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Author(s)	Ōtsubo, Shūji
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## ORDER OF LIOUVILLIAN ELEMENTS SATISFYING AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

SHŪJI ŌTSUBO

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**0. Introduction.** Let  $k$  be an algebraically closed ordinary differential field of characteristic 0, and  $\Omega$  be a universal extension of  $k$ . A finite chain of extending differential subfields  $k=L_0 \subset L_1 \subset \cdots \subset L_n$  in  $\Omega$  is called a Liouville chain over  $k$  if the following two conditions are satisfied:

- (i) The field of constants of  $L_n$  is  $k_0$ , where  $k_0$  is the field of constants of  $k$ ;
- (ii) For each  $i(1 \leq i \leq n)$  there exists a finite system of elements  $w_1, w_2, \dots, w_r$  of  $L_i$  which satisfies the following two conditions; either  $w'_j \in L_{i-1}$  or  $w'_j/w_j$  is the derivative of an element of  $L_{i-1}$  for each  $j(1 \leq j \leq r)$ ,  $L_i$  is an algebraic extension of  $L_{i-1}(w_1, w_2, \dots, w_r)$  of finite degree.

Let  $z$  be an element of  $\Omega$ . Then,  $z$  is called a liouvillian element over  $k$  if there exists a Liouville chain over  $k$  such that its end contains  $z$ . The following definition is due to Liouville [2] (cf. [8, p. 111]):

**DEFINITION.** A liouvillian element  $z$  over  $k$  is said to be of order  $m$  if  $m$  is the minimum of those  $n$  such that the end of a Liouville chain  $L_0 \subset \cdots \subset L_n$  over  $k$  contains  $z$ .

Let  $F$  be an algebraically irreducible element of the first order of the differential polynomial algebra  $k\{u\}$  in a single indeterminate  $u$  over  $k$ . Suppose that  $z$  is a solution of  $F=0$ . Then,  $z$  is a generic point of the general solution of  $F=0$  over  $k$  if and only if  $z$  is transcendental over  $k$ . Suppose that two liouvillian elements over  $k$  satisfy  $F=0$  and that they are transcendental over  $k$ . Then, their orders are the same.

**Theorem.** *The order of a liouvillian element over  $k$  satisfying  $F=0$  is at most three.*

For example, suppose that  $k$  is the algebraic closure of  $k_0(x)$  with  $x'=1$  and that  $F=u'-\alpha u/x$ , where  $\alpha \in k_0$ . Then, any non-trivial solution of  $F=0$  is of the second order if  $\alpha$  is not a rational number (cf. Liouville [2, pp. 94-98]).

**REMARK 1.** If we replace "liouvillian" by "generalized elementary" and

modify the definition of "order" to fit the replacement, then a similar result to our theorem can be derived from a theorem of Singer (cf. [7], [6, Theorem 1]).

In order to prove our theorem we shall prepare several lemmas: Suppose that  $y$  is a generic point of the general solution of  $F=0$  over  $k$ . Then,  $k(y, y')$  is a one-dimensional algebraic function field over  $k$  with  $F(y, y')=0$ . The following lemma is due to the author [3]:

**Lemma 1.** *Suppose that  $v_P(\tau') \leq 0$  for every prime divisor  $P$  of  $k(y, y')$ , where  $v_P$  is the normalized valuation belonging to  $P$  and  $\tau$  is a prime element in  $P$ . Then, the order of any liouvillian element over  $k$  satisfying  $F=0$  is 0.*

Let  $k^*$  be a differential subfield of  $\Omega$  containing  $k$  such that  $k^*$  is finitely generated over  $k$  and the field of constants of  $k^*$  is  $k_0$ , and  $\eta$  be a generic point of the general solution of  $F=0$  over  $k^*$ :

**Lemma 2.** *Suppose that there exists a liouvillian element  $z$  over  $k$  satisfying  $F=0$  whose order is not 0. Then, we have such  $k^*$  that  $z$  is algebraic over  $k^*$  and that*

$$(1) \quad k^*(\eta, \eta') \text{ contains a transcendental constant over } k^* .$$

**Lemma 3.** *Suppose that the condition (1) is satisfied by some  $k^*$  and that  $v_P(\tau') > 0$  for some prime divisor  $P$  of  $k(y, y')$ . Then, there exists in  $k(\eta, \eta')$  a transcendental element  $\phi$  over  $k$  such that  $\phi' = a\phi + b$ , where  $a, b \in k$ .*

Lemmas 2, 3 and Theorem will be proved in the sections 1, 3 and 4 respectively. In the section 2 we shall show the following:

**Proposition.** *Suppose that some  $k^*$  has the property (1). Then, in the algebraic closure of  $k^*$  there exists a liouvillian extension  $k^\#$  of  $k$  such that  $k^\#(\eta, \eta')$  has a transcendental constant over  $k^\#$ , if and only if  $v_P(\tau') > 0$  for some prime divisor  $P$  of  $k(y, y')$ .*

REMARK 2. Suppose that  $k$  is the algebraic closure of  $k_0(x)$  with  $x'=1$  and that

$$F = u' - \alpha u/x - 1/(1+x), \quad \alpha \in k_0 .$$

Then, any solution of  $F=0$  is of the third order if  $\alpha$  is not a rational number. This remark is due to M. Matsuda.

REMARK 3. The following theorem due to Rosenlicht [5] can be derived from Lemma 3: Assume that  $k=k_0$  and  $F=u'-f(u)$ , where  $f \in k(u)$ . Then, the condition (1) is satisfied by some  $k^*$  if and only if we are in one of the following three cases:  $f=0$ ,  $1/f=\partial g/\partial u$ ,  $1/f=c(\partial g/\partial u)/g$  with  $g \in k(u)$  and  $c \in k$ .

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**1. Proof of Lemma 2.** There exist in  $\Omega$  an element  $t$  and a differential subfield  $k_1$  containing  $k$  which satisfy the following conditions:  $k_1$  is finitely generated over  $k$ ; either  $t' \in k_1$  or  $t'/t \in k_1$ ; the field of constants of  $k_1(t)$  is  $k_0$ ;  $z$  is transcendental over  $k_1$  and algebraic over  $k_1(t)$ . Let us set  $k^* = k_1(t)$ . Then,  $\eta$  is a generic differential specialization of  $z$  over  $k_1$ . Hence, there exists an element  $u$  of  $\Omega$  such that  $(\eta, u)$  is a generic differential specialization of  $(z, t)$  over  $k_1$ . We have either  $u' \in k_1$  or  $u'/u \in k_1$ , and  $\eta$  is algebraic over  $k^*(u)$ . Since  $\eta$  is transcendental over  $k^*$ ,  $u$  is transcendental over  $k^*$ . Either  $u - t$  or  $u/t$  is a transcendental constant over  $k^*$ . Hence,  $k^*(\eta, \eta')$  contains a transcendental constant over  $k^*$ , since  $u$  is algebraic over  $k^*(\eta, \eta')$ .

**2. Proof of Proposition.** Firstly we shall prove the "only if" part. By the assumption there exists in  $k^*(\eta, \eta')$  a transcendental constant  $c$  over  $k^*$ . The solution  $\eta$  of  $F=0$  is algebraic over  $k^*(c)$ . Since  $k^*$  is a liouvillian extension of  $k$ ,  $\eta$  is a weakly liouvillian element over  $k$ . Hence,  $\nu_P(\tau') > 0$  for some prime divisor  $P$  of  $k(y, y')$  (cf. [3]).

Secondly we shall prove the "if" part. By the assumption there exists such a prime divisor  $P$  of  $k(\eta, \eta')$  that  $\nu_P(\tau') > 0$ . As  $\tau$  we can take an element of  $k(\eta, \eta')$ . In the completion of  $k(\eta, \eta')$  with respect to  $P$  we have

$$(2) \quad \tau' = \sum b_i \tau^i, \quad 1 \leq i < \infty, b_i \in k.$$

Let  $k_2$  denote the algebraic closure of  $k^*$  in  $\Omega$ . Then, there exists uniquely a prime divisor  $Q$  of  $k_2(\eta, \eta')$  such that the restriction of  $\nu_Q^*$  to  $k(\eta, \eta')$  is  $\nu_P$ , where  $\nu_Q^*$  is the normalized valuation belonging to  $Q$ . In this  $Q$ ,  $\tau$  is a prime element. In the completion of  $k_2(\eta, \eta')$  with respect to  $Q$  we have

$$(3) \quad \eta, \eta' \in k((\tau)),$$

because  $\tau \in k(\eta, \eta')$ . There exists in  $k_2(\eta, \eta')$  a transcendental constant  $c$  over  $k_2$  by the assumption (1). Since  $c^{-1}$  is a constant, we may assume that  $\nu_Q^*(c) \geq 0$ ;

$$(4) \quad c = \sum \gamma_i \tau^i, \quad 0 \leq i < \infty, \gamma_i \in k_2.$$

Differentiating both sides we have

$$0 = c' = \sum (\gamma_i' \tau^i + i \gamma_i \tau^{i-1} \tau'), \quad 0 \leq i < \infty.$$

Hence, for each  $i$  ( $0 \leq i < \infty$ )

$$(5) \quad \gamma_i' + i \gamma_i b_1 + \sum j \gamma_j b_{i-j+1} = 0 \quad (0 < j < i)$$

by (2). For  $i=0$  we have  $\gamma'_0=0$  and  $\gamma_0 \in k$ . There is a positive integer  $m$  such that  $\gamma_i=0$  for each  $i$  ( $1 \leq i < m$ ) and  $\gamma_m \neq 0$ , because  $c \notin k$ . We have

$$\gamma'_m + m\gamma_m b_1 = 0.$$

Let  $\delta$  be a root of  $\delta^m = \gamma_m$ . Then,  $\delta' + b_1 \delta = 0$ . For each  $i$  ( $m < i < \infty$ ) let us define an element  $u_i$  of  $k_2$  by  $\gamma_i = u_i \delta^i$ . Then,

$$\gamma'_i + i\gamma_i b_1 = u'_i \delta^i, \quad m < i < \infty,$$

and

$$u'_i \in k(\delta, u_{m+1}, \dots, u_{i-1}), \quad m < i < \infty$$

by (5). Since  $c \in k_2(\eta, \eta')$ , we have

$$(6) \quad c = S(\eta, \eta')/T(\eta), \quad (S, T) = 1, \deg_{\eta'} S < \deg_{\eta'} F,$$

where  $S(Y, Z)$  and  $T(Y)$  are polynomials over  $k_2$ :

$$\begin{aligned} S &= \sum \alpha_{ij} Y^i Z^j & (0 \leq i \leq p, 0 \leq j \leq q), \alpha_{ij} \in k_2, \\ T &= \sum \beta_i Y^i & (0 \leq i \leq r), \beta_r = 1, \beta_i \in k_2. \end{aligned}$$

Let  $L$  and  $M$  denote

$$k(\gamma_0, \gamma_1, \dots, \gamma_n, \dots), \quad 0 \leq n < \infty$$

and

$$k(\alpha_{00}, \dots, \alpha_{ij}, \dots, \alpha_{pq}; \beta_0, \dots, \beta_r), \quad 0 \leq i \leq p, 0 \leq j \leq q$$

respectively. We shall prove that

$$(7) \quad L = M.$$

For each  $n$  ( $0 \leq n < \infty$ ) we have

$$\gamma_n = \phi_n(\alpha_{00}, \dots, \alpha_{ij}, \dots, \alpha_{pq}; \beta_0, \dots, \beta_r)$$

by (3), (4) and (6), where  $\phi_n$  is a rational function of  $Y_{ij}$  ( $0 \leq i \leq p, 0 \leq j \leq q$ ) and  $Z_i$  ( $0 \leq i \leq r$ ) over  $k$ . Hence,  $L \subset M$ . Take an algebraic automorphism  $\sigma$  of  $k_2$  over  $L$ . Let  $S^\sigma$  and  $T^\sigma$  be the polynomials obtained from  $S$  and  $T$  respectively by operating  $\sigma$  on each of their coefficients. Then, we have

$$S^\sigma(\eta, \eta')/T^\sigma(\eta) = \sum \gamma_n \tau^n = S(\eta, \eta')/T(\eta), \quad 0 \leq n < \infty,$$

since each of  $\gamma_n$  ( $0 \leq n < \infty$ ) is left invariant by  $\sigma$ . Hence, each of  $\alpha_{ij}$  ( $0 \leq i \leq p, 0 \leq j \leq q$ ) and  $\beta_i$  ( $0 \leq i \leq r$ ) is left invariant by  $\sigma$ , and it is an element of  $L$ . Thus, we have (7). There exists a positive integer  $e$  such that  $L = k(\gamma_0, \dots, \gamma_e)$ . As  $k^*$  we can take  $L(\delta)$ .

**3. Proof of Lemma 3.** We may assume that the field of constants of  $k(\eta, \eta')$  is  $k_0$ : For, in the contrary case a transcendental constant of  $k(\eta, \eta')$  over  $k$  can be taken as  $\phi$ . By the discussions of the previous section, there exists such an extending chain  $k=N_{-1}\subset N_0\subset N_1\subset\cdots\subset N_n$  of differential sub-fields of the algebraic closure  $k_2$  of  $k^*$  in  $\Omega$  that satisfies the following three conditions:

- (iii) Each of the fields of constants of  $N_{n-1}(\eta, \eta')$  and  $N_n$  is  $k_0$ ;
- (iv) the field of constants of  $N_n(\eta, \eta')$  is not  $k_0$ ;
- (v) there exist elements  $t_0, \dots, t_n$  of  $k_2$  which satisfy the following conditions; for each  $i(0 < i \leq n)$ ,  $N_i=N_{i-1}(t_i)$  and  $t'_i \in N_{i-1}$ ;  $N_0=k(t_0)$  and  $t'_0=b_1t_0$ ,  $b_1 \in k$ . We may assume that  $t_i$  is transcendental over  $N_{i-1}$  for each  $i$  ( $1 \leq i \leq n$ ): For,  $t_i \in N_{i-1}$  if  $t_i$  is algebraic over  $N_{i-1}$ .

Firstly suppose that  $n$  is positive. By the induction on  $i$  we shall prove that for each  $i$  ( $0 < i \leq n$ ) there exists in  $N_{n-1}(\eta, \eta')$  a transcendental element  $\Phi_{n-i}$  over  $N_{n-i}$  such that the derivative of  $\Phi_{n-i}$  is an element of  $N_{n-i}$ . By (iii) and (iv) our statement is true for  $i=1$ , because  $N_n=N_{n-1}(t_n)$  and  $t'_n \in N_{n-1}$ . Suppose that our statement is true for  $i$  ( $1 \leq i < n$ ). For convenience let us represent  $\Phi_{n-i}$  by  $\Phi$ ,  $t_{n-i}$  by  $t$ ,  $N_{n-i-1}(\eta, \eta')$  by  $H$  and  $N_{n-i-1}$  by  $M$  respectively. Then,  $t$  is transcendental over  $H$ : For, in the contrary case  $\Phi$  is algebraic over  $M(t)$ ; this contradicts our assumption that  $\Phi$  is transcendental over  $N_{n-i}$ . Since  $\Phi \in H(t)$ , we have

$$\Phi = S/R, \quad (S, R) = 1, \quad S, R \in H[t];$$

here the coefficient of the highest degree in  $R$  is assumed to be 1. We shall prove that  $R \in M[t]$ . Let  $P_j$  run over all irreducible factors of  $R$  in which the coefficient of the highest degree is 1. Then,

$$\Phi = U + \sum Q_j/P_j^{\lambda_j} \quad (1 \leq j \leq \mu), \quad U, Q_j, P_j \in H[t];$$

here,

$$(8) \quad \deg Q_j < \lambda_j \deg P_j, \quad 1 \leq j \leq \mu.$$

Since  $t' \in M$ , we have

$$\deg(Q_j'P_j^{\lambda_j} - \lambda_j Q_j P_j^{\lambda_j-1} P_j') < 2\lambda_j \deg P_j, \quad 1 \leq j \leq \mu$$

by (8). Suppose that some  $P_j$  is not an element of  $M[t]$ . Then,

$$(Q_j/P_j^{\lambda_j})' = 0,$$

because  $\Phi' \in M(t)$ . This contradicts our assumption that the field of constants of  $H(t)$  is  $k_0$ . Hence,  $P_j \in M[t]$  for each  $j$ . Thus, we have  $R \in M[t]$  and

$$(9) \quad S'R - SR' \in M[t].$$

Since  $\Phi \notin M(t)$ , we have  $S \in M[t]$ . Set

$$S = s_0 + s_1 t + \dots + s_m t^m, \quad s_i \in H, s_m \neq 0.$$

Then, there exists an integer  $j$  ( $0 \leq j \leq m$ ) such that  $s_i \in M$  if  $i > j$  and  $s_j \notin M$ . We have  $s'_j \in M$  by (9). Since the field of constants of  $H$  is  $k_0$ ,  $s_j$  is transcendental over  $M$ . Hence,  $s_j$  can be taken as  $\Phi_{n-i-1}$ . Thus, the induction is completed. In particular, for  $i=n$  there exists in  $N_0(\eta, \eta')$  a transcendental element  $\Phi_0$  over  $N_0$  such that  $\Phi'_0 \in N_0$ . We are in one of the following two cases: In the first case  $t_0 \in k$ ; we have  $\Phi_0 \in k(\eta, \eta')$ ,  $\Phi'_0 \in k$  and  $\Phi_0 \notin k$ . In this case  $\Phi_0$  can be taken as  $\phi$ . In the second case  $t_0 \notin k$ ; let us set  $i=n$  in the above induction on  $i$ . Then, we have an element  $s_j$  of  $k(\eta, \eta')$  such that  $s_j \notin k$  and

$$s'_j + (j-r)b_1 s_j \in k, \quad r = \deg R,$$

because  $t' = b_1/t$ ,  $b_1 \in k$ . Hence,  $s_j$  can be taken as  $\phi$  in this case.

Secondly suppose that  $n=0$ . Then,  $t_0$  is transcendental over  $k(\eta, \eta')$ . By our assumption there exists in  $k(t_0, \eta, \eta')$  a transcendental constant over  $k(\eta, \eta')$ . Hence, in  $k(\eta, \eta')$  we have a nontrivial solution  $\phi$  of  $\phi' = hb_1\phi$  for some positive integer  $h$ , because  $t'_0 = b_1 t_0$  with  $b_1 \in k$ . Since the field of constants of  $k(t_0)$  is  $k_0$ ,  $\phi$  is transcendental over  $k$ .

**4. Proof of Theorem.** By Lemmas 1, 2 and 3 it is sufficient to prove the following: Suppose that  $k(y, y')$  contains a transcendental element  $\phi$  over  $k$  such that  $\phi' = a\phi + b$ ,  $a, b \in k$ . Then, any liouvillian element over  $k$  satisfying  $F=0$  is at most of the third order. We may set

$$\phi = Q(y, y')/P(y), \quad P, Q \in k\{u\}.$$

Let  $\Gamma$  be the set of all solutions of  $F=0$  contained in  $k$ . Firstly assume that  $\Gamma$  is infinite. In this case we shall prove that  $k(y, y')$  contains a transcendental constant over  $k$  and hence any liouvillian element over  $k$  satisfying  $F=0$  is of order 0. There exists an element  $J$  of  $k\{u\}$  satisfying  $J(y, y', \dots) \neq 0$  such that any differential specialization  $w$  of  $y$  over  $k$  with  $J(w, w', \dots) \neq 0$  can be extended to a differential specialization  $(w, \phi_0)$  of  $(y, \phi)$  over  $k$  (cf. Ritt [4], Koichin [1, p. 928]). Since  $\Gamma$  is infinite, there exists an element  $w$  of  $\Gamma$  such that  $J(w, w', \dots) \neq 0$  and  $P(w) \neq 0$ . Let  $(w, \phi_0)$  be a differential specialization of  $(y, \phi)$  over  $k$ . Then,  $\phi'_0 = a\phi_0 + b$ , and  $\phi_0 \in k$ . Set  $\psi = \phi - \phi_0$ . Then,  $\psi' = a\psi$ . In a similar way to the above we have an element  $\psi_0$  of  $k$  satisfying  $\psi'_0 = a\psi_0$  and  $\psi_0 \neq 0$ . The element  $\psi/\psi_0$  of  $k(y, y')$  is a transcendental constant over  $k$ . Secondly assume that  $\Gamma$  is finite. Take elements  $A, t$  of  $\Omega$  such that  $A' = a$ ,  $t' = at$  and  $t \neq 0$ . Let  $\Lambda$  be the prime differential ideal in  $k\{z_1, z_2, z_3, z_4\}$  whose generic zero over  $k$  is  $(A, t, \phi, y)$ . We define an element  $T$  of  $k[z_2, z_4]$  by

$$T = z_2 \prod (z_4 - w), \quad w \in \Gamma.$$

Then,  $T \notin \Lambda$ . There exists a zero  $(A_0, t_0, \phi_0, y_0)$  of  $\Lambda$  such that  $T(t_0, y_0) \neq 0$  and the field of constants of  $k\langle A_0, t_0, \phi_0, y_0 \rangle$  is  $k_0$  (cf. Kolchin [1]). We have  $A'_0 = a$ ,  $t'_0 = at_0$ ,  $t_0 \neq 0$ ,  $(\phi_0/t_0)' = b/t_0$ ,  $F(y_0, y'_0) = 0$  and  $y_0 \notin \Gamma$ . The element  $y_0$  is transcendental over  $k$  and algebraic over  $k(\phi_0)$ . Hence,  $y_0$  is a liouvillian element over  $k$  whose order is not 0 and at most 3.

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