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# SPECTRA OF LAPLACE-BELTRAMI OPERATORS ON $SO(n+2)/SO(2)\times SO(n)$ AND $Sp(n+1)/Sp(1)\times Sp(n)$

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**Introduction.** Let $M=G/K$ be a compact symmetric space with $G$ compact and semisimple. We assume that the Riemannian metric on $M$ is the metric induced from the Killing form sign-changed. We consider the Laplace-Beltrami operator $\Delta^p$ acting on $p$-forms and its spectrum $\mathrm{Spec}^p(M)$.

Ikeda and Taniguchi [3] computed $\mathrm{Spec}^p(M)$ for $M=S^n$ and $P^n(C)$, studying representations of $G$ and $K$. They showed that $\Delta^p=-$Casimir operator when we consider the space of $p$-forms $C^\infty(\Lambda^p M)$ as a $G$-module. Each irreducible $G$-submodule of $C^\infty(\Lambda^p M)$ is included in some eigenspace of $\Delta^p$ and the sum of irreducible $G$-submodules of $C^\infty(\Lambda^p M)$ equals to the sum of eigenspaces of $\Delta^p$. We can compute eigenvalues from Freudenthal's formula and multiplicities from Weyl's dimension formula. Thus to compute $\mathrm{Spec}^p(M)$, we have only to decompose $C^\infty(\Lambda^p M)$ into irreducible $G$-submodules and count out them.

But generally it is not easy. Though Beers and Millman [1] determined $\mathrm{Spec}^p(M)$ when $M$ is a Lie group of a low rank such as $SU(3)$ or $SO(5)$ by the similar method, these seem to be all we know.

Frobenius' reciprocity law enables us to reduce the problem into the following two: How does an irreducible $G$-module decompose into irreducible $K$-modules? How does the $p$-th exterior product of (complexified) cotangent space decompose into irreducible $K$-modules? The former is usually called a branching law.

In this paper, we give a branching law for $G=SO(n+2)$ and $K=SO(2)\times SO(n)$, which enables us to compute $\mathrm{Spec}^p(M)$. As a matter of fact, we should distinguish between the case $n=$odd and the case $n=$even. Almost in parallel, we get a branching law for $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$, which reproduces the result of Lepowsky [4] obtained in a different way.

The latter problem, i.e., the decomposition of an exterior power of an isotropy representation is a rather technical (but indispensable) part in computing $\mathrm{Spec}^p(M)$. We give a complete list of members in the decomposition for $G=SO(n+2)$ and $K=SO(2)\times SO(n)$. For $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$, we confine ourselves to indicating a procedure to determine the decomposition and giving lists for some $n$ and $p$.

Throughout this paper, modules are assumed to be over the complex number field $C$.

## 1. Branching laws

We state branching laws in terms of highest weights.

We denote by $M(n, C)$ the set of all $n\times n$-matrices of complex coefficients.

Let $G=SO(n+2)$ and $K=SO(2)\times SO(n)$. We adopt the following conventions:

$$\mathfrak{g}=\mathfrak{o}(n+2, C)=\{X\in M(n+2, C);\, {}^tX+X=0\}\,,$$
$$\mathfrak{k}=\mathfrak{o}(2, C)\times\mathfrak{o}(n, C)$$
$$=\left\{\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix};\ \begin{matrix} X\in M(2, C),\ {}^tX+X=0\\ Y\in M(n, C),\ {}^tY+Y=0\end{matrix}\right\},$$
$$\mathfrak{t}=\left\{\begin{pmatrix} R(\lambda_0) & & & & \\ & R(\lambda_1) & & & \\ & & \cdots\cdots & & \\ & & & R(\lambda_m) & \\ & & & & (0)\end{pmatrix};\ \begin{matrix} R(\lambda)=\begin{pmatrix} 0 & -\sqrt{-1}\lambda\\ \sqrt{-1}\lambda & 0\end{pmatrix}\\ \lambda_i\in C\end{matrix}\right\},$$

where $n=2m$ or $n=2m+1$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and also one of $\mathfrak{k}$. We regard $\lambda_i$ as a form on $\mathfrak{t}$ giving the value of $\lambda_i$. We take a Weyl chamber for $(\mathfrak{g}, \mathfrak{t})$ so that the simple roots of $\mathfrak{g}$ are $\alpha_0=\lambda_0-\lambda_1$, $\alpha_1=\lambda_1-\lambda_2$, $\cdots$, $\alpha_{m-1}=\lambda_{m-1}-\lambda_m$, and $\alpha_m=\lambda_{m-1}+\lambda_m$ when $n=2m$, $\alpha_m=\lambda_m$ when $n=2m+1$. We take a Weyl chamber for $(\mathfrak{k}, \mathfrak{t})$ so that the simple roots of $\mathfrak{k}$ are those of $\mathfrak{g}$ excluding $\alpha_0$.

We first treat the case $n=2m$.

Any dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of $G=SO(2m+2)$ is uniquely expressed as

$$\Lambda=h_0\lambda_0+h_1\lambda_1+\cdots+h_{m-1}\lambda_{m-1}+\varepsilon h_m\lambda_m\,, \tag{1.1}$$

where $\varepsilon=1$ or $-1$ and $h_0, h_1, \cdots, h_m$ are integers satisfying

$$h_0\geqslant h_1\geqslant\cdots\geqslant h_{m-1}\geqslant h_m\geqslant 0\,. \tag{1.2}$$

Any dominant integral form for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of $K=SO(2)\times SO(2m)$ is uniquely expressed as

$$\Lambda'=k_0\lambda_0+k_1\lambda_1+\cdots+k_{m-1}\lambda_{m-1}+\varepsilon' k_m\lambda_m\,, \tag{1.3}$$

where $\varepsilon'=1$ or $-1$ and $k_0, k_1, \cdots, k_m$ are integers satisfying

$$k_1\geqslant\cdots\geqslant k_{m-1}\geqslant k_m\geqslant 0\,. \tag{1.4}$$

For integers $h_0, h_1, \cdots, h_m$ and $k_1, k_2, \cdots, k_m$, we define integers $l_0, l_1, \cdots, l_m$ by

$$
\begin{aligned}
(1.5) \qquad & l_0 = h_0 - \max(h_1, k_1)\,, \\
& l_i = \min(h_i, k_i) - \max(h_{i+1}, k_{i+1}) \ \text{ for } 1 \leqslant i \leqslant m-1\,, \\
& l_m = \min(h_m, k_m)\,.
\end{aligned}
$$

**Theorem 1.1.** *Let $G=SO(2m+2)$ and $K=SO(2)\times SO(2m)$. Let $\Lambda$ be the highest weight of an irreducible $G$-module $V$. Then the irreducible decomposition of $V$ as a $K$-module contains an irreducible $K$-module $V'$ with the highest weight $\Lambda'$ if and only if;*

*a)*
$$
\begin{aligned}
& h_{i-1} \geqslant k_i \geqslant h_{i+1} \ \text{ for } 1 \leqslant i \leqslant m-1\,, \\
& h_{m-1} \geqslant k_m \ (\geqslant 0)\,,
\end{aligned}
$$

*expressing $\Lambda$ and $\Lambda'$ as* (1.1) *and* (1.3), *and*

*b) the coefficient of $X^{k_0}$ in the (finite) power series expansion in $X$ of*

$$X^{\varepsilon\varepsilon' l_m}\left(\prod_{i=0}^{m-1}\left((X^{l_i+1} - X^{-l_i-1})/(X - X^{-1})\right)\right)$$

*does not vanish.*

*Moreover, the number of the times $V'$ appearing in the decomposition is equal to the coefficient of $X^{k_0}$ in the expansion.*

REMARK. Suppose a) is satisfied. Then all the integers $l_0, l_1, \cdots, l_m$ are non-negative and all the coefficients in the power series are also non-negative.

The proof is given in the next section.

Next we treat the case $n=2m+1$.

Any dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of $G=SO(2m+3)$ is uniquely expressed as

$$(1.6) \qquad \Lambda = h_0\lambda_0 + h_1\lambda_1 + \cdots + h_m\lambda_m\,,$$

where $h_0, h_1, \cdots, h_m$ are integers satisfying (1.2). Any dominant integral form for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of $K=SO(2)\times SO(2m+1)$ is uniquely expressed as

$$(1.7) \qquad \Lambda' = k_0\lambda_0 + k_1\lambda_1 + \cdots + k_m\lambda_m\,,$$

where $k_0, k_1, \cdots, k_m$ are integers satisfying (1.4).

In this case we also define integers $l_0, l_1, \cdots, l_m$ by (1.5).

**Theorem 1.2.** *Let $G=SO(2m+3)$ and $K=SO(2)\times SO(2m+1)$. Let $\Lambda$ be the highest weight of an irreducible $G$-module $V$. Then the irreducible decomposition of $V$ as a $K$-module contains an irreducible $K$-module $V'$ with the highest weight $\Lambda'$ if and only if;*

*a)*
$$h_{i-1}\geqslant k_i\geqslant h_{i+1}\ \text{for}\ 1\leqslant i\leqslant m-1\,,$$
$$h_{m-1}\geqslant k_m\ (\geqslant 0)\,,$$

*expressing* $\Lambda$ *and* $\Lambda'$ *as* (1.6) *and* (1.7), *and*

*b) the coefficient of* $X^{k_0}$ *in the (finite) power series expansion in* $X$ *of*

$$(X-X^{-1})^{-m}(\textstyle\prod_{i=0}^{m-1}(X^{l_i+1}-X^{-l_i-1}))(X^{1/2}-X^{-1/2})^{-1}(X^{l_m+1/2}-X^{-l_m-1/2})$$

*does not vanish.*

*Moreover, the number of the times* $V'$ *appearing in the decomposition is equal to the coefficient of* $X^{k_0}$ *in the expansion.*

REMARK. Suppose a) is satisfied. Then all the integers $l_0, l_1, \cdots, l_m$ are non-negative and all the coefficients in the power series are also non-negative.

For the sake of completeness we state the branching law for $G=Sp(m+1)$ and $K=Sp(1)\times Sp(m)$. We adopt the following convensions:

$$\mathfrak{g}=\mathfrak{sp}(m+1, C)$$
$$=\left\{\begin{pmatrix} X & Z \\ Y & -{}^tX\end{pmatrix};\ \begin{matrix} X, Y, Z\in M(m+1, C) \\ {}^tY=Y,\ {}^tZ=Z\end{matrix}\right\},$$
$$\mathfrak{k}=\mathfrak{sp}(1, C)\times\mathfrak{sp}(m, C)$$
$$=\left\{\begin{pmatrix} x & 0 & z & 0 \\ 0 & X & 0 & Z \\ y & 0 & -x & 0 \\ 0 & Y & 0 & -{}^tX\end{pmatrix};\ \begin{matrix} x, y, z\in C \\ X, Y, Z\in M(m, C) \\ {}^tY=Y,\ {}^tZ=Z\end{matrix}\right\},$$
$$\mathfrak{t}=\{\mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_m, -\lambda_0, -\lambda_1, \cdots, -\lambda_m);\ \lambda_i\in C\}\,.$$

Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and also one of $\mathfrak{k}$. We regard $\lambda_i$ as a form on $\mathfrak{t}$. We take a Weyl chamber for $(\mathfrak{g}, \mathfrak{t})$ so that the simple roots of $\mathfrak{g}$ are $\alpha_0=\lambda_0-\lambda_1$, $\alpha_1=\lambda_1-\lambda_2, \cdots, \alpha_{m-1}=\lambda_{m-1}-\lambda_m$, $\alpha_m=2\lambda_m$. We take a Weyl chamber for $(\mathfrak{k}, \mathfrak{t})$ so that the simple roots of $\mathfrak{k}$ are $\alpha_0'=2\lambda_0$ and $\alpha_i$ $(1\leqslant i\leqslant m)$.

Since $G$ and $K$ are simply connected, each representation of their Lie algebras can be lifted to a group representation. Hence each dominant integral form corresponds to an irreducible representation and vice versa.

Any dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ is uniquely expressed as (1.6), where $h_0, h_1, \cdots, h_m$ are integers satisfying (1.2). Any dominant integral form for $(\mathfrak{k}, \mathfrak{t})$ is uniquely expressed as (1.7), where $k_0, k_1, \cdots, k_m$ are integers satisfying (1.4) and $k_0\geqslant 0$. We again define integers $l_0, l_1, \cdots, l_m$ by (1.5).

**Theorem 1.3** (Lepowsky). *Let* $G=Sp(m+1)$ *and* $K=Sp(1)\times Sp(m)$. *Let* $\Lambda$ *be the highest weight of an irreducible G-module* $V$. *Then the irreducible decomposition of* $V$ *as a K-module contains an irreducible K-module* $V'$ *with the* *highest weight* $\Lambda'$ *if and only if*;

*a*) $$h_{i-1} \geqslant k_i \geqslant h_{i+1} \text{ for } 1 \leqslant i \leqslant m-1\,,$$
$$h_{m-1} \geqslant k_m \ (\geqslant 0)\,,$$

*expressing* $\Lambda$ *and* $\Lambda'$ *as* (1.6) *and* (1.7), *and*
*b*) *the coefficient of* $X^{k_0+1}$ *in the* (*finite*) *power series expansion in* $X$ *of*

$$(X-X^{-1})^{-m}(\prod_{i=0}^{m}(X^{l_i+1}-X^{-l_i-1}))$$

*does not vanish.*

*Moreover, the number of the times* $V'$ *appearing in the decomposition is equal to the coefficient of* $X^{k_0+1}$ *in the expansion.*

REMARK. Suppose a) is satisfied. Then all the integers $l_0, l_1, \cdots, l_m$ are non-negative and all the coefficients of $X^k$ ($k>0$) are also non-negative. The coefficient of $X^{-k}$ is equal to the negation of the coefficient of $X^k$.

## 2. Proof of branching laws

Let $G$ be a compact connected semisimple Lie group, $K$ a closed subgroup of $G$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the complexified Lie algebras of $G$ and $K$. We assume that $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{t}$ which is also a Cartan subalgebra of $\mathfrak{k}$.

We consider a group algebra over $Z$ generated by an additive group of integral forms for $(\mathfrak{g}, \mathfrak{t})$ and one for $(\mathfrak{k}, \mathfrak{t})$. Since an integral form for $(\mathfrak{g}, \mathfrak{t})$ is also integral for $(\mathfrak{k}, \mathfrak{t})$, the group algebra for $(\mathfrak{g}, \mathfrak{t})$ is included in the group algebra for $(\mathfrak{k}, \mathfrak{t})$.

A formal character of a $G$-module $V$ is an element of the group algebra for $(\mathfrak{g}, \mathfrak{t})$ defined by the formal sum of all the weights of $V$. (See, for example, Humphreys [2].) For an irreducible $G$-module $V$ with the highest weight $\Lambda$, we denote its formal character by $\chi_G(\Lambda)$. We do the same for a $K$-module.

In terms of formal characters, a branching law for $G$ and $K$ means to determine the set $S$ (which counts multiplicities) in the following formula:

$$\chi_G(\Lambda) = \Sigma\chi_K(\Lambda') \quad (\Lambda' \in S)\,, \tag{2.1}$$

where $\Lambda$ is a dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ and $\Lambda'$ is one for $(\mathfrak{k}, \mathfrak{t})$.

We will rewrite (2.1). Let $W_G$ be the Weyl group of $(\mathfrak{g}, \mathfrak{t})$ acting on integral forms. We denote by $e(\Lambda)$ a generator of the group algebra corresponding to an integral form $\Lambda$. We define $\xi_G(\Lambda)$ by

$$\xi_G(\Lambda) = \Sigma(-1)^{\sigma}e(\sigma\Lambda) \quad (\sigma \in W_G)\,.$$

We set $\delta_G = (\Sigma\alpha)/2$ ($\alpha \in \Delta_G^+$), where $\Delta_G^+$ denotes the set of positive roots of $\mathfrak{g}$.

Then, by Weyl's character formula, we have

$$\xi_G(\Lambda+\delta_G) = \xi_G(\delta_G)\cdot\chi_G(\Lambda)\,.$$

We get in parallel

$$\xi_K(\Lambda'+\delta_K) = \xi_K(\delta_K)\cdot\chi_K(\Lambda')\,.$$

Now (2.1) is reduced to

$$\xi_G(\Lambda+\delta_G)\cdot\xi_K(\delta_K) = \xi_G(\delta_G)\cdot\Sigma\xi_K(\Lambda'+\delta_K) \quad (\Lambda'\in S)\,. \tag{2.2}$$

Our task is to devide $\xi_G(\Lambda+\delta_G)$ by $\xi_G(\delta_G)/\xi_K(\xi_K)$ and set it in the form $\Sigma\xi_K(\Lambda'+\delta_K)$. Since $\xi_K(\Lambda')$ for dominant integral forms $\Lambda'$ are linearly independent, the set $S$ is uniquely determined.

We may calculate in a larger group algebra generated by an additive group of forms. We can write

$$\xi_G(\delta_G) = \Pi(e(\alpha/2)-e(-\alpha/2)) \quad (\alpha\in\Delta_G^+)\,,$$
$$\xi_K(\delta_K) = \Pi(e(\alpha/2)-e(-\alpha/2)) \quad (\alpha\in\Delta_K^+)\,,$$

and so

$$\xi_G(\delta_G)/\xi_K(\delta_K) = \Pi(e(\alpha/2)-e(-\alpha/2)) \quad (\alpha\in\Delta_G^+\backslash\Delta_K^+)\,.$$

We will exhibit $\xi_G$ and $\xi_K$ in terms of $\lambda_i$ in the cases of our branching laws. We set $s(\Lambda)=e(\Lambda)-e(-\Lambda)$, $c(\Lambda)=e(\Lambda)+e(-\Lambda)$. We denote by $[a_{ij}]_{p:q}$ a square matrix whose suffixes $i, j$ range from $p$ to $q$.

a) $G=SO(2m+2)$, $K=SO(2)\times SO(2m)$.

When we express $\Lambda+\delta_G$ as in (1.1), $\varepsilon=1$ or $-1$ and $h_0, h_1, \cdots, h_m$ are integers satisfying

$$h_0>h_1>\cdots>h_{m-1}>h_m\geqslant 0\,.$$

When we express $\Lambda'+\delta_K$ as in (1.3), $\varepsilon'=1$ or $-1$ and $k_0, k_1, \cdots, k_m$ are integers satisfying

$$k_1>\cdots>k_{m-1}>k_m\geqslant 0\,.$$

We get

$$\xi_G(\Lambda+\delta_G) = (1/2)(\det[c(h_i\lambda_j)]_{0:m}+\varepsilon\det[s(h_i\lambda_j)]_{0:m})\,,$$
$$\xi_K(\Lambda'+\delta_K) = e(k_0\lambda_0)\cdot(1/2)(\det[c(k_i\lambda_j)]_{1:m}+\varepsilon'\det[s(k_i\lambda_j)]_{1:m})\,,$$
$$\xi_G(\delta_G)/\xi_K(\delta_K) = \Pi_{i=1}^m(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))\,.$$

b) $G=SO(2m+3)$, $K=SO(2)\times SO(2m+1)$.

When we express $\Lambda+\delta_G$ as in (1.6), $h_0, h_1, \cdots, h_m$ are integers $+1/2$ satisfying

$$h_0>h_1>\cdots>h_m>0\,.$$

When we express $\Lambda'+\delta_K$ as in (1.7), $k_0$ is an integer and $k_1, \cdots, k_m$ are integers $+1/2$ satisfying

$$k_1>k_2>\cdots>k_m>0\,.$$

We get

$$\xi_G(\Lambda+\delta_G)=\det[s(h_i\lambda_j)]_{0:m}\,,$$
$$\xi_K(\Lambda'+\delta_K)=e(k_0\lambda_0)\cdot\det[s(k_i\lambda_j)]_{1:m}\,,$$
$$\xi_G(\delta_G)/\xi_K(\delta_K)=s(\lambda_0/2)\cdot\prod_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))\,.$$

c) $G=Sp(m+1)$, $K=Sp(1)\times Sp(m)$.

When we express $\Lambda+\delta_G$ as in (1.6), $h_0, h_1, \cdots, h_m$ are integers satisfying

$$h_0>h_1>\cdots>h_{m-1}>h_m>0\,.$$

When we express $\Lambda'+\delta_K$ as in (1.7), $k_0, k_1, \cdots, k_m$ are integers satisfying

$$k_0>0,\ k_1>\cdots>k_{m-1}>k_m>0\,.$$

We get

$$\xi_G(\Lambda+\delta_G)=\det[s(h_i\lambda_j)]_{0:m}\,,$$
$$\xi_K(\Lambda'+\delta_K)=s(k_0\lambda_0)\cdot\det[s(k_i\lambda_j)]_{1:m}\,,$$
$$\xi_G(\delta_G)/\xi_K(\delta_K)=\prod_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s(\lambda_0-\lambda_i)/2))\,.$$

The crucial point in the proofs of our branching laws is that the quotient of $\det[s(h_i\lambda_j)]_{0:m}$ or $\det[c(h_i\lambda_j)]_{0:m}$ devided by $\prod_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))$ is a sum of (a finite power series in $e(\lambda_0)$) $\times$ ($\det[s(k_i\lambda_j)]_{1:m}$ or $\det[c(k_i\lambda_j)]_{1:m}$). The next lemma enables us to excute the division. The substitution of the obtained result in (2.2), using the above expressions, completes the proofs of the branching laws.

**Lemma 2.1.** *Let $(h_0, h_1, \cdots, h_m)$ be a set of integers satisfying $h_0>h_1>\cdots>h_m\geqslant 0$. Then*

$$\begin{aligned}(2.3)\qquad &\det[s(h_i\lambda_j)]_{0:m}/\prod_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))\\ &=(s(\lambda_0))^{-m}\Sigma(\prod_{i=0}^{m}s(l_i\lambda_0))\cdot\det[s(k_i\lambda_j)]_{1:m}\,,\end{aligned}$$

$$\begin{aligned}(2.4)\qquad &\det[c(h_i\lambda_j)]_{0:m}/\prod_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))\\ &=(s(\lambda_0))^{-m}\Sigma(\prod_{i=0}^{m-1}(s(l_i\lambda_0))\cdot H\cdot c(l_m\lambda_0)\cdot\det[c(k_i\lambda_j)]_{1:m}\,,\end{aligned}$$

*where the summation is taken over all the sets of integers $(k_1, k_2, \cdots, k_m)$ satisfying $k_1>k_2>\cdots>k_m\geqslant 0$ and*

$$\begin{aligned}(2.5)\qquad &h_{i-1}>k_i>h_{i+1}\ \textit{for}\ 1\leqslant i\leqslant m-1\,,\\ &h_{m-1}>k_m\ \ (\geqslant 0)\,,\end{aligned}$$

*and integers* $l_0, l_1, \cdots, l_m$ *are defined by* (1.5) *from* $h_0, h_1, \cdots, h_m$ *and* $k_1, k_2, \cdots, k_m$, *and further*

$$H = \begin{cases} 1 & \text{for } k_m > 0\,, \\ 1/2 & \text{for } k_m = 0\,. \end{cases}$$

*The equalities* (2.3) *and* (2.4) *are also valid when* $(h_0, h_1, \cdots, h_m)$ *is a set of integers*$+1/2$ *satisfying* $h_0 > h_1 > \cdots > h_m > 0$. *Then the summations should be taken over all the sets of integers*$+1/2$ $(k_1, k_2, \cdots, k_m)$ *satisfying* $k_1 > k_2 > \cdots > k_m > 0$ *and* (2.5).

REMARK. The assumption on $h_0, h_1, \cdots, h_m$ and $k_1, k_2, \cdots, k_m$ ensures us that $l_0, l_1, \cdots, l_{m-1}$ are positive integers.

Proof. We prove the case (2.4) where $(h_0, h_1, \cdots, h_m)$ is a set of integers. By sight changes, we can prove the other cases.

We transform $[c(h_i\lambda_j)]_{0:m}$ by subtracting "the $(i-1)$-th row $\times\, c(h_i\lambda_0)/c(h_{i-1}\lambda_0)$" from the $i$-th row in turn.

$$\begin{aligned} &\det[c(h_i\lambda_j)]_{0:m} \\ &= (\Pi_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \det[c(h_{i-1}\lambda_0)c(h_i\lambda_j) - c(h_i\lambda_0)c(h_{i-1}\lambda_j)]_{1:m} \\ &= (\Pi_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \\ &\quad \times \det[s((h_{i-1}+h_i)(\lambda_0+\lambda_j)/2)s((h_{i-1}-h_i)(\lambda_0-\lambda_j)/2) \\ &\qquad + s((h_{i-1}-h_i)(\lambda_0+\lambda_j)/2)s((h_{i-1}+h_i)(\lambda_0-\lambda_j)/2)]_{1:m}\,. \end{aligned}$$

We devide the $(i, j)$-element of the last matrix by $s((\lambda_0+\lambda_j)/2)\ s((\lambda_0-\lambda_j)/2)$. The result is

$$s(\lambda_0)^{-1} \Sigma P_i(k_i) c(k_i\lambda_j) \qquad (k_i \in Z)\,,$$

where $P_i(k)$ is given by

$$P_i(k) = \begin{cases} c(h_i\lambda_0)s((h_{i-1}-k)\lambda_0) & \text{if } h_{i-1} > k > h_i\,, \\ c(k\lambda_0)s((h_{i-1}-h_i)\lambda_0) & \text{if } h_i \geqslant k > 0\,, \\ s((h_{i-1}-h_i)\lambda_0) & \text{if } k = 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get

$$\begin{aligned} &\det[c(h_i\lambda_j)]_{0:m} / \Pi_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2)) \\ &= s(\lambda_0)^{-m}(\Pi_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \Sigma(\Pi_{i=1}^{m} P_i(k_i)) \det[c(k_i\lambda_j)]_{1:m} \\ &\qquad\qquad ((k_1, k_2, \cdots, k_m) \in Z^m) \\ &= s(\lambda_0)^{-m}(\Pi_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \Sigma \det[P_i(k_j)]_{1:m} \det[c(k_i\lambda_j)]_{1:m} \\ &\qquad\qquad (k_1 > k_2 > \cdots > k_m \geqslant 0)\,. \end{aligned}$$

Note that if $k_1, k_2, \cdots, k_m$ do not satisfy (2.5), then $\det[P_i(k_j)]_{1:m}$ vanishes. Indeed, if $k_i \geqslant h_{i-1}$ $(1 \leqslant i \leqslant m)$, then the first $i$ columns are linearly dependent. If $h_{i+1} \geqslant k_i$ $(1 \leqslant i \leqslant m-1)$, then the first $i+1$ rows are linearly dependent.

Assuming that $k_1, k_2, \cdots, k_m$ satisfy (2.5), we transform $[P_i(k_j)]_{1:m}$ by subtracting "the $(j-1)$-th column $\times c(k_j\lambda_0)/c(k_{j-1}\lambda_0)$" (or its half when $k_m=0$ and $j=m$) from the $j$-th column for $j=m, m-1, \cdots, 2$ in this order. The resulting matrix $[P_{ij}]_{1:m}$ is a tridiagonal matrix such that $P_{i,i+1}P_{i+1,i}=0$ for $1 \leqslant i \leqslant m-1$. This means that its determinant is equal to the product of the diagonal elements.

$$P_{ii} = c(h_{i-1}\lambda_0)s(l_{i-1}\lambda_0)c(p_i\lambda_0)/c(p_{i-1}\lambda_0)\,.$$

We defined $p_0, p_1, \cdots, p_m$ by $p_0=h_0$, $p_i=\min(h_i, k_i)$ for $1 \leqslant i \leqslant m$ $(p_m=l_m)$. Therefore

$$\begin{aligned}&\det[P_i(k_j)]_{1:m}\\ &= (\Pi_{i=0}^{m-1}c(h_i\lambda_0))\,(\Pi_{i=0}^{m-1}s(l_i\lambda_0))\cdot H\cdot c(p_m\lambda_0)/c(p_0\lambda_0)\\ &= (\Pi_{i=1}^{m-1}c(h_i\lambda_0))\,(\Pi_{i=0}^{m-1}s(l_i\lambda_0))\cdot H\cdot c(l_m\lambda_0)\,,\end{aligned}$$

which proves (2.4).

## 3. Decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$

We identify a complexified contangent space of $M=G/K$ at $o=[K]$ with $(\mathfrak{g}/\mathfrak{k})^*$, the dual space of $\mathfrak{g}/\mathfrak{k}$.

First we treat the case $G=SO(n+2)$ and $K=SO(2)\times SO(n)$.

The space $(\mathfrak{g}/\mathfrak{k})^*$ decomposes into two irreducible $K$-modules, $V_+$ and $V_-$, with the highest weights $\lambda_0+\lambda_1$ and $-\lambda_0+\lambda_1$. This decomposition of $(\mathfrak{g}/\mathfrak{k})^*$ gives a rough decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$:

$$\Lambda^p(\mathfrak{g}/\mathfrak{k})^* \cong \Sigma\Lambda^{r,s} \qquad (r+s=p)\,, \tag{3.1}$$

where $\Lambda^{r,s}=(\Lambda^r V_+)\otimes(\Lambda^s V_-)$. Then the $SO(2)$-parts of weights in $\Lambda^{r,s}$ are $(r-s)\lambda_0$. In order to decompose $\Lambda^{r,s}$ as a $K$-module, we should decompose it as an $SO(n)$-module.

Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_m$ be the fundamental weights of $SO(n)$ dual to the simple roots $\alpha_1, \alpha_2, \cdots, \alpha_m$. We set $\Lambda_0=0$. We denote by $V(\Lambda)$ an irreducible $SO(n)$-module with the highest weight $\Lambda$.

The space $\Lambda^{r,s}$ is isomorphic to $(\Lambda^r V(\Lambda_1))\otimes(\Lambda^s V(\Lambda_1))$ as an $SO(n)$-module. Since $\Lambda^{r,s}\cong\Lambda^{s,r}$ and $\Lambda^{r,s}\cong\Lambda^{n-r,s}$ as $SO(n)$-modules, we may restrict our attention to the case $0 \leqslant r \leqslant s \leqslant m$.

When $n=2m$, we define $V_{i,j}$ by

$$\begin{aligned}V_{i,j} &= V(\Lambda_i+\Lambda_j) \qquad \text{for } 0 \leqslant i \leqslant j \leqslant m-2\,,\\ V_{i,m-1} &= V(\Lambda_i+\Lambda_{m-1}+\Lambda_m) \qquad \text{for } 0 \leqslant i \leqslant m-2\,,\\ V_{m-1,m-1} &= V(2\Lambda_{m-1}+2\Lambda_m)\,,\end{aligned}$$

$$
\begin{aligned}
V_{i,m} &= V(\Lambda_i+2\Lambda_{m-1})\oplus V(\Lambda_i+2\Lambda_m) \qquad \text{for } 0\leqslant i\leqslant m-2\,,\\
V_{m-1,m} &= V(3\Lambda_{m-1}+\Lambda_m)\oplus V(\Lambda_{m-1}+3\Lambda_m)\,,\\
V_{m,m} &= V(4\Lambda_{m-1})\oplus V(4\Lambda_m)\,,\\
V_{i,j} &= V_{i,n-j} \qquad \text{for } m+1\leqslant j\leqslant n-i\,.
\end{aligned}
$$

When $n=2m+1$, we define $V_{i,j}$ by

$$
\begin{aligned}
V_{i,j} &= V(\Lambda_i+\Lambda_j) \qquad \text{for } 0\leqslant i\leqslant j\leqslant m-1\,,\\
V_{i,m} &= V(\Lambda_i+2\Lambda_m) \qquad \text{for } 0\leqslant i\leqslant m-1\,,\\
V_{m,m} &= V(4\Lambda_m)\,,\\
V_{i,j} &= V_{i,n-j} \qquad \text{for } m+1\leqslant j\leqslant n-i\,.
\end{aligned}
$$

**Proposition 3.1.** *An $SO(n)$-module $\Lambda^{r,s}(0\leqslant r\leqslant s\leqslant m)$ decomposes into irreducible modules as follows:*

$$\Lambda^{r,s}\cong\Sigma V_{i,j} \quad ((i,j)\in S)\,,$$

*where the set $S$ consists of pairs of non-negative integers $(i,j)$ satisfying $s-r\leqslant j-i$, $i+j\leqslant r+s$ and $i+j\equiv r+s$ (mod 2).*

This proposition and (3.1) give an $SO(n)$-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$, which is also the $K$-irreducible decomposition.

The proof of Proposition 3.1 resembles that of the primitive decomposition of $\Lambda^p(C^n+\bar{C}^n)$ via $U(n)$ and uses it.

The $SO(n)$-module $V(\Lambda_1)$ is isomorphic to $C^n$, the complexification of $R^n$ with a canonical $SO(n)$-action, and posesses a natural $SO(n)$-invariant symmetric inner product. We take an orthonormal basis $\{x_i\}$ $(1\leqslant i\leqslant n)$ in $R^n$. Then $\Omega=\Sigma_{i=1}^n x_i\otimes x_i$ is the unique $SO(n)$-invariant element in $V(\Lambda_1)\otimes V(\Lambda_1)$ up to a constant factor. We set $e_i=(x_{2i-1}-\sqrt{-1}x_{2i})/\sqrt{2}$, $e_{n-i+1}=(x_{2i-1}+\sqrt{-1}x_{2i})/\sqrt{2}$ for $1\leqslant i\leqslant m$ and $e_{m+1}=x_n$ when $n=2m+1$. Then we have for $H\in\mathfrak{t}\cap\mathfrak{o}(n,C)$

$$
\begin{aligned}
\rho(H)(e_i) &= \lambda_i(H)e_i \qquad \text{for } 1\leqslant i\leqslant m\,,\\
\rho(H)(e_{n-i+1}) &= -\lambda_i(H) \qquad \text{for } 1\leqslant i\leqslant m\,,\\
\rho(H)(e_{m+1}) &= 0 \qquad \text{when } n=2m+1\,,
\end{aligned}
$$

where $\rho$ denotes the action of $\mathfrak{o}(n,C)$. We can rewrite $\Omega$ as $\Sigma_{i=1}^n e_i\otimes e_{n-i+1}$. We define an $SO(n)$-homomorphism

$$L\colon\ \Lambda^{r,s}\to\Lambda^{r+1,s+1}$$

by $L\omega=\Omega\wedge\omega$ $(\omega\in\Lambda^{r,s})$.

**Lemma 3.2.** *For $r+s<n$ $(0\leqslant r,\ s\leqslant n)$, $L\colon\Lambda^{r,s}\to\Lambda^{r+1,s+1}$ is injective.*

In fact, $L^{n-r-s}\colon \Lambda^{r,s}\to\Lambda^{n-s,n-r}$ is an $SO(n)$-isomorphism. For the proof, see Weil [5]. Notice that the $SO(n)$-action on $C^n$ can be extended to $U(n)$-actions in two manners; a canonical one and a complex conjugate one. When we take a canonical action for a $U(n)$-action on $V_+$ and a complex conjugate action for a $U(n)$-action on $V_-$, $L$ is the same $U(n)$-homomorphism used in [5].

There is an $SO(n)$-isomorphism

$$*\colon\ \Lambda^p V(\Lambda_1)\to\Lambda^{n-p}V(\Lambda_1)$$

given by

$$(*\alpha,\beta)e_1\wedge e_2\wedge\cdots\wedge e_n=\alpha\wedge\beta,\ \alpha\in\Lambda^pV(\Lambda_1),\ \beta\in\Lambda^{n-p}V(\Lambda_1)\,,$$

where ( , ) denotes the symmetric inner product. If $(i_1,i_2,\cdots,i_n)$ is a permutation of $(1,2,\cdots,n)$,

$$*(e_{i_1}\wedge\cdots\wedge e_{i_r})=\mathrm{sgn}(i_1,i_2,\cdots,i_n)e_{n-i_{r+1}+1}\wedge\cdots\wedge e_{n-i_n+1}\,.$$

We define an $SO(n)$-homomorphism

$$T\colon\ \Lambda^{r,s}\to\Lambda^{r+1,s-1}$$

by the composition of the following three $SO(n)$-homomorphisms:

$$\begin{aligned}(-1)^{s-1}Id\otimes *\colon&\ \Lambda^{r,s}\to\Lambda^{r,n-s}\,,\\ L\colon&\ \Lambda^{r,n-s}\to\Lambda^{r+1,n-s+1}\,,\\ Id\otimes *^{-1}\colon&\ \Lambda^{r+1,n-s+1}\to\Lambda^{r+1,s-1}\,.\end{aligned}$$

An explicit formula for $T$ is given by

$$\begin{aligned}&T(e_{i_1}\wedge\cdots\wedge e_{i_r}\otimes e_{j_1}\wedge\cdots\wedge e_{j_s})\\ &=\textstyle\sum_{t=1}^{s}(-1)^{t-1}e_{j_t}\wedge e_{i_1}\wedge\cdots\wedge e_{i_r}\otimes e_{j_1}\wedge\cdots\wedge\widehat{e_{j_t}}\wedge\cdots\wedge e_{j_s}\,.\end{aligned}$$

The following lemmas are easily verified.

**Lemma 3.3.** *For $0\leqslant r<s\leqslant n$, $T\colon \Lambda^{r,s}\to\Lambda^{r+1,s-1}$ is an injective $SO(n)$-homomorphism.*

**Lemma 3.4.** *For $2\leqslant r\leqslant s\leqslant n-r$, the following diagram commutes:*

$$\begin{array}{ccc}\Lambda^{r-2,s} & \xrightarrow{\ L\ } & \Lambda^{r-1,s+1}\\ T\big\downarrow & & T\big\downarrow\\ \Lambda^{r-1,s-1} & \xrightarrow{\ L\ } & \Lambda^{r,s}\,.\end{array}$$

**Lemma 3.5.** *Let $T^*$ be the adjoint of $T$ with respect to the invariant symmetric inner product.*

*a) For $2\leqslant r\leqslant s\leqslant n-r$, the following diagram commutes:*

$$\begin{array}{ccc}
\Lambda^{r-2,s} & \xrightarrow{\ L\ } & \Lambda^{r-1,s+1} \\
T^*\uparrow & & T^*\uparrow \\
\Lambda^{r-1,s-1} & \xrightarrow{\ L\ } & \Lambda^{r,s} .
\end{array}$$

b) *For* $1 \leqslant s \leqslant r-1$, *the following diagram commutes*:

$$\begin{array}{ccc}
 & & \Lambda^{0,s+1} \\
 & \overset{0}{\nearrow} & T^*\uparrow \\
\Lambda^{0,s} & \xrightarrow{\ L\ } & \Lambda^{1,s}
\end{array} \qquad (0 \textit{ denotes } 0\textit{-map}).$$

Notice that an explicit formula for $T^*$ is given by

$$\begin{aligned} & T^*(e_{i_1}\wedge\cdots\wedge e_{i_r}\otimes e_{j_1}\wedge\cdots\wedge e_{j_s}) \\ &= \sum_{t=1}^{r}(-1)^{t-1} e_{i_1}\wedge\cdots\wedge\widehat{e_{i_t}}\wedge\cdots\wedge e_{i_r}\otimes e_{i_t}\wedge e_{j_1}\wedge\cdots\wedge e_{j_s} . \end{aligned}$$

From these lemmas we can deduce that $\Lambda^{r,s}$ contains submodules isomorphic to $\Lambda^{r-1,s-1}$ and $\Lambda^{r-1,s+1}$ with the intersection isomorphic to $\Lambda^{r-2,s}$ (or $\{0\}$ if $r=1$). The space $\Lambda^{r,s}$ must also contain $V_{r,s}$ which corresponds to the highest weight of $\Lambda^{r,s}$. It is obvious that $V_{r,s}$ can intersect with the sum of $\Lambda^{r-1,s-1}$ and $\Lambda^{r-1,s+1}$ only by $\{0\}$. Computing the dimension of the above modules, we can obtain

**Proposition 3.6.** *We have the following* $SO(n)$*-isomorphisms*:

$$\Lambda^{1,s} \cong V_{1,s} \oplus \Lambda^{0,s-1} \oplus \Lambda^{0,s+1} \quad (1 \leqslant s \leqslant m),$$
$$\Lambda^{r,s} \oplus \Lambda^{r-2,s} \cong V_{r,s} \oplus \Lambda^{r-1,s-1} \oplus \Lambda^{r-1,s+1} \quad (2 \leqslant r \leqslant s \leqslant m).$$

It is easy to see that this proposition is equivalent to Proposition 3.1.

REMARK. We may call $V_-$ the holomorphic part and $V_+$ the anti-holomorphic part by the following reason. Let $H_0$ be an element of $\mathfrak{t}$ satisfying $\lambda_0(H_0) = \sqrt{-1}$, $\lambda_i(H_0)=0$ for $1 \leqslant i \leqslant m$. Then $ad\ H_0$ defines a complex structure on $\mathfrak{g}/\mathfrak{k}$. The space $V_-$ is an eigenspace of $ad\ H_0$ in $(\mathfrak{g}/\mathfrak{k})^*$ with an eigenvalue $-\sqrt{-1}$ and the space $V_+$ is one with an eigenvalue $\sqrt{-1}$. Because $ad\ H_0$ commutes with the action of $K$, it defines on $M=G/K$ a $G$-invariant almost complex structure, with which the metric we assumed defines a Kaehler structure.

Note that Frobenius' reciprosity law gives an explicit correspondence between a $K$-submodule of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ and a $G$-submodule of $C^\infty(\Lambda^p M)$. In our case, the holomorphic [anti-holomorphic] part $V_-[V_+]$ corresponds to holomorphic [anti-holomorphic] forms and $V^{r,s}$ to forms of type $(s,r)$.

We proceed to the case $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$. The $K$-module $(\mathfrak{g}/\mathfrak{k})^*$ is an irreducible module with the highest weight $\lambda_0+\lambda_1$. We take a maximal torus $T$ in $Sp(1)$ whose complexified Lie algebra is contained in $\mathfrak{t}$.

We set $K'=T\times Sp(n)$. If we consider $(\mathfrak{g}/\mathfrak{k})^*$ as a $K'$-module, it decomposes into two irreducible $K'$-modules $V_+$ and $V_-$ with the highest weights $\lambda_0+\lambda_1$ and $-\lambda_0+\lambda_1$. We first study a $K'$-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ and next reconstruct a $K$-irreducible decomposition.

First we have the following rough decomposition as $K'$-modules:

$$\Lambda^p(\mathfrak{g}/\mathfrak{k})^* \cong \Sigma \Lambda^{r,s} \quad (r+s=p),$$

where $\Lambda^{r,s}=(\Lambda^r V_+)\otimes(\Lambda^s V_-)$. The $T$-parts of weights in $\Lambda^{r,s}$ are $(r-s)\lambda_0$. We should decompose $\Lambda^{r,s}$ as an $Sp(n)$-module. Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be the fundamental weights of $Sp(n)$ dual to the simple roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. We set $\Lambda_0=0$. Both $V_+$ and $V_-$ are irreducible $Sp(n)$-modules with the same highest weight $\Lambda_1=\lambda_1$. We denote by $V(\Lambda)$ an irreducible $Sp(n)$-module with the highest weight $\Lambda$.

**Proposition 3.7.** *For* $0\leqslant r\leqslant n$, *we have*

$$\begin{aligned}\Lambda^r V(\Lambda_1) &\cong V(\Lambda_r)\oplus V(\Lambda_{r-2})\oplus\cdots\oplus V(\Lambda_1) \quad \textit{when } r=\textit{odd},\\ &\cong V(\Lambda_r)\oplus V(\Lambda_{r-2})\oplus\cdots\oplus V(\Lambda_0) \quad \textit{when } r=\textit{even};\\ \Lambda^r V(\Lambda_1) &\cong \Lambda^{2n-r} V(\Lambda_1)\,.\end{aligned}$$

Proof. The $Sp(n)$-module $V(\Lambda_1)$ is isomorphic to $C^{2n}$, the complexification of $R^{2n}$ with a canonical $Sp(n)$-action, and posesses natural $Sp(n)$-invariant inner product and symplectic form $\omega$. We take an orthonormal basis $\{x_i\}$ $(1\leqslant i\leqslant 2n)$ in $R^{2n}$, which satisfies $\omega(x_i, x_j)=0$, $\omega(x_{n+i}, x_{n+j})=0$, $\omega(x_i, x_{n+j})=\delta_{ij}$ for $1\leqslant i, j\leqslant n$. We set $\Omega=\Sigma_{i=1}^n x_i\wedge x_{n+i}$, which is the unique $Sp(n)$-invariant element in $\Lambda^2 V(\Lambda_1)$ up to a constant factor. We define an $Sp(n)$-homomorphism $L\colon \Lambda^p V(\Lambda_1)\to\Lambda^{p+2}V(\Lambda_1)$ by $L\alpha=\Omega\wedge\alpha$ $(\alpha\in\Lambda^p V(\Lambda_1))$. Then $L$ is injective for $0\leqslant p<n$, as is seen in the proof of the primitive decomposition in [5]. The space $\Lambda^p V(\Lambda_1)$ includes a submodule isomorphic to $\Lambda^{p-2}V(\Lambda_1)$ and one isomorphic to $V(\Lambda_p)$, and they can intersect only by $\{0\}$. Computing the dimensions of these modules, we can prove

$$\Lambda^p V(\Lambda_1)\cong V(\Lambda_p)\oplus\Lambda^{p-2}V(\Lambda_1) \quad (2\leqslant p\leqslant n)\,,$$

which is equivalent to the top half of the proposition.

The remainder is obvious.

Thus to decompose $\Lambda^{r,s}$ as $Sp(n)$-modules, we have only to decompose $V(\Lambda_r)\otimes V(\Lambda_s)$ for $0\leqslant r\leqslant s\leqslant n$.

**Proposition 3.8.** *An* $Sp(n)$-*module* $V(\Lambda_r)\otimes V(\Lambda_s)$ $(0\leqslant r\leqslant s\leqslant n)$ *decomposes into irreducible modules as follows*:

$$V(\Lambda_r)\otimes V(\Lambda_s)\cong\Sigma V(\Lambda_i+\Lambda_j) \quad ((i,j)\in S)\,,$$

*where the set $S$ consists of pairs of non-negative integers $(i,j)$ satisfying* $s-r\leqslant j-i\leqslant 2n-s-r$, $i+j\leqslant r+s$ *and* $i+j\equiv r+s$ (mod 2).

Proof. As in the $SO(n)$ case, it is enough to prove

$$(V(\Lambda_p)\otimes V(\Lambda_q))\oplus(V(\Lambda_{p-2})\otimes V(\Lambda_q))$$
$$\cong V(\Lambda_p+\Lambda_q)\oplus(V(\Lambda_{p-1})\otimes V(\Lambda_{q-1}))\oplus(V(\Lambda_{p-1})\otimes V(\Lambda_{q+1}))$$
$$(0\leqslant p\leqslant q\leqslant n),$$

where the terms including $V(\Lambda_r)$ with $r<0$ or $r>n$ should be omitted. It is equivalent to the following relation among the formal characters: $(\chi=\chi_{Sp(n)})$

$$\chi(\Lambda_p)\chi(\Lambda_q)+\chi(\Lambda_{p-2})\chi(\Lambda_q)$$
$$=\chi(\Lambda_p+\Lambda_q)+\chi(\Lambda_{p-1})\chi(\Lambda_{q-1})+\chi(\Lambda_{p-1})\chi(\Lambda_{q+1}).$$

We can rewrite the above, using Weyl's character formula. We set $\xi=\xi_{Sp(n)}$ and $\delta=\delta_{Sp(n)}$ and factor out $(\xi(\delta))^2$. Then we have

$$\text{(3.2)}\qquad \xi(\Lambda_p+\delta)\xi(\Lambda_q+\delta)+\xi(\Lambda_{p-2}+\delta)\xi(\Lambda_q+\delta)$$
$$=\xi(\Lambda_p+\Lambda_q+\delta)\xi(\delta)+\xi(\Lambda_{p-1}+\delta)\xi(\Lambda_{q-1}+\delta)+\xi(\Lambda_{p-1}+\delta)\xi(\Lambda_{q+1}+\delta).$$

Let $\Lambda_1', \Lambda_2', \cdots, \Lambda_n'$ be the fundamental weights for $SO(2n)$, $\xi'=\xi_{SO(2n)}$ and $\delta'=\delta_{SO(2n)}$. We consider $\xi$ and $\xi'$ as finite power sereis in $e(\lambda_1), e(\lambda_2), \cdots, e(\lambda_n)$. We can represent $\xi$ as $(\prod_{i=1}^n s(\lambda_i))\times$(a linear combination of $\xi'$). For example, $(D=\prod_{i=1}^n s(\lambda_i))$

$$\text{(3.3)}\qquad \xi(\delta)=D\cdot\xi'(\delta'),$$
$$\xi(\Lambda_1+\delta)=D\cdot\xi'(\Lambda_1'+\delta'),$$
$$\xi(\Lambda_p+\delta)=D\cdot(\xi'(\Lambda_p'+\delta')-\xi'(\Lambda_{p-2}'+\delta'))\quad(2\leqslant p\leqslant n-2),$$
$$\xi(\Lambda_{n-1}+\delta)=D\cdot(\xi'(\Lambda_{n-1}'+\Lambda_n'+\delta')-\xi'(\Lambda_{n-3}'+\delta')),$$
$$\xi(\Lambda_n+\delta)=D\cdot(\xi'(2\Lambda_{n-1}'+\delta')+\xi'(2\Lambda_n+\delta')-\xi'(\Lambda_{n-2}'+\delta')),$$
$$\xi(\Lambda_p+\Lambda_q+\delta)=D\cdot(\xi'(\Lambda_p'+\Lambda_q'+\delta')-\xi'(\Lambda_{p-2}'+\Lambda_q'+\delta')$$
$$-\xi'(\Lambda_p'+\Lambda_{q-2}'+\delta')+\xi'(\Lambda_{p-2}'+\Lambda_{q-2}'+\delta))$$
$$(4\leqslant p+2\leqslant q\leqslant n-2).$$

On the other hand, Proposition 3.6 provides us relations among $\xi'$. For example,

$$\text{(3.4)}\qquad \xi'(\Lambda_p'+\delta')\xi'(\Lambda_q'+\delta')+\xi'(\Lambda_{p-2}'+\delta')\xi'(\Lambda_q'+\delta')$$
$$=\xi'(\Lambda_p'+\Lambda_q'+\delta')\xi'(\delta')+\xi'(\Lambda_{p-1}'+\delta')\xi'(\Lambda_{q-1}'+\delta')$$
$$+\xi'(\Lambda_{p-1}'+\delta')\xi'(\Lambda_{q+1}'+\delta')$$
$$(2\leqslant p\leqslant q\leqslant n-2).$$

By combining four equations of the (3.4) type, we can get an equation of the (3.2) type substituted the expressions (3.3).

Proposition 3.7 and 3.8 enables us to decompose $\Lambda^{r,s}$ as an $Sp(n)$-module, which completes the $K'$-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$.

Let us return to the $K$-irreducible decomposition. We note that an irreducible $K$-module with the highest weight $k\lambda_0+\Lambda$ decomposes into irreducible $K'$-modules with the highest weights $(k-2i)\lambda_0+\Lambda$ $(i=0, 1, \cdots, k)$. Conversely we gather the highest weights in the $K'$-irreducible decomposition in bunches of the above form: Take the highest weight $k\lambda_0+\Lambda$ of the biggest $T$-part for a fixed $Sp(n)$-part. Then make up the highest weights of the form $(k-2i)\lambda_0+\Lambda$ $(i=0, 1, \cdots, k)$ into a bunch. Next do the same in the remaining highest weights, and so on. This procedure exhausts the highest weights without fail and the member of the biggest $T$-part in each bunch gives the highest weight in the $K$-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$.

We present here a table of the highest weights of the irreducible $K$-submodules of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ when $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$ for some $p$.

| | | |
|---|---|---|
| $p=0$ | | 0. |
| $p=1$ | | $\lambda_0+\Lambda_1$. |
| $p=2$ | $n\geqslant 2$ | $2\lambda_0+\Lambda_2$, $2\lambda_0$, $2\Lambda_1$. |
| | $n=1$ | $2\lambda_0$, $2\Lambda_1$. |
| $p=3$ | $n\geqslant 3$ | $3\lambda_0+\Lambda_3$, $3\lambda_0+\Lambda_1$, $\lambda_0+\Lambda_1+\Lambda_2$, $\lambda_0+\Lambda_1$. |
| | $n=2$ | $3\lambda_0+\Lambda_1$, $\lambda_0+\Lambda_1+\Lambda_2$, $\lambda_0+\Lambda_1$. |
| $p=4$ | $n\geqslant 4$ | $4\lambda_0+\Lambda_4$, $4\lambda_0+\Lambda_2$, $4\lambda_0$, $2\lambda_0+\Lambda_1+\Lambda_3$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+\Lambda_2$, $2\Lambda_2$, $\Lambda_2$, 0. |
| | $n=3$ | $4\lambda_0+\Lambda_2$, $4\lambda_0$, $2\lambda_0+\Lambda_1+\Lambda_3$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+\Lambda_2$, $2\Lambda_2$, $\Lambda_2$, 0. |
| | $n=2$ | $4\lambda_0$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+\Lambda_2$, $2\Lambda_2$, $\Lambda_2$, 0. |

REMARK. For a compact symmetric space $M=G/K$ with $G$ compact and semisimple, $\Delta^p$ preserves a decomposition of $C^\infty(\Lambda^p M)$ corresponding to a decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ under Frobenius' reciprocity law.

## 4. Examples

In the cases $G=SO(n+2)$ and $K=SO(2)\times SO(n)$, the cases $n=1$ and 2 are exceptional. When $n=1$, all our computation becomes trivial, and when $n=2$, we need some modification, for $K$ is abelian. Anyway, since $M=G/M$ is homothetic to the standard sphere $S^2$ when $n=1$, and to $S^2\times S^2$ when $n=2$, the spectra are well-known.

Our first example is the case $G=SO(5)$ and $K=SO(2)\times SO(3)$ $(n=3, m=1)$. We set $\Lambda_1=\lambda_1/2$, $\bar{\Lambda}_0=\lambda_0$ and $\bar{\Lambda}_1=(\lambda_0+\lambda_1)/2$. We denote by $I(k, s)$ for non-negative integers $k$ and $s$ the irreducible $G$-module with the highest weight $k\bar{\Lambda}_0+2s\bar{\Lambda}_1$. The Casimir operator acts on $I(k, s)$ by the multiplication of $-\{(k+s)(k+2s+3)+s(s+1)\}/6$. The dimension of $I(k, s)$ is $(2k+2s+3)(k+2s+2)(k+1)(2s+1)/6$.

We give in Table A the highest weights that irreducible $K$-submodules of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ have and the $G$-modules which includes an irreducible $K$-submodule of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ at least once. We denote by $\mu$ the multiplicity, the number of the times a $K$-module appearing in the $K$-irreducible decomposition of a $G$-module. Integers $r$ and $s$ may take any non-negative value and $\mu=1$ unless otherwise denoted.

**Table A.**

| $p$ | H.W. | $G$-module | |
|---|---|---|---|
| 0 | 0 | $I(2r, s)$ | |
| 1 | $\lambda_0+2\Lambda_1$ | $I(2r, s)$ $r\geqslant 1$ or $s\geqslant 1$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 1$ |
| | $-\lambda_0+2\Lambda_1$ | $I(2r+1, s)$ $s\geqslant 1$ | |
| 2 | $2\lambda_0+2\Lambda_1$ | $I(2r, s)$ $r\geqslant 1$, $s\geqslant 1$ | |
| | $-2\lambda_0+2\Lambda_1$ | $I(2r+1, s)$ $r\geqslant 1$ or $s\geqslant 1$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 1$ |
| | $4\Lambda_1$ | $I(2r, s)$ $r\geqslant 1$ or $s\geqslant 2$ | $\mu=2$ if $s=1$ |
| | | | $\mu=3$ if $r\geqslant 1$, $s\geqslant 2$ |
| | | $I(2r+1, s)$ $s\geqslant 1$ | $\mu=2$ if $s\geqslant 2$ |
| | $2\Lambda_1$ | $I(2r, s)$ $s\geqslant 1$ | |
| | | $I(2r+1, s)$ | $\mu=2$ if $s\geqslant 1$ |
| | 0 | see above | |
| 3 | $3\lambda_0$<br>$-3\lambda_0$ | $I(2r+1, s)$ $r\geqslant 1$ | |
| | $\lambda_0+4\Lambda_1$ | $I(2r, s)$ $r\geqslant 1$ or $s\geqslant 2$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 2$ |
| | $-\Lambda_0+4\Lambda_1$ | $I(2r+1, s)$ $r\geqslant 1$ or $s\geqslant 1$ | $\mu=2$ if $r=0$, $s\geqslant 2$<br>or $r\geqslant 1$, $s=1$ |
| | | | $\mu=3$ if $r\geqslant 1$, $s\geqslant 2$ |
| | $\lambda_0+2\Lambda_1$<br>$-\lambda_0+2\Lambda_1$ | see above | |
| | $\lambda_0$<br>$-\lambda_0$ | $I(2r+1, s)$ | |

Next we give the information on $\mathrm{Spec}^p(M)$ for the case $G=SO(6)$ and $K=SO(2)\times SO(4)$ ($n=4$, $m=2$) in Table B. We set $\Lambda_1=(\lambda_1-\lambda_2)/2$, $\Lambda_2=(\lambda_1+\lambda_2)/2$, $\bar{\Lambda}_0=\lambda_0$, $\bar{\Lambda}_1=(\lambda_0+\lambda_1-\lambda_2)/2$ and $\bar{\Lambda}_2=(\lambda_0+\lambda_1+\lambda_2)/2$. We denote by $I_i(r,s)$ for non-negative integers $r$ and $s$ the irreducible $G$-module with the highest weight given in Table B-1. There we have also given the eigenvalue of the $(-8)\times$Casimir operator and the dimension of the module. We list the highest weight of irreducible submodules of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ in Table B-2. Table B-3 indicates $G$-modules which contain a $K$-module in their $K$-irreducible decomposition at least once and the number of times they contain the $K$-module. In Table B-3, integers $r$ and $s$ may take any non-negative value and multiplicity $\mu=1$ unless otherwise denoted.

**Table B-1.**

| Module | Highest Weight | |
|---|---|---|
| $I_0(r, s)$ | $2r\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})$ | e.v. $=(2r+s)(2r+s+4)+s(s+2)$<br>dim. $=(2r+s+2)^2(2r+2s+3)(2r+1)(s+1)^2/12$ |
| $I_1(r, s)$ | $(2r+1)\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+2\overline{\Lambda_1}$ | e.v. $=(2r+s+2)(2r+s+6)+(s+2)^2$ |
| $I_2(r, s)$ | $(2r+1)\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+2\overline{\Lambda_2}$ | dim. $=(2r+2s+6)(2r+s+5)(2r+s+3)(2r+2)$<br>$(s+3)(s+1)/12$ |
| $I_3(r, s)$ | $2r\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+4\overline{\Lambda_1}$ | e.v. $=(2r+s+3)(2r+s+5)+(s+3)^2$ |
| $I_4(r, s)$ | $2r\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+4\overline{\Lambda_2}$ | dim. $=(2r+2s+7)(2r+s+6)(2r+s+2)(2r+1)$<br>$(s+5)(s+1)/12$ |

**Table B-2.**

| $p$ | Highest Weight |
|---|---|
| 0 | 0 |
| 1 | $\lambda_0+\Lambda_1+\Lambda_2$, $-\lambda_0+\Lambda_1+\Lambda_2$. |
| 2 | $2\lambda_0+2\Lambda_1$, $2\lambda_0+2\Lambda_2$, $2\Lambda_1+2\Lambda_2$, $2\Lambda_1$, $2\Lambda_2$, 0, $-2\lambda_0+2\Lambda_1$, $-2\lambda_0+2\Lambda_2$. |
| 3 | $3\lambda_0+\Lambda_1+\Lambda_2$, $\lambda_0+3\Lambda_1+\Lambda_2$, $\lambda_0+\Lambda_1+3\Lambda_2$, $\lambda_0+\Lambda_1+\Lambda_2$ (twice), $-\lambda_0+\Lambda_1+\Lambda_2$ (twice), $-\lambda_0+\Lambda_1+3\Lambda_2$, $-\lambda_0+3\Lambda_1+\Lambda_2$, $-3\lambda_0+\Lambda_1+\Lambda_2$. |
| 4 | $4\lambda_0$, $2\lambda_0+2\Lambda_1+2\Lambda_2$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+2\Lambda_2$, $2\lambda_0$, $4\Lambda_1$, $4\Lambda_2$, $2\Lambda_1+2\Lambda_2$ (twice), $2\Lambda_1$, $2\Lambda_2$, 0 (twice), $-2\lambda_0+2\Lambda_1+2\Lambda_2$, $-2\lambda_0+2\Lambda_1$, $-2\lambda_0+2\Lambda_2$, $-2\lambda_0$, $-4\lambda_0$. |

**Table B-3.**

| H.W. | $G$-module | |
|---|---|---|
| 0 | $I_0(r, s)$ | |
| $\lambda_0+\Lambda_1+\Lambda_2$ | $I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 1$ | $\mu=2$ if $r\geqslant 2$, $s=1$ |
| $-\lambda_0+\Lambda_1+\Lambda_2$ | $I_1(r, s)$ | |
| | $I_2(r, s)$ | |
| $2\lambda_0+2\Lambda_1$ | $I_0(r, s)$ $r\geqslant 1$, $s\geqslant 1$ | |
| $-2\lambda_0+2\Lambda_2$ | $I_1(r, s)$ | |
| | $I_2(r, s)$ $r\geqslant 1$ | |

**Table B-3 (continued).**

| H.W. | G-module | |
|---|---|---|
| $2\lambda_0+2\Lambda_2$ <br> $-2\lambda_0+2\Lambda_1$ | $I_0(r, s)$ $r\geqslant 1$, $s\geqslant 1$ | |
| | $I_1(r, s)$ $r\geqslant 1$ | |
| | $I_2(r, s)$ | |
| $2\Lambda_1+2\Lambda_2$ | $I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 2$ | $\mu=2$ if $r\geqslant 2$, $s=1$ |
| | | $\mu=3$ if $r\geqslant 1$, $s\geqslant 2$ |
| $2\Lambda_1$ <br> $2\Lambda_2$ | $I_0(r, s)$ $s\geqslant 1$ | |
| | $I_1(r, s)$ | |
| | $I_2(r, s)$ | |
| $3\lambda_0+\Lambda_1+\Lambda_2$ <br> $-3\lambda_0+\Lambda_1+\Lambda_2$ | $I_0(r, s)$ $r=1$, $s\geqslant 1$ or $r\geqslant 2$ | $\mu=2$ if $r\geqslant 2$, $s\geqslant 1$ |
| | $I_1(r, s)$ $r\geqslant 1$ | |
| | $I_2(r, s)$ $r\geqslant 1$ | |
| $\lambda_0+3\Lambda_1+\Lambda_2$ <br> $-\lambda_0+\Lambda_1+3\Lambda_2$ | $I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 2$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 2$ |
| | $I_1(r, s)$ | $\mu=2$ if $s\geqslant 1$ |
| | $I_2(r, s)$ $r\geqslant 1$ or $s\geqslant 1$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 1$ |
| $\lambda_0+\Lambda_1+3\Lambda_2$ <br> $-\lambda_0+3\Lambda_1+\Lambda_2$ | $I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 2$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 2$ |
| | $I_1(r, s)$ $r\geqslant 1$ or $s\geqslant 1$ | $\mu=2$ if $r\geqslant 1$, $s\geqslant 1$ |
| | $I_2(r, s)$ | $\mu=2$ if $s\geqslant 1$ |
| $4\lambda_0$ <br> $-4\lambda_0$ | $I_0(r, s)$ $r\geqslant 2$ | |
| $2\lambda_0+2\Lambda_1+2\Lambda_2$ <br> $-2\lambda_0+2\Lambda_1+2\Lambda_2$ | $I_0(r, s)$ $r+s\geqslant 2$ | $\mu=2$ if $r\geqslant 2$, $s=1$ or $r=1$, $s\geqslant 2$ |
| $2\lambda_0$ <br> $-2\lambda_0$ | $I_0(r, s)$ $r\geqslant 1$ | |
| $4\Lambda_1$ <br> $4\Lambda_2$ | $I_0(r, s)$ $s\geqslant 2$ | |
| | $I_1(r, s)$ $s\geqslant 1$ | |
| | $I_2(r, s)$ $s\geqslant 1$ | |
| | $I_3(r, s)$ $r\geqslant 1$ | |
| | $I_4(r, s)$ $r\geqslant 1$ | |

Our last example is the case $G=Sp(3)$ and $K=Sp(1)\times Sp(2)$ $(n=2)$. Notice that in the case $G=\mathrm{Sp}(2)$ and $K=Sp(1)\times Sp(1)$ $(n=1)$, $M=G/K$ is homothetic to the standard sphere $S^4$ and therefore $\mathrm{Spec}^p(M)$ has already given in [3]. We set $\Lambda_1=\lambda_1$, $\Lambda_2=\lambda_1+\lambda_2$, $\bar{\Lambda}_0=\lambda_0$, $\bar{\Lambda}_1=\lambda_0+\lambda_1$ and $\bar{\Lambda}_2=\lambda_0+\lambda_1+\lambda_2$. We denote by $I(r,s,t)$ for non-negative integers $r$, $s$ and $t$ the irreducible $G$-module with the highest weight $r\bar{\Lambda}_0+s\bar{\Lambda}_1+t\bar{\Lambda}_2$. The eigenvalue of $(-16)\times$ Casimir operator on $I(r,s,t)$ is $2s(s+2t+r+5)+r(r+2t+6)+3t(t+2)$ and the dimension of $I(r,s,t)$ is $(2s+r+2t+5)(s+r+2t+4)(s+r+t+3)(s+r+2)(s+2t+3)(s+t+2)(s+1)\times(r+1)(t+1)/720$. The meaning of each column of Table C is similar to that of Table A. Integers $k$ may take any non-negative value and multiplicity $\mu=1$ unless otherwise denoted.

**Table C.**

| $p$ | H.W. | $G$-module |
|---|---|---|
| 0 | 0 | $I(0, k, 0)$. |
| 1 | $\lambda_0+\Lambda_1$ | $I(0, k, 0)$ $k\geqslant1$, $I(1, k, 1)$, $I(2, k, 0)$. |
| 2 | $2\lambda_0+\Lambda_2$ | $I(1, k, 1)$, $I(2, k, 0)$ $k\geqslant1$, $I(3, k, 1)$. |
| | $2\lambda_0$ | $I(2, k, 0)$. |
| | $2\Lambda_1$ | $I(0, k, 2)$, $I(1, k, 1)$, $I(2, k, 0)$. |
| 3 | $3\lambda_0+\Lambda_1$ | $I(2, k, 0)$ $k\geqslant1$, $I(3, k, 1)$, $I(4, k, 0)$. |
| | $\lambda_0+\Lambda_1+\Lambda_2$ | $I(0, k, 0)$ $k\geqslant2$, $I(0, k, 2)$, $I(1, k, 1)$ $\mu=2$ if $k\geqslant1$, $I(2, k, 0)$ $k\geqslant1$, $I(2, k, 2)$, $I(3, k, 1)$ $k\geqslant1$. |
| | $\lambda_0+\Lambda_1$ | see above |
| 4 | $4\lambda_0$ | $I(4, k, 0)$. |
| | $2\lambda_0+2\Lambda_1$ | $I(0, k, 0)$ $k\geqslant2$, $I(1, k, 1)$ $k\geqslant1$, $I(2, k, 0)$ $k\geqslant1$, $I(2, k, 2)$, $I(3, k, 1)$, $I(4, k, 0)$. |
| | $2\lambda_0+\Lambda_2$ | see above |
| | $2\Lambda_2$ | $I(0, k, 0)$ $k\geqslant2$, $I(1, k, 1)$ $k\geqslant1$, $I(2, k, 2)$. |
| | $\Lambda_2$ | $I(0, k, 0)$ $k\geqslant1$, $I(1, k, 1)$. |
| | 0 | see above |

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