Tsuchihashi, H.
Osaka J. Math.
36 (1999), 615–639

GALOIS COVERING SINGULARITIES II

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(Received September 10, 1997)

0. Introduction

Let Y be a simply connected open neighborhood of the origin in $\mathbf{C}^n$. For a finite Galois covering $\pi : X \to Y$ of Y, we denote by B_π, the divisor $r_1B_1 + r_2B_2 + \cdots + r_lB_l$, where $B_1, B_2, \cdots, B_l$ are the irreducible components of the branch locus $\{y \in Y | \#\pi^{-1}(y) < \deg\pi\}$ of π and r_j are the ramification indices of π along B_j, i.e., $r_j = \deg\pi/\max\{\#\pi^{-1}(y)|y \in B_j\}$. In the previous paper[4], we study Abel coverings $\pi : X \to Y$ of Y with $B_\pi = D$ for a given effective divisor D on Y. In this paper, we study Galois coverings $\pi : X \to Y$ such that $B_\pi = D$ and that the covering transformation groups $\mathrm{Gal}(X/Y) = \{g \in \mathrm{Aut}(X) | \pi\circ g = \pi\}$ are finite solvable groups (e.g. the dihedral groups D_{2s}, the quaternion group Q, the alternating group A_4 and the symmetric group S_4 of degree 4).

In Section 1, we show that for any Galois covering $\pi : X \to Y$ of Y, there exists a commutative diagram:

$$\begin{array}{ccccc} & \overline{X} & \xrightarrow{\lambda} & \overline{Y} & \\ \overline{\nu} & \downarrow & & \downarrow & \overline{\mu}\ , \\ & X & \xrightarrow{\pi} & Y & \end{array}$$

where $\overline{\mu} : \overline{Y} \to Y$ and $\overline{\nu} : \overline{X} \to X$ are Abel coverings such that $B_{\overline{\mu}} = B_\pi$ and that $\overline{\nu}$ does not ramify on $\pi^{-1}(Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l))$, and $\lambda : \overline{X} \to \overline{Y}$ is a Galois covering which does not ramify on $\overline{\mu}^{-1}(Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l))$. Moreover, the composite $\overline{\mu}\circ\lambda : \overline{X} \to Y$ of λ and $\overline{\mu}$ is a Galois covering and $[G_0, G_0] \simeq \mathrm{Gal}(\overline{X}/\overline{Y})$, where $G_0 = \mathrm{Gal}(\overline{X}/Y)$. Let $G_1 = [G_0, G_0]$, let $G_2 = [G_1, G_1]$, $\cdots$ and let $\overline{X_i} = \overline{X}/G_i$. Then $\overline{\mu}\circ\lambda$ induces Galois coverings $\overline{X_i} \to Y$ and Abel coverings $\overline{X_{i+1}} \to \overline{X_i}$ which do not ramify on $\overline{X_i} \setminus \mathrm{Sing}(\overline{X_i})$. Hence if the covering transformation group $\mathrm{Gal}(X/Y)$ is solvable, then there exists a Galois covering $\overline{X_m} \to Y$ which is the composite of Abel coverings $\overline{X_m} \to \overline{X_{m-1}}, \cdots, \overline{X_2} \to \overline{X_1}$ and $\overline{X_1} \to Y$ such that $B_{[\overline{X_1}\to Y]} = B_\pi$ and that $\overline{X_j} \to \overline{X_{j-1}}$ do not ramify along divisors for $2 \le j \le m$, and X is isomorphic to a quotient of $\overline{X_m}$. We also study the connection among B_π, $\mathrm{Gal}(X/Y)$ and the quasi-Gorensteinness. For example, if $\mathrm{Gal}(X/Y) \simeq Q$, then there exist at least two irreducible components B_j of B_π with $r_j = 4$ and X is not a quasi-Gorenstein singularity.

In Section 2, we study Abel coverings of normal Stein analytic spaces. First, we show that any Abel covering $Z' \to Z$ of a normal Stein space Z is isomorphic to the normalization of the analytic subspace of $Z \times \mathbf{C}^l$ defined by $\xi_1^{r_1} - f_1 = \cdots = \xi_l^{r_l} - f_l = 0$ for some holomorphic functions $f_1, \ldots, f_l$ on Z and for some positive integers $r_1, \ldots, r_l$, where $(\xi_1, \ldots, \xi_l)$ is a coordinate system of $\mathbf{C}^l$. Next, assume that Z is a Galois covering of a normal Stein space Y. Then we give a necessary and sufficient condition on these functions f_i and these integers r_i that the composite of the Abel covering $Z' \to Z$ and the Galois covering $Z \to Y$ becomes a Galois covering. Moreover, we show how to determine the structure of the covering transformation group $\mathrm{Gal}(Z'/Y)$.

In Section 3, we construct some examples of Galois coverings $\pi : X \to Y$ of an open neighborhood Y of the origin in $\mathbf{C}^n$ such that $\mathrm{Gal}(X/Y)$ are isomorphic to D_{2s}, Q, A_4 or S_4, applying the methods in §2. For example, if there exist holomorphic functions α, β, γ and $\delta (\not\equiv 0)$ on Y such that $\mathrm{codim}([\alpha] \cap [\beta]) \geq 2$ and that $\delta^2 = \gamma^3 + 2\alpha - 3\beta\gamma$, then there exists a Galois covering $\pi : X \to Y$ with $\mathrm{Gal}(X/Y) \simeq S_4$ and $B_\pi = 2D_1$, where D_1 is the divisor on Y defined by $\alpha^2 - \beta^3 = 0$. This Galois covering $\pi : X \to Y$ is constructed as the composite of three Abel coverings $X_3 \to X_2$, $X_2 \to X_1$ and $X_1 \to Y$, where X_1 (resp. X_2, X_3) is the normalization of the analytic subspace of $Y \times \mathbf{C}$ (resp. $X_1 \times \mathbf{C}$, $X_2 \times \mathbf{C}^2$) defined by $\xi^2 - (\alpha^2 - \beta^3) = 0$ (resp. $\zeta^3 - (\alpha - \xi) = 0$, $\eta_1^2 - (\gamma + \zeta + \beta/\zeta) = \eta_2^2 - (\gamma + \exp(2\pi\sqrt{-1}/3)\zeta + \exp(4\pi\sqrt{-1}/3)\beta/\zeta) = 0$).

In Section 4, we show that any Galois covering of a Stein analytic space whose covering transformation group is isomorphic to D_{2s}, Q, A_4 or S_4, is isomorphic to that constructed in the similar manner as in §3. For example, any Galois covering $\pi : X \to Y$ with $\mathrm{Gal}(X/Y) \simeq S_4$ is constructed as the composite of three Abel coverings as above.

Let p, q, r_1, s be positive integers with $g.c.d.(p,q) = 1$ and let D_1 be the irreducible divisor defined by the equation $z_1^p - z_2^q = 0$ on a simply connected open neighborhood Y of the origin in $\mathbf{C}^2$. In Section 5, we give a necessary and sufficient condition on these integers p, q, r_1 and s that there exists a Galois covering $\pi : X \to Y$ such that $\mathrm{Gal}(X/Y)$ is isomorphic to D_{2s}, A_4 or S_4 and that $B_\pi = r_1 D_1$. Moreover, we show the dual graph of the exceptional set of a resolution of X.

NOTATIONS. $\pi : X \to Y$: a Galois covering of an open neighborhood Y of $0 \in \mathbf{C}^n$

$B_1, B_2, \cdots, B_l$: the irreducible components of the branch locus of π

r_j : the ramification index of π along B_j

$B_\pi = r_1 B_1 + r_2 B_2 + \cdots + r_l B_l$

$H_j = [\pi_1(Y \setminus ((B_\pi)_{red})) \to \mathrm{Gal}(X/Y)](\{$lassos rounding B_j once in the positive direction$\})$

$\mu : \widetilde{Y} \to Y$: the projection ($\widetilde{Y} = \{(y, w_1, \ldots, w_l) \in Y \times \mathbf{C}^l | w_1^{r_1} - f_1 = \cdots = w_l^{r_l} - f_l = 0\}$)

$\sigma_i : (y, w_1, \dots, w_l) \mapsto (y, w_1, \dots, w_{i-1}, \rho_{r_i} w_i, w_{i+1}, \dots, w_l) \in \mathrm{Aut}(\widetilde{Y})$

$p : \mathrm{Gal}(\widetilde{Y}/Y) \to \mathbf{Z}_{r'_1} \oplus \cdots \oplus \mathbf{Z}_{r'_k}$: the homomorphism defined by $p(\sigma_{i_{j-1}+1}) = \cdots p(\sigma_{i_j}) = (0, \dots, \overset{j}{1}, \dots, 0)$

$\overline{\mu} : \overline{Y} \to Y$: the projection ($\overline{Y} = \widetilde{Y}/\ker(p)$)

$\widetilde{Y_0} := \mu^{-1}(Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l))$, $\overline{Y_0} := \overline{\mu}^{-1}(Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l))$

$\overline{X}'$: an irreducible component of $X \times_Y \overline{Y}$, $\overline{X}$: the normalization of $\overline{X}'$

$\overline{G}$: the subgroup of $\mathrm{Gal}(X/Y) \oplus \mathbf{Z}_{r'_1} \oplus \cdots \oplus \mathbf{Z}_{r'_k}$ generated by $(h_i, p(\sigma_i))$ for all $h_i \in H_i$

$p_1 : \overline{G} \to \mathrm{Gal}(X/Y)$, $p_2 : \overline{G} \to \mathbf{Z}_{r'_1} \oplus \cdots \oplus \mathbf{Z}_{r'_k}$: the projections

$\chi' : \mathrm{Gal}(\overline{Y}/Y) \to \mathbf{C}^*$: the homomorphism sending $p(\sigma_i)$ to ρ_{r_i}

$Z'_{\mathbf{f},\mathbf{r}} = \{(z, \xi_1, \dots, \xi_l) \in Z \times \mathbf{C}^l | \xi_1^{r_1} - f_1 = \cdots = \xi_l^{r_l} - f_l = 0\}$

$\pi_{\mathbf{f},\mathbf{r}} : Z_{\mathbf{f},\mathbf{r}} \to Z$: the composite of the normalization $Z_{\mathbf{f},\mathbf{r}} \to Z'_{\mathbf{f},\mathbf{r}}$ and the projection $Z \times \mathbf{C}^l \to Z$

$\tau^i_{\mathbf{f},\mathbf{r}} : (x, \xi_1, \dots, \xi_l) \mapsto (x, \xi_1, \dots, \xi_{i-1}, \rho_{r_i}\xi_i, \xi_{i+1}, \dots, \xi_l) \in \mathrm{Aut}(Z_{\mathbf{f},\mathbf{r}})$

$G_{\mathbf{f},\mathbf{r}} = \mathbf{Z}\tau^1_{\mathbf{f},\mathbf{r}} \oplus \cdots \oplus \mathbf{Z}\tau^l_{\mathbf{f},\mathbf{r}}$

$\widetilde{f}_i$: a holomorphic function on $Z_{\mathbf{f},\mathbf{r}}$ with $\widetilde{f}_i^{\,r_i} = \pi^*_{\mathbf{f},\mathbf{r}} f_i$

$\rho_r = \exp(2\pi\sqrt{-1}/r)$

1. Non-Abel Galois coverings

Let $\pi : X \to Y$ be a Galois covering of a simply connected open neighborhood Y of 0 in $\mathbf{C}^n$ and let $B_\pi = r_1 B_1 + \cdots + r_l B_l$ as in Introduction. Let H_j be the subset of $\mathrm{Gal}(X/Y)$ consisting of the images under the quotient map $\pi_1(Y \setminus (B_\pi)_{red}) \to \mathrm{Gal}(X/Y)$ of the lassos in $\pi_1(Y \setminus (B_\pi)_{red})$ rounding B_j once in the positive direction for $j = 1, \cdots, l$. Then $H_j = \{ghg^{-1} | g \in \mathrm{Gal}(X/Y)\}$ and $|h| = r_j$ for any h in H_j.

Proposition 1.1. *$\mathrm{Gal}(X/Y)$ is generated by $H_1 \cup H_2 \cup \cdots \cup H_l$.*

Proof. Let G be the subgroup of $\mathrm{Gal}(X/Y)$ generated by $H_1 \cup H_2 \cup \cdots \cup H_l$. Then the covering $X/G \to Y$ does not ramify on $Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l)$. Hence it is an isomorphism, because $Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l)$ is simply connected. Therefore, $G = \mathrm{Gal}(X/Y)$. □

We may assume that $H_1 = \cdots = H_{i_1}$, $H_{i_1+1} = \cdots = H_{i_2}$, $\cdots$, $H_{i_{k-1}+1} = \cdots = H_l$ and that $H_i \neq H_{i'}$ if $i \leq i_j < i'$ for some j. Then $r_1 = \cdots = r_{i_1}$, $r_{i_1+1} = \cdots = r_{i_2}$, $\cdots$ and $r_{i_{k-1}+1} = \cdots = r_l$. Let $\widetilde{Y}$ be the Abel covering of Y as in §3 of [4] for $D = B_\pi$, i.e., $\widetilde{Y}$ is the subvariety of $Y \times \mathbf{C}^l$ defined by the equations $w_i^{r_i} - f_i = 0$ ($i = 1, \cdots, l$), where f_i are defining equations of B_i. Recall that σ_i are the automorphisms of $\widetilde{Y}$ sending $(y, w_1, \dots, w_l)$ to $(y, w_1, \dots, w_{i-1}, \rho_{r_i} w_i, w_{i+1}, \dots, w_l)$, where $\rho_{r_i} = \exp(2\pi\sqrt{-1}/r_i)$. Let $p : \mathrm{Gal}(\widetilde{Y}/Y) \to \mathbf{Z}_{r'_1} \oplus \cdots \oplus \mathbf{Z}_{r'_k}$ be the homomorphism defined by $p(\sigma_1) = \cdots = p(\sigma_{i_1}) = (1, 0, \ldots, 0)$, $p(\sigma_{i_1+1}) = \cdots = p(\sigma_{i_2}) = (0, 1, 0, \ldots, 0)$, $\ldots$, $p(\sigma_{i_{k-1}+1}) = \cdots = p(\sigma_l) = (0, \ldots, 0, 1)$ and let $\overline{Y} = \widetilde{Y}/\ker(p)$. Then $\overline{Y}$ is isomorphic to the subvariety in $Y \times \mathbf{C}^k$ defined by the equations $w_j^{r'_j} - f_{i_{j-1}+1} \cdots f_{i_j} = 0$ ($j = 1, \cdots, k$), where $r'_j = r_i$ for $i_{j-1} < i \le i_j$, $i_0 = 0$, $i_k = l$. Moreover, $\overline{\mu} := \mu_{\ker(p)} : \overline{Y} \to Y$ is an Abel covering with $B_{\overline{\mu}} = B_\pi$ and $\mathrm{Gal}(\overline{Y}/Y) \simeq \mathbf{Z}_{r'_1} \oplus \cdots \oplus \mathbf{Z}_{r'_k}$. Let $\overline{X}$ be the normalization of an irreducible component $\overline{X}'$ of $X \times_Y \overline{Y}$. Then the projection $\overline{\pi} : \overline{X} \to Y$ to Y is a Galois covering with $B_{\overline{\pi}} = B_\pi$, because the projection $\overline{X} \to \overline{Y}$ to $\overline{Y}$ does not ramify on $\overline{Y_0} := \overline{\mu}^{-1}(Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l))$. Let $\overline{G}$ be the subgroup of $\mathrm{Gal}(X/Y) \oplus \mathrm{Gal}(\overline{Y}/Y)$ generated by $(h_i, p(\sigma_i))$ for all i and for all $h_i \in H_i$.

Proposition 1.2. $\overline{G} = \{g \in \mathrm{Gal}(X/Y) \oplus \mathrm{Gal}(\overline{Y}/Y) | g\overline{X}' = \overline{X}'\}$.

Proof. $g\overline{X}' = \overline{X}'$ for all g in $\overline{G}$, because $(h_i, p(\sigma_i))\overline{X}' = \overline{X}'$. Since $\overline{X}'/\overline{G} \to Y$ does not ramify on $Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l)$, we have $\overline{X}'/\overline{G} \simeq Y$. □

Let C be the commutators group of $G := \mathrm{Gal}(X/Y)$ and let $F = G/C$. Then we have the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \overline{G} & \xrightarrow{p_2} & \mathbf{Z}_{r'_1} \oplus \cdots \oplus \mathbf{Z}_{r'_k} & \longrightarrow & 0 \\ & & \| & & \downarrow p_1 & & \downarrow & & \\ 1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

where p_1 and p_2 are the restrictions to $\overline{G}$ of the projections from $G \oplus \mathrm{Gal}(\overline{Y}/Y)$ to G and $\mathrm{Gal}(\overline{Y}/Y)$, respectively. Here we note that $\ker(p_2) = C$, because $|h_i| = r_i$ for $h_i \in H_i$. Hence $\mathrm{Gal}(\overline{X}/\overline{Y}) \simeq C$. Also note that p_1 is surjective, by Proposition 1. On the other hand, $\mathrm{Gal}(\overline{X}/X) \simeq \ker(p_1)$ is isomorphic to a subgroup of $\mathrm{Gal}(\overline{Y}/Y)$. Hence it is an Abelian group.

As we see in §1 of [4], there exists a nowhere vanishing holomorphic n-form ϕ on $\widetilde{Y_0} := \mu^{-1}(Y \setminus \mathrm{Sing}(B_1 + \cdots + B_l))$ with $\sigma^*\phi = \chi(\sigma)\phi$ for $\sigma \in \mathrm{Gal}(\widetilde{Y}/Y)$, where $\chi : \mathrm{Gal}(\widetilde{Y}/Y) \to \mathbf{C}^*$ is the homomorphism sending σ_i to ρ_{r_i}. Since $\ker(p) \subset \ker(\chi)$, there exists a nowhere vanishing holomorphic n-form ϕ' on $\overline{Y_0}$ with $\overline{g}^*\phi' = \chi'(\overline{g})\phi'$ for all $\overline{g} \in \mathrm{Gal}(\overline{Y}/Y)$, where $\chi' : \mathrm{Gal}(\overline{Y}/Y) \to \mathbf{C}^*$ is the homomorphism sending $p(\sigma_i)$ to ρ_{r_i}. Then $\tilde{g}^*(\pi_2^*\phi') = ((\chi' \circ p_2)(\tilde{g}))(\pi_2^*\phi')$ for all $\tilde{g}$ in $\mathrm{Gal}(\overline{X}/Y)$, where $\pi_2 : \overline{X} \to \overline{Y}$ is the projection. Since $X \simeq \overline{X}/\ker(p_1)$, we have:

Proposition 1.3. *(X, x) is a quasi-Gorenstein singularity if and only if* $\ker(p_1)$ *is contained in* $\ker(\chi' \circ p_2)$*, where* $\{x\} = \pi^{-1}(0)$.

EXAMPLE 1.1. Assume that $\mathrm{Gal}(X/Y)$ is isomorphic to the symmetric group S_3 of degree 3, i.e., $\mathrm{Gal}(X/Y)$ is generated by two elements α and β enjoying the relations $\alpha^2 = \beta^2 = (\alpha\beta)^3 = e$. Then $H_i = \{\alpha, \beta\alpha\beta, \beta\}$ or $\{\beta\alpha, (\beta\alpha)^2\}$, accordingly as $r_i = 2$ or 3. Hence there exists at least one irreducible component B_i of $B_\pi = r_1B_1 + \cdots + r_lB_l$ with $r_i = 2$, by Proposition 1.1. Assume that there exists an irreducible component B_i with $r_i = 3$. Then $(\beta\alpha, p(\sigma_i))((\beta\alpha)^2, p(\sigma_i)) = (e, p(\sigma_i)^2) \in \ker(p_1)$. However, $(e, p(\sigma_i)^2) \notin \ker(\chi' \circ p_2)$. Hence (X, x) is not a quasi-Gorenstein singularity. Conversely, assume that all r_i are equal to 2. Then $\ker(p_1) = \{e\}$. Hence (X, x) is a quasi-Gorenstein singularity.

EXAMPLE 1.2. Assume that $\mathrm{Gal}(X/Y)$ is isomorphic to the quaternion group Q, i.e., $\mathrm{Gal}(X/Y)$ is generated by two elements α and β enjoying the relations $\alpha^4 = e$, $\beta^2 = \alpha^2$ and $\alpha\beta = \beta\alpha^3$. Then $H_i = \{\alpha^2\}$ $(r_i = 2)$, $\{\alpha, \alpha^3\}$ $(r_i = 4)$, $\{\beta, \alpha^2\beta\}$ $(r_i = 4)$ or $\{\alpha\beta, \alpha^3\beta\}$ $(r_i = 4)$. Hence there exist at least two irreducible components B_i of B_π with $r_i = 4$, by Proposition 1.1. Moreover, we see that (X, x) is not a quasi-Gorenstein singularity, because $\alpha\alpha^3 = \beta(\alpha^2\beta) = (\alpha\beta)(\alpha^3\beta) = e$.

2. Abel coverings of normal Stein spaces

Let Z be a normal Stein analytic space. We say that two Galois coverings $\pi_1 : Z_1 \to Z$ and $\pi_2 : Z_2 \to Z$ of Z are isomorphic over Z, if there exists an isomorphism $\phi : Z_1 \simeq Z_2$ with $\pi_1 = \pi_2 \circ \phi$.

DEFINITION 2.1. For an ordered set $\mathbf{f}$ of holomorphic functions $f_1, \ldots, f_l$ on Z and for an ordered set $\mathbf{r}$ of integers $r_1, \ldots, r_l$ greater than 1 with $|\mathbf{f}| = |\mathbf{r}|$, we denote by $Z_{\mathbf{f},\mathbf{r}}$, the normalization of the analytic subspace $Z'_{\mathbf{f},\mathbf{r}}$ in $Z \times \mathbf{C}^l$ defined by $\xi_1^{r_1} - f_1 = \cdots = \xi_l^{r_l} - f_l = 0$, where $(\xi_1, \ldots, \xi_l)$ is a coordinate system of $\mathbf{C}^l$. We also denote by $G_{\mathbf{f},\mathbf{r}}$, the Abelian group generated by the automorphisms $\tau^i_{\mathbf{f},\mathbf{r}}$ of $Z_{\mathbf{f},\mathbf{r}}$ sending $(x, \xi_1, \ldots, \xi_l)$ to $(x, \xi_1, \ldots, \xi_{i-1}, \rho_{r_i}\xi_i, \xi_{i+1}, \ldots, \xi_l)$, where $\rho_{r_i} = \exp(2\pi\sqrt{-1}/r_i)$, by $\pi_{\mathbf{f},\mathbf{r}}$, the composite of the normalization $Z_{\mathbf{f},\mathbf{r}} \to Z'_{\mathbf{f},\mathbf{r}}$ and the projection $Z \times \mathbf{C}^l \to Z$, and by $\widetilde{f}_i$, the holomorphic function on $Z_{\mathbf{f},\mathbf{r}}$ which is the pull-back of $\xi_{i|Z'_{\mathbf{f},\mathbf{r}}}$

Here we note that $\pi_{\mathbf{f},\mathbf{r}}$ does not ramify at a point x in Z, if there exist holomorphic functions $h_1, \ldots, h_l$ on a neighborhood U of x with $h_i^{r_i} = f_{i|U}$. Also note that $(\tau^i_{\mathbf{f},\mathbf{r}})^*\widetilde{f}_i = \rho_{r_i}\widetilde{f}_i$, $(\tau^i_{\mathbf{f},\mathbf{r}})^*\widetilde{f}_j = \widetilde{f}_j$ $(i \neq j)$ and $\widetilde{f}_i^{\,r_i} = \pi^*_{\mathbf{f},\mathbf{r}}f_i$.

Proposition 2.1. *For any Abel covering $\pi' : Z' \to Z$ of Z with $\mathrm{Gal}(Z'/Z) \simeq \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_l}$, there exist holomorphic functions $f_1, \ldots, f_l$ on Z such that $\pi_{\{f_1,\ldots,f_l\},\{r_1,\ldots,r_l\}}$ is isomorphic to π' over Z.*

Proof. Let $\tau_1, \ldots, \tau_l$ be generators of $\mathrm{Gal}(Z'/Z)$ with $|\tau_i| = r_i$. Take a holomorphic function f' on Z' and let $\bar{f}_i = \sum_{k_1=0}^{r_1-1} \cdots \sum_{k_l=0}^{r_l-1} (\tau_1^{k_1} \cdots \tau_l^{k_l})^* f'/\rho_{r_i}^{k_i}$. Then

$\tau_i^* \bar{f}_i = \rho_{r_i} \bar{f}_i$, $\tau_i^* \bar{f}_j = \bar{f}_j$ $(i \neq j)$ and $\bar{f}_i \not\equiv 0$ for a suitable f'. Since $\bar{f}_i^{r_i}$ are $\mathrm{Gal}(Z'/Z)$-invariant, there exist holomorphic functions $f_1, \cdots, f_l$ on Z with $\pi'^* f_i = \bar{f}_i^{r_i}$. Then the image of the holomorphic map $Z' \to Z \times \mathbf{C}^l$ sending x' to $(\pi'(x'), \bar{f}_1(x'), \ldots, \bar{f}_l(x'))$ is equal to $Z'_{\mathbf{f},\mathbf{r}}$, where $\mathbf{f} = \{f_1, \ldots, f_l\}$ and $\mathbf{r} = \{r_1, \ldots, r_l\}$. Since the surjective map $Z' \to Z'_{\mathbf{f},\mathbf{r}}$ is one to one on $Z' \setminus \pi'^{-1}(\cup_{i=1}^l [f_i])$, it induces a biholomorphic map $\phi : Z' \simeq Z_{\mathbf{f},\mathbf{r}}$ satisfying $\phi \circ \tau_i = \tau^i_{\mathbf{f},\mathbf{r}} \circ \phi$. □

Let $\mathbf{f} = \{f_1, \ldots, f_l\}$, $\mathbf{g} = \{g_1, \ldots, g_l\}$ be ordered sets of holomorphic functions on Z and let $\mathbf{r} = \{r_1, \ldots, r_l\}$ be an ordered set of integers greater than 1 with $|\mathbf{f}| = |\mathbf{g}| = |\mathbf{r}|$.

Proposition 2.2. *$Z_{\mathbf{f},\mathbf{r}}$ is reducible if and only if there exist non-negative integers s_i which are smaller than r_i and at least one of which is positive, and a holomorphic function h on Z with $f_1^{\frac{rs_1}{r_1}} \cdots f_l^{\frac{rs_l}{r_l}} = h^r$, where $r = l.c.m.(\frac{r_1}{g.c.d.(r_1,s_1)}, \ldots, \frac{r_l}{g.c.d.(r_l,s_l)})$. Especially, when $l = 1$, if $Z_{\mathbf{f},\mathbf{r}}$ is reducible, then there exist a holomorphic function h on Z and a divisor t of r_1 greater than 1 with $f_1 = h^t$.*

Proof. Assume that $Z'_{\mathbf{f},\mathbf{r}}$ is reducible and let Z_1 be an irreducible component of $Z'_{\mathbf{f},\mathbf{r}}$. Then $H := \{\sigma \in G_{\mathbf{f},\mathbf{r}} | \sigma Z_1 = Z_1\} \neq G_{\mathbf{f},\mathbf{r}}$. Hence there exist integers s_i such that $0 \leq s_i < r_i$, that $(s_1, \ldots, s_l) \neq (0, \ldots, 0)$ and that $h := \widetilde{f}_1^{s_1} \cdots \widetilde{f}_l^{s_l}$ is H-invariant. When $l = 1$, we may assume that s_1 is a divisor of r_1. Then $h^r = \pi^*_{\mathbf{f},\mathbf{r}} \left(f_1^{\frac{rs_1}{r_1}} \cdots f_l^{\frac{rs_l}{r_l}} \right)$ and the restriction of h to Z_1 is equal to the pull-back of a holomorphic function on Z, because $\mathrm{Gal}(Z_1/Z) = H$.

Conversely, let s_i and h be as in the proposition and let $\widetilde{h} = \widetilde{f}_1^{s_1} \cdots \widetilde{f}_l^{s_l}$. Then $\widetilde{h}^r = \pi^*_{\mathbf{f},\mathbf{r}} h^r$. Hence $\widetilde{h} = \epsilon \pi^*_{\mathbf{f},\mathbf{r}} h$ on each irreducible component Z_1 of $Z_{\mathbf{f},\mathbf{r}}$ for an r-th root ϵ of the unit. Therefore, $\sigma^* \widetilde{h} = \widetilde{h}$ for all elements σ in $\{\sigma \in G_{\mathbf{f},\mathbf{r}} | \sigma Z_1 = Z_1\}$. On the other hand, there exists an element σ in $G_{\mathbf{f},\mathbf{r}}$ with $\sigma^* \widetilde{h} \neq \widetilde{h}$. Hence $Z_1 \neq Z_{\mathbf{f},\mathbf{r}}$. □

Proposition 2.3. *Assume that $Z_{\mathbf{f},\mathbf{r}}$ and $Z_{\mathbf{g},\mathbf{r}}$ are irreducible. Then there exists an isomorphism $\phi : Z_{\mathbf{f},\mathbf{r}} \simeq Z_{\mathbf{g},\mathbf{r}}$ with $\pi_{\mathbf{g},\mathbf{r}} \circ \phi = \pi_{\mathbf{f},\mathbf{r}}$, if and only if there exist meromorphic functions h_j on Z $(1 \leq j \leq l)$ and integers s_{ij} $(1 \leq i, j \leq l)$ such that $h_j^{r_j} = g_j / (f_1^{s_{1j}} \cdots f_l^{s_{lj}})$ and that $r_i s_{ij} \equiv 0 (\mathrm{mod}\ r_j)$. Then $\phi \circ \tau^i_{\mathbf{f},\mathbf{r}} = (\tau^1_{\mathbf{g},\mathbf{r}})^{s_{i1}} \cdots (\tau^l_{\mathbf{g},\mathbf{r}})^{s_{il}} \circ \phi$.*

Proof. Assume that there exists an isomorphism $\phi : Z_{\mathbf{f},\mathbf{r}} \simeq Z_{\mathbf{g},\mathbf{r}}$ with $\pi_{\mathbf{g},\mathbf{r}} \circ \phi = \pi_{\mathbf{f},\mathbf{r}}$. Then there exist integers s_{ij} with $\phi \circ \tau^i_{\mathbf{f},\mathbf{r}} = (\tau^1_{\mathbf{g},\mathbf{r}})^{s_{i1}} \cdots (\tau^l_{\mathbf{g},\mathbf{r}})^{s_{il}} \circ \phi$. Since $|\tau^i_{\mathbf{f},\mathbf{r}}| = r_i$, we see that $r_i s_{ij} \equiv 0 (\mathrm{mod}\ r_j)$. Let $h_j = \phi^* \widetilde{g}_j / (\widetilde{f}_1^{r_1 s_{1j}/r_j} \cdots \widetilde{f}_l^{r_l s_{lj}/r_j})$. Then h_j are $G_{\mathbf{f},\mathbf{r}}$-invariant meromorphic functions on $Z_{\mathbf{f},\mathbf{r}}$ and $h_j^{r_j} = \pi^*_{\mathbf{f},\mathbf{r}} (g_j / (f_1^{s_{1j}} \cdots f_l^{s_{lj}}))$.

Conversely, assume that the condition of the proposition is satisfied. Then the restriction ϕ' to $Z'_{\mathbf{f},\mathbf{r}}$ of the meromorphic map $Z \times \mathbf{C}^l \to Z \times \mathbf{C}^l$ sending $(x, \xi_1, \ldots, \xi_l)$

to $(x, h_1\xi_1^{r_1s_{11}/r_1}\cdots\xi_l^{r_ls_{l1}/r_1}, \ldots, h_l\xi_1^{r_1s_{1l}/r_l}\cdots\xi_l^{r_ls_{ll}/r_l})$ is a meromorphic map onto $Z'_{\mathbf{g},\mathbf{r}}$. Hence ϕ' induces a meromorphic map $\phi : Z_{\mathbf{f},\mathbf{r}} \to Z_{\mathbf{g},\mathbf{r}}$ which is one to one on $Z_{\mathbf{f},\mathbf{r}} \setminus \pi_{\mathbf{f},\mathbf{r}}^{-1}(\cup_{i=1}^l[f_i] \cup \cup_{i=1}^l[g_i])$ and an irreducible component $\widetilde{Z}$ of $Z_{\mathbf{f},\mathbf{r}} \times_Z Z_{\mathbf{g},\mathbf{r}}$ is the graph of ϕ. Therefore, the projections $\widetilde{Z} \to Z_{\mathbf{f},\mathbf{r}}$ and $\widetilde{Z} \to Z_{\mathbf{g},\mathbf{r}}$ are isomorphic by Zariski's Main Theorem. □

Corollary 2.1. *Assume that $Z_{\mathbf{f},\mathbf{r}}$ is irreducible. Then for any automorphism σ of Z, there exists an automorphism $\tilde{\sigma}$ of $Z_{\mathbf{f},\mathbf{r}}$ with $\pi_{\mathbf{f},\mathbf{r}} \circ \tilde{\sigma} = \sigma \circ \pi_{\mathbf{f},\mathbf{r}}$, if and only if there exist meromorphic functions h_j on Z $(1 \le j \le l)$ and integers s_{ij} $(1 \le i, j \le l)$ such that $h_j^{r_j} = \sigma^* f_j/(f_1^{s_{1j}} \cdots f_l^{s_{lj}})$ and that $r_i s_{ij} \equiv 0 (\mathrm{mod}\ r_j)$. Then $\tilde{\sigma} \circ \tau_{\mathbf{f},\mathbf{r}}^i = (\tau_{\mathbf{f},\mathbf{r}}^1)^{s_{i1}} \cdots (\tau_{\mathbf{f},\mathbf{r}}^l)^{s_{il}} \circ \tilde{\sigma}$.*

Let σ_1, σ_2, $\cdots$ and σ_m be automorphisms of Z with $\sigma_1\sigma_2\cdots\sigma_m = id$. Assume that there exist meromorphic functions h_j^k on Z and integers s_{ij}^k $(1 \le i, j \le l,\ 1 \le k \le m)$ such that $(h_j^k)^{r_j} = \sigma_k^* f_j/(f_1^{s_{1j}^k} \cdots f_l^{s_{lj}^k})$ and that $r_i s_{ij}^k \equiv 0(\mathrm{mod}\ r_j)$. Then for each k there exist $r_1 r_2 \cdots r_l$ automorphisms $\widetilde{\sigma_k}$ of $Z_{\mathbf{f},\mathbf{r}}$ with $\pi_{\mathbf{f},\mathbf{r}} \circ \widetilde{\sigma_k} = \sigma_k \circ \pi_{\mathbf{f},\mathbf{r}}$, by the above corollary. Among those, for each k we can choose one so that $\widetilde{\sigma_k}^* \widetilde{f_j} = h_j^k \widetilde{f_1}^{r_1 s_{1j}^k/r_j} \cdots \widetilde{f_l}^{r_l s_{lj}^k/r_j}$. On the other hand, $\widetilde{\sigma_1} \cdots \widetilde{\sigma_m} = (\tau_{\mathbf{f},\mathbf{r}}^1)^{t_1} \cdots (\tau_{\mathbf{f},\mathbf{r}}^l)^{t_l}$ for some integers t_1, t_2, $\cdots$, t_l with $0 \le t_i < r_i$. Then $\widetilde{\sigma_m}^* \cdots \widetilde{\sigma_1}^* \widetilde{f_j} = \rho_{r_j}^{t_j} \widetilde{f_j}$. Hence we can determine these integers t_i by the data h_j^k and s_{ij}^k. For instance, if $m = 4$, $\sigma_3 = \sigma_1^{-1}$, $\sigma_4 = \sigma_2^{-1}$, i.e., $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and if $l = 1$, then

$$\rho_{r_1}^{t_1} = \frac{(\sigma_2^* h_1^1)\,(h_1^2)^{s_{11}^1}}{(\sigma_1^* h_1^2)\,(h_1^1)^{s_{11}^2}},$$

because $\widetilde{\sigma_2}^* \widetilde{\sigma_1}^* \widetilde{f_1} = (\sigma_2^* h_1^1)(h_1^2)^{s_{11}^1} \widetilde{f_1}^{s_{11}^2 s_{11}^1}$ and $\widetilde{\sigma_1}^* \widetilde{\sigma_2}^* \left((\tau_{\mathbf{f},\mathbf{r}}^1)^{t_1}\right)^* \widetilde{f_1} = \rho_{r_1}^{t_1} (\sigma_1^* h_1^2)(h_1^1)^{s_{11}^2} \widetilde{f_1}^{s_{11}^1 s_{11}^2}$.

Assume that Z is a Galois covering of a normal Stein space Y and let σ_1, σ_2, $\ldots$, σ_m be generators of $\mathrm{Gal}(Z/Y)$. If there exist meromorphic functions h_j^k and integers s_{ij}^k satisfying the condition of Corollary 2.1 for each σ_k $(k = 1, 2, \ldots, m)$, then the composite $Z_{\mathbf{f},\mathbf{r}} \to Z \to Y$ is a Galois covering of Y. Moreover, the structure of $\mathrm{Gal}(Z_{\mathbf{f},\mathbf{r}}/Y)$ is completely determined by the integers s_{ij}^k and t_j in the relations $\widetilde{\sigma_k} \tau_{\mathbf{f},\mathbf{r}}^i \widetilde{\sigma_k}^{-1} = (\tau_{\mathbf{f},\mathbf{r}}^1)^{s_{i1}^k} \cdots (\tau_{\mathbf{f},\mathbf{r}}^l)^{s_{il}^k}$ and $\widetilde{\sigma_{i_1}} \cdots \widetilde{\sigma_{i_p}} = (\tau_{\mathbf{f},\mathbf{r}}^1)^{t_1} \cdots (\tau_{\mathbf{f},\mathbf{r}}^l)^{t_l}$ for fundamental relations $\sigma_{i_1} \cdots \sigma_{i_p} = id$ of σ_1, σ_2, $\ldots$, σ_m.

REMARK 2.1. In the 2-dimensional case, we can construct resolutions $\lambda : \widetilde{Z} \to Z$ of Z, via embedded resolutions of B_π (see §3 of [4]), if $\pi : Z \to Y$ are Abel coverings of open sets Y of $\mathbf{C}^2$. Assume that $[\lambda^* f_i] = \sum c_{ij} E_j + r_i D'_i$ for some integers c_{ij} and divisors D'_i on $\widetilde{Z}$, where $\sum E_j$ are the exceptional sets of λ. Then $\pi_{\mathbf{f},\mathbf{r}} : Z_{\mathbf{f},\mathbf{r}} \to Z$ does not ramify on $Z \setminus \mathrm{Sing}(Z)$. If the dual graphs of $\sum E_j$ are tree, we can explicitly construct resolutions also of $Z_{\mathbf{f},\mathbf{r}}$, in the similar manner as in §3 of [4]. On the other hand, if there exists an integer j such that $\sum_{i=1}^{l} c_{ij} \frac{rs_i}{r_i}$ is not a multiple of $r = l.c.m.(\frac{r_1}{g.c.d.(r_1,s_1)}, \cdots, \frac{r_l}{g.c.d.(r_l,s_l)})$ for each combination of non-negative integers s_i which are smaller than r_i and at least one of which is positive, then by Proposition 2.2, $Z_{\mathbf{f},\mathbf{r}}$ are irreducible, because

$$\left[\lambda^*\left(f_1^{\frac{rs_1}{r_1}} \cdots f_l^{\frac{rs_l}{r_l}}\right)\right] = \sum \left(\sum_{i=1}^{l} c_{ij} \frac{rs_i}{r_i}\right) E_j + r \sum_{i=1}^{l} (s_1 D'_1 + \cdots + s_l D'_l).$$

3. Examples

Let $D = r_1 D_1 + r_2 D_2 + \cdots + r_k D_k$ be a divisor on a simply connected open neighborhood Y of 0 in $\mathbf{C}^n$, where r_j are integers greater than 1, D_j are reduced and may be reducible divisors on Y defined by $f_j = 0$. We assume that $0 \in D_j$, i.e., $f_j(0) = 0$ and that if $i \neq j$, then D_i and D_j have no common irreducible components, throughout this section. Let $\overline{Y}$ be the analytic subspace of $Y \times \mathbf{C}^k$ defined by $w_1^{r_1} - f_1 = \cdots = w_k^{r_k} - f_k = 0$, where $(w_1, w_2, \cdots, w_k)$ is a coordinate system of $\mathbf{C}^k$ and let $\overline{\mu} : \overline{Y} \to Y$ be the projection. Then we see in the same way as in Proposition 3 of [4] that $\overline{Y}$ is normal. Hence $\overline{Y} \simeq Y_{\{f_1,\ldots,f_k\},\{r_1,\ldots,r_k\}}$. Moreover, $\overline{\mu}$ is a Galois covering with $B_{\overline{\mu}} = D$ and $\mathrm{Gal}(\overline{Y}/Y) \simeq \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_k}$.

First, we construct some examples with the covering transformation groups whose commutators groups are cyclic. Let r be an integer greater than 1 and let g be a holomorphic function on $\overline{Y}$. We simply write $\overline{Y}_{g,r}$, $\pi_{g,r}$ and $\tau_{g,r}$ for $\overline{Y}_{\{g\},\{r\}}$, $\pi_{\{g\},\{r\}}$ and $\tau^1_{\{g\},\{r\}}$, respectively. Assume that there exist no holomorphic functions g_0 on $\overline{Y}$ with $g_0^s = g$ for any divisor s of r greater than 1. Then $\overline{Y}_{g,r}$ is irreducible, by Proposition 2.2. Furthermore, assume that there exist a meromorphic function h_j and an integer s_j satisfying $\overline{\sigma}_j^* g / g^{s_j} = h_j^r$ for each j, where $\overline{\sigma}_j$ is the automorphism of $\overline{Y}$ sending $(y, w_1, \ldots, w_j, \cdots, w_k)$ to $(y, w_1, \cdots, w_{j-1}, \rho_{r_j} w_j, w_{j+1}, \cdots, w_k)$. Then the composite $\overline{\mu} \circ \pi_{g,r} : \overline{Y}_{g,r} \to Y$ of $\pi_{g,r}$ and $\overline{\mu}$ is a Galois covering by Corollary 2.1. Moreover, if there exist an open neighborhood U of y_0 and a holomorphic function h on U with $h^r = g_{|U}$ for each point y_0 in $\overline{Y_0} := \overline{\mu}^{-1}(Y_0)$, then $\pi_{g,r}$ does not ramify on $\overline{Y_0}$ and hence $B_{\overline{\mu} \circ \pi_{g,r}} = D$, where $Y_0 = Y \setminus \mathrm{Sing}(D_1 + \cdots + D_k)$.

Proposition 3.1. *Let $D = 2D_1$. Assume that there exist holomorphic functions α, β on Y and an odd integer r greater than 2 such that $\alpha(0) = \beta(0) = 0$ and that $f_1 = \alpha^2 - \beta^r$. Then $\overline{\mu} \circ \pi_{g,r} : \overline{Y}_{g,r} \to Y$ is a Galois covering such that $B_{\overline{\mu} \circ \pi_{g,r}} = D$ and that $\mathrm{Gal}(\overline{Y}_{g,r}/Y) \simeq D_{2r}$, where $g = \alpha + w_1$.*

Proof. Suppose that there exists a holomorphic function g_0 on $\overline{Y}$ with $g_0^s = g$ for an integer s greater than 1. Then $g_0 = h_0 + w_1 h_1$ for some holomorphic functions h_0 and h_1 on Y. Since $g_0(0) = 0$, we have $h_0(0) = 0$. Then $g_0^s = (h_0^s + {}_sC_2 h_0^{s-2} h_1^2 f_1 + \cdots) + w_1({}_sC_1 h_0^{s-1} h_1 + {}_sC_3 h_0^{s-3} h_1^3 f_1 + \cdots)$ and $({}_sC_1 h_0^{s-1} h_1 + {}_sC_3 h_0^{s-3} h_1^3 f_1 + \cdots)(0) = 0$, because $f_1(0) = 0$. Hence $g_0^s \neq g$. Therefore, $\overline{Y}_{g,r}$ is irreducible, by Proposition 2.2. Since $g\overline{\sigma_1}^* g = \beta^r$ and $[\overline{\sigma_1}^* g] \cap [g] = \overline{\mu}^{-1}([\alpha] \cap [f_1]) = \overline{\mu}^{-1}([\alpha] \cap [\beta]) \subset \overline{\mu}^{-1}(\mathrm{Sing}(D_1))$, $\pi_{g,r}$ does not ramify on $\overline{Y_0}$. Hence $B_{\overline{\mu} \circ \pi_{g,r}} = D$. Let $h_1 = \beta/g$. Then $\overline{\sigma_1}^* g/g^{r-1} = h_1^r$. Hence there exists an automorphism $\widetilde{\sigma_1}$ of $\overline{Y}_{g,r}$ such that $\pi_{g,r} \circ \widetilde{\sigma_1} = \overline{\sigma_1} \circ \pi_{g,r}$ and that $\widetilde{\sigma_1} \tau_{g,r} = \tau_{g,r}^{-1} \widetilde{\sigma_1}$. Moreover, $\widetilde{\sigma_1}^2 = id$, because $(\widetilde{\sigma_1}^2)^* \widetilde{g} = \widetilde{\sigma_1}^* (\epsilon\beta/\widetilde{g}) = \widetilde{g}$, where ϵ is an r-th root of the unit. Hence $\mathrm{Gal}(\overline{Y}_{g,r}/Y) \simeq D_{2r}$. □

For example, $f_1 = z_1^{2q} - z_2^r$, $\alpha = z_1^q$ and $\beta = z_2$ satisfy the condition of the above proposition. Then we easily see that $\overline{Y}_{g,r}$ is non-singular, if $q = 1$. Hence if the divisors defined by $\alpha = 0$ and $\beta = 0$ cross normally at a point p in Y, then $\overline{Y}_{g,r}$ is non-singular at all points in $(\overline{\mu} \circ \pi_{g,r})^{-1}(p)$.

Let α and β be homogenous polynomials of degree r and 2, respectively. Then $f_1 = \alpha^2 - \beta^r$ is a homogenous polynomial of degree $2r$. Let $\lambda : Z \to Y$ be the blow up of Y at the origin and let $E = \lambda^{-1}(0)$. Then $[\lambda^* f_1] = \overline{D_1} + 2rE$, where $\overline{D_1}$ is the proper transformation of D_1. Hence the projection $\overline{Z} := \overline{Y} \times_Y Z \to Z$ does not ramify along E. Therefore, $\overline{Y}$ is a cone over the double covering of $E \simeq \mathbf{P}^{n-1}$ ramifying along the divisor defined by $f_1 = 0$. Moreover, $\overline{Y}$ has singularities along the inverse image of the intersection of the divisors defined by $\alpha = 0$ and $\beta = 0$. Since the vanishing order along $\overline{E} = \overline{\lambda}^{-1}(0)$ of the pull back $\overline{\lambda}^* g$ of $g = \alpha + w_1$ under the projection $\overline{\lambda} : \overline{Z} \to \overline{Y}$ is equal to r, the projection $\overline{Y}_{g,r} \times_{\overline{Y}} \overline{Z} \to \overline{Z}$ does not ramify along $\overline{E}$. Hence $\overline{Y}_{g,r}$ is also a cone over a covering space $\widetilde{E}$ of E whose covering transformation group is isomorphic to D_{2r}. If the divisors A and B of $\mathbf{P}^{n-1}$ defined by $\alpha = 0$ and $\beta = 0$, respectively, cross normally each other and the divisor defined by $\alpha^2 - \beta^r = 0$ has no singularities except those on the intersection of A and B, then $\widetilde{E}$ is non-singular and hence $\overline{Y}_{g,r}$ has only an isolated singularity.

Proposition 3.2. *Let $D = 2D_1 + 2D_2$. Assume that there exist a holomorphic function β on Y and an integer r greater than 1 with $f_1 - f_2 = \beta^r$. Then $\overline{\mu} \circ \pi_{g,r} : \overline{Y}_{g,r} \to Y$ is a Galois covering such that $B_{\overline{\mu} \circ \pi_{g,r}} = D$ and that $\mathrm{Gal}(\overline{Y}_{g,r}/Y) \simeq D_{4r}$, where $g = w_1 + w_2$.*

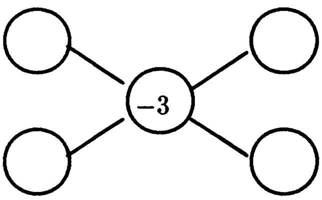
Proof. Suppose that there exists a holomorphic function g_0 on $\overline{Y}$ with $g_0^s = g$ for an integer s greater than 1. Then $g_0 = h_{00} + w_1 h_{10} + w_2 h_{01} + w_1 w_2 h_{11}$ for some holomorphic functions h_{00}, h_{10}, h_{01} and h_{11} on Y and $g_0^s = \overline{h_{00}} + w_1 \overline{h_{10}} + w_2 \overline{h_{01}} + w_1 w_2 \overline{h_{11}}$ for some holomorphic functions $\overline{h_{00}}$, $\overline{h_{10}}$, $\overline{h_{01}}$ and $\overline{h_{11}}$ on Y. Then $\overline{h_{10}}(0) = \overline{h_{01}}(0) = 0$, because h_{00}, $w_1^2 = f_1$ and $w_2^2 = f_2$ vanish at 0. Hence $g_0^s \neq g$. Therefore, $\overline{Y}_{g,r}$ is irreducible, by Proposition 2.2. Since $g\overline{\sigma_2}^* g = \beta^r$ and $[\overline{\sigma_2}^* g] \cap [g] = \overline{\mu}^{-1}(D_1 \cap D_2)$, $\pi_{g,r}$ does not ramify on $\overline{Y_0}$. Hence $B_{\overline{\mu} \circ \pi_{g,r}} = D$. Let $h_1 = \rho_{2r}\beta/g$ and let $h_2 = \beta/g$. Then $\overline{\sigma_i}^* g/g^{r-1} = h_i^r$ $(i = 1, 2)$. Among the automorphisms $\widetilde{\sigma_i}$ of $\overline{Y}_{g,r}$ satisfying $\pi_{g,r} \circ \widetilde{\sigma_i} = \overline{\sigma_i} \circ \pi_{g,r}$, we choose one so that $\widetilde{\sigma_i}^* \widetilde{g} = h_i \widetilde{g}^{r-1}$ $(i = 1, 2)$. Then we see by an easy calculation that $\widetilde{\sigma_1}^2 = \widetilde{\sigma_2}^2 = id$ and that $\widetilde{\sigma_1}\widetilde{\sigma_2}\widetilde{\sigma_1}\widetilde{\sigma_2} = \tau_{g,r}$. Therefore, $\mathrm{Gal}(\overline{Y}_{g,r}/Y) \simeq D_{4r}$. □

For example, $f_1 = z_1^q + z_2^r$ and $f_2 = z_1^q - z_2^r$ satisfy the condition of the above proposition. Then we easily see that $\overline{Y}_{g,r}$ is non-singular, if $q = 1$.

Proposition 3.3. *Let $D = 2D_1 + 2D_2$. Assume that there exists a holomorphic function β on Y with $f_1 - f_2 = \beta^2$. Then $\overline{\mu} \circ \pi_{g,2} : \overline{Y}_{g,2} \to Y$ is a Galois covering such that $B_{\overline{\mu}\circ\pi_{g,2}} = 4D_1 + 4D_2$ and that $\mathrm{Gal}(\overline{Y}_{g,2}/Y)$ is isomorphic to the quaternion group, where $g = w_1 w_2 (w_1 + w_2)$.*

Proof. Since $g\overline{\sigma_1}^* g = w_1^2 w_2^2 \beta^2$ and $g\overline{\sigma_2}^* g = -w_1^2 w_2^2 \beta^2$, there exist automorphisms $\widetilde{\sigma_i}$ of $\overline{Y}_{g,2}$ satisfying $\pi_{g,2} \circ \widetilde{\sigma_i} = \overline{\sigma_i} \circ \pi_{g,2}$ and $\widetilde{\sigma_i}\tau_{g,2} = \tau_{g,2}\widetilde{\sigma_i}$ $(i = 1, 2)$. If we choose one among those automorphisms for each $i = 1, 2$ so that $\widetilde{\sigma_1}^* \widetilde{g} = w_1 w_2 \beta / \widetilde{g}$ and that $\widetilde{\sigma_2}^* \widetilde{g} = \sqrt{-1} w_1 w_2 \beta / \widetilde{g}$, then $(\widetilde{\sigma_i}^*)^2 \widetilde{g} = -\widetilde{g}$, $\widetilde{\sigma_1}^* \widetilde{\sigma_2}^* \widetilde{g} = -\sqrt{-1}\widetilde{g}$ and $\widetilde{\sigma_2}^* \widetilde{\sigma_1}^* \widetilde{g} = \sqrt{-1}\widetilde{g}$. Hence $\widetilde{\sigma_i}^2 = \tau_{g,2}$ and $\widetilde{\sigma_1}\widetilde{\sigma_2} = \tau_{g,2}\widetilde{\sigma_2}\widetilde{\sigma_1}$. On the other hand, $\pi_{g,2}$ does not ramify along $[w_1 + w_2]$, because $(w_1 + w_2)(w_1 - w_2) = \beta^2$. While, $\pi_{g,2}$ ramifies along $[w_1] \cup [w_2]$. Hence $\widetilde{Y}_{g,2}$ is irreducible and $B_\pi = 4D_1 + 4D_2$. □

For example, $f_1 = z_1$ and $f_2 = z_1 - z_2^2$ satisfy the condition of the above proposition. Then the dual graph of the exceptional set of the minimal resolution of $\overline{Y}_{g,2}$ is as follows:



We can show the following proposition, in the manner similar to the proof of the above propositions.

Proposition 3.4. *Let $D = 2D_1 + r_2 D_2$, where r_2 is an odd integer greater than 2. Assume that there exist a holomorphic function α and an odd integer $q > 0$ with $f_1 + f_2^q = \alpha^2$. Let $g = \alpha + w_1$. Then $\overline{Y}_{g,qr_2} \to Y$ is a Galois covering such that $B_{\overline{\mu}\circ\pi_{g,qr_2}} = D$ and that $\mathrm{Gal}(\overline{Y}_{g,qr_2}/Y)$ is generated by three elements σ_1, σ_2 and τ enjoying the relations $\sigma_1^2 = \sigma_2^{r_2} = \tau^{qr_2} = e$, $\sigma_1\sigma_2 = \tau^q \sigma_2 \sigma_1$, $\sigma_1 \tau = \tau^{-1}\sigma_1$, $\sigma_2 \tau = \tau \sigma_2$.*

For example, $f_1 = z_1^2 - z_2^q$ and $f_2 = z_2$ satisfy the condition of the above proposition. When $q = 1$, $\langle \sigma_2 \tau^s \rangle$ is a normal subgroup of $\mathrm{Gal}(\overline{Y}_{g,r_2}/Y)$ and $\mathrm{Gal}(\overline{Y}_{g,r_2}/Y)/\langle \sigma_2 \tau^s \rangle \simeq D_{2r_2}$ for an integer s with $2s \equiv 1 (\mathrm{mod}\ r_2)$. Hence $\overline{Y}_{g,r_2}/\langle \sigma_2 \tau^s \rangle$ is a Galois covering of Y whose covering transformation group is isomorphic to D_{2r_2}.

Next, we construct examples whose covering transformation groups are isomorphic to the group $H_{q,r} = \langle \sigma, \tau_1, \dots, \tau_{q-1} \rangle$ generated by q elements σ, τ_1, … and τ_{q-1} enjoying the relations $\sigma^q = \tau_i^r = e$, $\tau_i \tau_j = \tau_j \tau_i$ $(1 \leq i, j \leq q-1)$, $\sigma \tau_1 = \tau_{q-1}^{-1} \sigma$, $\sigma \tau_i = \tau_{i-1} \tau_{q-1}^{-1} \sigma$ $(2 \leq i \leq q-1)$. Here we assume that $g.c.d.(q, r) = 1$. Then $[H_{q,r}, H_{q,r}] = \langle \tau_1, \dots, \tau_{q-1} \rangle$. Note that $H_{3,2}$ is isomorphic to the alternating group of degree 4.

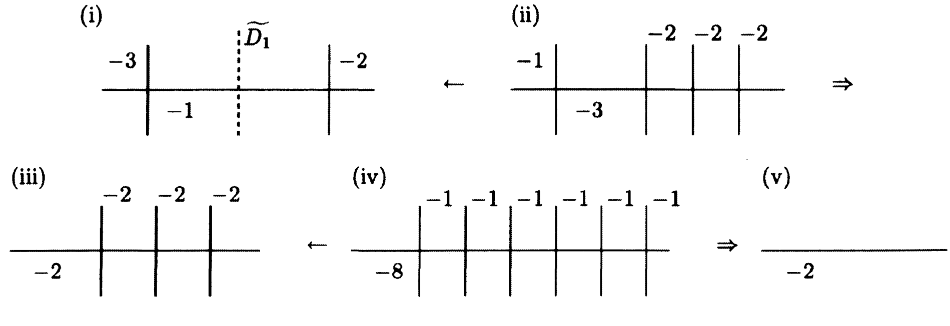
Proposition 3.5. *Let $D = qD_1$. Assume that there exist holomorphic functions α and β on Y such that $\alpha(0) = \beta(0) = 0$ and that $f_1 = \alpha^q - \beta^r$. Let $\mathbf{g} = \{g_1, \dots, g_{q-1}\}$, where $g_i = \alpha - \rho_q^{i-1} w_1$ and let $\mathbf{r} = \{\overbrace{r, \dots, r}^{q-1}\}$. If $\overline{Y}_{\mathbf{g},\mathbf{r}}$ is irreducible, then $\overline{\mu} \circ \pi_{\mathbf{g},\mathbf{r}} : \overline{Y}_{\mathbf{g},\mathbf{r}} \to Y$ is a Galois covering such that $B_{\overline{\mu} \circ \pi_{\mathbf{g},\mathbf{r}}} = D$ and that $\mathrm{Gal}(\overline{Y}_{\mathbf{g},\mathbf{r}}/Y) \simeq H_{q,r}$.*

Proof. Since $\overline{\sigma_1}^* g_i = g_{i+1}$ $(1 \leq i < q-1)$ and $\frac{\overline{\sigma_1}^* g_{q-1}}{g_1^{r-1} \cdots g_{q-1}^{r-1}} = \left(\frac{\beta}{g_1 \cdots g_{q-1}} \right)^r$, there exist r^{q-1} automorphisms $\widetilde{\sigma_1}$ of $\overline{Y}_{\mathbf{g},\mathbf{r}}$ such that $\pi_{\mathbf{g},\mathbf{r}} \circ \widetilde{\sigma_1} = \overline{\sigma_1} \circ \pi_{\mathbf{g},\mathbf{r}}$, that $\widetilde{\sigma_1} \tau_{\mathbf{g},\mathbf{r}}^1 = (\tau_{\mathbf{g},\mathbf{r}}^{q-1})^{-1} \widetilde{\sigma_1}$, and that $\widetilde{\sigma_1} \tau_{\mathbf{g},\mathbf{r}}^i = \tau_{\mathbf{g},\mathbf{r}}^{i-1} (\tau_{\mathbf{g},\mathbf{r}}^{q-1})^{-1} \widetilde{\sigma_1}$ $(2 \leq i \leq q-1)$, by Corollary 2.1. Among those, we can choose one so that $\widetilde{\sigma_1}^* \widetilde{g_i} = \widetilde{g_{i+1}}$ $(1 \leq i < q-1)$ and that $\widetilde{\sigma_1}^* \widetilde{g_{q-1}} = \frac{\beta}{\widetilde{g_1 \cdots g_{q-1}}}$. Then we see by an easy calculation that $\widetilde{\sigma_1}^q = id$. Hence $\langle \widetilde{\sigma_1}, \tau_{\mathbf{g},\mathbf{r}}^1, \dots, \tau_{\mathbf{g},\mathbf{r}}^{q-1} \rangle \simeq H_{q,r}$. On the other hand, $\pi_{\mathbf{g},\mathbf{r}}$ ramifies only along $[\overline{\mu}^* \alpha] \cap [\overline{\mu}^* \beta]$ $(\subset \overline{\mu}^{-1}(\mathrm{Sing}(D_1)))$, because $(\alpha - w_1)(\alpha - \rho_q w_1) \cdots (\alpha - \rho_q^{q-1} w_1) = \beta^r$. □

We see by the following example that if the divisors defined by $\alpha = 0$ and $\beta = 0$ cross normally at a point of Y, then $\overline{Y}_{\mathbf{g},\mathbf{r}}$ is irreducible.

EXAMPLE 3.1. We consider the case that $n = 2$, $\alpha = z_1$ and $\beta = z_2$. When $q = 3$ and $r = 2$, there exists an embedded resolution $Z \to Y$ of D_1 whose exceptional set is as (i) in the following picture. Then the projection $Z \times_Y \overline{Y} \to Z$ ramifies along thick lines in (i). Hence the exceptional set of $Z \times_Y \overline{Y}$ is as (ii) and we obtain a resolution $\overline{Z}$ of $\overline{Y}$ as (iii) contracting the exceptional curve of the first kind in $Z \times_Y \overline{Y}$. Then the projection $\overline{Z} \times_{\overline{Y}} \overline{Y}_{\mathbf{g},\mathbf{r}} \to \overline{Z}$ ramifies along thick lines in (iii), because the vanishing order of g_1, g_2 along the thin (resp. thick) lines are equal to 2 (resp. greater than 0) and that of $g_1 g_2 \widetilde{\sigma}^* g_2 = z_2^2$ along the thick lines is equal to 4. Thus we obtain a resolution of $\overline{Y}_{\mathbf{g},\mathbf{r}}$ as in (v) contracting the exceptional curves of the first kind in $\overline{Z} \times_{\overline{Y}} \overline{Y}_{\mathbf{g},\mathbf{r}}$. Here we note that the curve in (v) is rational, because $2 - 2g = 2 \cdot 4 - 3 \cdot \frac{1}{2} \cdot 4 = 2$.

In general, $\overline{Y}_{\mathbf{g},\mathbf{r}}$ is irreducible and isomorphic to the singularity obtained by contracting a non-singular curve E of the genus $1 + \frac{qr - 2r - q}{2} r^{q-2}$ with the self intersection number



$-r^{q-2}$. We can obtain the genus of E using Riemann-Hurwitz formula $2-2g = 2r^{q-1} - q\frac{r-1}{r}r^{q-1}$. For the calculation of the self-intersection number $-E^2$ of E, see the proof of Theorem 5.2 in §5. Let W be a small open neighborhood of 0 in $\mathbf{C}^2$ and let $W \to Y$ be the holomorphic map sending (x_1, x_2) to (x_1^a, x_2^b) for positive integers a and b. Then the normalization $\widetilde{W}$ of $W \times_Y \overline{Y}_{\mathbf{g},\mathbf{r}}$ is irreducible, because $\overline{\mu} \circ \pi_{g,r} : \overline{Y}_{g,r} \to Y$ does not ramify along $[z_1]$ and $[z_2]$. Hence the projection $\pi : \widetilde{W} \to W$ to W is a Galois covering such that $\mathrm{Gal}(\widetilde{W}/W) \simeq H_{q,r}$ and that $B_\pi = q\{x_1^{aq} - x_2^{br} = 0\}$.

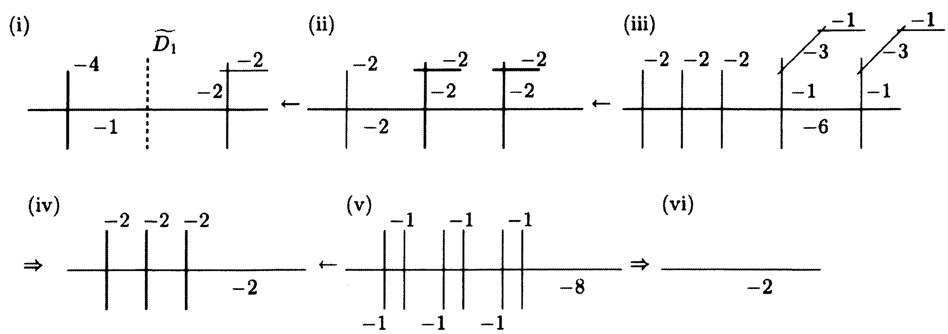
Finally, we construct examples whose covering transformation groups are isomorphic to the symmetric group S_4 of degree 4.

Proposition 3.6. *Let $D = 2D_1$. Assume that there exist holomorphic functions α, β, γ and δ on Y such that $\alpha(0) = \beta(0) = 0$, that $\delta \not\equiv 0$, that $f_1 = \alpha^2 - \beta^3$ and that $\delta^2 = \gamma^3 + 2\alpha - 3\beta\gamma$. Let $g = \alpha - w_1$, let $\mathbf{h} = \{h_1, h_2\}$, where $h_1 = \gamma + \widetilde{g} + \beta/\widetilde{g}$ and $h_2 = \gamma + \rho_3\widetilde{g} + \rho_3^2\beta/\widetilde{g}$, and let $\mathbf{s} = \{2, 2\}$. Then h_1, h_2 are holomorphic functions on $\overline{Y}_{g,3}$ and if $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is irreducible, then $\overline{\mu} \circ \pi_{g,3} \circ \pi_{\mathbf{h},\mathbf{s}} : (\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}} \to Y$ is a Galois covering such that $B_{\overline{\mu}\circ\pi_{g,3}\circ\pi_{\mathbf{h},\mathbf{s}}} = D$ and that $\mathrm{Gal}((\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}/Y) \simeq S_4$.*

Proof. Since $\overline{\sigma_1}^* g = \dfrac{\beta^3}{g} = \left(\dfrac{\beta}{\widetilde{g}}\right)^3$, $\dfrac{\beta}{\widetilde{g}}$ is a holomorphic function on $\overline{Y}_{g,3}$ and $\overline{Y}_{g,3}$ is a Galois covering of Y whose covering transformation group is generated by two elements $\tau_{g,3}$ and $\widetilde{\sigma_1}$ satisfying $\pi_{g,3} \circ \widetilde{\sigma_1} = \overline{\sigma_1} \circ \pi_{g,3}$ and $\widetilde{\sigma_1}\tau_{g,3} = \tau_{g,3}^2\widetilde{\sigma_1}$. Here we may assume that $\widetilde{\sigma_1}^*\widetilde{g} = \beta/\widetilde{g}$. Then $\widetilde{\sigma_1}^* h_1 = h_1$, $\tau_{g,3}^* h_1 = h_2$ and $\widetilde{\sigma_1}^* h_2 = \tau_{g,3}^* h_2 = \left(\frac{\delta}{h_1 h_2}\right)^2 h_1 h_2$, because $h_1 h_2 \tau_{g,3}^* h_2 = \gamma^3 + \widetilde{g}^3 + (\beta/\widetilde{g})^3 - 3\gamma\widetilde{g}(\beta/\widetilde{g}) = \gamma^3 + (\alpha - w_1) + (\alpha + w_1) - 3\beta\gamma = \delta^2$. Hence $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is a Galois covering of Y. Moreover, we easily see that $\mathrm{Gal}((\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}/Y)$ is isomorphic to the group $\langle\sigma, \tau, \lambda_1, \lambda_2\rangle$ generated by σ, τ, λ_1 and λ_2 enjoying the relations $\sigma^2 = \tau^3 = \lambda_1^2 = \lambda_2^2 = e$, $\sigma\tau = \tau^2\sigma$, $\sigma\lambda_1 = \lambda_1\lambda_2\sigma$, $\sigma\lambda_2 = \lambda_2\sigma$, $\tau\lambda_1 = \lambda_2\tau$, $\tau\lambda_2 = \lambda_1\lambda_2\tau$ and $\lambda_1\lambda_2 = \lambda_2\lambda_1$. On the other hand, $\pi_{g,3}$ ramifies only along $[\overline{\mu}^*\alpha] \cap [\overline{\mu}^*\beta]$ ($\subset \overline{\mu}^{-1}(\mathrm{Sing}(D_1))$), and $\pi_{\mathbf{h},\mathbf{s}}$ ramifies only along

$\{\pi_{g,3}^*\overline{\mu}^*\gamma = \widetilde{g} = \beta/\widetilde{g}\} \cup \{\pi_{g,3}^*\overline{\mu}^*\gamma = \rho_3\widetilde{g} = \rho_3^2\beta/\widetilde{g}\} \cup \{\pi_{g,3}^*\overline{\mu}^*\gamma = \rho_3^2\widetilde{g} = \rho_3\beta/\widetilde{g}\}$ which is contained in $[\pi_{g,3}^*\overline{\mu}^* f_1] \cap [\pi_{g,3}^*\overline{\mu}^*(\alpha - \gamma^3)] \cap [\delta]$. □

EXAMPLE 3.2. When $n = 2$, $\alpha = z_1^2$, $\beta = z_2$, $\gamma = 0$ and $\delta = \sqrt{2}z_1$ satisfy the relation $\delta^2 = \gamma^3 + 2\alpha - 3\beta\gamma$. There exists an embedded resolution $Z \to Y$ of D_1 whose exceptional set is as (i) in the following picture. Then $Z \times_Y \overline{Y} \to Z$ ramifies along thick lines. Hence there exists a resolution $\overline{Z}$ of $\overline{Y}$ whose exceptional set is as (ii). Then $\overline{Z} \times_{\overline{Y}} \overline{Y}_{g,3} \to \overline{Z}$ ramifies along thick lines in (ii). Hence contracting the exceptional curves of the first kind in (blowing up of $\overline{Z}$ at two points) $\times_{\overline{Y}} \overline{Y}_{g,3}$ (iii), we obtain a resolution $\widetilde{Z}$ of $\overline{Y}_{g,3}$ as (iv). Then $\widetilde{Z} \times_{\overline{Y}_{g,3}} (\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}} \to \widetilde{Z}$ ramifies along thick lines in (iv). Thus we obtain a resolution of $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ as (vi) contracting the exceptional curves of the first kind in $\widetilde{Z} \times_{\overline{Y}_{g,3}} (\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ (v).



Let W be a small open neighborhood of 0 in $\mathbf{C}^2$ and let $W \to Y$ be the holomorphic map sending (x_1, x_2) to (x_1^a, x_2^b) for positive integers a and b. Then the normalization $\widetilde{W}$ of $W \times_Y (\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is irreducible. Hence the projection $\pi : \widetilde{W} \to W$ to W is a Galois covering such that $\mathrm{Gal}(\widetilde{W}/W) \simeq S_4$ and that $B_\pi = 2\{x_1^{4a} - x_2^{3b} = 0\}$.

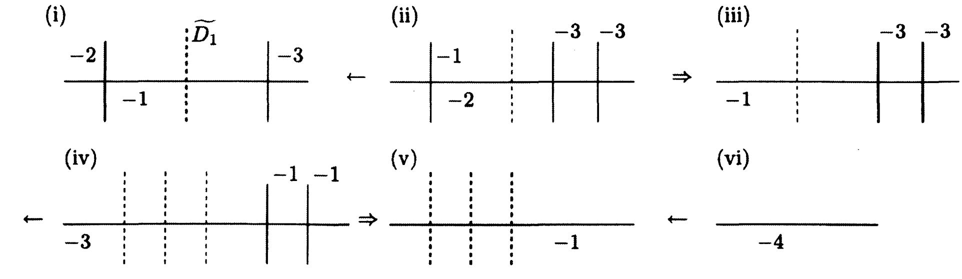
EXAMPLE 3.3. When $n = 4$, $\alpha = 2^5(2z_2^3 - 9z_1z_2z_3 + 27z_1^2z_4^a + 27z_3^2 - 72z_2z_4^a)$, $\beta = 2^4(z_2^2 - 3z_1z_3 + 12z_4^a)$, $\gamma = 3z_1^2 - 8z_2$ and $\delta = 3\sqrt{3}(z_1^3 - 4z_1z_2 + 8z_3)$ satisfy the relation $\delta^2 = \gamma^3 + 2\alpha - 3\beta\gamma$ for any positive integer a. Let X be the subvariety of $\mathbf{C}^5$ defined by $x_1x_2x_3x_4 - x_5^a = 0$ and let $\varphi : X \to \mathbf{C}^8$ be the holomorphic map sending $(x_1, x_2, x_3, x_4, x_5)$ to $(z_1, z_2, z_3, z_4, w_1, v_1, u_1, u_2)$, where $z_1 = x_1 + x_2 + x_3 + x_4$, $z_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$, $z_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$, $z_4 = x_5$, $w_1 = 2^5 3\sqrt{-3}\left(\prod_{\sigma \in A_4} x_{\sigma(1)}^3 x_{\sigma(2)}^2 x_{\sigma(3)} - \prod_{\sigma \in A_4} x_{\sigma(1)}^3 x_{\sigma(3)}^2 x_{\sigma(2)}\right)$, $v_1 = 4\left((x_1x_2 + x_3x_4) + \rho_3(x_1x_3 + x_2x_4) + \rho_3^2(x_1x_4 + x_2x_3)\right)$, $u_1 = \sqrt{3}(x_1 + x_2 - x_3 - x_4)$ and $u_2 = \sqrt{3}(x_1 - x_2 - x_3 + x_4)$. Then φ is one-to-one on a Zariski open set of X and its image is the closure of the subvariety of $\mathbf{C}^8 \setminus \{v_1 = 0\}$ defined by $w_1^2 - (\alpha^2 - \beta^3) =$

$v_1^3 - (\alpha + w_1) = u_1^2 - (\gamma + v_1 + \frac{\beta}{v_1}) = u_2^2 - (\gamma + \rho_3 v_1 + \frac{\beta}{\rho_3 v_1}) = 0$ for the above α, β, γ and δ. Hence $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is irreducible and isomorphic to X.

Proposition 3.7. *Let $D = 2D_1$. Assume that there exist holomorphic functions α and β on Y such that $\alpha(0) = \beta(0) = 0$ and that $f_1 = \alpha^2 - \beta^3$. Let $g = \alpha + w_1$, let $\mathbf{h} = \{h_1, h_2\}$, where $h_1 = w_1(\widetilde{g} - \beta/\widetilde{g})$ and $h_2 = w_1(\rho_3\widetilde{g} - \rho_3^2\beta/\widetilde{g})$, and let $\mathbf{s} = \{2, 2\}$. Then h_1, h_2 are holomorphic functions on $\overline{Y}_{g,3}$ and $\overline{\mu} \circ \pi_{g,3} \circ \pi_{\mathbf{h},\mathbf{s}} : (\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}} \to Y$ is a Galois covering such that $B_{\overline{\mu} \circ \pi_{g,3} \circ \pi_{\mathbf{h},\mathbf{s}}} = 4D_1$ and that* $\mathrm{Gal}((\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}/Y) \simeq S_4$.

Proof. We see in the same way as in the proof of Proposition 3.6 that h_1 and h_2 are holomorphic functions on $\overline{Y}_{g,3}$ and $\overline{Y}_{g,3}$ is a Galois covering of Y whose covering transformation group is generated by two elements $\tau_{g,3}$ and $\widetilde{\sigma_1}$ satisfying $\pi_{g,3} \circ \widetilde{\sigma_1} = \overline{\sigma_1} \circ \pi_{g,3}$, $\widetilde{\sigma_1}\tau_{g,3} = \tau_{g,3}^2\widetilde{\sigma_1}$ and $\widetilde{\sigma_1}^*\widetilde{g} = \beta/\widetilde{g}$. Then $\widetilde{\sigma_1}^* h_1 = h_1$, $\tau_{g,3}^* h_1 = h_2$ and $\widetilde{\sigma_1}^* h_2 = \tau_{g,3}^* h_2 = \left(\frac{\sqrt{2}f_1}{h_1 h_2}\right)^2 h_1 h_2$, because $h_1 h_2 \tau_{g,3}^* h_2 = w_1^3(\widetilde{g}^3 - (\beta/\widetilde{g})^3) = w_1^3((\alpha + w_1) - (\alpha - w_1)) = 2w_1^4$. Hence $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is a Galois covering of Y, if it is irreducible. Moreover, we see in the same way as in the proof of Proposition 3.6 that the covering transformation group is isomorphic to S_4 and $\pi_{g,3}$ ramifies only along $[\overline{\mu}^*\alpha] \cap [\overline{\mu}^*\beta]$ ($\subset \overline{\mu}^{-1}(\mathrm{Sing}(D_1))$). On the other hand, $\pi_{\mathbf{h},\mathbf{s}}$ ramifies only along $(\overline{\mu} \circ \pi_{g,3})^{-1}(D_1)$, because $h_1 h_2 \tau_{g,3}^* h_2 = 2w_1^4$. Let D' be an irreducible component of D_1. Then the number m of the irreducible components C_1, ... and C_m of $(\overline{\mu} \circ \pi_{g,3})^{-1}(D')$ is equal to 1 or 3. If $m = 1$, then the vanishing order of h_1 along C_1 is equal to $\frac{4}{3}$. Let c_i be the vanishing order of h_1 along C_i. Then the vanishing order of $2w_1^4 = h_1 h_2 \tau_{g,3}^* h_2$ along C_i is equal to $4 = c_1 + c_2 + c_3$, because $\tau_{g,3}C_1 = C_2$, $\tau_{g,3}C_2 = C_3$ or $\tau_{g,3}C_1 = C_3$, $\tau_{g,3}C_3 = C_2$. Hence two of c_1, c_2 and c_3 are equal to 1. Therefore, $\pi_{\mathbf{h},\mathbf{s}}$ ramifies along $C_1 + C_2 + C_3$ with the ramification index 2 and $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is irreducible. □

EXAMPLE 3.4. When $n = 2$, $\alpha = z_1$ and $\beta = z_2$, we see as in Example 3.2 that $(\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is isomorphic to the singularity obtained by contracting a non-singular rational curve with the self-intersection number -4.



Let W be a small open neighborhood of 0 in $\mathbf{C}^2$ and let $W \to Y$ be the holomorphic map sending (x_1, x_2) to (x_1^a, x_2^b) for positive integers a and b. Then the normalization $\widetilde{W}$ of $W \times_Y (\overline{Y}_{g,3})_{\mathbf{h},\mathbf{s}}$ is irreducible. Hence the projection $\pi : \widetilde{W} \to W$ to W is a Galois covering such that $\mathrm{Gal}(\widetilde{W}/W) \simeq S_4$ and that $B_\pi = 4\{x_1^{2a} - x_2^{3b} = 0\}$.

4. Galois coverings whose covering transformation groups are isomorphic to D_{2s}, Q, A_4 or S_4

Let $\pi : X \to Y$ be a Galois covering of a normal Stein analytic space Y.

Theorem 4.1. *Let r be an odd integer greater than 1. If $\mathrm{Gal}(X/Y) \simeq D_{2r}$ then there exist holomorphic functions α and β on Y such that $X \simeq (Y_{f,2})_{g,r}$, where $f = \alpha^2 - \beta^r$ and $g = \alpha + \widetilde{f}$. Moreover, if Y is non-singular and if B_π is irreducible, then there exists a holomorphic function γ on Y such that $f = \gamma^2 f_1$, where f_1 is a defining equation of $(B_\pi)_{red}$.*

Proof. $\mathrm{Gal}(X/Y)$ is generated by two elements σ and τ enjoying the relations $\sigma^2 = \tau^r = e$ and $\sigma\tau = \tau^{-1}\sigma$. Then there exists a holomorphic function $\widetilde{g_1}$ on X such that $\tau^*\widetilde{g_1} = \rho_r\widetilde{g_1}$ and that $X \simeq (X/\langle\tau\rangle)_{\widetilde{g_1}^r, r}$, by Proposition 2.1. Let $\widetilde{g_2} = \sigma^*\widetilde{g_1}$. Then $\sigma^*\widetilde{g_2} = \widetilde{g_1}$ and $\tau^*\widetilde{g_2} = \tau^*\sigma^*\widetilde{g_1} = \sigma^*(\tau^{-1})^*\widetilde{g_1} = \rho_r^{-1}\widetilde{g_2}$. Hence $\alpha = \frac{1}{2}(\widetilde{g_1}^r + \widetilde{g_2}^r)$ and $\beta = \widetilde{g_1}\widetilde{g_2}$ are $\mathrm{Gal}(X/Y)$-invariant. Let $\widetilde{f} = \frac{1}{2}(\widetilde{g_1}^r - \widetilde{g_2}^r)$. Then $\widetilde{f}^2 = \alpha^2 - \beta^r$ and $\widetilde{f} \not\equiv 0$. Otherwise, $\widetilde{g_1}/\widetilde{g_2}$ is a non-zero constant. However, it contradicts the fact that $\tau^*(\widetilde{g_1}/\widetilde{g_2}) = \rho_r^2(\widetilde{g_1}/\widetilde{g_2})$. Since $\tau^*\widetilde{f} = \widetilde{f}$ and $\sigma^*\widetilde{f} = -\widetilde{f}$, $X/\langle\tau\rangle \simeq Y_{\widetilde{f}^2,2}$.

If Y is non-singular and if $(B_\pi)_{red}$ is an irreducible divisor defined by $f_1 = 0$, then $X/\langle\tau\rangle$ is isomorphic to the hypersurface of $\mathbf{C} \times Y$ defined by $w_1^2 - f_1 = 0$. Hence $\gamma = \widetilde{f}/w_1$ is a $(\mathrm{Gal}(X/Y)/\langle\tau\rangle)$-invariant holomorphic function on $X/\langle\tau\rangle$. □

Since $g\tau_{f,2}^* g = (\alpha + \widetilde{f})(\alpha - \widetilde{f}) = \beta^r$, $\pi_{g,r}$ ramifies only along $[\pi_{f,2}^*\alpha] \cap [\widetilde{f}] = [\pi_{f,2}^*\alpha] \cap [\pi_{f,2}^*\beta]$. Let D be an irreducible divisor on Y contained in $[\alpha] \cap [\beta]$, let a and b be the vanishing order along D of α and β, respectively. When $2a < br$, the vanishing order of f along D is equal to $2a$. Hence then $\pi_{f,2}$ does not ramify along D. Moreover, the vanishing orders along $\pi_{f,2}^{-1}(D)$ of g and $\tau_{f,2}^* g$ are not smaller than a and at least one of them is equal to a. Hence $\pi_{g,r}$ ramifies along $\pi_{f,2}^{-1}(D)$ with the index $\frac{r}{g.c.d.(a,r)}$. When $2a > br$, $\pi_{f,2}$ ramifies along D with the index $\frac{2}{g.c.d.(2,br)}$ and $\pi_{g,r}$ does not ramify along $\pi_{f,2}^{-1}(D)$. In the case of the example $\overline{Y}_{g,r_2}/\langle\sigma_2\tau^s\rangle$ following Proposition 3.4 , $\alpha = z_1 z_2^{\frac{r-1}{2}}$ and $\beta = z_2$ satisfy the condition of the above theorem and $[\overline{Y}_{g,r_2}/\langle\sigma_2\tau^s\rangle \to Y]$ ramifies along the divisors defined by $z_1^2 - z_2 = 0$ and $z_2 = 0$ with the indices 2 and r, respectively.

In the manner similar to the proof of the above theorem, we can show the following.

Theorem 4.2. *If* $\mathrm{Gal}(X/Y) \simeq D_{4r}$, *then there exist holomorphic functions* f_1, f_2 *and* β *on* Y *such that* $f_1 - f_2 = \beta^r$ *and that* $X \simeq (Y_{\{f_1,f_2\},\{2,2\}})_{g,r}$, *where* $g = \widetilde{f_1} + \widetilde{f_2}$.

Since $g\left(\tau^2_{\{f_1,f_2\},\{2,2\}}\right)^* g = (\widetilde{f_1} + \widetilde{f_2})(\widetilde{f_1} - \widetilde{f_2}) = f_1 - f_2 = \beta^r$, $\pi_{g,r}$ ramifies only along $[\widetilde{f_1}] \cap [\widetilde{f_2}]$. Let D be an irreducible divisor on Y contained in $[f_1] \cap [f_2]$, let a_1 and a_2 be the vanishing orders along D of f_1 and f_2, respectively. If $a_1 < a_2$, then a_1 is a multiple of r, because $f_1 - f_2 = \beta^r$. Assume that $a_1 \le a_2$. If at least one of a_1 and a_2 is odd, then $\pi_{\{f_1,f_2\},\{2,2\}}$ ramifies along D and $\pi_{g,r}$ ramifies along $\pi^{-1}_{\{f_1,f_2\},\{2,2\}}(D)$ with the index $\frac{r}{g.c.d.(a_1,r)}$. While, if a_1 and a_2 are both even, then $\pi_{\{f_1,f_2\},\{2,2\}}$ does not ramify along D and $\pi_{g,r}$ ramifies along $\pi^{-1}_{\{f_1,f_2\},\{2,2\}}(D)$ with the index $\frac{r}{g.c.d.(a_1/2,r)}$. For example, if $Y = \mathbf{C}^2$, if $f_1 = z_1 z_2^2$ and if $f_2 = (z_1 - z_2^{r-2})z_2^2$, then π ramifies along the divisors defined by $z_1 = 0$, $z_1 - z_2^{r-2} = 0$ and $z_2 = 0$ with the indices 2, 2 and r, respectively.

Theorem 4.3. *If* $\mathrm{Gal}(X/Y) \simeq Q$, *then there exist holomorphic functions* f_1, f_2 *and* β *on* Y *such that* $\beta^2 = f_1 f_2 (f_2 - f_1)$ *and that* $X \simeq \left(Y_{\{f_1,f_2\},\{2,2\}}\right)_{g,2}$, *where* $g = \widetilde{f_1} + \widetilde{f_2}$.

Proof. $\mathrm{Gal}(X/Y)$ is generated by two elements σ and τ enjoying the relations $\sigma^4 = e$, $\tau^2 = \sigma^2$ and $\sigma\tau = \tau\sigma^3$. There exists a holomorphic function h_1 such that $\sigma^* h_1 = \sqrt{-1}h_1$. Let $h_2 = \tau^* h_1$. Then $\sigma^* h_2 = -\sqrt{-1}h_2$, $\tau^* h_2 = -h_1$. Hence $\beta = \frac{1}{4}h_1 h_2 (h_1^4 - h_2^4)$ is a $\mathrm{Gal}(X/Y)$-invariant holomorphic function on X. Let $\widetilde{g} = \frac{1}{\sqrt{2}}(h_1 + h_2)$. Then $(\sigma^2)^* \widetilde{g} = -\widetilde{g}$. Hence $X \simeq (X/\langle\sigma^2\rangle)_{\widetilde{g}^2,2}$. Let $\widetilde{f_1} = h_1 h_2$ and let $\widetilde{f_2} = \frac{1}{2}(h_1^2 + h_2^2)$. Then $\sigma^* \widetilde{f_1} = \widetilde{f_1}$, $\tau^* \widetilde{f_1} = -\widetilde{f_1}$, $\sigma^* \widetilde{f_2} = -\widetilde{f_2}$ and $\tau^* \widetilde{f_2} = \widetilde{f_2}$. Hence $(X/\langle\sigma^2\rangle) \simeq Y_{\{\widetilde{f_1}^2,\widetilde{f_2}^2\},\{2,2\}}$. On the other hand, $\widetilde{g}^2 = \widetilde{f_1} + \widetilde{f_2}$ and $\beta^2 = \widetilde{f_1}^2 \widetilde{f_2}^2 (\widetilde{f_2}^2 - \widetilde{f_1}^2)$. □

Since $g\left(\tau^1_{\{f_1,f_2\},\{2,2\}}\right)^* g = f_2 - f_1 = \left(\frac{\beta}{\widetilde{f_1}\widetilde{f_2}}\right)^2$, $\pi_{g,2}$ ramifies only along $[\widetilde{f_1}] \cap [\widetilde{f_2}]$. Let D be an irreducible divisor on Y contained in $[f_1] \cap [f_2]$, let a_1 and a_2 be the vanishing orders along D of f_1 and f_2, respectively. If $a_1 < a_2$, then a_2 is even, because $f_1 f_2 (f_1 - f_2) = \beta^2$. Assume that $a_1 \le a_2$. If a_1 is odd, then $\pi_{\{f_1,f_2\},\{2,2\}}$ and $\pi_{g,2}$ ramifies along D and $\pi^{-1}_{\{f_1,f_2\},\{2,2\}}(D)$, respectively, i.e., $\pi_{\{f_1,f_2\},\{2,2\}} \circ \pi_{g,2}$ ramifies along D with the index 4. While, if a_1 is even, then $\pi_{\{f_1,f_2\},\{2,2\}}$ does not ramify along D and $\pi_{g,2}$ does not ramify or ramifies along $\pi^{-1}_{\{f_1,f_2\},\{2,2\}}(D)$, accordingly as a_1 is a multiple of 4 or not. For example, if $Y = \mathbf{C}^3$, if $f_1 = z_1^2 z_2 (z_1 - z_2) z_3^2$ and if $f_2 = z_1 z_2^2 (z_1 - z_2) z_3^2$, then $\pi_{\{f_1,f_2\},\{2,2\}} \circ \pi_{g,2}$ ramifies along the divisors defined by $z_1 = 0$, $z_2 = 0$, $z_1 - z_2 = 0$ and $z_3 = 0$ with the indices 4, 4, 4 and 2, respectively.

Theorem 4.4. *If* $\mathrm{Gal}(X/Y) \simeq A_4$, *then there exist holomorphic functions* f_1,

f_2, α, β and γ on Y such that $\beta^2 = \alpha^3 + f_1 + f_2 - 3\alpha\gamma$, that $\gamma^3 = f_1 f_2$ and that $X \simeq (Y_{f_1,3})_{\{g_1,g_2\},\{2,2\}}$, where $g_1 = \alpha + \widetilde{f_1} + \gamma/\widetilde{f_1}$ and $g_2 = \alpha + \rho_3\widetilde{f_1} + \rho_3^2\gamma/\widetilde{f_1}$.

Proof. $\mathrm{Gal}(X/Y)$ is generated by three elements τ, λ_1 and λ_2 enjoying the relations $\tau^3 = \lambda_1^2 = \lambda_2^2 = e$, $\tau\lambda_1 = \lambda_2\tau$, $\tau\lambda_2 = \lambda_1\lambda_2\tau$ and $\lambda_1\lambda_2 = \lambda_2\lambda_1$. Let $\widetilde{g_1} = \widetilde{g_0} + \lambda_2^*\widetilde{g_0} - \lambda_1^*\widetilde{g_0} - \lambda_1^*\lambda_2^*\widetilde{g_0}$ for a suitable holomorphic function $\widetilde{g_0}$ on X. Then $\widetilde{g_1} \not\equiv 0$, $\lambda_1^*\widetilde{g_1} = -\widetilde{g_1}$ and $\lambda_2^*\widetilde{g_1} = \widetilde{g_1}$. Let $\widetilde{g_2} = \tau^*\widetilde{g_1}$ and let $\widetilde{g_3} = \tau^*\widetilde{g_2}$. Then $\lambda_1^*\widetilde{g_2} = \widetilde{g_2}$, $\lambda_2^*\widetilde{g_2} = -\widetilde{g_2}$ and $\lambda_1^*\widetilde{g_3} = \lambda_2^*\widetilde{g_3} = -\widetilde{g_3}$. Hence $X \simeq (X/\langle\lambda_1,\lambda_2\rangle)_{\{\widetilde{g_1}^2,\widetilde{g_2}^2\},\{2,2\}}$ and $\alpha = \frac{1}{3}(\widetilde{g_1}^2+\widetilde{g_2}^2+\widetilde{g_3}^2)$, $\beta = \widetilde{g_1}\widetilde{g_2}\widetilde{g_3}$ are $\mathrm{Gal}(X/Y)$-invariant. Let $\widetilde{f_1} = \frac{1}{3}(\widetilde{g_1}^2+\rho_3^2\widetilde{g_2}^2+\rho_3\widetilde{g_3}^2)$ and $\widetilde{f_2} = \frac{1}{3}(\widetilde{g_1}^2 + \rho_3\widetilde{g_2}^2 + \rho_3^2\widetilde{g_3}^2)$. Then $\tau^*\widetilde{f_1} = \rho_3\widetilde{f_1}$ and $\tau^*\widetilde{f_2} = \rho_3^2\widetilde{f_2}$. Hence $\gamma = \widetilde{f_1}\widetilde{f_2}$, $f_1 = \widetilde{f_1}^3$ and $f_2 = \widetilde{f_2}^3$ are $\mathrm{Gal}(X/Y)$-invariant and $X/\langle\lambda_1,\lambda_2\rangle \simeq Y_{f_1,3}$. On the other hand, $\beta^2 = \widetilde{g_1}^2\widetilde{g_2}^2\widetilde{g_3}^2 = (\alpha+\widetilde{f_1}+\widetilde{f_2})(\alpha+\rho_3\widetilde{f_1}+\rho_3^2\widetilde{f_2})(\alpha+\rho_3^2\widetilde{f_1}+\rho_3\widetilde{f_2}) = \alpha^3 + f_1 + f_2 - 3\alpha\gamma$. □

For example, if $Y = \mathbf{C}^2$, then $f_1 = -z_1^2(z_2^2 - z_1)$, $f_2 = -z_1(z_2^2 - z_1)^2$, $\alpha = z_2^2$, $\beta = z_2(2z_1 - z_2^2)$ and $\gamma = z_1(z_2^2 - z_1)$ satisfy the relations $\beta^2 = \alpha^3 + f_1 + f_2 - 3\alpha\gamma$ and $\gamma^3 = f_1 f_2$. In this case, X is isomorphic to the singularity obtained by contracting a non-singular rational curve with the self-intersection number -6.

Theorem 4.5. *If $\mathrm{Gal}(X/Y) \simeq S_4$, then there exist holomorphic functions α, β, γ and δ on Y such that $\delta^2 = \gamma^3 + 2\alpha - 3\beta\gamma$ and that $X \simeq ((Y_{f,2})_{g,3})_{\{h_1,h_2\},\{2,2\}}$, where $f = \alpha^2 - \beta^3$, $g = \alpha + \widetilde{f}$, $h_1 = \gamma + \widetilde{g} + \beta/\widetilde{g}$ and $h_2 = \gamma + \rho_3\widetilde{g} + \rho_3^2\beta/\widetilde{g}$.*

Proof. $\mathrm{Gal}(X/Y)$ is generated by four elements σ, τ, λ_1 and λ_2 enjoying the relations $\sigma^2 = \tau^3 = \lambda_1^2 = \lambda_2^2 = e$, $\sigma\tau = \tau^{-1}\sigma$, $\sigma\lambda_1 = \lambda_1\lambda_2\sigma$, $\sigma\lambda_2 = \lambda_2\sigma$, $\tau\lambda_1 = \lambda_2\tau$, $\tau\lambda_2 = \lambda_1\lambda_2\tau$ and $\lambda_1\lambda_2 = \lambda_2\lambda_1$. Let $\widetilde{h_1} = h_0 + \sigma^*h_0 + \lambda_2^*h_0 + \sigma^*\lambda_2^*h_0 - \lambda_1^*h_0 - \lambda_1^*\sigma^*h_0 - \lambda_1^*\lambda_2^*h_0 - \lambda_1^*\sigma^*\lambda_2^*h_0$ for a suitable holomorphic function h_0 on X. Then $\widetilde{h_1} \not\equiv 0$, $\lambda_1^*\widetilde{h_1} = -\widetilde{h_1}$ and $\sigma^*\widetilde{h_1} = \lambda_2^*\widetilde{h_1} = \widetilde{h_1}$. Let $\widetilde{h_2} = \tau^*\widetilde{h_1}$ and let $\widetilde{h_3} = \tau^*\widetilde{h_2}$. Then $\lambda_1^*\widetilde{h_2} = \widetilde{h_2}$, $\lambda_2^*\widetilde{h_2} = -\widetilde{h_2}$ and $\lambda_1^*\widetilde{h_3} = \lambda_2^*\widetilde{h_3} = -\widetilde{h_3}$. Hence $X \simeq (X/\langle\lambda_1,\lambda_2\rangle)_{\{\widetilde{h_1}^2,\widetilde{h_2}^2\},\{2,2\}}$. Moreover, since $\sigma^*\widetilde{h_2} = \sigma^*\tau^*\widetilde{h_1} = (\tau^2)^*\sigma^*\widetilde{h_1} = \widetilde{h_3}$ and $\sigma^*\widetilde{h_3} = \widetilde{h_2}$, we see that $\gamma = \frac{1}{3}(\widetilde{h_1}^2 + \widetilde{h_2}^2 + \widetilde{h_3}^2)$ and $\delta = \widetilde{h_1}\widetilde{h_2}\widetilde{h_3}$ are $\mathrm{Gal}(X/Y)$-invariant. Let $\widetilde{g_1} = \frac{1}{3}(\widetilde{h_1}^2 + \rho_3^2\widetilde{h_2}^2 + \rho_3\widetilde{h_3}^2)$ and $\widetilde{g_2} = \frac{1}{3}(\widetilde{h_1}^2 + \rho_3\widetilde{h_2}^2 + \rho_3^2\widetilde{h_3}^2)$. Then $\lambda_i^*\widetilde{g_j} = \widetilde{g_j}$ $(1 \le i,j \le 2)$, $\sigma^*\widetilde{g_1} = \widetilde{g_2}$, $\tau^*\widetilde{g_1} = \rho_3\widetilde{g_1}$ and $\tau^*\widetilde{g_2} = \rho_3^2\widetilde{g_2}$. Hence $\alpha = \frac{1}{2}(\widetilde{g_1}^3 + \widetilde{g_2}^3)$ and $\beta = \widetilde{g_1}\widetilde{g_2}$ are $\mathrm{Gal}(X/Y)$-invariant. Suppose that $\widetilde{g_1} \equiv 0$. Then $\widetilde{g_2} \equiv 0$ and hence $\widetilde{h_1}^2 = \widetilde{h_2}^2 = \widetilde{h_3}^2$. It contradicts the fact that $\lambda_1^*(\widetilde{h_1}/\widetilde{h_2}) = -\widetilde{h_1}/\widetilde{h_2}$. Therefore, $X/\langle\lambda_1,\lambda_2\rangle \simeq (X/\langle\tau,\lambda_1,\lambda_2\rangle)_{\widetilde{g_1}^3,3}$. Let $\widetilde{f} = \frac{1}{2}(\widetilde{g_1}^3 - \widetilde{g_2}^3)$. Then $\widetilde{f}$ is $\langle\tau,\lambda_1,\lambda_2\rangle$-invariant and $\sigma^*\widetilde{f} = -\widetilde{f}$. Hence $X/\langle\tau,\lambda_1,\lambda_2\rangle \simeq Y_{\widetilde{f}^2,2}$. On the other hand, $\delta^2 = \widetilde{h_1}^2\widetilde{h_2}^2\widetilde{h_3}^2 = (\gamma + \widetilde{g_1} + \widetilde{g_2})(\gamma + \rho_3\widetilde{g_1} + \rho_3^2\widetilde{g_2})(\gamma + \rho_3^2\widetilde{g_1} + \rho_3\widetilde{g_2}) = \gamma^3 + \widetilde{g_1}^3 +$

$\widetilde{g_2}^3 - 3\gamma\widetilde{g_1}\widetilde{g_2} = \gamma^3 + 2\alpha - 3\beta\gamma$, $\widetilde{f}^2 = \alpha^2 - \beta^3$ and $\widetilde{g_1}^3 = \alpha + \widetilde{f}$. □

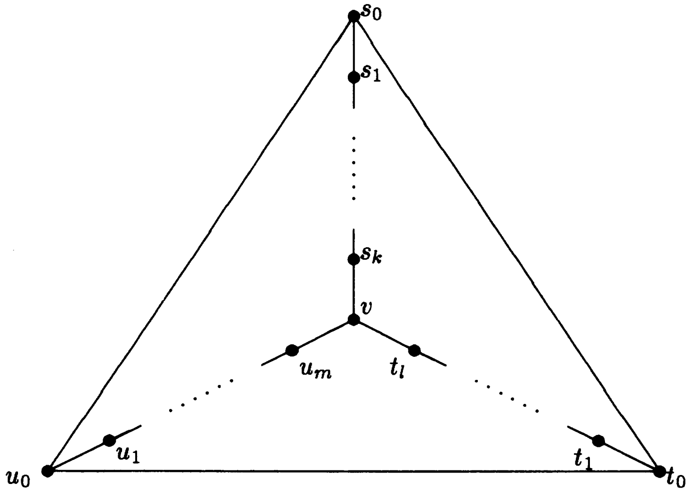
In the case of Example 3.4, $\alpha = (z_1^2 - z_2^3)^2$, $\beta = -(z_1^2 - z_2^3)z_2$, $\gamma = 0$ and $\delta = \sqrt{2}(z_1^2 - z_2^3)$ satisfy the condition of the above theorem.

5. Two dimensional Galois covering singularities

First, we show the following:

Theorem 5.1. *Let p, q, r be positive integers with* $g.c.d.(p,q) = g.c.d.(q,r) = g.c.d.(r,p) = 1$ *and let* D_1 *be the divisor defined by* $z_1^p - z_2^q = 0$ *on a simply connected open neighborhood* Y *of* 0 *in* $\mathbf{C}^2$. *If* $\pi : X \to Y$ *is a Galois covering of* Y *with* $B_\pi = rD_1$, *then* $\mathrm{Gal}(X/Y)$ *is cyclic or unsolvable.*

Proof. Let $\widetilde{Y}$, $\overline{Y}$, $\overline{X}$, $\overline{G}$ and C be as in Section 1. Then $\overline{X}/C \simeq \overline{Y} \simeq \widetilde{Y}$, because B_π is irreducible. Hence if $\mathrm{Gal}(X/Y)$ is an Abelian group, then it is cyclic. Suppose that $\mathrm{Gal}(X/Y)$ is solvable and not cyclic. Then C is a non trivial solvable group. Hence by Proposition 2.1, there exist holomorphic functions $g_1, \ldots, g_j$ on $\overline{Y}$ and positive integers $r_1, \ldots, r_j$ such that $C/[C,C] \simeq \mathbf{Z}_{r_1} \oplus \cdots \oplus \mathbf{Z}_{r_j}$, that $\overline{Y}_{\mathbf{g},\mathbf{r}} \simeq \overline{X}/[C,C]$ and that $\pi_{\mathbf{g},\mathbf{r}} : \overline{Y}_{\mathbf{g},\mathbf{r}} \to \overline{Y}$ does not ramify on $\overline{Y} \setminus \overline{\mu}^{-1}(0)$, where $\mathbf{g} = \{g_1, \ldots, g_j\}$ and $\mathbf{r} = \{r_1, \ldots, r_j\}$. On the other hand, since $\overline{Y}$ is isomorphic to the hypersurface of $Y \times \mathbf{C}$ defined by $w^r - (z_1^p - z_2^q) = 0$, we can describe a resolution of $\overline{Y}$ in the following way (see §6 of [3] and 1.6 of [2]). There exist integral vectors $s_0 = (1,0,0)$, $s_1, \cdots, s_k$, $t_0 = (0,1,0)$, $t_1, \cdots, t_l$, $u_0 = (0,0,1)$, $u_1, \cdots, u_m$, $v = (qr, rp, pq)$ and negative integers $a_1, \cdots, a_k$, $b_1, \cdots, b_l$, $c_1, \cdots, c_m$, d such that $s_{i-1} + a_i s_i + s_{i+1} = 0$ $(1 \le i \le k)$, that $t_{i-1} + b_i t_i + t_{i+1} = 0$ $(1 \le i \le l)$, that $u_{i-1} + c_i u_i + u_{i+1} = 0$ $(1 \le i \le m)$ and that $s_k + t_l + u_m + dv = 0$, where $s_{k+1} = t_{l+1} = u_{m+1} = v$. Then there exists a resolution $\nu : \overline{Z} \to \overline{Y}$ of $\overline{Y}$ such that the exceptional set $\nu^{-1}(\overline{\mu}^{-1}(0)) = D + \sum_{i=1}^k A_i + \sum_{i=1}^l B_i + \sum_{i=1}^m C_i$ consists of rational curves, where $A_i^2 = a_i$, $B_i^2 = b_i$, $C_i^2 = c_i$, $D^2 = d$ and $A_iA_{i+1} = B_iB_{i+1} = C_iC_{i+1} = A_kD = B_lD = C_mD = 1$. Since $\pi_{\mathbf{g},\mathbf{r}}$ does not ramify on $\overline{Y} \setminus \overline{\mu}^{-1}(0)$, we can express $[\nu^* g_1] = D' + r_1 D''$, where $D' = d'D + \sum_{i=1}^k a_i' A_i + \sum_{i=1}^l b_i' B_i + \sum_{i=1}^m c_i' C_i$ for some positive integers a_i', b_i', c_i', d'. Then $A_i D' (= a_{i-1}' + a_i a_i' + a_{i+1}') \equiv B_i D' (= b_{i-1}' + b_i b_i' + b_{i+1}') \equiv C_i D' (= c_{i-1}' + c_i c_i' + c_{i+1}') \equiv DD' (= a_k' + b_l' + c_m' + dd') \equiv 0 (\mathrm{mod}\ r_1)$, where $a_0' = b_0' = c_0' = 0$ and $a_{k+1}' = b_{l+1}' = c_{m+1}' = d'$. Let $\langle\ ,\ \rangle$ be the ordinary inner product of $\mathbf{R}^2$, i.e., $\langle (s,t),(u,v) \rangle = su + tv$ and let x be the element in $\mathbf{Q}^3$ defined by $\langle x, v \rangle = d'$, $\langle x, s_k \rangle = a_k'$ and $\langle x, t_l \rangle = b_l'$. Then $\langle x, u_m \rangle \equiv c_m' (\mathrm{mod}\ r_1)$, because $0 = \langle x, s_k + t_l + u_m + dv \rangle = a_k' + b_l' + \langle x, u_m \rangle + dd'$. In the same way, we have $\langle x, s_i \rangle \equiv a_i'$, $\langle x, t_i \rangle \equiv b_i'$ and $\langle x, u_i \rangle \equiv c_i' (\mathrm{mod}\ r_1)$. Since $\langle x, s_0 \rangle \equiv \langle x, t_0 \rangle \equiv \langle x, u_0 \rangle \equiv 0 (\mathrm{mod}\ r_1)$, we see that $x \in r_1 \mathbf{Z}^3$. Hence $a_i' \equiv b_i' \equiv c_i' \equiv d' \equiv 0 (\mathrm{mod}\ r_1)$. Therefore, the projection $\overline{Z} \times_{\overline{Y}} \overline{Y}_{g_1, r_1} \to \overline{Z}$ does not ramify. However, $\overline{Z}$ is simply connected, a



contradiction. □

We make preparations for later convenience. Let N be a free $\mathbf{Z}$-module of rank 2 and let $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^2$. For a rational cone σ in $N_{\mathbf{R}}$, we denote by $X(N, \sigma)$, the torus embedding $T_N\text{emb}(\{\text{faces of } \sigma\})$ corresponding to the fan consisting of the faces of σ. Let u and v be primitive elements in N, let $\square$ be the convex hull of $((\mathbf{R}_{\geq 0}u + \mathbf{R}_{\geq 0}v) \setminus \{0\}) \cap N$ and let $v_0 = u$, v_1, …, $v_k = v$ be in this order the points of N on the compact faces of $\square$. Then there exist integers a_i smaller than -1 satisfying $v_{i-1} + a_i v_i + v_{i+1} = 0$ for $i = 1, \ldots, k-1$. We denote by $LB(N, u, v)$ and $CF(N, u, v)$, the ordered sets $\{v_0, v_1, \ldots, v_k\}$ and $\{a_1, a_2, \ldots, a_{k-1}\}$, respectively. There exists a resolution of $X(N, \mathbf{R}_{\geq 0}u + \mathbf{R}_{\geq 0}v)$ such that the exceptional set is a chain of rational curves with the self-intersection numbers a_1, a_2, …, a_{k-1} (see Proposition 1.19 of [2]). Moreover, we note that if $N = \mathbf{Z}^2$, $v_0 = u = (1, 0)$, $v_{k-1} = (s, t)$ and $v_k = v = (q, p)$, then

$$[[a_1, a_2, \ldots, a_{k-1}]] = \frac{p}{p - \left(q - \left[\frac{q}{p}\right] p\right)} \quad \text{and} \quad [[a_{k-1}, a_{k-2}, \ldots, a_1]] = \frac{p}{t}$$

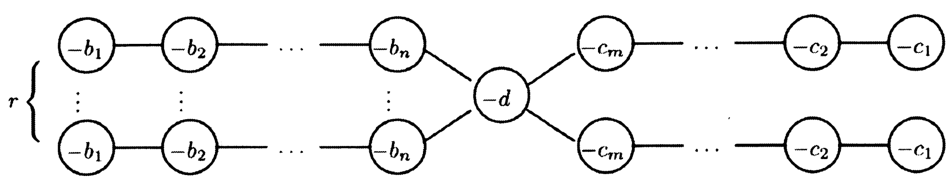
(see p25 of [2]). On the other hand, $\pi_1(X(\mathbf{Z}^2, \sigma) \setminus orb(\sigma)) \simeq \mathbf{Z}_p$, where $\sigma = \mathbf{R}_{\geq 0}(1, 0) + \mathbf{R}_{\geq 0}(q, p)$ (see Proposition 1.9 of [2]).

Lemma 5.1. *Assume that p is a multiple of a positive integer r and let $N = \mathbf{Z}(1, 0) \oplus \mathbf{Z}(0, r)$. Then $\widetilde{X} := X(N, \sigma) \simeq X(\mathbf{Z}^2, \mathbf{R}_{\geq 0}(1, 0) + \mathbf{R}_{\geq 0}(q, p/r))$ is a cyclic covering of $X(\mathbf{Z}^2, \sigma)$ with the index r ramifying only at $orb(\sigma)$. Conversely, such a covering does not exist except those isomorphic to $\widetilde{X}$.*

Proof. The linear map sending $(1,0)$ and $(0,1)$ to $(1,0)$ and $(0,r)$, respectively, induces an isomorphism of fans from $(\mathbf{Z}^2, \{\text{faces of } \mathbf{R}_{\geq 0}(1,0)+\mathbf{R}_{\geq 0}(q,p/r)\})$ to $(N, \{\text{faces of } \sigma\})$. Hence $X(N,\sigma) \simeq X(\mathbf{Z}^2, \mathbf{R}_{\geq 0}(1,0)+\mathbf{R}_{\geq 0}(q,p/r))$. The inclusion $i : N \hookrightarrow \mathbf{Z}^2$ gives rise to a covering map $i_* : X(N,\sigma) \to X(\mathbf{Z}^2,\sigma)$ with $\mathrm{Gal}(X(N,\sigma)/X(\mathbf{Z}^2,\sigma)) \simeq \mathbf{Z}^2/N \simeq \mathbf{Z}_r$ (see Theorem 1.13 and Proposition 1.25 in [2]). This map i_* does not ramify along $T_{\mathbf{Z}^2}$, $orb(\mathbf{R}_{\geq 0}(1,0))$ and $orb(\mathbf{R}_{\geq 0}(q,p))$, because $(1,0)$ and (q,p) are points in N. Hence i_* ramifies only along $orb(\sigma)$, because $X(\mathbf{Z}^2,\sigma) = T_{\mathbf{Z}^2} \sqcup orb(\mathbf{R}_{\geq 0}(1,0)) \sqcup orb(\mathbf{R}_{\geq 0}(q,p)) \sqcup orb(\sigma)$. Next, assume that $p = r$. Then $\widetilde{X}$ is a cyclic covering of $X(\mathbf{Z}^2,\sigma)$ with the index p and $\widetilde{X} \setminus orb(\sigma)$ is simply connected. Hence any Galois covering $\overline{X}$ of $X(\mathbf{Z}^2,\sigma)$ ramifying only along $orb(\sigma)$, is isomorphic to the quotient of $\widetilde{X}$ by a subgroup of $\mathrm{Gal}(\widetilde{X}/X(\mathbf{Z}^2,\sigma)) \simeq \mathbf{Z}_p$, which is isomorphic to $X(\mathbf{Z}(1,0)\oplus\mathbf{Z}(0,s),\sigma)$, where $s = |\mathrm{Gal}(\overline{X}/X(\mathbf{Z}^2,\sigma))|$. □

Let $\pi : X \to Y$ be a Galois covering of a simply connected open neighborhood Y of 0 in $\mathbf{C}^n$ such that $\mathrm{Gal}(X/Y)$ is isomorphic to the dihedral group D_{2r} of order $2r$. Then there exists at least one irreducible component B_i of $B_\pi = r_1B_1 + \cdots + r_lB_l$ along which the ramification index r_i of π is equal to 2, by Proposition 1.1. Hence if B_π is irreducible, then $r_1 = 2$.

Theorem 5.2. *Let $D = 2D_1$, where D_1 is the divisor on a simply connected open neighborhood Y of 0 in $\mathbf{C}^2$ defined by $z_1^p - z_2^q = 0$ for an integer $p > 0$ and for an odd integer $q > 0$ with g.c.d.$(p,q) = 1$. Then there exists a Galois covering $\pi : X \to Y$ such that $B_\pi = D$ and that $\mathrm{Gal}(X/Y) \simeq D_{2r}$, if and only if p is even and r is a divisor of q. Moreover, then there exists a resolution of X, the dual graph of whose exceptional set is as follows:*



Here,

$$[[b_1,b_2,\ldots,b_n]] = \frac{\frac{p}{2}}{\frac{p}{2} - \left(q - \left[\frac{2q}{p}\right]\frac{p}{2}\right)}, \quad [[c_1,c_2,\ldots,c_m]] = \frac{\frac{q}{r}}{\frac{q}{r} - \left(p - \left[\frac{pr}{q}\right]\frac{q}{r}\right)},$$

$$d = 2\alpha + r\beta + \frac{2r}{pq}, \ \alpha = \begin{cases} 0 & \text{if } q = r \\ [[c_m,\cdots,c_1]]^{-1} & \text{if } q > r \end{cases},$$

$$\beta = \begin{cases} 0 & \text{if } p = 2 \\ [[b_n, \cdots, b_1]]^{-1} & \text{if } p > 2 \end{cases}.$$

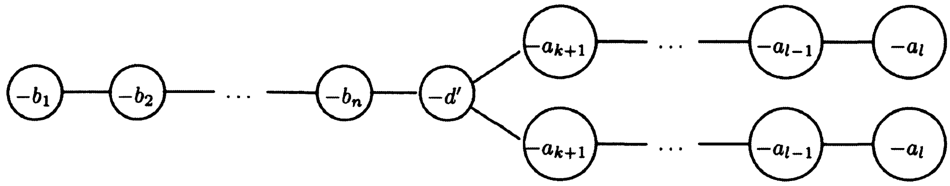
REMARK 5.1. The value of d in the above theorem may be equal to 1. For example, if $p = 2$ and $q = r$, or if $p = 2$, $q = 15$ and $r = 3$, then $d = 1$. Hence the resolution in the above theorem may not be minimal. Especially, if $p = 2$ and $q = r$, then X is non-singular. Also in the following two theorems, the resolutions may not be minimal.

Proof of Theorem 5.2. If the condition in the theorem is satisfied, then $\alpha = z_1^{p/2}$ and $\beta = z_2^{q/r}$ satisfy the condition of Proposition 3.1.

Suppose that p is odd. Then $g.c.d.(p, q) = g.c.d.(q, 2) = g.c.d.(2, p) = 1$. Hence no Galois coverings satisfying the condition of the theorem exist, by Theorem 5.1. Let $LB(\mathbf{Z}^2, (1,0), (q,p)) = \{v_0, v_1, \ldots, v_k\}$, $CF(\mathbf{Z}^2, (1,0), (q,p)) = \{a_1, a_2, \ldots, a_{k-1}\}$, $LB(\mathbf{Z}^2, (q,p), (0,1)) = \{v_k, v_{k+1}, \ldots, v_{l+1}\}$, $CF(\mathbf{Z}^2, (q,p), (0,1)) = \{a_{k+1}, a_{k+2}, \ldots, a_l\}$. Here we recall that $v_0 = (1,0)$, $v_k = (q,p)$, $v_{l+1} = (0,1)$ and that a_i are integers smaller than -1 and satisfy $v_{i-1} + a_i v_i + v_{i+1} = 0$ for $1 \leq i \leq l, i \neq k$. Since $\{v_{k-1}, v_k\}$ and $\{v_k, v_{k+1}\}$ are bases of $\mathbf{Z}^2$, there exists an integer a_k satisfying $v_{k-1} + a_k v_k + v_{k+1} = 0$. Hence by Proposition 1.1. 10 of [2] and [1], we obtain an embedded resolution $\lambda : Z \to Y$ of D_1 such that the exceptional set $E = \sum_{i=1}^{l} E_i$ is a chain of non-singular rational curves with the self-intersection numbers $E_i^2 = a_i$ for $1 \leq i \leq l$ and that $\widetilde{D_1} \cdot E_k = 1$, where $\widetilde{D_1}$ is the proper transformation of D_1 and $E_i E_{i+1} = 1$ for $1 \leq i \leq l-1$. Since at least one of a_1, a_2, $\ldots$ and a_l is equal to -1, we see that $a_k = -1$. On the other hand, the vanishing order c_i of $\lambda^*(z_1^p - z_2^q)$ along E_i is equal to $\langle v_i, (0,q)\rangle$ or $\langle v_i, (p,0)\rangle$, accordingly as $i \leq k$ or $i \geq k$. Let $\widetilde{Y}$ be the hypersurface of $Y \times \mathbf{C}$ defined by $w_1^2 - (z_1^p - z_2^q) = 0$, let $\widetilde{Z}$ be the normalization of $\widetilde{Y} \times_Y Z$, let $\widetilde{\lambda} : \widetilde{Z} \to \widetilde{Y}$, $\nu : \widetilde{Z} \to Z$ be the projections and let $\widetilde{E} = \nu^{-1}(E)$. Since c_i are even for all $i \geq k$, we see that ν does not ramify along E_i for all $i \geq k$. Hence for each $i \geq k+1$, the inverse image $\nu^{-1}(E_i)$ of E_i consists of two irreducible components, which we denote by F_i and F_i'. Then $\sigma_1 F_i = F_i'$ and we may assume that $F_i F_{i+1} = F_i' F_{i+1}' = 1$ and that $F_i F_{i+1}' = F_i' F_{i+1} = 0$ for $k+1 \leq i \leq l-1$, where σ_1 is the automorphism of $\widetilde{Z}$ induced by the automorphism of $\widetilde{Y}$ sending (z_1, z_2, w_1) to $(z_1, z_2, -w_1)$. On the other hand, for each $i \leq k-1$, we see that at least one of c_i and c_{i+1} is odd. Otherwise $\{v_i, v_{i+1}\}$ is not a basis of $\mathbf{Z}^2$, because q is odd and $v_i = (*, c_i/q)$ for $1 \leq i \leq k$. Hence ν ramifies along at least one of E_i and E_{i+1}. Therefore, $F_i = \nu^{-1}(E_i)$ is irreducible for each $i \leq k$. Clearly $\sigma_1 F_i = F_i$. Now assume that there exists a Galois covering $\pi : X \to Y$ such that $B_\pi = D$ and that $\mathrm{Gal}(X/Y) = D_{2r}$. Then there exist holomorphic functions g and h on $\widetilde{Y}$ such that X is isomorphic to $\widetilde{Y}_{g,r}$ and that $(\sigma_1^* g)g = h^r$, by Proposition 2.1 and Corollary 2.1. Since $\pi_{g,r} : \widetilde{Y}_{g,r} \to \widetilde{Y}$ does not ramify on $\widetilde{Y} \setminus \{0\}$, $C := [\widetilde{\lambda}^* g] \equiv \sum_{i=1}^{k} d_i F_i + \sum_{i=k+1}^{l} (d_i F_i + d_i' F_i') (\mathrm{mod}\ r)$ for some integers d_i and d_i'. Note that since D is irreducible, r is odd, by Proposition 1.1 and that $\sigma_1^* C + C = [\widetilde{\lambda}^*(g \sigma_1^* g)] \equiv \sum_{i=1}^{k} 2 d_i F_i + \sum_{i=k+1}^{l} (d_i + d_i')(F_i + F_i') (\mathrm{mod}\ r)$.

Hence we see that $d_i \equiv 0 (\mathrm{mod}\ r)$ for $i \leq k$. Let $s = g.c.d.(d_{k+1}, \cdots, d_l, r)$. Then $d'_i \equiv 0 (\mathrm{mod}\ s)$ for $k+1 \leq i \leq l$, because $d_i + d'_i \equiv 0 (\mathrm{mod}\ r)$. Hence the cyclic covering $\widetilde{Z} \times_{\widetilde{Y}} \widetilde{Y}_{g,r}/\langle \tau^s_{g,r} \rangle \to \widetilde{\widetilde{Z}}$ with the index s does not ramify. However, $\widetilde{E}$ is simply connected. Hence $s = 1$. On the other hand, since $CF_i \equiv 0 (\mathrm{mod}\ r)$ and $F_i^2 = E_i^2$, we have $d_{i-1} + a_i d_i + d_{i+1} \equiv 0 (\mathrm{mod}\ r)$ for $k+1 \leq i \leq l$, where $d_{l+1} = 0$. Since $\langle (d_l, 0), v_{l+1} \rangle = 0$ and $\langle (d_l, 0), v_l \rangle = d_l$, we see that $\langle (d_l, 0), v_i \rangle \equiv d_i (\mathrm{mod}\ r)$ for $k \leq i \leq l-1$. Hence $g.c.d.(d_l, r) = 1$. Therefore, $q \equiv 0 (\mathrm{mod}\ r)$, because $qd_l \equiv d_k \equiv 0 (\mathrm{mod}\ r)$.

Let $v_{k-1} = (s,t)$ and let $v_{k+1} = (u,v)$. Then $sp - tq = qv - pu = 1$, $s + u = q$ and $t + v = p$, because $v_{k-1} - v_k + v_{k+1} = 0$. Let $LB(N, (1,0), (q,p)) = \{v'_0 = (1,0), v'_1, \ldots, v'_n = (s',t'), v'_{n+1} = (q,p)\}$ and let $CF(N, (1,0), (q,p)) = \{b_1, b_2, \ldots, b_n\}$, where $N = \mathbf{Z}(1,0) \oplus \mathbf{Z}(0,2)$. Then $[[b_1, b_2, \ldots, b_n]] = \frac{\frac{p}{2}}{\frac{p}{2} - (q - [\frac{2q}{p}]\frac{p}{2})}$ by Lemma 5.1, and $s'p - t'q = 2$. Hence $\frac{2s-s'}{q} = \frac{2t-t'}{p}$ and $2v_{k-1} = \frac{2t-t'}{p} v_k + v'_n$. Here we note that $\frac{2t-t'}{p} \in \mathbf{Z}$, because $\{v_k, v'_n\}$ is a basis of N and $2v_{k-1} \in N$. Moreover, $\frac{2t-t'}{p} \geq 0$. Otherwise, $v'_n = 2v_{k-1} + (-\frac{2t-t'}{p})v_k$ is contained in the interior of the convex hull of $((\mathbf{R}_{\geq 0}(1,0) + \mathbf{R}_{\geq 0}(q,p)) \setminus \{0\}) \cap N$, because $2v_{k-1}, v_k \in N$. Also note that $t' = 0$ (, i.e., $n = 0$) if and only if $p = 2$. While, if $t' > 1$, then $\frac{p}{t'} = [[b_n, \ldots, b_1]]$. Let $w_j = jv_k + v'_n$ for $j = 0, \ldots, \frac{2t-t'}{p}$. Then $v_k + w_{\frac{2t-t'}{p}-1} = w_{\frac{2t-t'}{p}}$ and $w_{j-1} + w_{j+1} = 2w_j$ for $j = 1, \ldots, \frac{2t-t'}{p} - 1$. Let $\Sigma = \{\text{faces of } \mathbf{R}_{\geq 0}v_i + \mathbf{R}_{\geq 0}v_{i+1} | 0 \leq i \leq k-1\}$, let $\Lambda = \{\text{faces of } \mathbf{R}_{\geq 0}v'_i + \mathbf{R}_{\geq 0}v'_{i+1} | 0 \leq i \leq n\}$, and let $\Theta = \{\text{faces of } \mathbf{R}_{\geq 0}v'_i + \mathbf{R}_{\geq 0}v'_{i+1}, \text{faces of } \mathbf{R}_{\geq 0}w_j + \mathbf{R}_{\geq 0}w_{j+1} | 0 \leq i \leq n-1, 0 \leq j \leq \frac{2t-t'}{p}\}$, where $w_{\frac{2t-t'}{p}+1} = v_k$. Then some neighborhood U of $E_1 + E_2 + \cdots + E_{k-1}$ is isomorphic to an open set of $T_{\mathbf{Z}^2}\mathrm{emb}(\Sigma)$, the point $E_{k-1}E_k$ corresponds to $orb(\mathbf{R}_{\geq 0}v_{k-1} + \mathbf{R}_{\geq 0}v_k)$ by this isomorphism and $\nu^{-1}(U)$ is isomorphic to an open set of $T_N\mathrm{emb}(\Sigma)$, by Lemma 5.1. Since $2v_{k-1} = w_{\frac{2t-t'}{p}}$, Θ as well as Σ contains $\mathbf{R}_{\geq 0}v_{k-1} + \mathbf{R}_{\geq 0}v_k$. Hence the restriction of the birational map $T_N\mathrm{emb}(\Theta) \to T_N\mathrm{emb}(\Sigma)$ to the neighborhood $T_N\mathrm{emb}(\{\text{faces of } \mathbf{R}_{\geq 0}v_{k-1} + \mathbf{R}_{\geq 0}v_k\})$ of $orb(\mathbf{R}_{\geq 0}v_{k-1} + \mathbf{R}_{\geq 0}v_k)$ is biholomorphic. While, we have a holomorphic map $T_N\mathrm{emb}(\Theta) \to T_N\mathrm{emb}(\Lambda)$, because Θ is a subdivision of Λ. Hence replacing $\nu^{-1}(U)$ by an open set of $T_N\mathrm{emb}(\Lambda)$, we obtain a resolution of $\widetilde{Y}$ the dual graph of whose exceptional set is as follows.



Since the map $T_N\text{emb}(\Theta) \to T_N\text{emb}(\Lambda)$ is the composite of $\frac{2t-t'}{p}$ times blow up at $orb(\mathbf{R}_{\geq 0}v'_n + \mathbf{R}_{\geq 0}v'_{n+1})$, we have $d' = 2 - \frac{2t-t'}{p}$. Let $LB(\mathbf{Z}(r,0) \oplus \mathbf{Z}(0,1), (q,p), (0,1)) = \{(q,p), (u',v'), \ldots, (0,1)\}$ and let $CF(\mathbf{Z}(r,0) \oplus \mathbf{Z}(0,1), (q,p), (0,1)) = \{c_1, c_2, \ldots, c_m\}$. Then $[[c_1, c_2, \ldots, c_m]] = \frac{\frac{q}{r}}{\frac{q}{r} - (p - [\frac{pr}{q}]\frac{q}{r})}$ and $r(u,v) = \frac{rv-v'}{p}(q,p) + (u',v')$, because $qv' - pu' = r$ and $qv - pu = 1$. Hence we obtain a resolution of X the dual graph of whose exceptional set is as in the proposition and

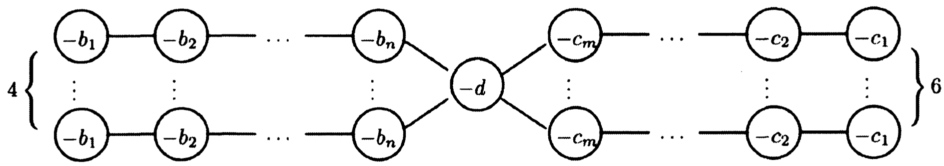
$$\begin{aligned} d &= (2 - \frac{2t-t'}{p})r - \frac{rv-v'}{p}2 = 2r - \frac{2t-t'}{p}r - \frac{2rv}{p} + \frac{2v'}{p} \\ &= 2r - \frac{2}{p}(t+v)r + \frac{t'r}{p} + 2\left(\frac{u'}{q} + \frac{r}{pq}\right) = r\frac{t'}{p} + 2\frac{u'}{q} + \frac{2r}{pq}. \end{aligned}$$

Here we note that $u' = 0$ if and only if $q = r$ and that if $u' > 1$, then $\frac{q}{u'} = [[c_m, \ldots, c_1]]$. □

Next, we consider the case that $\text{Gal}(X/Y) \simeq A_4$. Let $D = r_1 D_1$, where r_1 is an integer greater than 1 and D_1 is a divisor on a simply connected open neighborhood Y of 0 in $\mathbf{C}^2$ defined by $z_1^p - z_2^q = 0$ for positive integers p and q with $g.c.d.(p,q) = 1$. Here we may assume that q is not a multiple of 3. If $r_1 = 3$, if p and q are multiples of 3 and 2, respectively, then there exists a Galois covering $\pi : X \to Y$ such that $B_\pi = D$ and that $\text{Gal}(X/Y) \simeq A_4$, as we see in Example 3.1. Conversely, let $\pi : X \to Y$ be such a Galois covering. Then $X/[G,G]$ is isomorphic to the hypersurface $\widetilde{Y}$ of $Y \times \mathbf{C}$ defined by $w_1^3 - (z_1^p - z_2^q) = 0$, where $G = \text{Gal}(X/Y)$, and $r_1 = 3$, because $|g| = 3$ for any element g in $G \setminus [G,G]$. Hence there exist holomorphic functions g_1, g_2 and meromorphic functions h_1, h_2 on $\widetilde{Y}$ such that $\widetilde{Y}_{\{g_1,g_2\},\{2,2\}} \simeq X$, that $\frac{\sigma_1^* g_1}{g_2} = h_1^2$ and that $\frac{\sigma_1^* g_2}{g_1 g_2} = h_2^2$, by Proposition 2.1 and Corollary 2.1, where σ_1 is the automorphism of $\widetilde{Y}$ sending (z_1, z_2, w_1) to $(z_1, z_2, \rho_3 w_1)$. Then $g_1(\sigma_1^* g_1)\left((\sigma_1^2)^* g_1\right) = (g_1 g_2 h_1 (\sigma_1^* h_1) h_2)^2$. On the other hand, we can construct a $\text{Gal}(\widetilde{Y}/Y)$-equivariant resolution $\lambda : \widetilde{Z} \to \widetilde{Y}$ of $\widetilde{Y}$ (see §3 of [4]). Then $[\lambda^* g_1] = \sum c_i E_i + 2\widetilde{C}$ for some positive integers c_i and for a certain divisor $\widetilde{C}$ on $\widetilde{Z}$, where $\sum E_i$ is the exceptional set of λ and each irreducible component of $\widetilde{C}$ is not contained in $\sum E_i$, because $\pi_{\{g_1,g_2\},\{2,2\}}$ does not ramify along divisors. Hence $(\sum c_i E_i) E_j \equiv 0 (\text{mod } 2)$. Moreover, we have $[\lambda^* g_1] + [\lambda^* \sigma_1^* g_1] + [\lambda^* (\sigma_1^2)^* g_1] \equiv 0 (\text{mod } 2)$. Hence we can show the following theorem, in the manner similar to the proof of Theorem 5.2.

Theorem 5.3. *Let $D = r_1 D_1$, where r_1 is an integer greater than* 1 *and D_1 is the divisor on a simply connected open neighborhood Y of* 0 *in $\mathbf{C}^2$ defined by $z_1^p - z_2^q = 0$ for positive integers p and q with $g.c.d.(p,q) = 1$ the latter of which is not a multiple of* 3. *Then there exists a Galois covering $\pi : X \to Y$ such that $B_\pi = D$ and that* $\text{Gal}(X/Y) \simeq A_4$, *if and only if $r_1 = 3$, p and q are multiples of* 3 *and* 2, *respectively.*

The dual graph of the exceptional set of a resolution of X in the above theorem is as follows:



Here,

$$[[b_1, b_2, \dots, b_n]] = \frac{\frac{p}{3}}{\frac{p}{3} - \left(q - \left[\frac{3q}{p}\right]\frac{p}{3}\right)}, \quad [[c_1, c_2, \dots, c_m]] = \frac{\frac{q}{2}}{\frac{q}{2} - \left(p - \left[\frac{p2}{q}\right]\frac{q}{2}\right)},$$

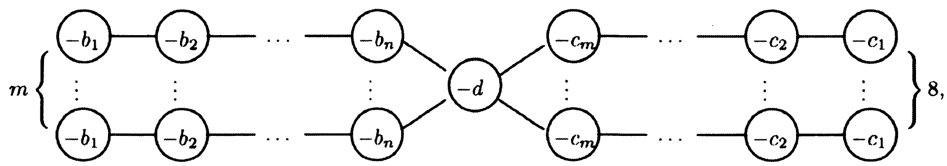
$$d = 6\alpha + 4\beta + \frac{12}{pq}, \ \alpha = \begin{cases} 0 & \text{if } q = 2 \\ [[c_m, \cdots, c_1]]^{-1} & \text{if } q > 2 \end{cases},$$

$$\beta = \begin{cases} 0 & \text{if } p = 3 \\ [[b_n, \cdots, b_1]]^{-1} & \text{if } p > 3 \end{cases}.$$

Finally, we consider the case that $\mathrm{Gal}(X/Y) \simeq S_4$. Let $D = r_1 D_1$, where r_1 is an integer greater than 1 and D_1 is a divisor on a simply connected open neighborhood Y of 0 in $\mathbf{C}^2$ defined by $z_1^p - z_2^q = 0$ for positive integers p and q with $g.c.d.(p, q) = 1$. Here we may assume that q is odd. If $r_1 = 2$ (resp. 4), if p and q are multiples of 4 (resp. 2) and 3, respectively, then there exists a Galois covering $\pi : X \to Y$ such that $B_\pi = D$ and that $\mathrm{Gal}(X/Y) \simeq S_4$, as we see in Example 3.2 (resp. 3.4). Conversely, let $\pi : X \to Y$ be such a Galois covering of Y, let $G = \mathrm{Gal}(X/Y)$, let $G_1 = [G, G]$ and let $G_2 = [G_1, G_1]$. Then X/G_2 is a Galois covering of Y such that $\mathrm{Gal}((X/G_2)/Y) \simeq D_6$ and that $B_{[X/G_2 \to Y]} = 2D_1$, and $r_1 = 2$ or 4, because $B_{[X/G_1 \to Y]} = 2D_1$ and $|g| = 2$ or 4 for any element g in $G \setminus G_1$. Hence p is even and q is a multiple of 3, by Theorem 5.2. Moreover, there exist holomorphic functions g_1 and g_2 on X/G_2 such that $(X/G_2)_{\{g_1, g_2\}, \{2,2\}} \simeq X$. We see by a similar consideration as in the proof of Theorem 5.2 that such functions do not exist, if $r_1 = 2$ and if p is not a multiple of 4. Thus we have:

Theorem 5.4. *Let D_1 be the divisor on a simply connected open neighborhood Y of* 0 *in* $\mathbf{C}^2$ *defined by $z_1^p - z_2^q = 0$ for an integer $p > 0$ and for an odd integer $q > 0$ with $g.c.d.(p, q) = 1$. Then there exists a Galois covering $\pi : X \to Y$ such that $B_\pi = 2D_1$ (resp. $4D_1$) and that* $\mathrm{Gal}(X/Y) \simeq S_4$, *if and only if p and q are multiples of* 4 *(resp.* 2 *) and* 3, *respectively.*

When $B_\pi = 2D_1$ (resp. $4D_1$), the dual graph of the exceptional set of a resolution of X in the above theorem is as follows:



where $m = 6$ (resp. 12),

$$[[b_1, b_2, \ldots, b_n]] = \frac{\frac{p}{4}}{\frac{p}{4} - \left(q - \left[\frac{4q}{p}\right]\frac{p}{4}\right)} \left(\text{resp. } \frac{\frac{p}{2}}{\frac{p}{2} - \left(q - \left[\frac{2q}{p}\right]\frac{p}{2}\right)}\right),$$

$$[[c_1, c_2, \ldots, c_m]] = \frac{\frac{q}{3}}{\frac{q}{3} - \left(p - \left[\frac{p3}{q}\right]\frac{q}{3}\right)}, \quad d = 8\alpha + m\beta + \frac{24}{pq},$$

$$\alpha = \begin{cases} 0 & \text{if } q = 3 \\ [[c_m, \cdots, c_1]]^{-1} & \text{if } q > 3 \end{cases}, \quad \beta = \begin{cases} 0 & \text{if } p = 4 \text{ (resp. 2)} \\ [[b_n, \cdots, b_1]]^{-1} & \text{if } p > 4 \text{ (resp. 2)} \end{cases}.$$

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