NORMAL EMBEDTHING OF SPHERES INTO $\mathbb{C}^n$

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1. Introduction

The notion of normal submanifold was introduced by J.C. Sikorav ([6]) as a weaker version of Lagrangian submanifold.

Polterovich ([5]) showed that if $L$ is a closed normal non-Lagrangian submanifold of a symplectic manifold $M$ and the Euler characteristic of $L$ vanishes then its displacement energy $e(L)$ vanishes.

The basic notions such as ‘normal’, ‘symplectic’, ‘weakly Lagrangian’ etc. are explained in Section 2 below and the definition of the displacement energy is provided in the later part of this section.

It is well known that $S^1$ and $S^3$ are totally real submanifolds of $\mathbb{C}^1$ and $\mathbb{C}^3$, respectively. L. Polterovich ([5]) showed that if $L$ is a totally real submanifold of a symplectic manifold $(V, \omega)$ and $L$ is parallelizable then $L$ is normal. So $S^1$ and $S^3$ are normal submanifold of $\mathbb{C}^1$ and $\mathbb{C}^3$, respectively. In fact $S^1$ is a Lagrangian submanifold of $\mathbb{C}^1$ and it follows that it is a normal submanifold. As for $S^3$ we consider the standard embedding and explicitly construct in Section 4 below the Lagrangian subbundle of $T^*\mathbb{C}^3|S^3$ which is transverse to the tangent bundle.

The following two theorems are our main results which respectively answer the two questions: (a) Which $S^n$ admits a normal embedding into $\mathbb{C}^n$? (b) When the product of spheres admits a normal embedding into the complex Euclidean space?

**Theorem 1.1.** $S^n$ admits a normal embedding into $\mathbb{C}^n$ if and only if $n$ is 1 or 3.

**Theorem 1.2.** $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_i \geq 1$, $i = 1, 2, \ldots, k$, $k \geq 2$, admits a normal embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ if and only if some $n_i$ is odd.

Note that H. Hofer ([3]) defined the displacement energy of a subset $A$ of a sym-
plectic manifold $M$ as

$$\inf \left\{ \text{Max}_{M \times I} H - \text{Min}_{M \times I} H \mid H \in \mathcal{C} \text{ such that } g^t_H A \cap A = \emptyset \right\}$$

where $\mathcal{C}$ is the set of all smooth real valued functions which attain both maximum and minimum on the product $M \times I$ of $M$ with the closed unit interval $I$ and $g^t_H$ is the Hamiltonian flow at time 1 determined by $H$.

The normal embeddings of Theorem 1.2 are not necessarily the product of the standard embeddings (see [7]) and therefore their images may not be contained in a codimension 1 plane. Also the embedding is not Lagrangian unless some $n_i$ is 1. Even if some $n_i$ is 1 and the embedding is Lagrangian, we recall the fact that any Lagrangian embedding of a manifold of dimension greater than 1 and with vanishing Euler characteristic can be $C^\ell$-approximated for any $\ell \geq 1$, by non-Lagrangian normal embeddings ([5]). Therefore, Theorem 1.12 in [5] by L. Polterovich implies:

**Corollary 1.3.** Assume $k > 1$. If some $n_i$, $i = 1, 2, \ldots, k$, is odd, the product of spheres, $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_i \geq 1$, $i = 1, 2, \ldots, k$, $k \geq 2$, admits a normal embedding into $\mathbb{C}^{n_1+\cdots+n_k}$, for which the displacement energy vanishes.

2. Basic notions and facts

A smooth manifold $M$ is called symplectic if there is a nondegenerate closed 2-form $\omega$ on $M$. Such a 2-form is called a symplectic form or a symplectic structure on $M$. It follows that dim $M$ should be even if $M$ is symplectic.

On the other hand a vector bundle of finite rank is referred to as a symplectic vector bundle if it is considered with a fixed symplectic two form. A subbundle $\eta$ of a symplectic vector bundle $\xi$ is a Lagrangian subbundle if 2 (rank $\eta$) = rank $\xi$ and the restriction of the symplectic form to $\eta$ is the zero form.

Let $M$ be a symplectic manifold of dimension $2n$ with a symplectic structure $\omega$. Let $L$ be a smooth manifold of dimension $n$ and let $f: L \to M$ be an embedding (resp. immersion). We call $f$ a Lagrangian embedding (resp. immersion) if the tangent bundle $TL$ of $L$ is a Lagrangian subbundle of the symplectic vector bundle $f^*TM$ with the symplectic form $f^*\omega$. We call $f$ a normal embedding (resp. immersion) if there is a Lagrangian subbundle $\mathcal{L}$ of $f^*TM$ which is transverse to $TL$. Note that every Lagrangian submanifold $L$ of $M$ is normal.

We say that an embedding $f: L \to M$ is weakly Lagrangian if $TL \subset f^*TM$ is homotopic through $n$-dimensional subbundles to a Lagrangian subbundle ([4]) in $f^*TM$.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or normal embedding must be understood as ‘into $\mathbb{C}^n$’ unless otherwise specified.
3. Proofs

First of all we need the following.

**Lemma 3.1.** Let $f$ be a normal embedding of a smooth oriented $n$-dimensional manifold $L$ into a symplectic $2n$-dimensional manifold $M$. Then

$$TL \cong \nu_f$$

where $TL$ is the tangent bundle of $L$ and $\nu_f$, the normal bundle of $f$.

Proof. Since $f$ is a normal embedding, there is a Lagrangian subbundle $\mathbb{L}^n \subset f^*TM$ which is transverse to $TL \subset f^*TM$. In particular, we have: $f^*TM = TL + \mathbb{L}$. Since the quotient bundle $f^*TM/TL$ is none other than $\nu_f$, we have $\mathbb{L} \cong \nu_f$.

Now let $J$ be an almost complex structure on $M$ compatible with the symplectic structure. Then we have $TL + \mathbb{L} = f^*TM = JL + \mathbb{L}$ and it follows that $TL \cong f^*TM/\mathbb{L} \cong JL$. Thus we conclude that

$$TL \cong JL \cong \mathbb{L} \cong \nu_f,$$

$\square$

**Corollary 3.2.** If a smooth oriented closed $n$-manifold $L$ admits a normal embedding into $\mathbb{C}^n$, then we have

$$\chi(L) = 0$$

where $\chi(L)$ is the Euler number of $L$.

Proof. Regard $L$ as a normal submanifold of $\mathbb{C}^n$ and let $\nu$ denote the normal bundle. Consider the normal neighborhood $N$ of $L$. Let $D\nu$ and $S\nu$ denote respectively the disk and the sphere bundles of $\nu$. Then one of the generator $U$ of the integral cohomology group

$$H^n(D\nu, S\nu; \mathbb{Z}) \cong H^n(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}$$

pulled back to $H^n(N; \mathbb{Z}) \cong H^n(L; \mathbb{Z})$ is the Euler class of $TL$, presuming a suitable orientation of $L$, since $\nu \cong TL$ by Lemma 3.1 above. The Euler class evaluated at the fundamental class of $L$ is the Euler number of $L$. However $U$ when pulled back to $H^n(N)$ is the zero element, for we have the following commutative diagram:

$$\begin{array}{ccc}
H^n(\mathbb{C}^n, \mathbb{C}^n - \text{int}N; \mathbb{Z}) & \longrightarrow & H^n(N, \partial N; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^n(\mathbb{C}^n; \mathbb{Z}) & \longrightarrow & H^n(N; \mathbb{Z})
\end{array}$$

where all the arrows come from the inclusions. $\square$
Proof of Theorem 1.1 If \( f : S^n \to \mathbb{C}^n \) is an embedding, then the normal bundle \( \nu_f \) of \( f \) must be trivial since it is stably trivial and its Euler class vanishes. So by Lemma 3.1 the tangent bundle \( TS^n \) is trivial. Thus if \( n \neq 1, 3, 7 \), \( S^n \) does not admit any normal embedding into \( \mathbb{C}^n \).

On the other hand, \( S^1 \) admits a Lagrangian embedding. Also by applying an observation of Polterovich ([5]), \( S^3 \) has a normal embedding since \( S^3 \) admits totally real embedding and it is parallelizable.

It remains to show that \( S^7 \) does not admit any normal embedding, which is the assertion of Corollary 3.4 below.

The following is needed to show that \( S^7 \) admits no normal embedding into \( \mathbb{C}^7 \), which however seems worth an observation on its own right.

**Theorem 3.3.** Let \( M \) be a symplectic \( 2n \)-manifold and \( L \) be a smooth \( n \)-manifold which admits a normal embedding into \( M \). If \( L \) is parallelizable, then the embedding is weakly Lagrangian.

Proof. We regard \( L \) as a normal submanifold of \( M \). Let \( \mathbb{L} \) be a Lagrangian subbundle of \( TM|_L \) which is transverse to \( TL \subset TM|_L \). Let \( J \) denote an almost complex structure of \( M \) compatible with the symplectic structure.

Then we have that \( TL \cong TM|_L/\mathbb{L} \) and \( JL \cong TM|_L/\mathbb{L} \) (See the proof of Lemma 3.1). Thus we have: \( TL \cong JL \cong \mathbb{L} \).

In particular, \( \mathbb{L} \) is trivial.

Let \( \{e_1, e_2, \ldots, e_n\} \) and \( \{f_1, f_2, \ldots, f_n\} \) be global frames respectively of \( TL \) and \( \mathbb{L} \). Then define a homotopy \( \mathbb{L}_t, 0 \leq t \leq 1 \), in \( TM|_L \) from \( TL \) to \( \mathbb{L} \) by defining \( \mathbb{L}_t \) as the subbundle generated by the frame:

\[
\{\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\}, \quad \gamma_i(t) = (1 - t)e_i + tf_i, i = 1, 2, \ldots, n.
\]

It is straightforward to see that \( \{\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\} \) is indeed a frame, that is, \( \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t) \) are linearly independent at any point of \( L \) for all \( t \in [0, 1] \).

**Corollary 3.4.** \( S^7 \) does not admit any normal embedding into \( \mathbb{C}^7 \).

Proof. Assume that \( S^7 \) admits a normal embedding into \( \mathbb{C}^7 \). Then, since \( S^7 \) is parallelizable, the normal embedding is weakly Lagrangian by Theorem 3.3 above. But according to Kawashima ([4]), \( S^7 \) admits a weakly Lagrangian embedding if and only if \( n = 1, 3 \). This means that \( S^7 \) does not admit any normal embedding.

**Remark.** (i) A totally real submanifold \( L \) of a symplectic manifold which is parallelizable is normal according to Polterovich. Theorem 3.3 further means that \( L \) is weakly Lagrangian.
(ii) Note that Theorem 3.3 together with our explicit construction in the next section of the Lagrangian subbundle transverse to $T S^3$ for the standard embedding of $S^3$ into $\mathbb{C}^3$ proves that the standard embedding is weakly Lagrangian (cf. [4]).

Proof of Theorem 1.2. We prove the case when $k = 2$ and the general case follows by an inductive argument.

If both $m$ and $n$ are even, then $\chi(S^m \times S^n) \neq 0$, by Corollary 3.2, $S^m \times S^n$ does not admit any normal embedding.

If $m$ or $n$ is odd, then $S^m \times S^n$ admits a totally real embedding into $\mathbb{C}^{mn}$ (cf. Example 1, [7]) and $S^m \times S^n$ is parallelizable. Therfore, according to Polterovich ([5]), $S^m \times S^n$ admits a normal embedding into $\mathbb{C}^{mn}$.

4. A Lagrangian subbundle transverse to the tangent bundle of $S^3$

Three linearly independent tangent vector fields $X_1$, $X_2$, $X_3$ of

$$S^3 = \{(x_1, x_2, x_3, x_4, 0, 0) \in \mathbb{C}^3 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

are defined as follows:

$$X_1(x) = (-x_2, x_1, -x_4, x_3, 0, 0)$$
$$X_2(x) = (-x_3, x_4, x_1, -x_2, 0, 0)$$
$$X_3(x) = (-x_4, -x_3, x_2, x_1, 0, 0) .$$

Now the three linearly independent normal vector fields on $S^3$ are defined as follows:

$$N_1(x) = (x_1, x_2, x_3, x_4, 0, 0)$$
$$N_2(x) = (-x_1 - x_4, -x_2 - x_3, x_2 - x_3, x_1 - x_4, 1, 0)$$
$$N_3(x) = (-x_1 + x_3, -x_2 - x_4, -x_1 - x_3, x_2 - x_4, 0, 1) .$$

Then clearly $N_1$, $N_2$, $N_3$ are not in the tangent space $T_x S^3$. In fact, we have that the determinant of the matrix $(X_1, X_2, X_3, N_1, N_2, N_3)$ is $-1$ and the standard symplectic form vanishes on the subspace generated by $N_1$, $N_2$, $N_3$. Thus the subbundle of $T \mathbb{C}^3|_{S^3}$ generated by $N_1$, $N_2$, $N_3$ is a Lagrangian subbundle transverse to the tangent bundle.

References


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