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DIRECT SUMS OF INDECOMPOSABLE MODULES

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1. Introduction. This paper studies direct sums $M=\oplus_{i\in I}M_i$ of indecomposable modules. Specifically, we give a number of necessary and sufficient conditions for such a sum to be quasi-continuous or continuous. This question was settled in [6], in a very satisfactory way, in case the index set I is finite, or the ring is right-noetherian, but the general case dealt with here is much more complicated.

Such sums $M=\oplus_{i\in I}M_i$ have been investigated in great detail, in a long series of papers since about 1970, by M. Harada and his collaborators, usually under the additional hypothesis that the M_i have local endomorphism rings (so that the Krull-Schmidt-Azumaya Theorem applies). One of the central results is the following:

Theorem 1 ([3], p. 22). *For a module with a decomposition $M=\oplus_{i\in I}M_i$, and with all* endo(M_i) *local, the following statements are equivalent*:

(1) *The decomposition is locally semi-T-nilpotent*;

(2) *it complements direct summands*;

(3) *any local direct summand of M is a direct summand.*

(The relevent terms are defined later on in this section.)

The present paper owes a great deal to the work of these authors. In particular, we refer to [5], [4] and the manuscript [7], results of which are announced in [8]. The reader will notice considerable overlap with the paper by K. Oshiro, and some of our arguments are lifted from it with little modification. Our main original contribution is the application of [1] (cf. our Lemma 1), from which we derive that any quasi-continuous $M=\oplus_{i\in I}M_i$ with indecomposable M_i is locally semi-T-nilpotent, and which allows us to free the results of [7] from their relativization with respect to uniform dimension (ie. the conditions $(\alpha-C_i)$ defined in Section 4), and to simplify the proofs. Moreover, applications of the main theorem of [6] yield a short proof of Theorem 8.

All our modules are right-modules over a ring R. m^0 denotes the annihilator in R, of the element $m\in M$. $X\subset' M$ and $Y\subset^{\oplus} M$ signify that X is an essential submodule, and Y a direct summand, of M. The sum of an independent family of submodules of M is called a local direct summand if every finite subsum is a direct summand. For a given decomposition $M=\oplus_{i\in I}M_i$ and a subset J of the index set I, $M(J)$ stands for $\oplus_{i\in J}M_i$.

A module M is called continuous if it satisfies
(C_1): for all $X\subset M$ there exists X^* with $X\subset' X^*\subset^{\oplus}M$; and
(C_2): for every $X\subset M$ which is isomorphic to a direct summand of M, $X\subset^{\oplus}M$ holds.
A module M is quasi-continuous if it satisfies (C_1) and
(C_3): for all X, $Y\subset^{\oplus}M$ with $X\cap Y=0$, $X\oplus Y\subset^{\oplus}M$ holds.

For details about these concepts, we refer to [6] and the literature cited there.

For a module with a given decomposition $M=\oplus_{i\in I}M_i$, one is also interested in the following conditions:
(C_3'): for every $X\subset^{\oplus}M$ and $J\subset I$ with $X\cap M(J)=0$, $X\oplus M(J)\subset^{\oplus}M$ holds. (This is obviously a consequence of (C_3).)
(C_5') (relative injectivity, the lettering is taken from [7]): $M(J)$ is $M(I-J)$-injective, for all $J\subset I$.
(lsTn) (local semi-T-nilpotency): for every sequence f_n: $M_{i_n}\to M_{i_{n+1}}$ $(n\in \boldsymbol{N})$ of non-isomorphisms, with all i_n distinct, and every $x\in M_{i_0}$, there exists $m\in \boldsymbol{N}$ with $f_m\cdots f_0(x)=0$.

We note that, if the conclusion of the Krull-Schmidt-Azumaya Theorem holds for M, and in particular if all endo(M_i) are local, then these conditions do not depend on the specific decomposition.

2. Ascending chain conditions.

In this section, we study an arbitrary decomposition $M=\oplus_{i\in I}M_i$. Our first lemma is a reformulation of ([1], Theorem 2.4):

Lemma 1. *$M=\oplus_{i\in I}M_i$ is B-injective if and only if each M_i is B-injective, and for every choice of $m_i\in M_i$ and $b\in B$ with $m_i^0\supset b^0$ $(i\in I)$ the net $\bigcap_{i\notin F}m_i^0$ (F any finite subset of I) becomes stationary.*

Motivated by this result and later applications, we introduce three ascending chain conditions, with respect to the given decomposition $M=\oplus_{i\in I}M_i$:
(A′): For any choice of $m_i\in M_i$ $(i\in I)$ the net $\bigcap_{i\notin F}m_i^0$ becomes stationary;
(A″): for any choice of $m_i\in M_i$ $(i\in I)$, with the additional condition that $m_j^0\subset m_i^0$ holds for a suitable j and all i, the net $\bigcap_{i\notin F}m_i^0$ becomes stationary;
(A‴): for any choice of $x_n\in M_{i_n}$ $(n\in \boldsymbol{N}$, i_n distinct) such that the sequence x_n^0 is increasing, this sequence becomes stationary.

It is clear that (A′) implies (A″), and (A″) implies (A‴) (in the second case, take $x_i=0$ for all $i\neq i_n$). It is also easy to see that (A′) and (A″) hold if and only if they hold for every countable subset of I.

The first corollary is important for later applications:

Corollary 2. *$M=\oplus_{i\in I}M_i$ satisfies* (C_5') *if and only if M_i is M_j-injective for all $i\neq j$, and* (A″) *holds. In particular, this is true if M is quasi-continuous.*

Proof. According to Lemma 1, (A″) together with the relative injectivities means that $M(I-j)$ is M_j-injective for all j. It follows trivially that $M(I-J)$ is M_j-injective for all $j\in J$, and any $J\subset I$. But then $M(I-J)$ is $M(J)$-injective, by ([1], Proposition 1.16 (2)). The converse is obvious; and it is well known that every quasi-continuous module $M=\oplus_{i\in I}M_i$ satisfies (C_5').

The first part of the second corollary extends a well known result of Faith ([2], Proposition 3):

Corollary 3. (1) *$M=\oplus_{i\in I}M_i$ is injective if and only if each M_i is injective, and* (A′) *holds.*

(2) *$M=\oplus_{i\in I}M_i$ is quasi-injective if and only if M_i is M_j-injective for all i and j, and* (A″) *holds.*

Proof. (1) Recall Baer's Lemma, and use $b=1\in R=B$ in Lemma 1.

(2) As in the previous proof, we obtain that $M(I-j)$ is M_j-injective for each j. As M_j is quasi-injective, that is M_j-injective, we conclude that $M=M(I-j)\oplus M_j$ is M_j-injective. As before we deduce that M is M-injective.

3. (Quasi-) Continuity and the condition (A″). In this section, we consider a decomposition $M=\oplus_{i\in I}M_i$ into indecomposable M_i. We start with a useful observation:

Theorem 4. *Let $M=\oplus_{i\in I}M_i$, where all M_i are uniform and M_j-injective for all $j\neq i$. Then* (lsTn) *is equivalent to* (A‴). *In particular both hold if M is quasi-continuous.*

Proof. Let (A‴) be given, and consider the situation of local semi-T-nilpotency, namely non-isomorphisms f_n: $M_{i_n}\to M_{i_{n+1}}$ with distinct i_n, and $x\in M_{i_0}$. Put $x_0=x$, $x_{n+1}=f_n\cdots f_0(x)$. Then obviously $x_n^0\subset x_{n+1}^0$, and therefore $x_m^0=x_{m+1}^0$ holds for some m, by (A‴). Thus $f_m\,|\,x_mR$: $x_mR\to x_{m+1}R$ is an isomorphism, and consequently f_m is a monomorphism since M_{i_m} is uniform (provided $x_m\neq 0$). As M_{i_m} is $M_{i_{m+1}}$-injective, f_m^{-1} extends to a homomorphism g: $M_{i_{m+1}}\to M_{i_m}$, which is again injective since $M_{i_{m+1}}$ is uniform. We conclude that g and f_m^{-1} coincide. Consequently f_m is surjective, contrary to the assumption that it is a non-isomorphism. We deduce $x_m=0$, that is $f_m\cdots f_0(x)=0$. This proves local semi-T-nilpotency.

Conversely suppose the decomposition $M=\oplus_{i\in I}M_i$ is locally semi-T-nilpotent, and consider elements $x_n\in M_{i_n}$ as in condition (A‴). If the sequence x_n^0 does not become stationary, then passing to a subsequence we may assume $x_n^0 \subsetneqq x_{n+1}^0$ for all n. Then the natural maps $x_n R \to x_{n+1} R$ are not injective, and hence their extensions by relative injectivity, f_n: $M_{i_n} \to M_{i_{n+1}}$, are non-isomorphisms. By local semi-T-nilpotency, $x_{m+1} = f_m \cdots f_0(x_0) = 0$ holds for some m. We conclude $x_{m+1}^0 = R$, in contradiction to $x_{m+1}^0 \subsetneqq x_{m+2}^0$.

Finally, if M is quasi-continuous, then it satisfies (A″) by Corollary 2, and this implies (A‴) as noted earlier.

Necessary conditions, for $M = \oplus_{i \in I} M_i$ to be (quasi-)continuous, are that the M_j are (quasi-)continuous and M_j-injective for all $j \neq i$. In [6], these conditions were shown to be also sufficient, provided the index set I is finite, or the ring R is right-noetherian. Here we shall show that in general, one has to add the condition (A″), in complete analogy with Corollary 3 (2). The next lemma was observed in [7]:

Lemma 5. *Let $M = \oplus_{i \in I} M_i$, with all M_i uniform. Suppose $X \subset M$ with $X \cap M(J) = 0$ for some $J \subset I$. Then there exists $J \subset K \subset I$ with $X \oplus M(K) \subset' M$.*

Proof. We take, by Zorn's Lemma, $J \subset K \subset I$ maximal with $X \cap M(K) = 0$. Then $X \oplus M(K)$ is essential in M, since the M_i are uniform.

Corollary 6. *Let $M = \oplus_{i \in I} M_i$, with all M_i uniform. Then* (C_3) *and* (C_3') *are equivalent; and in this situation, the decomposition complements direct summands.*

Proof. Let (C_3') be given, and let X, Y be two summands of M with $X \cap Y = 0$. By Lemma 5 there is $K \subset I$ with $X \oplus M(K) \subset' M$. Then (C_3') implies $X \oplus M(K) = M$, hence $X \cong M(I-K)$. Thus we can write $X = \oplus_{i \in I-K} X_i$ with $X_i \cong M_i$ for $i \in I-K$, as well as $X_i = M_i$ for $i \in K$. Then the decomposition $M = \oplus_{i \in I} X_i$ is isomorphic to the original one, hence inherits (C_3'). We conclude $X \oplus Y \subset^{\oplus} M$, that is ($C_3$). The converse is trivial.

The initial consideration has shown that $X \oplus M(K) = M$ if $X \subset^{\oplus} M$, demonstrating that the decomposition complements direct summands.

The next theorem improves ([7], Theorem A):

Theorem 7. *The following statements are equivalent, for $M = \oplus_{i \in I} M_i$ with indecomposable M_i:*

(1) *M is quasi-continuous;*

(2) *the M_i are uniform, and* (C_5') *holds;*

(3) *the M_i are quasi-continuous and M_j-injective for all $j \neq i$, and* (A″) *holds.*

Proof. (2) and (3) are equivalent, by Corollary 2, and since an indecomposable module is quasi-continuous if and only if it is uniform. That (1) implies (2), is well known.

Given (2), we show first the following claim: If $X\cap M(J)=0$ holds for $X\subset M$ and $J\subset I$, then there is $X\subset' X^*\subset M$ and $J\subset K\subset I$ such that $X^*\oplus M(K)=M$.

Indeed, Lemma 5 yields $J\subset K\subset I$ with $X\oplus M(K)\subset' M$. We write $M=M(I-K)\oplus M(K)$, with projections π_1 and π_2. Then ker $\pi_1=M(K)$ has zero intersection with X, and consequently $X\cong\pi_1 X\subset M(I-K)$ holds. This inclusion is easily seen to be essential. The mapping $\pi_1 X\ni\pi_1 x\to\pi_2 x\in M(K)$ is well defined, and extends by (C_5') to f: $M(I-K)\to M(K)$. We define $X^*=\{z+fz: z\in M(I-K)\}$. One readily checks that X is essentially contained in X^*, and that $X^*\oplus M(K)=M$ holds true. This completes the proof of our claim.

The special case $J=\emptyset$ of the claim yields (C_1). In another special case, namely when $X\subset^{\oplus} M$, we obtain $X=X^*$ hence (C_3'). As (C_3') is equivalent to (C_3) by Corollary 6, we have shown (1).

The following analogous characterization of continuity is related to ([7], Theorem 2.5):

Theorem 8. *The following statements are equivalent, for $M=\oplus_{i\in I}M_i$ with indecomposable M_i:*
(1) *M is continuous;*
(2) *M is quasi-continuous, and the M_i are continuous;*
(3) *the M_i are continuous and M_j-injective for all $j\neq i$, and* (A″) *holds.*

Proof. The equivalence of (2) and (3) follows from Theorem 7, and (1) implies (2) trivially.

To deduce (1) from (2), it suffices, by Lemma 11 of [6], to show that every essential monomorphism f: $M\to M$ is surjective.

We obtain $f(M)=\oplus_{i\in I}f(M_i)\subset M$, and hence by ($C_1$) $f(M_i)\subset' P_i\subset^{\oplus} M$. As $f(M_i)\cong M_i\subset' M_i\subset^{\oplus} M$ is trivially true, Theorem 4 of [6] yields $P_i\cong M_i$. We derive essential monomorphisms $M_i\cong f(M_i)\subset' P_i\cong M_i$, which must be isomorphisms since the M_i are (indecomposable and) continuous. We conclude $f(M_i)=P_i\subset^{\oplus} M$.

By (C_3), $f(M)=\oplus_{i\in I}f(M_i)$ is now a local direct summand of M. As $M=\oplus_{i\in I}M_i$ is locally semi-T-nilpotent by Theorem 4, Harada's Theorem quoted in the introduction yields $f(M)\subset^{\oplus} M$. (Alternatively, we could refer to our Theorem 13 in Section 5.) From $f(M)\subset' M$ we conclude now $f(M)=M$.

We conclude the section by listing examples which separate the ascending chain conditions (A′), (A″) and (A‴) from Section 2. Each of these examples is of the type $M=\oplus_{n\in N}M_n$, with indecomposable injective (and projective) M_n.

(1) Let $R=\hat{\boldsymbol{Z}}_p\ltimes C_{p^\infty}$, the split extension of the ring $\hat{\boldsymbol{Z}}_p$ of p-adic integers by the Prüfer group C_{p^∞}; and $M_n=R$ for all $n\in\boldsymbol{N}$.

This decomposition is not locally semi-T-nilpotent (since multiplication by p^n is nonzero on R for all n), hence it does not satisfy (A‴).

(2) ([9], p. 314) Let R be any perfect ring such that the injective hull $E(R_R)$ is projective, but $E({}_RR)$ is not. Write $E(R_R)=\oplus P_j$, a finite direct sum of indecomposables. Let the family of M_n consist of countably many copies of each P_j.

Then $M=\oplus_{n\in N}M_n$ is locally semi-T-nilpotent (since R is perfect) hence satisfies (A‴), but is not quasi-continuous hence does not satisfy (A″).

(3) Let $R=\Pi_{n\in N}K_n$, a product of fields; and $M_n=K_n$.

Here M is semisimple hence quasi-injective, hence satisfies (A″), but is not injective (the injective hull being R) hence does not satisfy (A′).

(4) Let R be any quasi-Frobenius ring. Write $R=\oplus P_j$, a finite direct sum of indecomposables, and let the family of M_n consist again of countably many copies of each P_j.

Then M is injective, and satisfies (A′).

4. The condition $(1-C_i)$. In this section we study direct sums $M=\oplus_{i\in I}M_i$ of uniform modules M_i.

K. Oshiro [7] introduced relaxations $(\alpha-C_i)$ ($i=1, 2, 3$) and $(\alpha-C_3')$, of the conditions (C_i), (C_3') defined in the introduction, and obtained by restricting to submodules X (and Y in case of (C_3)) of uniform dimension $\leqq\alpha$, for any cardinal α. We actually employ only the weakest of these restrictions, the one for $\alpha=1$, which is concerned with uniform submodules. Our results strengthen those in [7].

Theorem 9. *Let $M=\oplus_{i\in I}M_i$, with all M_i uniform, Then M is quasi-continuous if and only if $(1-C_1)$ and $(1-C_3')$ hold.*

Proof. The necessity of the two conditions is obvious. As to sufficiency, in view of Theorem 7, it is enough to show that $M(I-j)$ is M_j-injective for each j. Thus we consider a homomorphism f: $A\rightarrow M(I-j)$ from a non-zero submodule A of M_j. The module $B=\{a+fa:\ a\in A\}$ is isomorphic to A, hence non-zero and uniform. It is easily verified that $B\cap M(I-j)=0$ holds. By $(1-C_1)$ there exists $B\subset' B^*\subset^{\oplus}M$. Clearly B^* is again uniform and satisfies $B^*\cap M(I-j)=0$. Thus $(1-C_3')$ implies $B^*\oplus M(I-j)=M$. One checks that the restriction $-\pi|M_j$ of the projection π: $B^*\oplus M(I-j)\rightarrow M(I-j)$ extends f.

REMARK. Using Lemma 5, one readily obtains the following reformulation of $(1-C_3')$, in the present situation where the M_i are uniform: whenever X is a non-zero uniform summand of M with $X\cap M(I-j)=0$ for some j, then $X\oplus M(I-j)=M$.

If we add the assumption that endo(M_i) is local for all i, then $(1-C_3')$ can be replaced by the weaker $(1-C_3)$, as well as by a condition which follows from pairwise relative injectivity according to the next remark.

REMARK. If M_i, M_j are uniform, and M_i is M_j-injective, then every monomorphism $M_i \to M_j$ is surjective.

(Indeed, a monomorphism f: $M_i \to M_j$ extends to an isomorphism g: $E(M_i) \to E(M_j)$ between the injective hulls. As M_i is M_j-injective, $g^{-1}(M_j) \subset M_i$ holds. One deduces $M_j = gg^{-1}(M_j) \subset g(M_i) = f(M_i) \subset M_j$, hence $f(M_i) = M_j$.)

Theorem 10. *Let* $M = \oplus_{i \in I} M_i$, *with all* M_i *uniform and* endo(M_i) *local. Then the following statements are equivalent*:

(1) *M is quasi-continuous*;

(2) $(1-C_1)$ *and* $(1-C_3)$ *hold*;

(3) $(1-C_1)$ *holds, and monomorphisms* $M_i \to M_j$ *are surjective for* $i \neq j$.

Proof. (1) implies (2) trivially. Let (2) be given, and let f: $M_i \to M_j$ be injective, where $i \neq j$. Then $M_i^* = \{x + fx \colon x \in M_i\}$ is isomorphic to M_i hence uniform. One easily checks $M_i^* \oplus M_j = M_i \oplus M_j$ and $M_i \cap M_i^* = 0$. Therefore one has M_i, $M_i^* \subset^{\oplus} M$, and obtains $M_i \oplus M_i^* \subset^{\oplus} M$ from $(1-C_3)$. On the other hand, a straightforward calculation shows $M_i \oplus M_i^* \subset' M_i \oplus M_j$, and one concludes $M_i \oplus M_i^* = M_i \oplus M_j$. Thus any $y \in M_j$ can be written as $y = x_1 + (x_2 + fx_2)$; x_1, $x_2 \in M_i$. One concludes $y = fx_2$, demonstrating that f is surjective.

Finally, let (3) be given. The following lemma shows that $M(I-j)$ is M_j-injective for each j, and Theorem 7 yields (1).

The following lemma is one of the implications of ([5], Theorem 12). We include a relatively quick and direct proof, for the reader's convenience.

Lemma 11. *The condition* (3) *of* Theorem 10 *implies* (C_5').

Proof. We start by recalling that an element $x = \sum x_i \in M = \oplus_{i \in I} M_i$ is called smooth if it is non-zero, and all its non-zero components x_i have the same annihilator. It is easily seen that every non-zero element has a smooth multiple.

To verify (C_5'), we have to extend an arbitrary homomorphism f: $A \to M(I-j)$ from a non-zero submodule A of M_j, to all of M_j. We define $F = \{i \in I \colon \pi_i f \text{ injective}\}$, where the π_i are the natural projections of $M = \oplus_{i \in I} M_i$. We claim that for every $a \in A$ such that $a + fa$ is smooth, $F = \{i \in I \colon \pi_i fa \neq 0\}$ holds.

Indeed, any such a is non-zero. If $\pi_i fa = 0$, then $\pi_i f$ is certainly not injective. If $\pi_i fa \neq 0$, but $\pi_i fb = 0$ for some $0 \neq b \in A$, then $ar = bs \neq 0$ holds by uniformity of A. This implies $0 = \pi_i fbs = \pi_i far$, hence $r \in (\pi_i fa)^0 = a^0$ by smoothness, and hence $ar=0$, a contradiction.

In particular, since smooth elements $a+fa$ ($a\in A$) exist, by the initial remark, F is finite, and the support of any such smooth element $a+fa$ is $F\cup\{j\}$.

As $\pi_j f=0$, we have $f=\sum_{i\in F}\pi_i f+f^*$, where $f^*=\sum_{i\notin F\cup\{j\}}\pi_i f$. Thus it suffices to extend the $\pi_i f$ ($i\in F$) and f^*. We denote anyone of these maps by g.

We define $N=\{a+ga\colon a\in A\}$, isomorphic to A hence non-zero uniform, and obtain $N\subset' N^*\subset^{\oplus} M$ by $(1-C_1)$. N^* is uniform hence indecomposable, thus isomorphic to some M_k by the Krull-Schmidt-Azumaya Theorem. Consequently it has local endomorphism ring and the exchange property, and we obtain $M=N^*\oplus M(I-k)$.

In the case $g=f^*$, we have $k=j$ (since otherwise $N^*\cap M(I-k)$ contains $\ker f^*$; and $\ker f^*$ is non-zero as it contains each $a\in A$ for which $a+fa$ is smooth, by our description of F). In the other case $g=\pi_i f$ ($i\in F$), we have $k=i$ or j (since otherwise $N^*\cap M(I-k)$ contains the graph of $\pi_i f$).

If $k=j$, then the restriction $-\pi\,|\,M_j$ of the projection $\pi\colon N^*\oplus M(I-j)\to M(I-j)$ extends g. Similarly, if $k=i$ (whence $g=\pi_i f$ is injective), the restriction $-\mu\,|\,M_i$ of the projection $\mu\colon N^*\oplus M(I-i)\to M(I-i)\to M_j$ extends g^{-1}. This extension is injective by uniformity of M_i, hence bijective by assumption, so that it has an inverse which extends g.

The last and easy theorem of the section is similar to Theorem 10, but it concerns continuous modules.

Theorem 12. *The following statements are equivalent for* $M=\oplus_{i\in I}M_i$ *with all* M_i *uniform*:

(1) *M is continuous*;

(2) $(1-C_1)$ *and* $(1-C_2)$ *hold*;

(3) $(1-C_1)$ *holds, and monomorphisms* $M_i\to M_j$ *are surjective for all* i, j.

Proof. (1) implies (2) trivially. If (2) is given, then $(1-C_2)$ implies that the M_i are continuous and hence have local endomorphism ring. Then (3) follows from Theorem 10, if we can verify $(1-C_3)$.

But this also follows from $(1-C_2)$: If X, Y are two uniform summands of M with $X\cap Y=0$, consider the projection $\pi\colon M=X\oplus U\to U$. Then $Y\cap \ker\pi=Y\cap X=0$ yields $Y\cong\pi Y\subset U$, and $(1-C_2)$ shows $\pi Y\subset^{\oplus}M$ hence $\pi Y\oplus V=U$. One obtains $M=X\oplus U=X\oplus\pi Y\oplus V=X\oplus Y\oplus V$, that is $(1-C_3)$.

Finally, (3) implies again that the M_i are continuous hence have local endomorphism ring; and the Theorems 10 and 8 yield (1).

5. The extending property. We shall be concerned with the following property of a module M:

(E) (extending property for independent families of submodules): $\oplus_{j\in J}A_j\subset M$ implies the existence of $A_j\subset' A_j^*\subset M$ with $\oplus_{j\in J}A_j^*\subset^{\oplus}M$.

Our theorem slightly improves ([7], Theorem 3.5), cf. also ([8], Theorem 2). For completeness sake we include a full proof.

Theorem 13. *A module has the extending property* (E) *if and only if it is quasi-continuous, and is the direct sum of indecomposable submodules.*

Proof. (1) Let the module M satisfy (E). The special cases $|J|=1$ and $|J|=2$ of (E) yield (C_1) and (C_3), and show that M is quasi-continuous.

We choose, by Zorn's Lemma, a maximal direct sum of cyclic submodules, $\oplus C_j \subset M$. Clearly the inclusion is essential. By (E) we obtain $C_j \subset' C_j^* \subset M$ with $\oplus C_j^* = M$.

Each C_j^* is of finite uniform dimension. Indeed, otherwise one could find an infinite direct sum $\oplus S_k$ which is essential in C_j^*. As C_j^* is quasi-continuous, we obtain $S_k \subset' S_k^* \subset^{\oplus} C_j^*$, and (E) yields $\oplus S_k^* \subset^{\oplus} M$ hence $\oplus S_k^* = C_j^*$. This contradicts the fact that C_j^* is essential over the cyclic module C_j.

Thus each C_j^* contains an essential finite direct sum $\oplus S_k$ of uniform submodules S_k. By the argument just given, we obtain that each $C_j^* = \oplus S_k^*$ is a finite direct sum of indecomposables.

(2) Conversely we have to show the extending property for any $A = \oplus_{\alpha \in J} A_\alpha \subset M$. Adding A_0, maximal with $A_0 \cap A = 0$, we may assume $A \subset' M$. Using (C_1), we may replace the A_α by direct summands. Applying Corollary 6, we then obtain $A_\alpha \cong M(K_\alpha)$, and we may assume that the A_α are uniform. Thus we have reduced to the situation where all A_α are uniform direct summands of M, and where A is essential in M, and we have to show $A = M$.

Suppose $A \neq M$. Inductively we construct an increasing sequence of finite subsets F_n of J and subsets I_n of I such that $M = A(F_n) \oplus M(I_n)$, as well as distinct $i_n \in I_n$ and $x_n \in M_{i_n} - A$ such that the sequence of annihilators x_n^0 is strictly increasing, and finally $r_n \in R$ such that $0 \neq x_n r_n \in A(F_{n+1})$. Once this is done, we have a contradiction with (A'''), which is valid according to Theorem 4.

To begin with $n=0$, we take $F_0 = \emptyset$ and $I_0 = I$. The supposition $A \neq M$ allows us to pick $i_0 \in I = I_0$ and $x_0 \in M_{i_0} - A$. From $A \subset' M$ we obtain $r_0 \in R$ with $0 \neq x_0 r_0 \in A$.

To construct the corresponding quantities for $n+1$, we observe first that $x_n r_n \in A$ yields a finite $F_{n+1} \subset J$ with $x_n r_n \in A(F_{n+1})$. Obviously we may choose F_{n+1} to contain F_n. As $A(F_{n+1}) \subset^{\oplus} M$ holds by (C_3), Corollary 6 yields $I_{n+1} \subset I$ with $A(F_{n+1}) \oplus M(I_{n+1}) = M$.

We write $x_n = a + \sum y_i \in A(F_{n+1}) \oplus \oplus_{i \in I_{n+1}} M_i$. We note $x_n^0 \subsetneqq y_i^0$ for all $i \in I_{n+1}$ (since $x_n r = 0$ yields $y_i r = 0$, and since $r_n \notin x_n^0$ but $r_n \in y_i^0$ due to $x_n r_n \in A(F_{n+1})$). $x_n \notin A$ implies $y_k \notin A$ for some $k \in I_{n+1}$. We put $i_{n+1} = k$ and $x_{n+1} = y_k$, for such k. The validity of $i_{n+1} \in I_{n+1}$, $x_{n+1} \in M_{i_{n+i}} - A$ and $x_n^0 \subsetneqq x_{n+1}^0$ is then clear. If $i_{n+1} = i_m$ would hold for some $m \leqq n$, then we would obtain $x_m r_m \in M_{i_m} = M_{i_{n+1}} \subset M(I_{n+1})$ and $x_m r_m \in A(F_{m+1}) \subset A(F_{n+1})$, in contradiction to $x_m r_m \neq 0$. Finally, $A \subset' M$ yields the existence of $r_{n+1} \in R$ with $0 \neq x_{n+1} r_{n+1} \in A$.

REMARK. It seems worthwhile to point out here, that a quasi-continuous module which is the direct sum of indecomposable submodules, behaves in many ways as if these submodules had local endomorphism rings, though this need not be the case. Specifically, the assertion of the Krull-Schmidt-Azumaya Theorem holds true (this follows from [6], Proposition 9), as do the three statements of Harada's Theorem cited in the introduction ((lsTn) holds by Theorem 4, complementation of direct summands works by Corollary 6, and local direct summands become direct summands by Theorem 13).

We end by giving an explicit example of such a module: Let R be a commutative noetherian ring, and P_i $(i \in I)$ a family of pairwise incomparable non-maximal prime ideals. Then $M = \oplus_{i \in I} R/P_i$ is quasi-continuous by Theorem 7, as there are no non-zero maps between distinct R/P_i. In many instances, the R/P_i are not local (for example, for the polynomial ring R in two variables over a field).

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