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Osaka University
1. Introduction

The Markov partition in dynamical systems supplies us important informations (for examples, for studies of equilibrium states [5] and zeta functions [24]).

Such a partition was first constructed for Anosov diffeomorphisms by Ja.G. Sinai [35]. After that, R. Bowen [5] showed the existence of Markov partition on nonwandering sets of Axiom A diffeomorphisms. In these papers the notion of canonical coordinate play an important role to construct Markov partitions. K. Hiraide [20], in purely topological setting, proved the existence of Markov partition for expansive homeomorphisms with POTP by constructing canonical coordinates. For example, every expansive automorphism of a solenoidal group has POTP, and hence a cononical coordinate as well as a Markov partition (N. Aoki [2], [3] and [20]).

However homeomorphisms with Markov partitions do not necessarily have canonical coordinates. In fact, every pseudo-Anosov map has a Markov partition and does not have cononical coordinates (see paragraphs 9 and 10 of [17]).

Thus it is natural to ask what kind of expanisive homeomorphisms have Markov partitions. The purpose of this paper is to give necessary and sufficient conditions for expansive homeomorphisms to have Markov partitions. More precisely we can describe our result as follows;

Theorem. Let X be a compact metric space and f be an expansive homeomorphism of X with expansive constant $c^*$. Then the following conditions are equivalent;

(I) there exists $c > 0$ with $2c < c^*$ such that for every $x \in X$ there exists an $\eta = \eta(x) > 0$ such that $\{ Y_c(y) \cap B_\eta(x) \mid y \in B_\eta(x) \}$ is finite,

(II) there exists $c < 0$ with $2c < c^*$ such that for every $x \in X$ there exists a $\delta = \delta(x) > 0$ such that $\{ Z_c(y) \cap B_\delta(x) \mid y \in B_\delta(x) \}$ is finite,

(III) $(X, f)$ has SPOTP,

(IV) $(X, f)$ has a Markov partition.

The proof will be give in section 3 and the auxiliary results used in the proof will be prepared in section 2. We shall describe in section 4 some applications
of our result.

In the remainder of this section, we shall give some definitions which are
used in the proof of our result.

Let $X$ be a compact metric space with metric $d$, and $f$ be a homeomorphism
of $X$. For $x \in X$, $B_{\epsilon}(x)$ will denote the closed $\epsilon$-ball in $X$ centered at $x$.
For $x \in X$ and $\epsilon > 0$ define subsets $W_{s}^{*}(x)$ and $W_{u}^{*}(x)$ of $B_{\epsilon}(x)$ by
$W_{s}^{*}(x) = \bigcap_{n \geq 0} f^{-n}B_{\epsilon}(f^{n}x)$ and $W_{u}^{*}(x) = \bigcap_{n \geq 0} f^{-n}B_{\epsilon}(f^{n}x)$. Then we have

\begin{align}
(1.1) & \quad fW_{s}^{*}(x) \subset W_{s}^{*}(fx), \quad f^{-1}W_{s}^{*}(x) \subset W_{s}^{*}(f^{-1}x), \\
(1.2) & \quad y \in W_{s}^{*}(x) \quad \text{if and only if} \quad x \in W_{s}^{*}(y) \quad (\sigma = s, u), \\
\end{align}

and

\begin{align}
(1.3) & \quad z \in W_{s}^{*}(x) \quad \text{whenever} \quad y \in W_{s}^{*}(x) \quad \text{and} \quad z \in W_{u}^{*}(y) \quad (\sigma = s, u). \\
\end{align}

**Definition 1.** $(X, f)$ is said to be expansive if there exists a constant $c^{*} > 0$
such that

\[ \{x\} = \bigcap_{n \in \mathbb{Z}} f^{-n}(B_{\epsilon}(f^{n}x)) = W_{s}^{*}(x) \cap W_{s}^{*}(x) \]

for all $x \in X$, and such a $c^{*}$ is said to be an expansive constant for $f$.

For every $\epsilon > 0$ define subsets $Y_{\epsilon}$ and $Z_{\epsilon}$ of $X \times X$ by

\[ Y_{\epsilon} = \{(x, y) \in X \times X \mid W_{s}^{*}(x) \cap W_{s}^{*}(y) \neq \emptyset\} \]

and

\[ Z_{\epsilon} = \{(x, y) \in X \times X \mid (x, y) \in Y_{\epsilon} \quad \text{and} \quad (y, x) \in Y_{\epsilon}\}. \]

For $x \in X$ and $\epsilon > 0$, subsets $Y_{\epsilon}(x)$ and $Z_{\epsilon}(x)$ of $X$ are defined by

\[ Y_{\epsilon}(x) = \{y \in X \mid (x, y) \in Y_{\epsilon}\} \]

and

\[ Z_{\epsilon}(x) = \{y \in X \mid (x, y) \in Z_{\epsilon}\}. \]

Then we have

\begin{align}
(1.4) & \quad y \in Z_{\epsilon}(x) \quad \text{if and only if} \quad x \in Z_{\epsilon}(y) \\
(1.5) & \quad W_{s}^{*}(x) \cup W_{s}^{*}(x) \subset Z_{\epsilon}(x) \subset Y_{\epsilon}(x). \\
\end{align}

**Definition 2.** Let $\mathcal{D}$ be a finite partition of $X$; i.e., a finite family of
subsets of whose elements are mutually disjoint and $\cup_{D \in \mathcal{D}} D = X$. A sequence
\[ \{x_{i}\}_{i \in \mathbb{Z}} \]

of points in $X$ is said to be an $\alpha$-pseudo orbit with respect to $\mathcal{D}$ if
\[ d(x_{i}, x_{i+1}) \leq \alpha \quad \text{and} \quad x_{i} \sim_{\mathcal{D}} x_{i+1} \quad \text{for all} \quad i \in \mathbb{Z} \]
where $x_{i} \sim_{\mathcal{D}} y$ denotes that $x$ and $y$ are
in the same element of $\mathcal{D}$. A sequence $\{x_{i}\}_{i \in \mathbb{Z}}$ of points in $X$ is said to be
\( \beta \)-traced if \( \cap_{i \in \mathbb{Z}} f^{-i}(B(x)) \neq \emptyset \). The following notion was introduced by Y. Takahashi. \((X, f)\) is said to have \textit{special pseudo orbit tracing property} (abbrev. \textit{SPOTP}) if there exists a finite partition \( \mathcal{D} \) such that for every \( \beta > 0 \), there is \( \alpha > 0 \) such that every \( \alpha \)-pseudo orbit with respect to \( \mathcal{D} \) is \( \beta \)-traced. Especially \((X, f)\) is said to have the \textit{pseudo orbit tracing property} (abbrev. \textit{POTP}) if \( \mathcal{D} \) can be chosen so that \( \mathcal{D} = \{X\} \). The notion of \textit{SPOTP} was firstly used in M. Yuri [39]. It seems likely that for every homeomorphism of a torus, \textit{SPOTP} implies \textit{POTP}. However the author does not have the proof.

Let us denote \( A_{\delta} = \{(x, y) \in X \times X | d(x, y) \leq \delta\} \) for \( \delta > 0 \). If \((X, f)\) has \textit{POTP}, then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( A_{\delta} \subset Z_{\varepsilon} \).

Let \((X, f)\) be expansive and \( c > 0 \) be a number such that \( 2c \) is an expansive constant. Then for \((x, y) \in Y_{\varepsilon}, W_{\varepsilon}(x) \cap W_{\varepsilon}(y) \neq \emptyset \) and the set \( W_{\varepsilon}(x) \cap W_{\varepsilon}(y) \) consists only of one point by expansiveness. Therefore we can define the map \( [\ , \ ] : Y_{\varepsilon} \rightarrow X \) by \( (x, y) \mapsto [x, y] \in W_{\varepsilon}(x) \cap W_{\varepsilon}(y) ((x, y) \in Y_{\varepsilon}) \). Then the following statement holds:

\begin{align}
(1.6) & \quad [x, x] = x , \\
(1.7) & \quad y \in W_{\varepsilon}(x) \text{ and } z \in W_{\varepsilon}(x) \text{ imply that } (y, z) \in Y_{\varepsilon} \text{ and } [y, z] = x , \\
(1.8) & \quad [x, [y, z]] = [x, z] \text{ if } (y, z), (x, z) \in Y_{\varepsilon} \text{ and } (x, [y, z]) \in Y_{\varepsilon} , \\
(1.9) & \quad [[x, y], z] = [x, z] \text{ if } (x, y), (x, z) \in Y_{\varepsilon} \text{ and } ([x, y], z) \in Y_{\varepsilon} , \\
\end{align}

and

\begin{align}
(1.10) & \quad W_{\varepsilon}(x) \cap W_{\varepsilon}(y) = \{[x, y]\} \text{ if } (x, y) \in Y_{\varepsilon} \text{ for } \varepsilon \leq c .
\end{align}

**Definition 3.** Under the above notations, a subset \( E \) of \( X \) is said to be a \textit{rectangle} if \((x, y) \in Y_{\varepsilon} \) and \([x, y] \in E \) for all \( x, y \in E \).

**Definition 4.** A finite family \( \mathcal{P} \) of closed rectangles of \( X \) is said to be a \textit{Markov partition} for \((X, f)\) if \( \mathcal{P} \) satisfies the following conditions;

1) \( P = \overline{\text{int } P} \) for all \( P \in \mathcal{P} \),
2) \( \bigcup_{P \in \mathcal{P}} P = X \),
3) \( \text{int } P \cap \text{int } Q = \emptyset \) for all \( P, Q \in \mathcal{P} \) with \( P \neq Q \),
4) for every sequence \( \{P_{n}\}_{n \in \mathbb{Z}} \) of elements of \( \mathcal{P} \), \( \bigcap_{n \in \mathbb{Z}} f^{-n} P_{n} \) consists at most of one point,
5) \( f(W_{\varepsilon}(x) \cap \text{int } P) \subset W_{\varepsilon}(fx) \cap \text{int } Q \)

and

\[ f^{-1}(W_{\varepsilon}(fx) \cap \text{int } Q) \subset W_{\varepsilon}(x) \cap \text{int } P \]

whenever \( x \in \text{int } P \cap f^{-1} \text{ int } Q \) (\( P, Q \in \mathcal{P} \)),

6) there exists subsets \( B' \) and \( B'' \) of \( X \) such that

\[ fB' \subset B', \ f^{-1}B'' \subset B'' \text{, and } B' \cup B'' = \bigcup_{P \in \mathcal{P}} \partial P . \]
Markov partitions are not partitions in strict sense. However we use the word conventionally.

2. Auxiliary results

In this section fundamental results are described. Throughout, let \((X, f)\) be as in our theorem, and \(c > 0\) be a number with \(2c \leq c^*\).

(L. 1) For every \(\rho > 0\) there exists an integer \(N > 0\) such that for every \(x \in X\)
\[
y \in \bigcap_{i=-N}^{n} f^i B_\varepsilon(f^i x) \text{ implies } y \in B_\rho(x).
\]

Proof. See p. 109 of [15].

By (L. 1) the following statement is clear;

(L. 2) For every \(\rho > 0\), there exists an integer \(N > 0\) such that for every \(x \in X\) and every \(n > N\)
\[
f^n W^s(x) \subset W^s(f^n x) \quad \text{and} \quad f^{-n} W^u(x) \subset W^u(f^{-n} x).
\]

The following statement is clear from (L. 2) and uniform continuity of \(f\).

(L. 3) For every \(\rho > 0\) there exists \(\varepsilon > 0\) such that for every \(x \in X\)
\[
W^s(x) \cap B_\varepsilon(x) \subset W^s_\rho(x) \quad \text{and} \quad W^u(x) \cap B_\varepsilon(x) \subset W^u_\rho(x).
\]

(L. 4) \(Y_\varepsilon, Z_\varepsilon, Y_\varepsilon(x)\) and \(Z_\varepsilon(x)\) are compact, and \([\ , \ ]\) is uniformly continuous on \(Y_\varepsilon\).

Proof. For any sequence \(\{(x_n, y_n)\}_{n \in N}\) of points in \(Y_\varepsilon\) which converges to a point \((x, y)\), there is a subsequence \(\{(x_{n_j}, y_{n_j})\}_{j \in N}\) such that \([x_{n_j}, y_{n_j}]\) converges to \(z \in X\). Since \([x_{n_j}, y_{n_j}] \subset W^s_{c_j}(x_{n_j})\) for all \(j \in N\), we have \(z \in f^{-n} B_\varepsilon(f^n x)\) for all \(n \geq 0\), and so \(z \in W^s_\varepsilon(x)\). Similarly we have \(z \in W^u_\varepsilon(y)\) and hence \(z = [x, y]\). Therefore \(Y_\varepsilon\) is closed and \([\ , \ ]\) is continuous on \(Y_\varepsilon\). Since \(X \times X\) is compact, \(Y_\varepsilon\) is compact and so \([\ , \ ]\) is uniformly continuous on \(Y_\varepsilon\). Since \(Y_\varepsilon\) is compact, \(Y_\varepsilon' = \{(x, y) \in X \times X \mid (y, x) \in Y_\varepsilon\}\) is compact. Thus \(Z_\varepsilon = Y_\varepsilon \cap Y_\varepsilon'\) is compact. It is clear that \(Y_\varepsilon(x)\) and \(Z_\varepsilon(x)\) are compact.

(L. 5) For every \(\varepsilon > 0\) there exists \(\rho > 0\) such that \(Z_\varepsilon \cap \Delta_{\rho} \subset Z_\varepsilon\).

Proof. For given \(\varepsilon > 0\), by (L. 3) there is a \(\gamma > 0\) such that \(W^s_\varepsilon(x) \cap B_\gamma(x) \subset W^s_\varepsilon(x)\) and \(W^u_\varepsilon(x) \cap B_\gamma(x) \subset W^u_\varepsilon(x)\) for all \(x \in X\). Since \([\ , \ ]\) is uniformly continuous on \(Y_\varepsilon\) by (L. 4), there exists \(\rho > 0\) such that \(d([x, y], x) = d([x, y], [x, x]) \leq \gamma\) and \(d([x, y], y) = d([x, y], [y, y]) \leq \gamma\) for all \((x, y) \in Z_\varepsilon \cap \Delta_{\rho}\). Therefore \((x, y) \in Z_\varepsilon \cap \Delta_{\rho}\) implies \((x, y) \in Z_\varepsilon\).

(L. 6) Let \(\mathcal{D}\) be a finite partition of \(X\). Then every \(\alpha\)-pseudo orbit with respect
to \(D\) is \(\beta\)-traced if there is a strictly increasing sequence \(\{M_n\}_{n \in \mathbb{N}}\) of positive integers such that for every \(\alpha\)-pseudo orbit \(\{x_i\}_{i \in \mathbb{Z}}\) with respect to \(D\), the following holds:

\[
\bigcap_{i=0}^{M_n} f^{-i} B_\beta(x_i) \neq \emptyset \quad \text{for all } n \in \mathbb{N}.
\]

Proof. Let \(\{x_i\}_{i \in \mathbb{Z}}\) be an \(\alpha\)-pseudo orbit with respect to \(D\) and \(\{M_n\}_{n \in \mathbb{N}}\) be a strictly increasing sequence of positive integers. For every \(n \in \mathbb{N}\) define an \(\alpha\)-pseudo orbit \(\{y_i\}_{i \in \mathbb{Z}}\) with respect to \(D\) by \(y_i = x_i - [M_{i+1}/2]\) for all \(i \in \mathbb{Z}\). ([M] \((M \geq 0)\) denotes the maximal integer which does not exceed \(M\).) By the assumption we have

\[
E_n = \bigcap_{i=0}^{M_n} f^{-i} B_\beta(y_i) \neq \emptyset
\]

for all \(n \in \mathbb{N}\), then \(\bigcap_{i \in \mathbb{Z}} f^{-i} B_\beta(x_i) = \bigcap_{n \in \mathbb{N}} E_n \neq \emptyset\) since \(E_n\) are closed and decreasing.

The following (L.7) and (L.8) are general properties of topological spaces and we omit the proofs since they are easily checked.

(L.7) If \(A_i, \ldots, A_k \subseteq X\) are closed, then one has

\[
\bigcup_{i=1}^k \text{int} A_i = \text{int} \left(\bigcup_{i=1}^k A_i\right)
\]

(L.8) If \(A_1, A_2 \subseteq X\) are closed and \(A_1 \supseteq A_2\), then we have

\[
\text{int} A_1 = \text{int} A_2 \cup \text{int} \left(A_1 \setminus A_2\right)
\]

(L.9) If \((X, f)\) has a Markov partition, then \((X, f)\) has Markov partitions with arbitrary small diameters.

Proof. Let \(P\) be a Markov partition for \((X, f)\). Define \(P^n = \{\bigcap_{i=-n}^{n} f^{-i} \text{int} P\} / P \in P\) for \(-n \leq i \leq n\) for \(n \geq 1\). Then the maximal diameter of the elements of \(P^n\) converges to 0 as \(n \to \infty\) by condition 4) of the definition of Markov partitions. We also have that \(P^n (n \geq 1)\) is a Markov partition. Indeed, we have \(\bigcup_{P \in P^n} P = X\) by (L.7). Other properties for \(P^n\) to be a Markov partition for \((X, f)\) is easily checked from the fact that \(P\) is a Markov partition.

3. Proof of Theorem

Theorem will be obtained in proving the following claims.

Claim 1 \((I) \iff (II)\).

Claim 2 \((II) \iff (III)\).

Claim 3 \((III) \iff (IV)\).

Claim 4 \((IV) \iff (I)\).
Proof of Claim 1 As in condition (I) let \(0<c<\varepsilon/2\) and take \(\eta=\eta(x)\) for \(x\in X\). Then there are finite number of subsets \(Y^1, \ldots, Y^n\) of \(B_\varepsilon(x)\) such that for every \(y\in B_\varepsilon(x)\) there exists \(1\leq i\leq n\) such that \(Y_i(y)\cap B_\varepsilon(x)=Y^i\). Put \(Y^i=\{z\in Y_i|Y_i(z)\cap B_\varepsilon(x)=Y^i\}\). Then \(Z_\varepsilon(y)\cap B_\varepsilon(x)=Y(y)\cup \bigcup_i Y^i\) where \(\Lambda=\{1\leq i\leq n|y\in Y^i\}\). Therefore \(\{Z_\varepsilon(y)\cap B_\varepsilon(x)|y\in B_\varepsilon(x)\}\) is finite.

Proof of Claim 2 The proof will be done along the following five steps.

Step 2.1. Let \(c\) be as in condition (II). Then there exist \(\delta_0>0\) and a finite partition \(\mathcal{P}\) such that \(x\sim y\) implies \(Z_\varepsilon(x)\cap B_{\delta_0}(x)\subset Z_\varepsilon(y)\).

Proof. For \(x\in X\) let \(\delta=\delta(x)>0\) be as in condition (II). Since \(X\) is compact, we can find finite points \(x_1, \ldots, x_k\) \(\subset X\) such that \(X=\bigcup_{i=1}^k \text{int}S_{\delta(x)}(x_i)\).

Put \(R_i=B_{\delta(x)}(x_i)\) for \(1\leq i\leq k\). By condition (II), for every \(1\leq i\leq k\) there exist finite number of subsets \(Z^1_i, \ldots, Z^n_i\subset R_i\) such that for every \(y\in R_i\), \(Z_\varepsilon(y)\cap R_i=Z^j_i\) holds for some \(1\leq j\leq n_i\). We can assume that \(Z^j_i\)'s are different if \(j\)'s are different. Denoting that \(\mathcal{Z}_i^j=\{x\in R_i|Z_\varepsilon(x)\cap R_i=Z^j_i\}\) for \(1\leq j\leq n_i\) and \(1\leq i\leq k\), we have \(R_i=\bigcup_{j=1}^{n_i} \mathcal{Z}_i^j\) (disjoint union) for \(1\leq i\leq k\). For every \(x\in X\) and \(1\leq i\leq k\), define \(D_i(x)\subset X\) by

\[
D_i(x) = \begin{cases} \mathcal{Z}_i^j & \text{if } x\in R_i \\ X\setminus R_i & \text{if } x\in R_i \end{cases}
\]

Put \(D(x)=\bigcap_{i=1}^k D_i(x)\) for \(x\in X\). Then \(\mathcal{D}=\{D(x)|x\in X\}\) is a finite partition. For every \(x\in X\) there exists \(1\leq i\leq k\) such that \(x\in B_{\delta(x)/2}(x_i)\) by the choice of \(x_1, \ldots, x_k\), and then we have \(B_{\delta_0}(x)\subset R_i\) where \(\delta_0=\min_{1\leq i\leq k} \delta(x_i)/2\). Since \(x\in \mathcal{Z}^j_i\) for some \(1\leq i\leq n_i\), we have \(y\in D(y)=D(x)\subset \mathcal{Z}^j_i\) for \(y\in X\) with \(x\sim y\).

Let \(\gamma\) be a number such that \(0<\gamma<\delta_0\) and

\(B_\varepsilon(x)\subset \bigcap_{i=1}^k f^{-i}B_\varepsilon(f^i(x))\).

It is enough to show that for small \(\beta>0\) there exists \(\alpha>0\) such that every \(\alpha\)-pseudo orbit with respect to \(\mathcal{D}\) is \(\beta\)-traced.

Let \(\beta>0\) be small enough. Then \(\beta<\gamma\) and \(Z_\varepsilon\cap \Delta_\beta\subset Z_\gamma\) by (L.5). Similarly there is \(\varepsilon\) with \(0<\varepsilon<\beta/6\) such that \(Z_\varepsilon\cap \Delta_\varepsilon\subset Z_{\beta/3}\) by (L.2). We can find \(M\in \mathcal{N}\) such that \(f^M W^\varepsilon_\gamma(x)\subset W^\varepsilon_{\beta/3}(f^M x)\) and \(f^{-M} W^\varepsilon_\gamma(x)\subset W^\varepsilon_{\beta/3}(f^{-M} x)\) for all \(x\in X\). Let \(\alpha>0\) be a number such that \(B_\varepsilon(x)\subset \bigcap_{i=0}^\infty f^{-i}B_{\varepsilon 2M}(f^ix)\) for all \(x\in X\).

By (L.6), it is enough to show that for every \(\alpha\)-pseudo orbit \(\{x_i\}_{i\in \mathbb{Z}}\) with respect to \(\mathcal{D}\) the following holds:

\[
(\ast) \quad \cap_{i=0}^n f^{-i}B_\varepsilon(x_i) \neq \emptyset \quad \text{for all } n \in \mathbb{N}.
\]
To prove (*) we shall prepare steps 2.2, 3, 4, and 5. From now on, we fix any $\alpha$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ with respect to $\mathcal{D}$.

**Step 2.2.** For every $i \geq 0$, $0 \leq j \leq M$ and $0 \leq k \leq M - j$,
\[ d(f^{j+k}x_i, f^kx_{i+j}) \leq e/2. \]

Proof. Since $\{x_i\}_{i \in \mathbb{Z}}$ is an $\alpha$-pseudo orbit with respect to $\mathcal{D}$, by the choice of $\alpha$ we have
\[ d(f^{j+k}x_i, f^{j+k-l}x_{i+l}) \leq e/2M \]
for all $j, k$ with $0 \leq j + k - l \leq M$. Therefore
\[ d(f^{j+k}x_i, f^kx_{i+j}) \leq \sum_{l=0}^{j-1} d(f^{j+k-l}x_{i+l}, f^{j+k-l}x_{i+l}) \leq e/2 \]
for all $0 \leq j \leq M$ and $0 \leq k \leq M - j$.

**Step 2.3.** If $y \in W^s_{\mathcal{D}}(x_i)$, then $(f^jy, x_{i+j}) \in Z_\gamma$ for all $0 \leq j \leq M$.

Proof. Since $y \in W^s_{\mathcal{D}}(x_i)$, we have $d(f^jy, f^jx_i) \leq \beta/3$ for all $j \geq 0$. By Step 2.2, we have $d(f^jx_i, x_{i+j}) \leq e/2$ for all $0 \leq j \leq M$. Thus $d(f^jy, x_{i+j}) \leq \beta/3 + e/2 < \beta/2$. It is clear that $(y, x_i) \in Z_\gamma$. Assume that $(f^jy, x_{i+j}) \in Z_\gamma$ for some $0 \leq j \leq M - 1$. By the choice of $\gamma$ we have that $(f^{j+1}y, f^{j+1}x_{i+j}) \in Z_\varepsilon$. Since $f^{j+1}x_{i+j} \sim x_{i+j+1}$, $(f^{j+1}y, x_{i+j+1}) \in Z_\varepsilon$ by Step 2.1. Thus the fact $d(f^{j+1}y, x_{i+j+1}) < \beta$ implies $(f^{j+1}y, x_{i+j+1}) \in Z_\gamma$.

**Step 2.4.** For $n \geq 0$ and $y \in W^s_{\mathcal{D}}(x_{nM})$,
\[ (f^M y, x_{(n+1)M}) \in Z_{\mathcal{D}/3}. \]

Proof. By Step 2.3 we have $(f^M y, x_{(n+1)M}) \in Z_\gamma$. By the choice of $M$, $f^M y \in W^s_{\mathcal{D}/3}(f^M x_{nM})$. Since $d(f^M x_{nM}, x_{(n+1)M}) \leq e/2$ by Step 2.2, we have that $d(f^M y, x_{(n+1)M}) \leq e$ and so $(f^M y, x_{(n+1)M}) \in Z_{\mathcal{D}/3}$.

Put $y_0 = x_0$ and $y_n = [x_{nM}, f^M y_{n-1}]$ for $n \geq 1$ (Remark that $[x_{nM}, f^M y_{n-1}]$ is always defined by Step 2.4).

**Step 2.5.** For every $u \geq 0$ and $w \geq 0$,
\[ f^{-w}y_{u+w} \in W^s_{\varepsilon/3}(y_u). \]

Proof. Take and fix any $u \geq 0$ and $w \geq 0$. Since $y_{u+w} \in W^s_{\varepsilon/3}(f^M y_{u+w-1})$, it is easily checked that $f^{-M}y_{u+w} \in W^s_{\varepsilon/2}(y_{u+w-1})$. Thus the fact $y_{u+w-1} \in W^s_{\varepsilon/3}$ ($f^M y_{u+w-2}$) implies
\[ f^{-M}y_{u+w} \in W^s_{\varepsilon/2+\varepsilon/3}(f^M y_{u+w-2}) \subset W^s_{\varepsilon}(f^M y_{u+w-2}), \]
from which $f^{-2M}y_{u+w} \in W^s_{\varepsilon/2}(y_{u+w-2})$. Repeating this procedure, we get the con-
Take and fix any \( n \leq 0 \). Put \( y = f^{-n}y_\ast \). For \( 0 \leq i \leq nM - 1 \) there exist \( 0 \leq j \leq n - 1 \) and \( 0 \leq k \leq M - 1 \) such that \( i = jM + k \). By Step 2.5 we have \( f^iy \in W_r^u(f^{jM}y_{i+1}) \), and by the fact that \( y_{i+1} \in W_r^u(f^{jM}y_i) \), \( f^iy \in W_r^u(f^{jM}y_{i+1}) \). Since \( y_i \in W_r^u(x_{iM}) \), we have \( f^iy \in W_r^u(f^ix_{iM}) \) by (1.1). Similarly \( d(f^ix_{iM}, x_{iM+k}) < \varepsilon/2 \) by Step 2.2 and \( f^iy \in B_{\varepsilon/2}B_p(x_i) \). Therefore (*) is proved.

Proof of Claim 3. Let \( \mathcal{D} \) be a finite partition in the definition of SPOTP and \( \varepsilon < 0 \) be a number such that \( 2\varepsilon \leq \varepsilon^\ast \). Take a \( \varepsilon < \varepsilon/2 \) so small that \((x, y) \in Z_\varepsilon \cap \Delta_\varepsilon \) implies \((fx, fy), (f^{-1}x, f^{-1}y) \in Z_\varepsilon \) (such a \( \beta \) exists by (L.5)). Let \( \alpha > 0 \) be a number such that every \( \alpha \)-pseudo orbit with respect to \( \mathcal{D} \) is \( \beta/2 \)-traced. Take \( 0 < \gamma < \alpha/2 \) such that \( d(x, y) < \gamma \) implies \( d(fx, fy) < \alpha/2 \). Since \( \mathcal{D} \) is finite, we can find a finite set \( T = \{t_1, \ldots, t_r\} \subset X \) such that for every \( x \in X \) there exists \( t_i \in T \) such that \( x \sim^\varepsilon t_i \), \( fx \sim^\varepsilon ft_i \), and \( d(x, t_i) < \gamma \).

For every \( v \in T^z \) we denote by \( v_i \) the \( i \)-th component of \( v \). Let us put

\[ \Sigma(T) = \{v \in T^z \mid \{v_i\}_{i \in \mathbb{Z}} \text{ is an } \alpha \text{-pseudo orbit with respect to } \mathcal{D}\} . \]

Since \( \beta \) is an expansive constant, we can define \( \theta: \Sigma(T) \rightarrow X \) by

\[ \theta(v) \cap \cap_{i \in \mathbb{Z}} f^{-i}B_{\beta/2}(v_i) \quad \text{for all } v \in \Sigma(T) . \]

For every \( x \in X \) and \( n \in \mathbb{Z} \), take \( v_n \in T \) such that \( f^nx \sim^\varepsilon v_n, f^{n+1}x \sim^\varepsilon f v_n \) and \( d(f^nx, v_n) < \gamma \). Then we have \( v = (v_n)_{n \in \mathbb{Z}} \in \Sigma(T) \) and \( \theta(v) = x \). Thus \( \theta \) is surjective. It is easy to check that \( f \circ \theta = \theta \circ \sigma \) where \( \sigma: \Sigma(T) \rightarrow \Sigma(T) \) is the shift automorphism, i.e., \( \sigma(v)_n = v_{n+1} \) for all \( n \in \mathbb{Z} \) and \( v \in \Sigma(T) \).

For \( 1 \leq i \leq r \), put \( \text{cyl}(t_i) = \{v \in \Sigma(T) \mid v_0 = t_i \} \) and \( T_i = \theta(\text{cyl}(t_i)) \). Then \( \text{diam } T_i \leq \beta \) and \( \cup_{i=1}^r T_i = X \). Since \( f \) is expansive, \( \theta \) is continuous by (L.1) and so \( T_i \) (\( 1 \leq i \leq r \)) is closed. For every \( v, w \in \Sigma(T) \) with \( v_0 = w_0 \), we can define \( [v, w] \in \Sigma(T) \) by \( [v, w]_n = v_n \) for \( n \geq 0 \) and \( [v, w]_n = w_n \) for \( n \leq 0 \). Then it is easy to check that \( (\theta(v), \theta(w)) \in \gamma \) and \( \theta([v, w]) = [\theta(v), \theta(w)] \).

To prove Claim 3, we shall prepare steps that will lead us to this end goal.

Step 3.1. \( T_i \)’s are closed rectangles.

Proof. It was already shown that \( T_i \)’s are closed. Thus it is only to show that \( T_i \)’s are rectangles. Take any \( 1 \leq i \leq r \) and any \( x, y \in T_i \). Then there exist \( v \in \theta^{-1}(x) \) and \( w \in \theta^{-1}(y) \) with \( v_0 = w_0 = t_i \). Since \( \theta([v, w]) = [\theta(v), \theta(w)] = [x, y] \) and \( [v, w]_n = t_i \), we have \( [x, y] \in T_i \) and hence \( T_i \) is a rectangle.

Step 3.2. For \( v \in \Sigma(T) \), we have the following: (1) for every \( y \in W_r^s(\theta(v)) \cap \theta(\text{cyl}(v_0)) \) there exists \( w \in \theta^{-1}(y) \) such that \( w_n = v_n \) for all \( n \geq 0 \), (2) for every \( y \in W_r^s(\theta(v)) \cap \theta(\text{cyl}(v_0)) \) there exists \( w \in \theta^{-1}(y) \) such that
Proof. For \( y \in W^s_\varepsilon(\theta(v)) \cap \theta(\text{cyl}(v_0)) \), there exists \( v' \in \theta^{-1}(y) \) such that \( v'_0 = v_0 \). Therefore we can define \( w = [v, v'] \in \Sigma(T) \) and \( \theta(w) = \theta([v, v']) = [\theta(v), y] = y \). Then \( w \in \theta^{-1}(y) \) and \( w_n = v_n \) for all \( n \geq 0 \). Thus (1) of Step 3.2 holds. Similarly we have (2).

**Step 3.3.** For every \( x \in X \) and \( T_i \) \((1 \leq i \leq r)\) with \( x \in T_i \) we have

\[
W^s_{\varepsilon}(x) \cap T_i \subset W^s_\beta(x)
\]
for \( \sigma = s, u \).

Proof. For \( y \in W^s_\varepsilon(x) \cap T_i \) and \( n \geq 0 \), there exists \( T_j \) \((1 \leq j \leq r)\) such that \( f^ny, f^nx \in T_j \) by Step 3.2. Since \( \text{diam} T_j \leq \beta \), we have \( d(f^n y, f^n x) \leq \beta \) for all \( n \geq 0 \) and so \( y \in W^s_\beta(x) \). Therefore \( W^s_{\varepsilon}(x) \cap T_i \subset W^s_\beta(x) \). Similarly we have \( W^s_\beta(x) \cap T_i \subset W^s_\beta(x) \).

For \( x \in X \), define subsets \( \Lambda(x), \Lambda'(x) \), and \( \Lambda^s(x) \) of \( \{1, \ldots, r\} \) by

\[
\Lambda(x) = \{1 \leq i \leq r \mid x \in T_i\},
\]
\[
\Lambda'(x) = \bigcup_{j \in \Lambda(x)} \{1 \leq i \leq r \mid T_i \cap T_j \cap W^s_{\varepsilon}(x) + \emptyset\},
\]
and

\[
\Lambda^s(x) = \bigcup_{i \in \Lambda(x)} \{1 \leq i \leq r \mid T_i \cap T_j \cap W^s_{\varepsilon}(x) + \emptyset\}.
\]

**Step 3.4.** For every \( i \in \Lambda^s(x) (\sigma = s, u) \) there exists \( j \in \Lambda(x) \) such that \( T_i \cap T_j \cap W^s_{\varepsilon}(x) + \emptyset \).

Proof. We give the proof for \( \sigma = s \). For \( i \in \Lambda^s(x) \) there exists \( j \in \Lambda(x) \) such that \( T_i \cap T_j \cap W^s_{\varepsilon}(x) + \emptyset \). Take \( y \in T_i \cap T_j \cap W^s_{\varepsilon}(x) \). By Step 3.3 we have \( y \in W^s_\beta(x) \). Thus \( T_i \cap T_j \cap W^s_\beta(x) + \emptyset \).

**Step 3.5.** \( \Lambda'(x) \cap \Lambda^s(x) = \Lambda(x) \) for all \( x \in X \).

Proof. It is clear from definition that \( \Lambda(x) \subset \Lambda'(x) \cap \Lambda^s(x) \) for all \( x \in X \). To prove that \( \Lambda'(x) \cap \Lambda^s(x) \subset \Lambda(x) \), we use Step 3.4. Indeed, for every \( i \in \Lambda'(x) \cup \Lambda^s(x) \) there exists \( y \in T_i \cap W^s_\beta(x) \) and \( z \in T_i \cap W^s_\beta(x) \). Since \( y, z \in T_i \), we have \([y, z] \subset T_i \). Therefore \([y, z] \subset W^s_\beta(y) \cap W^s_\beta(z) \subset W^s_\beta(x) \cap W^s_\beta(x) \). Since \( 2\beta < c \) and \( 2c \) is an expansive constant, we have \( x = [y, z] \subset T_i \). Thus \( i \in \Lambda(x) \).

For \( x \in X \), put \( L(x) = \{y \in X \mid \Lambda^s(y) = \Lambda^s(x) \} \) for \( \sigma = s, u \). Clearly \( x \in L(x) \) for all \( x \in X \), and \( y \in L(x) \) implies \( L(y) = L(x) \). Put \( \mathcal{L} = \{L(x) \mid x \in X\} \). Then \( \mathcal{L} \) is a finite partition. For \( L \in \mathcal{L} \), denote \( \Lambda(L) = \Lambda(x) \), \( \Lambda'(L) = \Lambda'(x) \), and \( \Lambda^*(L) = \Lambda^*(x) \) for \( x \in L \). Notice that \( \Lambda(L), \Lambda^*(L), \) and \( \Lambda^*(L) \) are independent of the choice of \( x \in L \).

**Step 3.6.** Every \( L \in \mathcal{L} \) is a rectangle.
Proof. For \( x, y \in L, [x, y] \) is defined and \([x, y] \in T_i \) for all \( i \in \Lambda(L) \) since \( x, y \in T_i \) for all \( i \in \Lambda(L) \). Therefore we have \( \Lambda([x, y]) \subseteq \Lambda(L) \). For every \( i \in \Lambda'(L) \) there exists \( j \in \Lambda(x) \) such that \( T_i \cap T_j \cap W^*([x, y]) = \emptyset \) by Step 3.4. Since \( \Lambda(x) = \Lambda(L) \subseteq \Lambda([x, y]) \), we have \( j \in \Lambda([x, y]) \). Since \( x \in W^*([x, y]) \), \( T_i \cap T_j \cap W^*([x, y]) = \emptyset \) and so \( i \in \Lambda'(x, y) \). Thus \( \Lambda'([x, y]) \subseteq \Lambda'(L) \). Conversely for every \( i \in \Lambda'(x, y) \) there exists \( j \in \Lambda([x, y]) \) such that \( T_i \cap T_j \cap W^*([x, y]) = \emptyset \). For every \( k \in \Lambda(x) \), we have \( [x, y] \in T_k \cap T_j \cap W^*([x, y]) \) since \( \Lambda([x, y]) \supseteq \Lambda(x) \), and so \( j \in \Lambda'([x, y]) \). Similarly \( j \in \Lambda'(L) \). We obtain \( j \in \Lambda(L) \) by Step 3.5. Since \([x, y] \in W^*([x, y]) \), we have \( T_i \cap T_j \cap W^*(x) = \emptyset \). Hence \( i \in \Lambda'(L) \). Therefore \( \Lambda'([x, y]) = \Lambda'(L) \). Similarly we have \( \Lambda'(x, y) = \Lambda'(L) \).

**Step 3.7.** For every \( x \in X \),

\[ f(W^*_i(x) \cap L(x)) \subseteq W^*_i(fx) \cap L(fx) \]

and

\[ f^{-1}(W^*_i(fx) \cap L(fx)) \subseteq W^*_i(x) \cap L(x) \]

Proof. Take any \( y \in W^*_i(x) \cap L(x) \). For any \( i \in \Lambda'(fx) \) there exist \( j \in \Lambda(fx) \) and \( w \in \theta^{-1}(x) \) such that \( w = t_i \). Let \( 1 \leq k \leq r \) be the number such that \( v_0 = t_k \). Since \( z \in W^*_i(fx) \cap T_j \), we have \( f^{-1}z \in T_k \) by Step 3.2 (2). It is clear that \( y \in T_k \cap W^*_i(x) \), and \( fy \in T_j \) by Step 3.2 (1). Since \( fy, z \in T_j \), we have \([z, fy] \in T_j \). Since \( z \in T_i \), there exists \( \omega \in \theta^{-1}(x) \) such that \( \omega = t_i \). Let \( 1 \leq l \leq r \) be the number such that \( \omega = t_l \). Since \( f^{-1}x \in T_j \cap W^*_i(x) \), we have \( l \in \Lambda^*(x) \) and so \( l \in \Lambda^*(y) \). By the fact that \( l \in \Lambda^*(x) \cap \Lambda^*(y) \), we have \([f^{-1}_x, y] \in T_l \). Thus \( f[f^{-1}_x, y] = [x, fy] \in T_i \) by Step 3.2 (1). Hence \([x, fy] \in T_i \cap T_j \cap W^*_i(fy) \), and so \( i \in \Lambda^*(fy) \). Therefore \( \Lambda^*(fy) \subseteq \Lambda^*(fx) \). Since \( x \in W^*_i(y) \cap L(y) \), by symmetry \( \Lambda^*(fy) \subseteq \Lambda^*(fx) \). Thus \( \Lambda^*(fy) = \Lambda^*(fx) \), and similarly \( \Lambda^*(fy) = \Lambda^*(fx) \). Since \( fW^*_i(x) \subseteq W^*_i(fx) \), we have \( f(W^*_i(x) \cap L(x)) \subseteq W^*_i(fx) \cap L(fx) \). Similarly we see that \( f^{-1}(W^*_i(fx) \cap L(fx)) \subseteq W^*_i(x) \cap L(x) \).

**Step 3.8.** For every \( x \in X \) there exists a neighborhood \( U \) of \( x \) such that \( \Lambda(x) \supseteq \Lambda(y) \) for all \( y \in U \).

Proof. For \( x \in X \), put \( U = X \setminus \bigcup_{i \in \Lambda(x)} T_i \). Then \( U \) is open, \( x \in U \) and \( \Lambda(y) \subseteq \Lambda(x) \) for \( y \in U \).

By Step 3.8, the following is clear.

**Step 3.9.** Let \( \sigma = s, u \) and \( L \subseteq L \). For every \( x \in X \) with \( \Lambda^*(x) \supseteq \Lambda^*(L) \) there exists a neighborhood \( U \) of \( x \) such that \( \Lambda^*(y) \supseteq \Lambda^*(L) \) for all \( y \in U \).

**Step 3.10.** \( \bigcup_{L \subseteq L} \text{int}^\infty L = X \).
Proof. For $L \in \mathcal{L}$, define $\mathcal{L}(L) = \{M \in \mathcal{L} | \Lambda^*(M) \supset \Lambda^*(L) \text{ for } \sigma = s, u\}$ and $L^* = \bigcup M \in \mathcal{L}(L) M$. Then $L^*$ is closed by Step 3.9. Since $\bigcup L \in \mathcal{L}^* = \bigcup L \in \mathcal{L} L = X$, we have $\bigcup L \in \mathcal{L} \text{ int } L^* = X$ by (L.7). Therefore it is sufficient to show that $\text{ int } M^* \subset \bigcup L \in \mathcal{L} \text{ int } L$ for all $M \in \mathcal{L}$.

Put $\mathcal{L}_0 = \{L \in \mathcal{L} | \mathcal{L}(L) = \{L\}\}$ and $\mathcal{L}_n = \{L \in \mathcal{L} | M \in \mathcal{L}_{n-1} \} \setminus \{L\}$ ($n \geq 1$) inductively. If $M \in \mathcal{L}_0$, then it is trivial that $\text{ int } M^* = \text{ int } M \subset \bigcup L \in \mathcal{L} \text{ int } L$. Assume that $\text{ int } M^* \subset \bigcup L \in \mathcal{L} \text{ int } L$ for all $M \in \mathcal{L}_{n-1}$. For $M \in \mathcal{L}_n$ put $A_1 = M^*$ and $A_2 = \bigcup N \in \mathcal{L}(M) \{M\} N^*$. By (L.8), we have

$$\text{ int } M^* = \text{ int } M \cup \bigcup N \in \mathcal{L}(M) \{M\} \text{ int } N^*.$$ 

By (L.7),

$$\text{ int } \bigcup N \in \mathcal{L}(M) \{M\} \text{ int } N^* = \bigcup N \in \mathcal{L}(M) \{M\} \text{ int } N^*,$$

and so

$$\text{ int } M^* = \text{ int } M \cup \left( \bigcup N \in \mathcal{L}(M) \{M\} \text{ int } N^* \right).$$

Since $\mathcal{L}(M) \{M\} \subset \mathcal{L}_{n-1}$, we have $\text{ int } M^* \subset \bigcup L \in \mathcal{L} \text{ int } L$ for all $M \in \mathcal{L}_n$. By the fact that every $M \in \mathcal{L}$ satisfies $M \in \mathcal{L}_n$ for some $n \geq 0$, we get the proof.

**Step 3.11.** Let $E$ be a rectangle. Then $x \in \text{ int } E$ if and only if there exists a neighborhood $U$ of $x$ such that $U \cup W^s(x) \subset E$ for $\sigma = s, u$.

Proof. Suppose that there is a neighborhood $U$ of $x$ such that $U \cap W^s(x) \subset E$ for $\sigma = s, u$. By Step 3.8, there is an neighborhood $V$ of $x$ such that $\Lambda(y) \subset \Lambda(x)$ for all $y \in V$. Then for $y \in V$ and $t \in \Lambda(y)$, we have $x, y \in T_t$, and so $(x, y) \in \mathcal{L}$. Since $x = [x, x]$ and $[x, y]$ is continuous, there exists a neighborhood $W \subset V$ such that $[x, y] \in U \cap W^s(x)$ and $[y, x] \in U \cap W^u(x)$ for all $y \in W$. Then we have $y = [(y, x), [x, y]] \in E$ for all $y \in W$. Therefore $x \in \text{ int } E$. The “only if” part is clear and so we omit the proof.

**Step 3.12.** For every $L \in \mathcal{L}$, $\text{ int } L$ is a rectangle.

Proof. For $x, y \in \text{ int } L$, it is clear that $[x, y] \in L$. By Step 3.8, there exists a neighborhood $U$ of $[x, y]$ such that $\Lambda([x, y]) \supset \Lambda(x)$ for all $z \in U$. Take $z \in U$ and $i \in \Lambda(z)$. Then $i \in \Lambda([x, y]) = \Lambda(x)$ (= $\Lambda(L)$). Thus we have $x, z \in T_i$. Therefore we can define $[z, x]$ for all $z \in U$. Notice that $[x, y]$ is continuous. Since $x = [(x, y), [x, y]]$ and $x \in \text{ int } L$, there exists a neighborhood $V \subset U$ of $[x, y]$ such that $[z, x] \in L$ for all $z \in V$. Hence for $z \cap V \cap W^s([x, y])$ we have $z = [(z, x), [x, y]] \in L$. Similarly there exists a neighborhood $V'$ of $[x, y]$ such that $z \in L$ for all $z \in V' \cap W^u([x, y])$. Thus $W \cap W^u([x, y]) \subset L$ ($\sigma = s, u$) where $W = V \cap V'$. Therefore $[x, y] \in \text{ int } L$ by Step 3.11.

We prove that $\mathcal{P} = \{\text{ int } L | \text{ int } L \neq \emptyset, L \in \mathcal{L}\}$ is a Markov partition. Obviously $\bigcup P \in \mathcal{P} = X$ (by Step 3.10) and $\text{ int } P = P$ for all $P \in \mathcal{P}$.
diam \( P \leq \beta \) for \( P \in \mathcal{P} \), we have 4) in the definition of Markov partitions. Since \( \text{int} \, L (L \in \mathcal{L}) \) is a rectangle by Step 3.12, so is \( \overline{\text{int} \, L} \) by (L.4). Therefore \( \mathcal{P} \) consists of closed rectangles. We remark that \( \text{int} \, L \cap \text{int} \, L' = \emptyset \) for \( L, L' \in \mathcal{L} \) with \( L \neq L' \). Then for any \( P \in \mathcal{P} \), there is a unique \( L \in \mathcal{L} \) such that \( P = \overline{\text{int} \, L} \) and so we write \( L(P) \) for such the \( L \). For \( P, Q \in \mathcal{P} \) with \( P \neq Q \), we have \( \text{int} \, L(P) \cap \text{int} \, L(Q) = \emptyset \) and so \( P \cap Q \) has no interior. Therefore \( \text{int} \, P \cap \text{int} \, Q = \emptyset \).

**Step 3.13.** For every \( x \in \text{int} \, L \cap f^{-1} \text{int} \, M \) \((L, M \in \mathcal{L})\),
\[
f(W^s_t(x) \cap \text{int} \, L) \subset W^s_t(fx) \cap \text{int} \, M
\]
and
\[
f^{-1}(W^s_t(fx) \cap \text{int} \, M) \subset W^s_t(x) \cap \text{int} \, L.
\]

Proof. Take \( y \in W^s_t(x) \cap \text{int} \, L \). Then \( fy \in M \) by Step 3.7 and there exists a neighborhood \( U \) of \( y \) such that \( \Lambda(z) \subset \Lambda(M) \) for all \( z \in U \) by Step 3.8. For \( z \in U \) and \( i \in \Lambda(z) \), we have \( fx, z \in T \), and so \( (fx, z) \in Z \). Since \( fx \in \text{int} \, M \) and \( fx = [fy, fx] \), there exists a neighborhood \( V \subset U \) of \( fy \) such that \( [x, fx] \in M \) and \( [fx, z] \in M \) for all \( z \in V \cap f(\text{int} \, L) \). Thus \( z = [fx, z] \in [fx, z] \in M \) for all \( z \in V \cap f(\text{int} \, L) \), and so \( fy \in \text{int} \, M \). Clearly \( fy \in W^s_t(fx) \). Hence \( f(W^s_t(x) \cap L) \subset W^s_t(fx) \cap \text{int} \, M \). Similarly we have \( f^{-1}(W^s_t(fx) \cap M) \subset W^s_t(x) \cap \text{int} \, L \).

**Step 3.14.** \( \overline{\text{int}} \, P \cap f^{-1} \text{int} \, Q = \overline{\text{int}} \, L(P) \cap f^{-1} \text{int} \, L(Q) \) for all \( P, Q \in \mathcal{P} \).

Proof. We have \( \overline{\text{int}} \, A \cap \overline{\text{int}} \, B = \overline{\text{int}} \, (A \cap B) \) for open sets \( A \) and \( B \). Therefore \( \overline{\text{int}} \, A \cap \overline{\text{int}} \, B = A \cap B \) for open sets \( A \) and \( B \). Put \( A = \overline{\text{int}} \, L(P) \) and \( B = f^{-1} \text{int} \, L(Q) \). Then
\[
\overline{\text{int}} \, P \cap f^{-1} \text{int} \, Q = \overline{\text{int}} \, L(P) \cap f^{-1} \text{int} \, L(Q) \text{.}
\]

**Step 3.15.** For every \( x \in \overline{\text{int}} \, P \cap f^{-1} \text{int} \, Q \),
\[
f(W^s_t(x) \cap P) \subset W^s_t(fx) \cap Q
\]
and
\[
f^{-1}(W^s_t(fx) \cap Q) \subset W^s_t(x) \cap P.
\]

Proof. If \( x \in \overline{\text{int}} \, P \cap f^{-1} \text{int} \, Q \), by Step 3.14 there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \) of points in \( \text{int} \, L(P) \cap f^{-1} \text{int} \, L(Q) \) such that \( x_n \to x \) as \( n \to \infty \). For \( y \in W^s_t(x) \subset P \), there exists a sequence \( \{y_n\}_{n \in \mathbb{N}} \) of the points in \( \text{int} \, L(P) \) such that \( y_n \to y \) as \( n \to \infty \). Since \( x_n \) and \( y_n \) are in a rectangle \( \text{int} \, L(P) \), we have that \( [x_n, y_n] \in W^s_t(x) \cap \text{int} \, L(P) \) for all \( n \in \mathbb{N} \). Then by Step 3.13, we have that \( f[x_n, y_n] \in W^s_t(fx_n) \cap \text{int} \, L(Q) \) for all \( n \in \mathbb{N} \). Since \( \cdot \) is continuous, \( [x_n, y_n] \to [x, y] \) as \( n \to \infty \). Thus \( fy \in W^s_t(fx) \cap \overline{\text{int}} \, L(Q) = W^s_t(fx) \cap Q \) and so \( f(W^s_t(x) \cap P) \subset W^s_t(fx) \cap Q \). Similarly, we obtain that \( f^{-1}(W^s_t(fx) \cap Q) \subset W^s_t(x) \cap P \).
Step 3.16. For every \( x \in \text{int } P \cap f^{-1} \text{int } Q \),
\[
f(W^*_s(x) \cap \text{int } P) \subset W^*_s(fx) \cap \text{int } Q
\]
and
\[
f^{-1}(W^*_u(fx) \cap \text{int } Q) \subset W^*_u(x) \cap \text{int } P.
\]

Proof. For \( y \in W^*_s(x) \cap \text{int } P \), int \( P \) is a neighborhood of \( y \) and so \( f(\text{int } P) \)
is a neighborhood of \( fy \). By the choice of \( \beta \), we have \( f(W^*_s(x) \cap \text{int } P) = f(W^*_s(y) \cap \text{int } P) \supset W^*_\beta(fy) \cap f(\text{int } P) \). By (L.3) there exists a neighborhood \( V \subset f(\text{int } P) \) of \( fy \) such that \( V \cap W^*_\beta(fy) \subset f(W^*_s(x) \cap \text{int } P) \subset Q \) by Step 3.15. Since \( fy \in Q \), there exists a neighborhood \( W \) of \( fy \) such that \( \Lambda(z) \subset \Lambda(fy) = \Lambda(fx) \) for all \( z \in W \) by Step 3.8. For \( z \in W \) and \( i \in \Lambda(z) \), we have \( z, fx \in T \), and so \( (z, fx) \in Z^* \). Since \( fx \in \text{int } Q \) and \([fy, fx] = fx\), there exists a neighborhood \( W' \subset W \) of \( fy \) such that \( [z, fx] \in Q \). For \( z \in W^*_s(fy) \cap W' \) we have \( z - [z, fx] \in Q \). Put \( U = V \cap W' \), then \( W^*_s(fy) \cap U \subset Q \) for \( \sigma = s, u \). Thus the conclusion of Claim 3 is followed by the following Step.

Step 3.17. \( fB' \subset B' \) and \( f^{-1}B^u \subset B^u \).

Proof. For \( x \in B' \) there exists \( P \in \mathcal{P} \) such that \( x \in \partial'P \). By (L.7) we have
\[
\bigcup_{Q \in \mathcal{Q}} \text{int } (P \cap f^{-1}Q) = \text{int } \left( \bigcup_{Q \in \mathcal{Q}} (P \cap f^{-1}Q) \right).
\]
Therefore \( \bigcup_{Q \in \mathcal{Q}} \text{int } P \cap f^{-1}Q = \text{int } P = P \), and so there exists \( Q \in \mathcal{Q} \) such that \( x \in \text{int } P \cap f^{-1}Q \). Assume that \( fx \notin Q \). Then there exists a neighborhood \( U \) of \( fx \) such that \( U \cap W^*_u(fx) \subset Q \). Clearly \( f^{-1}U \) is a neighborhood of \( x \). By Step 3.15, \( f^{-1}U \cap W^*_\beta(x) \subset f^{-1}(U \cap W^*_\beta(fx)) \subset P \). By (L.3) there exists a neighborhood \( V \) of \( x \) such that \( W^*_s(x) \cap V \subset W^*_\beta(x) \). Thus \( (V \cap f^{-1}U) \cap W^*_s(x) \subset P \) and so \( x \in \partial'P \). This is a contradiction. So we have \( fx \in \partial'Q \subset B^u \) and \( fB' \subset B' \). Similarly we have \( f^{-1}B^u \subset B^u \).
Proof of Claim 4. Let $c>0$ be a number such that $2c$ is an expansive constant for $f$ and $\mathcal{P}$ be a Markov partition for $(X,f)$. By (L.9) we can assume that $\max\{\text{diam } P | P \in \mathcal{P}\} \leq c/2$. It is easy to check that $W^s_{\epsilon} (x) \cap P \in W^s_{\epsilon} (x)$ for all $x \in P \in (\mathcal{P})$ and for $\sigma = s, u$. Thus $W^s_{\epsilon} (x) \cap W^u_{\epsilon} (y) \neq \emptyset$ for all $x, y \in P \in (\mathcal{P})$. Fix $x \in X$ and define

$$\Gamma^*(x) = \{ P \in \mathcal{P} | P \cap W^s_{\epsilon} (x) \neq \emptyset \}$$

and

$$Y^*(x) = \{ y \in X | W^s_{\epsilon} (y) \cap P \neq \emptyset \text{ for some } P \in \Gamma^*(x) \}.$$

It is easily checked that $Y^s_{\epsilon} (x) \subseteq Y^*(x) \subseteq Y^s_{\epsilon} (x)$ for all $x \in X$. Since $\mathcal{P}$ is finite, so is $\{ \Gamma^*(x) | x \in X \}$. It is clear that $Y^*(x) = Y^*(y)$ whenever $\Gamma^*(x) = \Gamma^*(y)$. Therefore $\{ Y^*(x) | x \in X \}$ is finite. By (L.5), there exists $\eta > 0$ such that $\Delta_\eta \cap Y^s_{\epsilon} \subseteq Y^s_{\epsilon}$. Then $Y^s_{\eta} (y) \cap B_s (x) = Y^s_{\epsilon} (y) \cap B_s (x)$ for all $x \in X$ and all $y \in B_s (x)$. Clearly we have $Y^s_{\eta} (y) \cap B_s (x) = Y^s_{\eta} (y) \cap B_s (x)$ for all $x \in X$ and all $y \in B_s (x)$. Therefore $\{ Y^s_{\eta} (y) \cap B_s (x) | y \in B_s (x) \} = \{ Y^s_{\eta} (y) \cap B_s (x) | y \in B_s (x) \}$ is finite for all $x \in X$.

4. Applications

This section contains some applications of our theorem.

Let $S$ be a finite set, and $\sigma$ be the shift automorphism of $S^\mathbb{Z}$. The usual product topology is given to $S^\mathbb{Z}$ and elements of $S^\mathbb{Z}$ will be written as $x = (x_n)_{n \in \mathbb{Z}}$. Let $\Sigma$ be a $\sigma$-invariant closed subset of $S^\mathbb{Z}$. A system $(\Sigma, \sigma)$ is said to be a subshift. A subshift $(\Sigma, \sigma)$ is said to be of finite type if there exist $n \in \mathbb{N}$ and a subset $B$ of $S^n$ such that

$$\Sigma = \{ x \in S^\mathbb{Z} | (x_i, x_{i+1}, \ldots, x_{i+n-1}) \in B \text{ for all } i \in \mathbb{Z} \}.$$

A subshift $(\Sigma, \sigma)$ is said to be sofic if there exists a subshift $(\Sigma', \sigma')$ of finite type such that $(\Sigma, \sigma)$ is a factor of $(\Sigma', \sigma')$, i.e., there exists a continuous surjective map $\phi: \Sigma' \to \Sigma$ such that $\sigma \circ \phi = \phi \circ \sigma'$. Remark that a sofic subshift does not have POTP unless it is of finite type ([37]).

Application 1. A subshift $(\Sigma, \sigma)$ of $(S^\mathbb{Z}, \sigma)$ has Markov partitions if and only if it is sofic.

Proof. If $(\Sigma, \sigma)$ of $(S^\mathbb{Z}, \sigma)$ has Markov partitions, then $(\Sigma, \sigma)$ is a factor of a subshift of finite type (c.f., see [15]), and so $(\Sigma, \sigma)$ is sofic. Conversely, let $(\Sigma, \sigma)$ be sofic and $W$ be the set of words which occur in $\Sigma$. For $w \in W$, define $F(w) = \{ w' \in W | w w' \in W \}$. Since $(\Sigma, \sigma)$ is sofic, $\{ F(w) | w \in W \}$ is finite, i.e., $(\Sigma, \sigma)$ is $F$-finitary ([38]). For $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma$, $F_n = F(x_{-n}, x_{-(n-1)}, \ldots, x_0)$ ($n \geq 0$) is a decreasing sequence of $W$. Since $\{ F(w) | w \in W \}$ is finite, there exists $N \geq 0$
such that \( F_n = F_N \) for all \( n \geq N \). Put \( \hat{F}(x) = F_N \). Clearly \( \{ \hat{F}(x) | x \in \Sigma \} \subset \{ F(w) | w \in W \} \). Let \( G(x) = \{ y \in \Sigma | (y_1, \ldots, y_n) \in \hat{F}(x) \) for all \( n \geq 1 \} \) for \( x \in \Sigma \). For \( x, y \in \Sigma \), define \( z = (z_i)_{i \in \mathbb{Z}} \in S^\mathbb{Z} \) by \( z_i = x_i \) for \( i \leq 0 \) and \( z_i = y_i \) for \( i \geq 1 \). Then \( z \in \Sigma \) if and only if \( y \in G(x) \). It is clear that \( \{ G(x) | x \in \Sigma \} \) is finite. Define the metric on \( \Sigma \) by \( d(x, y) = \max_{i \in \mathbb{Z}} \delta(x_i, y_i)/2^{|i|} \) for \( x, y \in \Sigma \) where \( \delta(x_i, y_i) = 0 \) if \( x_i = y_i \) and \( 1 \) if \( x_i \neq y_i \). Put \( c = 1/3 \). Then \( 2c \) is an expansive constant and for \( \sigma \),

\[
W^s(x) = \{ y \in \Sigma | y_i = x_i \quad \text{for all} \quad i \leq 0 \}
\]

\[
W^u(x) = \{ y \in \Sigma | y_i = x_i \quad \text{for all} \quad i \geq 1 \}
\]

Therefore we have that \( Y_c(x) = \{ y \in \Sigma | y \in G(x), y_0 = x_0 \) and \( y_{-1} = x_{-1} \} \) for all \( x \in \Sigma \). Since \( \{ G(x) | x \in \Sigma \} \) is finite and \( S \) is finite, we obtain that \( \{ Y_c(x) | x \in \Sigma \} \) is finite. This shows that \( (\Sigma, \sigma) \) satisfies condition (I) of our theorem and so \( (\Sigma, \sigma) \) has Markov partitions.

Let \( f \) be an expansive homeomorphism of a compact metric space \( X \), and \( c > 0 \) be a number such that \( 2c \) is an expansive constant for \( f \). A point \( x \in X \) is said to be a singular point if there exists a sequence \( \{ x_n \}_{n \in \mathbb{N}} \) such that \( x_n \to x \) as \( n \to \infty \) and \( x \in \text{int} Y_c(x_n) \) for all \( n \in \mathbb{N} \). There are no singular points if \( (X, f) \) has POTP.

**Application 2.** Pseudo-Anosov maps (for the definition, see [18]) have Markov partitions, but not have POTP.

Proof. It is sketched in [17] that pseudo-Anosov maps have Markov partitions. But this is easily obtained by our theorem. For, from the definition pseudo-Anosov maps are expansive and have condition (I) of our theorem. Pseudo-Anosov maps do not have POTP since they have singular points by definition.

Let \( f \) be a hyperbolic automorphism of \( r \)-dimensional torus \( T^r \). It is known that \( (T^r, f) \) is expansive and has POTP. It is known also that for every fixed point \( p \in T^r \) of \( f \), there is a point \( p' \neq p \) \( T^r \) such that \( f^n p' \to p \) as \( n \to \pm \infty \).

**Application 3.** Let \( f \) be a hyperbolic automorphism of \( T^r \) with fixed points \( p \) and \( q \). Let \( X \) be the quotient space of \( T^r \) induced by identifying \( p \) with \( q \), and \( g \) be the homeomorphism of \( X \) induced from \( f \). Then \( (X, g) \) has Markov partitions, but it does not have POTP.

Proof. Clearly \( (X, g) \) is expansive and has condition (I) of our theorem, and so it has Markov partitions. Since \( p (= q) \) is singular point in \( X \), \( (X, g) \) does not have POTP.
In Applications 2 and 3, the number of singular points are finite. However we can consider the case with infinite number of singular points as follows;

**Application 4.** Let \( f \) be a hyperbolic automorphism of \( T_r \) with fixed points \( p \) and \( q \), and \( p' (\neq p) \in T_r \) be a point such that \( f^n p' \to p \) as \( n \to \pm \infty \). Let \( X \) be the quotient space of \( T_r \) induced by identifying \( p \) with \( q \) and \( f^n p' \) with \( f^n p' + (q - p) \) \((n \in \mathbb{Z})\), and \( g \) be the homeomorphism of \( X \) induced from \( f \). Then \((X, g)\) has Markov partitions and has infinite number of singular points.

Proof. It is easy to check that \((X, g)\) is expansive. \((X, g)\) has Markov partitions as in the proof of Application 3. Since \( f^n p' \)'s are different singular points, the conclusion is obtained.

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