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Supplement to my Paper
“On the Homogeneous Linear Partial
Differential Equation of the First Order”

By Takashi KASUGA

In our paper [2] above-mentioned (in the following, we shall cite it as “H”), we treated the following homogeneous partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_{\mu}(x, y_1, \dots, y_n) \frac{\partial z}{\partial y_{\mu}} = 0 \quad (n \geq 1)$$

without the usual condition of the total differentiability on the solution $z(x, y_1, \dots, y_n)$.

Here we remark that we can treat the non-homogeneous linear partial differential equation of a rather general type

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_{\mu}(x, y_1, \dots, y_n) \frac{\partial z}{\partial y_{\mu}} = h(x, y_1, \dots, y_n) z + k(x, y_1, \dots, y_n)$$

in a similar way by the use of Theorem 1 of “H”.

1. We shall use the same notations and abbreviations as explained in § 1.1 of “H”. We add only a new abbreviation for points in R^{n+2} : $(x; y; z) = (x, y_1, \dots, y_n, z)$.

In the following, we shall denote by G a fixed open set in R^{n+1} , by $h(x; y)$, $k(x; y)$ and $f_{\lambda}(x; y)$ ($\lambda = 1, \dots, n$) $n+2$ fixed continuous functions defined on G which have continuous $\partial h / \partial y_{\mu}$, $\partial k / \partial y_{\mu}$, $\partial f_{\lambda} / \partial y_{\mu}$ ($\lambda, \mu = 1, \dots, n$).

Under the above conditions, we shall consider the partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_{\mu}(x; y) \frac{\partial z}{\partial y_{\mu}} = h(x; y) z + k(x; y). \quad (1)$$

With (1), we shall associate the simultaneous ordinary differential equations

$$\begin{cases} \frac{dy_{\lambda}}{dx} = f_{\lambda}(x; y) & (\lambda = 1, \dots, n) \\ \frac{dz}{dx} = h(x; y) z + k(x; y). \end{cases} \quad (2)$$

We denote by \tilde{G} , the open set in R^{n+2} defined by

$$(x; y; z) : (x; y) \in G \quad +\infty > z > -\infty .$$

Then the continuous curves in R^{n+2} representing the solutions of (2) which are prolonged as far as possible on both sides in \tilde{G} , will be called *characteristic curves of (2) in \tilde{G}* . Through any point $(\xi; \eta; \zeta)$ in \tilde{G} , there passes one and only one characteristic curve of (2) in \tilde{G} ¹⁾. We represent it by $\tilde{C}(\xi; \eta; \zeta)$.

A continuous function $z(x; y)$ defined on G will be called a *quasi-solution of (1) on G* , if it has $\partial z/\partial x$, $\partial z/\partial y_\lambda$ ($\lambda=1, \dots, n$) except at most at the points of an enumerable set in G and satisfies (1) almost everywhere in G . Here $\partial z/\partial x$, $\partial z/\partial y_\lambda$ need not necessarily be continuous.

On the other hand, a continuous function $z(x; y)$ defined on G will be called a *solution of (1) in G in the ordinary sense*, if it is totally differentiable and satisfies (1) everywhere in G .

We consider also the homogeneous partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = 0 \quad (3)$$

which was treated in "H". We define the characteristic curve $C(\xi; \eta|G)$ of (3) passing through the point $(\xi; \eta)$ of G , quasi-solutions of (3), and solutions of (3) in the ordinary sense as in "H".

For the proof of Theorem 1, we shall also consider the non-homogeneous partial differential equation

$$\frac{\partial z}{\partial x} + \sum_{\mu=1}^n f_\mu(x; y) \frac{\partial z}{\partial y_\mu} = h(x; y) . \quad (4)$$

We represent the characteristic curve of (4) in \tilde{G} which passes through the point $(\xi; \eta; \zeta)$ of \tilde{G} by $C^*(\xi; \eta; \zeta)$.

We shall prove the following theorem.

Theorem 1. *Let S be a hypersurface in R^{n+2} representing a quasi-solution $z=z(x; y)$ of (1) on G and $(\xi; \eta; \zeta) \in S$, then $\tilde{C}(\xi; \eta; \zeta) \subset S$.*

By Theorem 1, we can easily prove, as Theorem 2 of "H", the following:

Theorem 2. *If for a fixed number $\xi^{(0)}$, the family of all the characteristic curves $C(\xi^{(0)}; \eta|G)$ of (3) such that $\eta \in G[\xi^{(0)}]$ covers G and $\psi(\eta)$ is a totally differentiable function defined on $G[\xi^{(0)}]$, then there is*

1) cf. Kamke [1] §16, Nr. 79, Satz 4.

one and only one quasi-solution of (1) on G such that $z(\xi^{(0)}; \eta) = \psi(\eta)$ on $G[\xi^{(0)}]$ and this quasi-solution $z(x; y)$ is also a solution of (1) on G in the ordinary sense.

The proof of this theorem goes in a similar way as in "H". Thus we shall omit it.

2. Proof of Theorem 1.

Let $(\xi'; \eta'; \zeta')$ be any point which $\tilde{C}(\xi; \eta; \zeta)$ has in common with S . Then

$$\tilde{C}(\xi; \eta; \zeta) = \tilde{C}(\xi'; \eta'; \zeta') \quad \text{and} \quad \zeta' = z(\xi'; \eta'). \quad (5)$$

We represent the characteristic curve $C(\xi'; \eta' | G)$ of (3) by

$$\begin{aligned} y_\lambda &= \varphi_\lambda(x) & (\lambda = 1, \dots, n) \\ \beta &> x > \alpha. \end{aligned} \quad (6)$$

Then $\tilde{C}(\xi'; \eta'; \zeta' - a)$ where a is a positive number, can be represented in the form

$$\begin{cases} y_\lambda = \varphi_\lambda(x) & (\lambda = 1, \dots, n) \\ z = \tilde{\psi}(x) & \beta > x > \alpha. \end{cases} \quad (7)$$

Also $C^*(\xi'; \eta'; \log a)$ can be represented in the form

$$\begin{cases} y_\lambda = \varphi_\lambda(x) & (\lambda = 1, \dots, n) \\ z = \psi^*(x) & \beta > x > \alpha. \end{cases} \quad (8)$$

Then by the well known theory of the characteristics²⁾, there is a solution $z = \tilde{z}(x; y)$ of (1) in the ordinary sense defined in a neighbourhood of $(\xi'; \eta')$ such that

$$\tilde{z}(\xi'; \eta') = \zeta' - a \quad (9)$$

and

$$\tilde{z}(x; \varphi(x)) = \tilde{\psi}(x) \quad (10)$$

in a neighbourhood of ξ' .

Also there is a solution $z = z^*(x; y)$ of (4) in the ordinary sense defined in a neighbourhood of $(\xi'; \eta')$ such that

$$z^*(\xi'; \eta') = \log a \quad (11)$$

and

$$z^*(x; \varphi(x)) = \psi^*(x) \quad (12)$$

in a neighbourhood of ξ' .

2) cf. Kamke [1] §32, Nr. 171, Satz 1 and §32, Nr. 173, Satz 4.

If we put

$$z_1(x; y) = \log \{z(x; y) - \bar{z}(x; y)\} - z^*(x; y), \quad (13)$$

then by an easy calculation we can prove that $z_1(x; y)$ is a quasi-solution of (3) in a neighbourhood of $(\xi'; \eta')$. Also by (5), (9) and (11)

$$z_1(\xi'; \eta') = 0.$$

Hence by Theorem 1 of "H",

$$z_1(x; \varphi(x)) = 0$$

in a neighbourhood of ξ' .

Therefore by (10), (12) and (13)

$$\begin{aligned} 0 &= z_1(x; \varphi(x)) = \log \{z(x; \varphi(x)) - \bar{z}(x; \varphi(x))\} - z^*(x; \varphi(x)) \\ &= \log \{z(x; \varphi(x)) - \tilde{\psi}(x)\} - \psi^*(x) \end{aligned}$$

and so

$$z(x; \varphi(x)) = \tilde{\psi}(x) + \exp \psi^*(x) \quad (14)$$

in a neighbourhood of ξ' .

Hence, by the definition of $\tilde{\psi}(x)$ and $\psi^*(x)$, $z(x; \varphi(x))$ is differentiable and

$$\begin{aligned} \frac{d}{dx} z(x; \varphi(x)) &= \frac{d\tilde{\psi}(x)}{dx} + \frac{d\psi^*(x)}{dx} \exp \psi^*(x) \\ &= h(x; \varphi(x))\tilde{\psi}(x) + k(x; \varphi(x)) + h(x; \varphi(x)) \exp \psi^*(x) \\ &= h(x; \varphi(x))(\tilde{\psi}(x) + \exp \psi^*(x)) + k(x; \varphi(x)) \end{aligned}$$

and so by (14)

$$\frac{d}{dx} z(x; \varphi(x)) = h(x; \varphi(x)) z(x; \varphi(x)) + k(x; \varphi(x))$$

in a neighbourhood of ξ' .

Therefore by the definition of $\varphi_\lambda(x)$ and of $\tilde{C}(\xi'; \eta'; \zeta')$, considering (5), it follows that S contains the portion of $\tilde{C}(\xi'; \eta'; \zeta')$ ($= \tilde{C}(\xi; \eta; \zeta)$) in a neighbourhood of $(\xi'; \eta'; \zeta')$.

We can represent $\tilde{C}(\xi; \eta; \zeta)$ in the form

$$\begin{cases} y = \varphi_\lambda(x) & (\lambda = 1, \dots, n) \\ z = \psi(x) & \alpha < x < \beta. \end{cases}$$

We have shown above that the set E of points x in the interval $\alpha < x < \beta$ such that $z(x; \varphi(x)) = \psi(x)$, is open in the interval $\alpha < x < \beta$.

Also by the continuity of $\varphi_\lambda(x)$, $\psi(x)$ and $z(x; y)$, E is closed in the interval $\alpha < x < \beta$. Furthermore E is not empty since $\xi \in E$. Hence E is identical with the interval $\alpha < x < \beta$. This completes the proof of Theorem 1.

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References

- [1] E. Kamke: Differentialgleichungen reeller Funktionen, (1930).
- [2] T. Kasuga: On the homogeneous linear partial differential equation of the first order, Osaka Math. J. 7, 39-67 (1955).

