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Supplement to my Paper

"On the Homogeneous Linear Partial Differential Equation of the First Order"

By Takashi KASUGA

In our paper [2] above-mentioned (in the following, we shall cite it as "H"), we treated the following homogeneous partial differential equation

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_\mu(x, y_1, \cdots, y_n)\frac{\partial z}{\partial y_\mu}=0 \qquad (n \geqq 1)$$

without the usual condition of the total differentiability on the solution $z(x, y_1, \cdots, y_n)$.

Here we remark that we can treat the non-homogeneous linear partial differential equation of a rather general type

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_\mu(x, y_1, \cdots, y_n)\frac{\partial z}{\partial y_\mu}=h(x, y_1, \cdots, y_n)\, z+k(x, y_1, \cdots, y_n)$$

in a similar way by the use of Theorem 1 of "H".

1. We shall use the same notations and abbreviations as explained in § 1. 1 of "H". We add only a new abbreviation for points in R^{n+2}: $(x\,;\, y\,;\, z)=(x, y_1, \cdots, y_n, z)$.

In the following, we shall denote by G a fixed open set in R^{n+1}, by $h(x\,;\,y)$, $k(x\,;\,y)$ and $f_\lambda(x\,;\,y)$ $(\lambda=1, \cdots, n)$ $n+2$ fixed continuous functions defined on G which have continuous $\partial h/\partial y_\mu$, $\partial k/\partial y_\mu$, $\partial f_\lambda/\partial y_\mu$ $(\lambda,\ \mu=1, \cdots, n)$.

Under the above conditions, we shall consider the partial differential equation

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_\mu(x\,;\,y)\frac{\partial z}{\partial y_\mu}=h(x\,;\,y)\, z+k(x\,;\,y)\,. \tag{1}$$

With (1), we shall associate the simultaneous ordinary differential equations

$$\begin{cases} \dfrac{dy_\lambda}{dx}=f_\lambda(x\,;\,y) \qquad (\lambda=1, \cdots, n) \\ \dfrac{dz}{dx}=h(x\,;\,y)z+k(x\,;\,y)\,. \end{cases} \tag{2}$$

We denote by $\tilde{G}$, the open set in R^{n+2} defined by

$$(x\,;\,y\,;\,z):\,(x:\,y)\in G \quad +\infty > z > -\infty\,.$$

Then the continuous curves in R^{n+2} representing the solutions of (2) which are prolonged as far as possible on both sides in $\tilde{G}$, will be called *characteristic curves of* (2) *in* $\tilde{G}$. Through any point $(\xi\,;\,\eta\,;\,\zeta)$ in $\tilde{G}$, there passes one and only one characteristic curve of (2) in $\tilde{G}$[1]. We represent it by $\tilde{C}(\xi\,;\,\eta\,;\,\zeta)$.

A continuous function $z(x\,;\,y)$ defined on G will be called a *quasi-solution of* (1) *on* G, if it has $\partial z/\partial x$, $\partial z/\partial y_\lambda$ $(\lambda=1, \cdots, n)$ except at most at the points of an enumerable set in G and satisfies (1) almost everywhere in G. Here $\partial z/\partial x$, $\partial z/\partial y_\lambda$ need not necessarily be continuous.

On the other hand, a continuous function $z(x\,;\,y)$ defined on G will be called a *solution of* (1) *in* G *in the ordinary sense*, if it is totally differentiable and satisfies (1) everywhere in G.

We consider also the homogeneous partial differential equation

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_\mu(x\,;\,y)\frac{\partial z}{\partial y_\mu}=0 \tag{3}$$

which was treated in "H". We define the characteristic curve $C(\xi\,;\,\eta\,|\,G)$ of (3) passing through the point $(\xi\,;\,\eta)$ of G, quasi-solutions of (3), and solutions of (3) in the ordinary sense as in "H".

For the proof of Theorem 1, we shall also consider the non-homogeneous partial differential equation

$$\frac{\partial z}{\partial x}+\sum_{\mu=1}^{n} f_\mu(x\,;\,y)\frac{\partial z}{\partial y_\mu}=h(x\,;\,y)\,. \tag{4}$$

We represent the characteristic curve of (4) in $\tilde{G}$ which passes through the point $(\xi\,;\,\eta\,;\,\zeta)$ of $\tilde{G}$ by $C^*(\xi\,;\,\eta\,;\,\zeta)$.

We shall prove the following theorem.

Theorem 1. *Let S be a hypersurface in R^{n+2} representing a quasi-solution $z=z(x\,;\,y)$ of* (1) *on G and $(\xi\,;\,\eta\,;\,\zeta)\in S$, then $\tilde{C}(\xi\,;\,\eta\,;\,\zeta)\subset S$.*

By Theorem 1, we can easily prove, as Theorem 2 of "H", the following:

Theorem 2. *If for a fixed number $\xi^{(0)}$, the family of all the characteristic curves $C(\xi^{(0)}\,;\,\eta\,|\,G)$ of* (3) *such that $\eta\in G[\xi^{(0)}]$ covers G and $\psi(\eta)$ is a totally differentiable function defined on $G[\xi^{(0)}]$, then there is*

1) cf. Kamke [1] §16, Nr. 79, Satz 4.

one and only one quasi-solution of (1) *on* G *such that* $z(\xi^{(0)}; \eta)=\psi(\eta)$ *on* $G[\xi^{(0)}]$ *and this quasi-solution* $z(x; y)$ *is also a solution of* (1) *on* G *in the ordinary sense.*

The proof of this theorem goes in a similar way as in "H". Thus we shall omit it.

2. **Proof of Theorem 1.**

Let $(\xi'; \eta'; \zeta')$ be any point which $\tilde{C}(\xi; \eta; \zeta)$ has in common with S. Then

$$\tilde{C}(\xi; \eta; \zeta)=\tilde{C}(\xi'; \eta'; \zeta') \quad \text{and} \quad \zeta'=z(\xi'; \eta'). \tag{5}$$

We represent the characteristic curve $C(\xi'; \eta' | G)$ of (3) by

$$\begin{aligned} y_\lambda &= \varphi_\lambda(x) \qquad (\lambda=1, \cdots, n) \\ & \qquad\qquad \beta > x > \alpha. \end{aligned} \tag{6}$$

Then $\tilde{C}(\xi'; \eta'; \zeta'-a)$ where a is a positive number, can be represented in the form

$$\begin{cases} y_\lambda = \varphi_\lambda(x) & (\lambda=1, \cdots, n) \\ z = \tilde{\psi}(x) & \beta > x > \alpha. \end{cases} \tag{7}$$

Also $C^*(\xi'; \eta'; \log a)$ can be represented in the form

$$\begin{cases} y_\lambda = \varphi_\lambda(x) & (\lambda=1, \cdots, n) \\ z = \psi^*(x) & \beta > x > \alpha. \end{cases} \tag{8}$$

Then by the well known theory of the characteristics[2], there is a solution $z=\tilde{z}(x; y)$ of (1) in the ordinary sense defined in a neighbourhood of $(\xi'; \eta')$ such that

$$\tilde{z}(\xi'; \eta')=\zeta'-a \tag{9}$$

and

$$\tilde{z}(x; \varphi(x))=\tilde{\psi}(x) \tag{10}$$

in a neighbourhood of ξ'.
Also there is a solution $z=z^*(x; y)$ of (4) in the ordinary sense defined in a neighbourhood of $(\xi'; \eta')$ such that

$$z^*(\xi'; \eta')=\log a \tag{11}$$

and

$$z^*(x; \varphi(x))=\psi^*(x) \tag{12}$$

in a neighbourhood of ξ'.

2) cf. Kamke [1] §32, Nr. 171, Satz 1 and §32, Nr. 173, Satz 4.

If we put

$$z_1(x\,;\,y)=\log\,\{z(x\,;\,y)-\tilde{z}(x\,;\,y)\}-z^*(x\,;\,y)\,, \tag{13}$$

then by an easy calculation we can prove that $z_1(x\,;\,y)$ is a quasi-solution of (3) in a neighbourhood of $(\xi'\,;\,\eta')$. Also by (5), (9) and (11)

$$z_1(\xi'\,;\,\eta')=0\,.$$

Hence by Theorem 1 of "H",

$$z_1(x\,;\,\varphi(x))=0$$

in a neighbourhood of ξ'.

Therefore by (10), (12) and (13)

$$\begin{aligned}0=z_1(x\,;\,\varphi(x))&=\log\,\{z(x\,;\,\varphi(x))-\tilde{z}(x\,;\,\varphi(x)\}-z^*(x\,;\,\varphi(x))\\&=\log\,\{z(x\,;\,\varphi(x))-\tilde{\psi}(x)\}-\psi^*(x)\end{aligned}$$

and so

$$z(x\,;\,\varphi(x))=\tilde{\psi}(x)+\exp\psi^*(x) \tag{14}$$

in a neighbourhood of ξ'.

Hence, by the definition of $\tilde{\psi}(x)$ and $\psi^*(x)$, $z(x\,;\,\varphi(x))$ is differentiable and

$$\begin{aligned}\frac{d}{dx}z(x\,;\,\varphi(x))&=\frac{d\tilde{\psi}(x)}{dx}+\frac{d\psi^*(x)}{dx}\exp\psi^*(x)\\&=h(x\,;\,\varphi(x))\tilde{\psi}(x)+k(x\,;\,\varphi(x))+h(x\,;\,\varphi(x))\exp\psi^*(x)\\&=h(x\,;\,\varphi(x))(\tilde{\psi}(x)+\exp\psi^*(x))+k(x\,;\,\varphi(x))\end{aligned}$$

and so by (14)

$$\frac{d}{dx}z(x\,;\,\varphi(x))=h(x\,;\,\varphi(x))\,z(x\,;\,\varphi(x))+k(x\,;\,\varphi(x))$$

in a neighbourhood of ξ'.

Therefore by the definition of $\varphi_\lambda(x)$ and of $\tilde{C}(\xi'\,;\,\eta'\,;\,\zeta')$, considering (5), it follows that S contains the portion of $\tilde{C}(\xi'\,;\,\eta'\,;\,\zeta')$ $(=\tilde{C}(\xi\,;\,\eta\,;\,\zeta))$ in a neighbourhood of $(\xi'\,;\,\eta'\,;\,\zeta')$.

We can represent $\tilde{C}(\xi\,;\,\eta\,;\,\zeta)$ in the form

$$\begin{cases}y=\varphi_\lambda(x) & (\lambda=1,\cdots,n)\\ z=\psi(x) & \alpha<x<\beta\,.\end{cases}$$

We have shown above that the set E of points x in the interval $\alpha<x<\beta$ such that $z(x\,;\,\varphi(x))=\psi(x)$, is open in the interval $\alpha<x<\beta$.

Also by the continuity of $\varphi_\lambda(x)$, $\psi(x)$ and $z(x;y)$, E is closed in the interval $\alpha<x<\beta$. Furthermore E is not empty since $\xi\in E$. Hence E is identical with the interval $\alpha<x<\beta$. This completes the proof of Theorem 1.

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References

[1] E. Kamke: Differentialgleichungen reeller Funktionen, (1930).
[2] T. Kasuga: On the homogeneous linear partial differential equation of the first order, Osaka Math. J. **7**, 39–67 (1955).