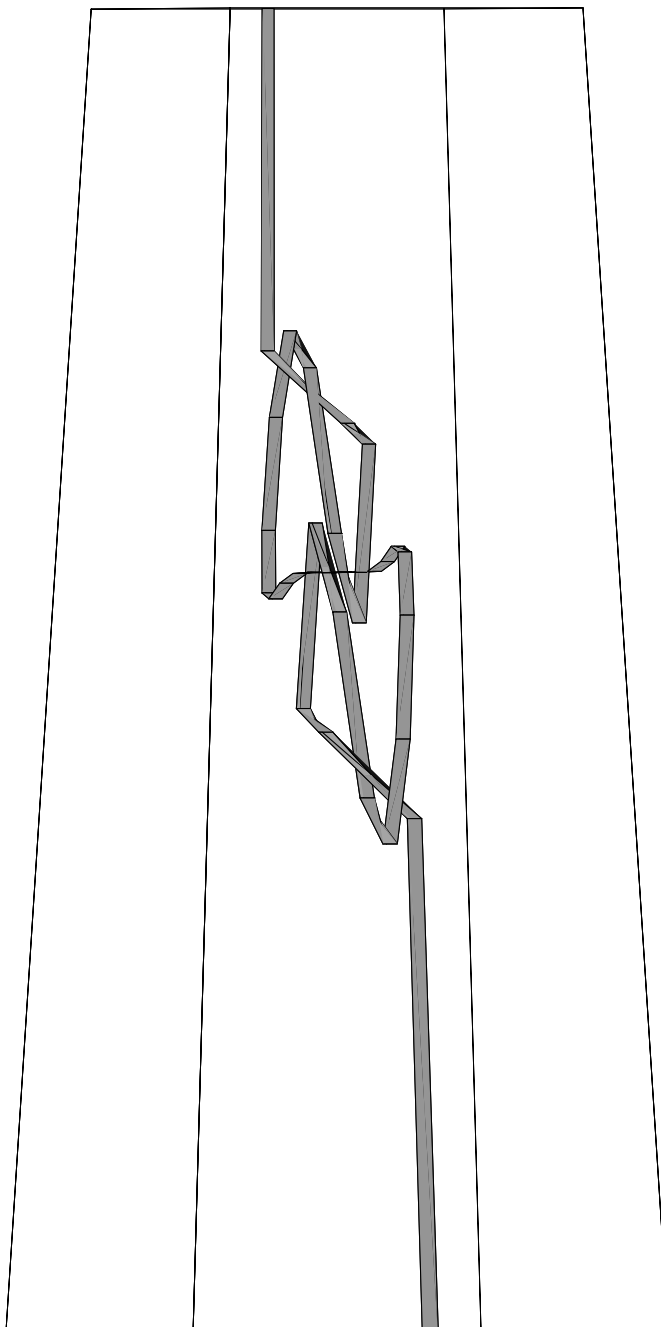


Fig. 7. Faithfully strongly negative amphicheiral representation of 8_{17}

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