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## NORMAL EMBEDDING OF SPHERES INTO $\mathbb{C}^n$

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### 1. Introduction

The notion of normal submanifold was introduced by J.C. Sikorav ([6]) as a weaker version of Lagrangian submanifold.

Polterovich ([5]) showed that if  $L$  is a closed normal non-Lagrangian submanifold of a symplectic manifold  $M$  and the Euler characteristic of  $L$  vanishes then its displacement energy  $e(L)$  vanishes.

The basic notions such as ‘normal’, ‘symplectic’, ‘weakly Lagrangian’ etc. are explained in Section 2 below and the definition of the displacement energy is provided in the later part of this section.

It is well known that  $S^1$  and  $S^3$  are totally real submanifolds of  $\mathbb{C}^1$  and  $\mathbb{C}^3$ , respectively. L. Polterovich ([5]) showed that if  $L$  is a totally real submanifold of a symplectic manifold  $(V, \omega)$  and  $L$  is parallelizable then  $L$  is normal. So  $S^1$  and  $S^3$  are normal submanifold of  $\mathbb{C}^1$  and  $\mathbb{C}^3$ , respectively. In fact  $S^1$  is a Lagrangian submanifold of  $\mathbb{C}^1$  and it follows that it is a normal submanifold. As for  $S^3$  we consider the standard embedding and explicitly construct in Section 4 below the Lagrangian subbundle of  $T\mathbb{C}^3|_{S^3}$  which is transverse to the tangent bundle.

The following two theorems are our main results which respectively answer the two questions: (a) Which  $S^n$  admits a normal embedding into  $\mathbb{C}^n$ ? (b) When the product of spheres admits a normal embedding into the complex Euclidean space?

**Theorem 1.1.**  $S^n$  admits a normal embedding into  $\mathbb{C}^n$  if and only if  $n$  is 1 or 3.

**Theorem 1.2.**  $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ ,  $n_i \geq 1$ ,  $i = 1, 2, \dots, k$ ,  $k \geq 2$ , admits a normal embedding into  $\mathbb{C}^{n_1+n_2+\cdots+n_k}$  if and only if some  $n_i$  is odd.

Note that H. Hofer ([3]) defined the *displacement energy* of a subset  $A$  of a sym-

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plectic manifold  $M$  as

$$\inf \left\{ \max_{M \times I} H - \min_{M \times I} H \mid H \in \mathcal{C} \text{ such that } g_H^1 A \cap A = \emptyset \right\}$$

where  $\mathcal{C}$  is the set of all smooth real valued functions which attain both maximum and minimum on the product  $M \times I$  of  $M$  with the closed unit interval  $I$  and  $g_H^1$  is the Hamiltonian flow at time 1 determined by  $H$ .

The normal embeddings of Theorem 1.2 are not necessarily the product of the standard embeddings (see [7]) and therefore their images may not be contained in a codimension 1 plane. Also the embedding is not Lagrangian unless some  $n_i$  is 1. Even if some  $n_i$  is 1 and the embedding is Lagrangian, we recall the fact that any Lagrangian embedding of a manifold of dimension greater than 1 and with vanishing Euler characteristic can be  $C^l$ -approximated for any  $l \geq 1$ , by non-Lagrangian normal embeddings ([5]). Therefore, Theorem 1.12 in [5] by L. Polterovich implies:

**Corollary 1.3.** *Assume  $k > 1$ . If some  $n_i$ ,  $i = 1, 2, \dots, k$ , is odd, the product of spheres,  $S^{n_1} \times S^{n_2} \times \dots \times S^{n_k}$ ,  $n_i \geq 1$ ,  $i = 1, 2, \dots, k$ ,  $k \geq 2$ , admits a normal embedding into  $\mathbb{C}^{n_1+n_2+\dots+n_k}$ , for which the displacement energy vanishes.*

## 2. Basic notions and facts

A smooth manifold  $M$  is called *symplectic* if there is a nondegenerate closed 2-form  $\omega$  on  $M$ . Such a 2-form is called a *symplectic form* or a *symplectic structure* on  $M$ . It follows that  $\dim M$  should be even if  $M$  is symplectic.

On the other hand a vector bundle of finite rank is referred to as a *symplectic vector bundle* if it is considered with a fixed symplectic two form. A subbundle  $\eta$  of a symplectic vector bundle  $\xi$  is a *Lagrangian subbundle* if  $2(\text{rank } \eta) = \text{rank } \xi$  and the restriction of the symplectic form to  $\eta$  is the zero form.

Let  $M$  be a symplectic manifold of dimension  $2n$  with a symplectic structure  $\omega$ . Let  $L$  be a smooth manifold of dimension  $n$  and let  $f: L \rightarrow M$  be an embedding (resp. immersion). We call  $f$  a *Lagrangian embedding* (resp. *immersion*) if the tangent bundle  $TL$  of  $L$  is a Lagrangian subbundle of the symplectic vector bundle  $f^*TM$  with the symplectic form  $f^*\omega$ . We call  $f$  a *normal embedding* (resp. *immersion*) if there is a Lagrangian subbundle  $\mathbb{L}$  of  $f^*TM$  which is transverse to  $TL$ . Note that every Lagrangian submanifold  $L$  of  $M$  is normal.

We say that an embedding  $f: L \rightarrow M$  is weakly Lagrangian if  $TL \subset f^*TM$  is homotopic through  $n$ -dimensional subbundles to a Lagrangian subbundle ([4]) in  $f^*TM$ .

We will consider  $\mathbb{C}^n$  with the usual symplectic structure. A Lagrangian embedding or normal embedding must be understood as ‘into  $\mathbb{C}^n$ ’ unless otherwise specified.

### 3. Proofs

First of all we need the following.

**Lemma 3.1.** *Let  $f$  be a normal embedding of a smooth oriented  $n$ -dimensional manifold  $L$  into a symplectic  $2n$ -dimensional manifold  $M$ . Then*

$$TL \cong \nu_f$$

where  $TL$  is the tangent bundle of  $L$  and  $\nu_f$ , the normal bundle of  $f$ .

Proof. Since  $f$  is a normal embedding, there is a Lagrangian subbundle  $\mathbb{L}^n \subset f^*TM$  which is transverse to  $TL \subset f^*TM$ . In particular, we have:  $f^*TM = TL + \mathbb{L}$ . Since the quotient bundle  $f^*TM/TL$  is none other than  $\nu_f$ , we have  $\mathbb{L} \cong \nu_f$ .

Now let  $J$  be an almost complex structure on  $M$  compatible with the symplectic structure. Then we have  $TL + \mathbb{L} = f^*TM = J\mathbb{L} + \mathbb{L}$  and it follows that  $TL \cong f^*TM/\mathbb{L} \cong J\mathbb{L}$ . Thus we conclude that

$$TL \cong J\mathbb{L} \cong \mathbb{L} \cong \nu_f . \quad \square$$

**Corollary 3.2.** *If a smooth oriented closed  $n$ -manifold  $L$  admits a normal embedding into  $\mathbb{C}^n$ , then we have*

$$\chi(L) = 0$$

where  $\chi(L)$  is the Euler number of  $L$ .

Proof. Regard  $L$  as a normal submanifold of  $\mathbb{C}^n$  and let  $\nu$  denote the normal bundle. Consider the normal neighborhood  $N$  of  $L$ . Let  $D\nu$  and  $S\nu$  denote respectively the disk and the sphere bundles of  $\nu$ . Then one of the generator  $U$  of the integral cohomolgy group

$$H^n(D\nu, S\nu; \mathbb{Z}) \cong H^n(N, \partial N : \mathbb{Z}) \cong \mathbb{Z}$$

pulled back to  $H^n(N; \mathbb{Z}) \cong H^n(L; \mathbb{Z})$  is the Euler class of  $TL$ , presuming a suitable orientation of  $L$ , since  $\nu \cong TL$  by Lemma 3.1 above. The Euler class evaluated at the fundamental class of  $L$  is the Euler number of  $L$ . However  $U$  when pulled back to  $H^n(N)$  is the zero element, for we have the following commutative diagram:

$$\begin{array}{ccc} H^n(\mathbb{C}^n, \mathbb{C}^n - \text{int}N; \mathbb{Z}) & \longrightarrow & H^n(N, \partial N; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^n(\mathbb{C}^n; \mathbb{Z}) & \longrightarrow & H^n(N; \mathbb{Z}) \end{array}$$

where all the arrows come from the inclusions. □

Proof of Theorem 1.1 If  $f: S^n \rightarrow \mathbb{C}^n$  is an embedding, then the normal bundle  $\nu_f$  of  $f$  must be trivial since it is stably trivial and its Euler class vanishes. So by Lemma 3.1 the tangent bundle  $TS^n$  is trivial. Thus if  $n \neq 1, 3, 7$ ,  $S^n$  does not admit any normal embedding into  $\mathbb{C}^n$ .

On the other hand,  $S^1$  admits a Lagrangian embedding. Also by applying an observation of Polterovich ([5]),  $S^3$  has a normal embedding since  $S^3$  admits totally real embedding and it is parallelizable.

It remains to show that  $S^7$  does not admit any normal embedding, which is the assertion of Corollary 3.4 below.

The following is needed to show that  $S^7$  admits no normal embedding into  $\mathbb{C}^7$ , which however seems worth an observation on its own right.

**Theorem 3.3.** *Let  $M$  be a symplectic  $2n$ -manifold and  $L$  be a smooth  $n$ -manifold which admits a normal embedding into  $M$ . If  $L$  is parallelizable, then the embedding is weakly Lagrangian.*

Proof. We regard  $L$  as a normal submanifold of  $M$ . Let  $\mathbb{L}$  be a Lagrangian subbundle of  $TM|_L$  which is transverse to  $TL \subset TM|_L$ . Let  $J$  denote an almost complex structure of  $M$  compatible with the symplectic structure.

Then we have that  $TL \cong TM|_L/\mathbb{L}$  and  $J\mathbb{L} \cong TM|_L/\mathbb{L}$  (See the proof of Lemma 3.1). Thus we have:  $TL \cong J\mathbb{L} \cong \mathbb{L}$ .

In particular,  $\mathbb{L}$  is trivial.

Let  $\{e_1, e_2, \dots, e_n\}$  and  $\{f_1, f_2, \dots, f_n\}$  be global frames respectively of  $TL$  and  $\mathbb{L}$ . Then define a homotopy  $\mathbb{L}_t$ ,  $0 \leq t \leq 1$ , in  $TM|_L$  from  $TL$  to  $\mathbb{L}$  by defining  $\mathbb{L}_t$  as the subbundle generated by the frame:

$$\{\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)\}, \quad \gamma_i(t) = (1-t)e_i + tf_i, i = 1, 2, \dots, n.$$

It is straightforward to see that  $\{\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)\}$  is indeed a frame, that is,  $\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)$  are linearly independent at any point of  $L$  for all  $t \in [0, 1]$ .  $\square$

**Corollary 3.4.**  *$S^7$  does not admit any normal embedding into  $\mathbb{C}^7$ .*

Proof. Assume that  $S^7$  admits a normal embedding into  $\mathbb{C}^7$ . Then, since  $S^7$  is parallelizable, the normal embedding is weakly Lagrangian by Theorem 3.3 above. But according to Kawashima ([4]),  $S^n$  admits a weakly Lagrangian embedding if and only if  $n = 1, 3$ . This means that  $S^7$  does not admit any normal embedding.  $\square$

REMARK. (i) A totally real submanifold  $L$  of a symplectic manifold which is parallelizable is normal according to Polterovich. Theorem 3.3 further means that  $L$  is weakly Lagrangian.

(ii) Note that Theorem 3.3 together with our explicit construction in the next section of the Lagrangian subbundle transverse to  $TS^3$  for the standard embedding of  $S^3$  into  $\mathbb{C}^3$  proves that the standard embedding is weakly Lagrangian (cf. [4]).

Proof of Theorem 1.2. We prove the case when  $k = 2$  and the general case follows by an inductive argument.

If both  $m$  and  $n$  are even, then  $\chi(S^m \times S^n) \neq 0$ , by Corollary 3.2,  $S^m \times S^n$  does not admit any normal embedding.

If  $m$  or  $n$  is odd, then  $S^m \times S^n$  admits a totally real embedding into  $\mathbb{C}^{m+n}$  (cf. Example 1, [7]) and  $S^m \times S^n$  is parallelizable. Therefore, according to Polterovich ([5]),  $S^m \times S^n$  admits a normal embedding into  $\mathbb{C}^{m+n}$ .  $\square$

#### 4. A Lagrangian subbundle transverse to the tangent bundle of $S^3$

Three linearly independent tangent vector fields  $X_1, X_2, X_3$  of

$$S^3 = \{(x_1, x_2, x_3, x_4, 0, 0) \in \mathbb{C}^3 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

are defined as follows:

$$\begin{aligned} X_1(x) &= (-x_2, x_1, -x_4, x_3, 0, 0) \\ X_2(x) &= (-x_3, x_4, x_1, -x_2, 0, 0) \\ X_3(x) &= (-x_4, -x_3, x_2, x_1, 0, 0) . \end{aligned}$$

Now the three linearly independent normal vector fields on  $S^3$  are defined as follows:

$$\begin{aligned} N_1(x) &= (x_1, x_2, x_3, x_4, 0, 0) \\ N_2(x) &= (-x_1 - x_4, -x_2 - x_3, x_2 - x_3, x_1 - x_4, 1, 0) \\ N_3(x) &= (-x_1 + x_3, -x_2 - x_4, -x_1 - x_3, x_2 - x_4, 0, 1) . \end{aligned}$$

Then clearly  $N_1, N_2, N_3$  are not in the tangent space  $T_x S^3$ . In fact, we have that the determinant of the matrix  $(X_1, X_2, X_3, N_1, N_2, N_3)$  is  $-1$  and the standard symplectic form vanishes on the subspace generated by  $N_1, N_2, N_3$ . Thus the subbundle of  $T\mathbb{C}^3|_{S^3}$  generated by  $N_1, N_2, N_3$  is a Lagrangian subbundle transverse to the tangent bundle.

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