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ORDER OF LIOUVILLIAN ELEMENTS SATISFYING AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

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0. Introduction. Let k be an algebraically closed ordinary differential field of characteristic 0, and Ω be a universal extension of k. A finite chain of extending differential subfields $k=L_0\subset L_1\subset\cdots\subset L_n$ in Ω is called a Liouville chain over k if the following two conditions are satisfied:

(i) The field of constants of L_n is k_0, where k_0 is the field of constants of k;

(ii) For each $i(1\leqq i\leqq n)$ there exists a finite system of elements $w_1, w_2, \cdots, w_r$ of L_i which satisfies the following two conditions; either $w'_j\in L_{i-1}$ or w'_j/w_j is the derivative of an element of L_{i-1} for each j $(1\leqq j\leqq r)$, L_i is an algebraic extension of $L_{i-1}(w_1, w_2, \cdots, w_r)$ of finite degree.

Let z be an elemen of Ω. Then, z is called a liouvillian element over k if there exists a Liouville chain over k such that its end contains z. The following definition is due to Liouville [2] (cf. [8, p. 111]):

DEFINITION. A liouvillian element z over k is said to be of order m if m is the minimum of those n such that the end of a Liouville chain $L_0\subset\cdots\subset L_n$ over k contains z.

Let F be an algebraically irreducible element of the first order of the differential polynomial algebra $k\{u\}$ in a single indeterminate u over k. Suppose that z is a solution of $F=0$. Then, z is a generic point of the general solution of $F=0$ over k if and only if z is transcendental over k. Suppose that two liouvillian elements over k satisfy $F=0$ and that they are transcendental over k. Then, their orders are the same.

Theorem. *The order of a liouvillian element over k satisfying $F=0$ is at most three.*

For example, suppose that k is the algebraic closure of $k_0(x)$ with $x'=1$ and that $F=u'-\alpha u/x$, where $\alpha\in k_0$. Then, any non-trivial solution of $F=0$ is of the second order if α is not a rational number (cf. Liouville [2, pp. 94–98]).

REMARK 1. If we replace "liouvillian" by "generalized elementary" and modify the definition of "order" to fit the replacement, then a similar result to our theorem can be derived from a theorem of Singer (cf. [7], [6, Theorem 1]).

In order to prove our theorem we shall prepare several lemmas: Suppose that y is a generic point of the general solution of $F=0$ over k. Then, $k(y, y')$ is a one-dimensional algebraic function field over k with $F(y, y')=0$. The following lemma is due to the author [3]:

Lemma 1. *Suppose that $\nu_P(\tau')\leqq 0$ for every prime divisor P of $k(y, y')$, where ν_P is the normalized valuation belonging to P and τ is a prime element in P. Then, the order of any liouvillian element over k satisfying $F=0$ is 0.*

Let k^* be a differential subfield of Ω containing k such that k^* is finitely generated over k and the field of constants of k^* is k_0, and η be a generic point of the general solution of $F=0$ over k^*:

Lemma 2. *Suppose that there exists a liouvillian element z over k satisfying $F=0$ whose order is not 0. Then, we have such k^* that z is algebraic over k^* and that*

(1) $\qquad k^*(\eta, \eta')$ *contains a transcendental constant over* k^*.

Lemma 3. *Suppose that the condition* (1) *is satisfied by some k^* and that $\nu_P(\tau')>0$ for some prime divisor P of $k(y, y')$. Then, there exists in $k(\eta, \eta')$ a transcendental element ϕ over k such that $\phi'=a\phi+b$, where $a, b\in k$.*

Lemmas 2, 3 and Theorem will be proved in the sections 1, 3 and 4 respectively. In the section 2 we shall show the following:

Proposition. *Suppose that some k^* has the property* (1). *Then, in the algebraic closure of k^* there exists a liouvillian extension $k^\sharp$ of k such that $k^\sharp(\eta, \eta')$ has a transcendental constant over $k^\sharp$, if and only if $\nu_P(\tau')>0$ for some prime divisor P of $k(y, y')$.*

Remark 2. Suppose that k is the algebraic closure of $k_0(x)$ with $x'=1$ and that

$$F = u'-\alpha u/x-1/(1+x)\,, \qquad \alpha\in k_0\,.$$

Then, any solution of $F=0$ is of the third order if α is not a rational number. This remark is due to M. Matsuda.

Remark 3. The following theorem due to Rosenlicht [5] can be derived from Lemma 3: Assume that $k=k_0$ and $F=u'-f(u)$, where $f\in k(u)$. Then, the condition (1) is satisfied by some k^* if and only if we are in one of the following three cases: $f=0$, $1/f=\partial g/\partial u$, $1/f=c(\partial g/\partial u)/g$ with $g\in k(u)$ and $c\in k$.

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1. Proof of Lemma 2. There exist in Ω an element t and a differential subfield k_1 containing k which satisfy the following conditions: k_1 is finitely generated over k; either t' or $t'/t \in k_1$; the field of constants of $k_1(t)$ is k_0; z is transcendental over k_1 and algebraic over $k_1(t)$. Let us set $k^* = k_1(t)$. Then, η is a generic differential specialization of z over k_1. Hence, there exists an element u of Ω such that (η, u) is a generic differential specialization of (z, t) over k_1. We have either $u' \in k_1$ or $u'/u \in k_1$, and η is algebraic over $k^*(u)$. Since η is transcendental over k^*, u is transcendental over k^*. Either $u-t$ or u/t is a transcendental constant over k^*. Hence, $k^*(\eta, \eta')$ contains a transcendental constant over k^*, since u is algebraic over $k^*(\eta, \eta')$.

2. Proof of Proposition. Firstly we shall prove the "only if" part. By the assumption there exists in $k^\sharp(\eta, \eta')$ a transcendental constant c over $k^\sharp$. The solution η of $F=0$ is algebraic over $k^\sharp(c)$ Since $k^\sharp$ is a liouvillian extension of k, η is a weakly liouvillian element over k Hence, $\nu_P(\tau')>0$ for some prime divisor P of $k(y, y')$ (cf. [3]).

Secondly we shall prove the "if" part. By the assumption there exists such a prime divisor P of $k(\eta, \eta')$ that $\nu_P(\tau')>0$. As τ we can take an element of $k(\eta, \eta')$. In the completion of $k(\eta\ \eta')$ with respect to P we have

$$\tau' = \sum b_i \tau^i, \qquad 1 \leqq i < \infty,\ b_i \in k. \tag{2}$$

Let k_2 denote the algebraic closure of k^* in Ω. Then, there exists uniquely a prime divisor Q of $k_2(\eta, \eta')$ such that the restriction of ν_Q^* to $k(\eta, \eta')$ is ν_P, where ν_Q^* is the normalized valuation belonging to Q. In this Q, τ is a prime element. In the completion of $k_2(\eta, \eta')$ with respect to Q we have

$$\eta,\ \eta' \in k((\tau)), \tag{3}$$

because $\tau \in k(\eta, \eta')$. There exists in $k_2(\eta, \eta')$ a transcendental constant c over k_2 by the assumption (1). Since c^{-1} is a constant, we may assume that $\nu_Q^*(c) \geqq 0$;

$$c = \sum \gamma_i \tau^i, \qquad 0 \leqq i < \infty,\ \gamma_i \in k_2. \tag{4}$$

Differentiating both sides we have

$$0 = c' = \sum (\gamma_i' \tau^i + i\gamma_i \tau^{i-1}\tau'), \qquad 0 \leqq i < \infty.$$

Hence, for each i $(0 \leqq i < \infty)$

$$\gamma_i' + i\gamma_i b_1 + \sum j\gamma_j b_{i-j+1} = 0 \qquad (0<j<i) \tag{5}$$

by (2). For $i=0$ we have $\gamma_0'=0$ and $\gamma_0\in k$. There is a positive integer m such that $\gamma_i=0$ for each $i\ (1\leqq i<m)$ and $\gamma_m\neq 0$, because $c\notin k$. We have

$$\gamma_m'+m\gamma_m b_1=0\,.$$

Let δ be a root of $\delta^m=\gamma_m$. Then, $\delta'+b_1\delta=0$. For each $i(m<i<\infty)$ let us define an element u_i of k_2 by $\gamma_i=u_i\delta^i$. Then,

$$\gamma_i'+i\gamma_i b_1=u_i'\delta^i\,,\qquad m<i<\infty,$$

and

$$u_i'\in k(\delta,\,u_{m+1},\,\cdots,\,u_{i-1})\,,\qquad m<i<\infty$$

by (5). Since $c\in k_2(\eta,\,\eta')$, we have

$$(6)\qquad c=S(\eta,\,\eta')/T(\eta)\,,\qquad (S,\,T)=1,\ \deg_{\eta'}S<\deg_{y'}F\,,$$

where $S(Y,\,Z)$ and $T(Y)$ are polynomials over k_2:

$$\begin{aligned} S&=\sum\alpha_{ij}Y^iZ^j && (0\leqq i\leqq p,\ 0\leqq j\leqq q),\ \alpha_{ij}\in k_2\,,\\ T&=\sum\beta_iY^i && (0\leqq i\leqq r),\ \beta_r=1,\ \beta_i\in k_2\,. \end{aligned}$$

Let L and M denote

$$k(\gamma_0,\,\gamma_1,\,\cdots,\,\gamma_n,\,\cdots)\,,\quad 0\leqq n<\infty$$

and

$$k(\alpha_{00},\,\cdots,\,\alpha_{ij},\,\cdots,\,\alpha_{pq};\,\beta_0,\,\cdots,\,\beta_r),\qquad 0\leqq i\leqq p,\ 0\leqq j\leqq q$$

respectively. We shall prove that

$$(7)\qquad L=M\,.$$

For each $n\ (0\leqq n<\infty)$ we have

$$\gamma_n=\phi_n(\alpha_{00},\,\cdots,\,\alpha_{ij},\,\cdots,\,\alpha_{pq};\,\beta_0,\,\cdots,\,\beta_r)$$

by (3), (4) and (6), where ϕ_n is a rational function of $Y_{ij}\ (0\leqq i\leqq p,\ 0\leqq j\leqq q)$ and $Z_i\ (0\leqq i\leqq r)$ over k. Hence, $L\subset M$. Take an algebraic automorphism σ of k_2 over L. Let S^σ and T^σ be the polynomials obtained from S and T respectively by operating σ on each of their coefficients. Then, we have

$$S^\sigma(\eta,\eta')/T^\sigma(\eta)=\sum\gamma_n\tau^n=S(\eta,\,\eta')/T(\eta),\qquad 0\leqq n<\infty,$$

since each of $\gamma_n(0\leqq n<\infty)$ is left invariant by σ. Hence, each of $\alpha_{ij}(0\leqq i\leqq p,\ 0\leqq j\leqq q)$ and $\beta_i(0\leqq i\leqq r)$ is left invariant by σ, and it is an element of L. Thus, we have (7). There exists a positive integer e such that $L=k(\gamma_0,\,\cdots,\,\gamma_e)$. As k^* we can take $L(\delta)$.

3. Proof of Lemma 3. We may assume that the field of constants of $k(\eta, \eta')$ is k_0: For, in the contrary case a transcendental constant of $k(\eta, \eta')$ over k can be taken as ϕ. By the discussions of the previous section, there exists such an extending chain $k=N_{-1}\subset N_0\subset N_1\subset\cdots\subset N_n$ of differential subfields of the algebraic closure k_2 of k^* in Ω that satisfies the following three conditions:

(iii) Each of the fields of constants of $N_{n-1}(\eta, \eta')$ and N_n is k_0;

(iv) the field of constants of $N_n(\eta, \eta')$ is not k_0;

(v) there exist elements $t_0, \cdots, t_n$ of k_2 which satisfy the following conditions; for each $i(0<i\leqq n)$, $N_i=N_{i-1}(t_i)$ and $t_i'\in N_{i-1}$; $N_0=k(t_0)$ and $t_0'=b_1t_0$, $b_1\in k$. We may assume that t_i is transcendental over N_{i-1} for each $i\ (1\leqq i\leqq n)$: For, $t_i\in N_{i-1}$ if t_i is algebraic over N_{i-1}.

Firstly suppose that n is positive. By the induction on i we shall prove that for each $i\ (0<i\leqq n)$ there exists in $N_{n-1}(\eta, \eta')$ a transcendental element Φ_{n-i} over N_{n-i} such that the derivative of Φ_{n-i} is an element of N_{n-i}. By (iii) and (iv) our statement is true for $i=1$, because $N_n=N_{n-1}(t_n)$ and $t_n'\in N_{n-1}$. Suppose that our statement is true for $i\ (1\leqq i<n)$. For convenience let us represent Φ_{n-i} by Φ, t_{n-i} by t, $N_{n-i-1}(\eta, \eta')$ by H and N_{n-i-1} by M respectively. Then, t is transcendental over H: For, in the contrary case Φ is algebraic over $M(t)$; this contradicts our assumption that Φ is transcendental over N_{n-i}. Since $\Phi\in H(t)$, we have

$$\Phi = S/R\,,\ (S, R)=1\,, \qquad S,\ R\in H[t];$$

here the coefficient of the highest degree in R is assumed to be 1. We shall prove that $R\in M[t]$. Let P_j run over all irreducible factors of R in which the coefficient of the highest degree is 1. Then,

$$\Phi = U+\sum Q_j/P_j^{\lambda_j} \qquad (1\leqq j\leqq \mu),\ U,\ Q_j,\ P_j\in H[t];$$

here,

$$\deg Q_j<\lambda_j \deg P_j\,, \qquad 1\leqq j\leqq \mu\,. \tag{8}$$

Since $t'\in M$, we have

$$\deg (Q_j'P_j^{\lambda_j}-\lambda_jQ_jP_j^{\lambda_j-1}P_j')<2\lambda_j \deg P_j\,, \qquad 1\leqq j\leqq \mu$$

by (8). Suppose that some P_j is not an element of $M[t]$. Then,

$$(Q_j/P_j^{\lambda_j})'=0\,,$$

because $\Phi'\in M(t)$. This contradicts our assumption that the field of constants of $H(t)$ is k_0. Hence, $P_j\in M[t]$ for each j. Thus, we have $R\in M[t]$ and

$$S'R-SR'\in M[t]\,. \tag{9}$$

Since $\Phi\notin M(t)$, we have $S\notin M[t]$. Set

$$S = s_0+s_1t+\cdots+s_mt^m, \qquad s_i\in H,\ s_m\neq 0.$$

Then, there exists an integer j $(0\leqq j\leqq m)$ such that $s_i\in M$ if $i>j$ and $s_j\notin M$. We have $s_j'\in M$ by (9). Since the field of constants of H is k_0, s_j is transcendental over M. Hence, s_j can be taken as Φ_{n-i-1}. Thus, the induction is completed. In particular, for $i=n$ there exists in $N_0(\eta, \eta')$ a transcendental element Φ_0 over N_0 such that $\Phi_0'\in N_0$. We are in one of the following two cases: In the first case $t_0\in k$; we have $\Phi_0\in k(\eta, \eta')$, $\Phi_0'\in k$ and $\Phi_0\notin k$. In this case Φ_0 can be taken as ϕ. In the second case $t_0\notin k$; let us set $i=n$ in the above induction on i. Then, we have an element s_j of $k(\eta, \eta')$ such that $s_j\notin k$ and

$$s_j'+(j-r)b_1s_j\in k, \quad r = \deg R,$$

because $t'=b_1/t$, $b_1\in k$. Hence, s_j can be taken as ϕ in this case.

Secondly suppose that $n=0$. Then, t_0 is transcendental over $k(\eta, \eta')$. By our assumption there exists in $k(t_0, \eta, \eta')$ a transcendental constant over $k(\eta, \eta')$. Hence, in $k(\eta, \eta')$ we have a nontrivial solution ϕ of $\phi'=hb_1\phi$ for some positive integer h, because $t_0'=b_1t_0$ with $b_1\in k$. Since the field of constants of $k(t_0)$ is k_0, ϕ is transcendental over k.

4. Proof of Theorem. By Lemmas 1, 2 and 3 it is sufficient to prove the following: Suppose that $k(y, y')$ contains a transcendental element ϕ over k such that $\phi'=a\phi+b$, a, $b\in k$. Then, any liouvillian element over k satisfying $F=0$ is at most of the third order. We may set

$$\phi = Q(y, y')/P(y), \qquad P, Q\in k\{u\}.$$

Let Γ be the set of all solutions of $F=0$ contained in k. Firstly assume that Γ is infinite. In this case we shall prove that $k(y, y')$ contains a transcendental constant over k and hence any liouvillian element over k satisfying $F=0$ is of order 0. There exists an element J of $k\{u\}$ satisfying $J(y, y', \cdots)\neq 0$ such that any differential specialization w of y over k with $J(w, w', \cdots)\neq 0$ can be extended to a differential specialization (w, ϕ_0) of (y, ϕ) over k(cf. Ritt [4], Koichin [1, p. 928]). Since Γ is infinite, there exists an element w of Γ such that $J(w, w', \cdots)\neq 0$ and $P(w)\neq 0$. Let (w, ϕ_0) be a differential specialization of (y, ϕ) over k. Then, $\phi_0'=a\phi_0+b$, and $\phi_0\in k$. Set $\psi=\phi-\phi_0$. Then, $\psi'=a\psi$. In a similar way to the above we have an element ψ_0 of k satisfying $\psi_0'=a\psi_0$ and $\psi_0\neq 0$. The element ψ/ψ_0 of $k(y, y')$ is a transcendental constant over k. Secondly assume that Γ is finite. Take elements A, t of Ω such that $A'=a$, $t'=at$ and $t\neq 0$. Let Λ be the prime differential ideal in $k\{z_1, z_2, z_3, z_4\}$ whose generic zero over k is (A, t, ϕ, y). We define an element T of $k[z_2, z_4]$ by

$$T = z_2 \prod(z_4 - w), \qquad w \in \Gamma .$$

Then, $T \notin \Lambda$. There exists a zero (A_0, t_0, ϕ_0, y_0) of Λ such that $T(t_0, y_0) \neq 0$ and the field of constants of $k\langle A_0, t_0, \phi_0, y_0\rangle$ is k_0 (cf. Kolchin [1]). We have $A_0' = a$, $t_0' = at_0$, $t_0 \neq 0$, $(\phi_0/t_0)' = b/t_0$, $F(y_0, y_0') = 0$ and $y_0 \notin \Gamma$. The element y_0 is transcendental over k and algebraic over $k(\phi_0)$. Hence, y_0 is a liouvillian element over k whose order is not 0 and at most 3.

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