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HOMEOMORPHISMS WITH MARKOV PARTITIONS

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1. Introduction

The Markov partition in dynamical systems suplies us important informations (for examples, for studies of equilibrium states [5] and zeta functions [24]).

Such a partition was first constructed for Anosov diffeomorphisms by Ja.G. Sinai [35]. After that, R. Bowen [5] showed the existence of Markov partition on nonwandering sets of Axiom A diffeomorphisms. In these papers the notion of canonical coordinate play an important role to construct Markov partitions. K. Hiraide [20], in purely topological setting, proved the existence of Markov partition for expansive homeomorphisms with POTP by constructing canonical coordinates. For example, every expansive automorphism of a solenoidal group has POTP, and hence a cononical coordinate as well as a Markov partition (N. Aoki [2], [3] and [20]).

However homeomorphisms with Markov partitions do not necessarily have canonical coordinates. In fact, every pseudo-Anosov map has a Markov partition and does not have cononical coordinates (see paragraphs 9 and 10 of [17]).

Thus it is natural to ask what kind of expanisive homeomorphisms have Markov partitions. The purpose of this paper is to give necessary and sufficient conditions for expansive homeomorphisms to have Markov partitions. More precisely we can describe our result as follows;

Theorem. *Let X be a compact metric space and f be an expansive homeomorphism of X with expansive constant c^*. Then the following conditions are equivalent;*

(I) *there exists $c>0$ with $2c\leqq c^*$ such that for every $x\in X$ there exists an $\eta=\eta(x)>0$ such that $\{Y_c(y)\cap B_\eta(x) \mid y\in B_\eta(x)\}$ is finite,*

(II) *there exists $c<0$ with $2c\leq c^*$ such that for every $x\in X$ there exists a $\delta=\delta(x)>0$ such that $\{Z_c(y)\cap B_\delta(x) \mid y\in B_\delta(x)\}$ is finite,*

(III) *(X,f) has SPOTP,*

(IV) *(X,f) has a Markov partition.*

The proof will be give in section 3 and the auxiliary results used in the proof will be prepared in section 2. We shall describe in section 4 some applications of our result.

In the remainder of this section, we shall give some definitions which are used in the proof of our result.

Let X be a compact metric space with metric d, and f be a homeomorphism of X. For $x\in X$, $B_\varepsilon(x)$ will denote the closed ε-ball in X centered at x. For $x\in X$ and $\varepsilon>0$ define subsets $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ of $B_\varepsilon(x)$ by $W^s_\varepsilon(x)=\bigcap_{n\geq 0}f^{-n}B_\varepsilon(f^n x)$ and $W^u_\varepsilon(x)=\bigcap_{n\leq 0}f^{-n}B_\varepsilon(f^n x)$. Then we have

$$(1.1)\qquad fW^s_\varepsilon(x)\subset W^s_\varepsilon(fx),\quad f^{-1}W^u_\varepsilon(x)\subset W^u_\varepsilon(f^{-1}x)\,,$$

$$(1.2)\qquad y\in W^\sigma_\varepsilon(x)\quad\text{if and only if}\quad x\in W^\sigma_\varepsilon(y)\qquad(\sigma=s,u)\,,$$

and

$$(1.3)\qquad z\in W^\sigma_{\varepsilon_1+\varepsilon_2}(x)\ \text{whenever}\ y\in W^\sigma_{\varepsilon_1}(x)\ \text{and}\ z\in W^\sigma_{\varepsilon_2}(y)\qquad(\sigma=s,u)\,.$$

DEFINITION 1. (X,f) is said to be *expansive* if there exists a constant $c^*>0$ such that

$$\{x\}=\bigcap_{n\in\mathbf{Z}}f^{-n}(B_{c^*}(f^n x))\qquad(=W^s_{c^*}(x)\cap W^u_{c^*}(x))$$

for all $x\in X$, and such a c^* is said to be an *expansive constant* for f.

For every $\varepsilon>0$ define subsets Y_ε and Z_ε of $X\times X$ by

$$Y_\varepsilon=\{(x,y)\in X\times X\,|\,W^s_\varepsilon(x)\cap W^u_\varepsilon(y)\neq\emptyset\}$$

and

$$Z_\varepsilon=\{(x,y)\in X\times X\,|\,(x,y)\in Y_\varepsilon\quad\text{and}\quad(y,x)\in Y_\varepsilon\}\,.$$

For $x\in X$ and $\varepsilon>0$, subsets $Y_\varepsilon(x)$ and $Z_\varepsilon(x)$ of X are defined by

$$Y_\varepsilon(x)=\{y\in X\,|\,(x,y)\in Y_\varepsilon\}$$

and

$$Z_\varepsilon(x)=\{y\in X\,|\,(x,y)\in Z_\varepsilon\}\,.$$

Then we have

$$(1.4)\qquad y\in Z_\varepsilon(x)\quad\text{if and only if}\ x\in Z_\varepsilon(y)$$

and

$$(1.5)\qquad W^s_\varepsilon(x)\cup W^u_\varepsilon(x)\subset Z_\varepsilon(x)\subset Y_\varepsilon(x)\,.$$

DEFINITION 2. Let $\mathscr{D}$ be a finite partition of X; i.e., a finite family of subsets of whose elements are muturally disjoint and $\bigcup_{D\in\mathscr{D}}D=X$. A sequence $\{x_i\}_{i\in\mathbf{Z}}$ of points in X is said to be an *α-pseudo orbit* with respect to $\mathscr{D}$ if $d(fx_i,x_{i+1})\leqq\alpha$ and $fx_i\underset{\mathscr{D}}{\sim}x_{i+1}$ for all $i\in\mathbf{Z}$ where $x\underset{\mathscr{D}}{\sim}y$ denotes that x and y are in the same element of $\mathscr{D}$. A sequence $\{x_i\}_{i\in\mathbf{Z}}$ of points in X is said to be

β-traced if $\bigcap_{i\in Z} f^{-i}(B_\beta(x_i))\neq\emptyset$. The following notion was introduced by Y. Takahashi. (X, f) is said to have *special pseudo orbit tracing property* (abbrev. *SPOTP*) if there exists a finite partition $\mathcal{D}$ such that for every $\beta>0$, there is $\alpha>0$ such that every α-pseudo orbit with respect to $\mathcal{D}$ is β-traced. Especially (X, f) is said to have the *pseudo orbit tracing property* (abbrev. *POTP*) if $\mathcal{D}$ can be chosen so that $\mathcal{D}=\{X\}$. The notion of *SPOTP* was firstly used in M. Yuri [39]. It seems likely that for every homeomorphism of a torus, SPOTP implies POTP. However the author does not have the proof.

Let us denote $\Delta_\delta=\{(x, y)\in X\times X\mid d(x, y)\leqq\delta\}$ for $\delta>0$. If (X, f) has POTP, then for every $\varepsilon>0$ there exists $\delta>0$ such that $\Delta_\delta\subset Z_\varepsilon$.

Let (X, f) be expansive and $c>0$ be a number such that $2c$ is an expanisve constant. Then for $(x, y)\in Y_c$, $W_c^s(x)\cap W_c^u(y)\neq\emptyset$ and the set $W_c^s(x)\cap W_c^u(y)$ consists only of one point by expansiveness. Therefore we can define the map $[\ ,\ ]: Y_c\to X$ by $(x, y)\mapsto[x, y]\in W_c^s(x)\cap W_c^u(y)$ $((x, y)\in Y_c)$. Then the following statement holds:

$$(1.6)\qquad [x, x]=x\,,$$

$$(1.7)\qquad y\in W_c^s(x) \text{ and } z\in W_c^u(x) \text{ imply that } (y, z)\in Y_c \text{ and } [y, z]=x\,,$$

$$(1.8)\qquad [x, [y, z]]=[x, z] \text{ if } (y, z), (x, z)\in Y_c \text{ and } (x, [y, z])\in Y_c,$$

$$(1.9)\qquad [[x, y], z]=[x, z] \text{ if } (x, y), (x, z)\in Y_c \text{ and } ([x, y], z)\in Y_c,$$

and

$$(1.10)\qquad W_\varepsilon^s(x)\cap W_\varepsilon^u(y)=\{[x, y]\} \text{ if } (x, y)\in Y_\varepsilon \quad \text{for } \varepsilon\leqq c\,.$$

DEFINITION 3. Under the above notations, a subset E of X is said to be a *rectangle* if $(x, y)\in Y_c$ and $[x, y]\in E$ for all $x, y\in E$.

DEFINITION 4. A finite family $\mathcal{P}$ of closed rectangles of X is said to be a *Markov partition* for (X, f) if $\mathcal{P}$ satisfies the following conditions;

1) $P=\overline{\operatorname{int} P}$ for all $P\in\mathcal{P}$,
2) $\bigcup_{P\in\mathcal{P}} P=X$,
3) $\operatorname{int} P\cap\operatorname{int} Q=\emptyset$ for all $P, Q\in\mathcal{P}$ with $P\neq Q$,
4) for every sequence $\{P_n\}_{n\in Z}$ of elements of $\mathcal{P}$, $\bigcap_{n\in Z} f^{-n}P_n$ consists at most of one point,
5) $f(W_c^s(x)\cap\operatorname{int} P)\subset W_c^s(fx)\cap\operatorname{int} Q$

and

$$f^{-1}(W_c^u(fx)\cap\operatorname{int} Q)\subset W_c^u(x)\cap\operatorname{int} P$$

whenever $x\in\operatorname{int} P\cap f^{-1}\operatorname{int} Q$ $(P, Q\in\mathcal{P})$,

6) there exists subsets B^s and B^u of X such that

$$fB^s\subset B^s,\ f^{-1}B^u\subset B^u,\quad\text{and}\quad B^s\cup B^u=\bigcup_{P\in\mathcal{P}}\partial P\,.$$

Markov partitions are not partitions in strict sense. However we use the word conventionally.

2. Auxiliary results

In this section fundamental results are described. Throughout, let (X, f) be as in our theorem, and $c>0$ be a number with $2c\leqq c^*$.

(L. 1) *For every $\rho>0$ there exists an integer $N>0$ such that for every $x\in X$*

$$y\in \bigcap_{i=-N}^{N} f^{-i}B_c(f^i x) \quad \textit{implies} \quad y\in B_\rho(x)\,.$$

Proof. See p. 109 of [15].

By (L. 1) the following statement is clear;

(L. 2) *For every $\rho>0$, there exists an integer $N>0$ such that for every $x\in X$ and every $n>N$*

$$f^n W_c^s(x)\subset W_\rho^s(f^n x) \quad \textit{and} \quad f^{-n} W_c^u(x)\subset W_\rho^u(f^{-n}x)\,.$$

The following statement is clear from (L. 2) and uniform continuity of f.

(L. 3) *For every $\rho>0$ there exists $\varepsilon>0$ such that for every $x\in X$*

$$W_c^s(x)\cap B_\varepsilon(x)\subset W_\rho^s(x) \quad \textit{and} \quad W_c^u(x)\cap B_\varepsilon(x)\subset W_\rho^u(x)\,.$$

(L.4) *Y_c, Z_c, $Y_c(x)$ and $Z_c(x)$ are compact, and $[\ ,\ ]$ is uniformly continuous on Y_c.*

Proof. For any sequence $\{(x_n, y_n)\}_{n\in N}$ of points in Y_c which converges to a point (x, y), there is a subsequence $\{(x_{n_j}, y_{n_j})\}_{j\in N}$ such that $[x_{n_j}, y_{n_j}]$ converges to $z\in X$. Since $[x_{n_j}, y_{n_j}]\in W_c^s(x_{n_j})$ for all $j\in \boldsymbol{N}$, we have $z\in f^{-n}B_c(f^n x)$ for all $n\geqq 0$, and so $z\in W_c^s(x)$. Similarly we have $z\in W_c^u(y)$ and hence $z=[x, y]$. Therefore Y_c is closed and $[\ ,\ ]$ is continuous on Y_c. Since $X\times X$ is compact, Y_c is compact and so $[\ ,\ ]$ is uniformly continuous on Y_c. Since Y_c is compact, $Y_c'=\{(x, y)\in X\times X\,|\,(y, x)\in Y_c\}$ is compact. Thus $Z_c=Y_c\cap Y_c'$ is compact. It is clear that $Y_c(x)$ and $Z_c(x)$ are compact.

(L. 5) *For every $\varepsilon>0$ there exists $\rho>0$ such that $Z_c\cap\Delta_\rho\subset Z_\varepsilon$.*

Proof. For given $\varepsilon>0$, by (L.3) there is a $\gamma>0$ such that $W_c^s(x)\cap B_\gamma(x)\subset W_\varepsilon^s(x)$ and $W_c^u(x)\cap B_\gamma(x)\subset W_\varepsilon^u(x)$ for all $x\in X$. Since $[\ ,\ ]$ is uniformly continuous on Y_c by (L.4), there exists $\rho>0$ such that $d([x, y], x)=d([x, y], [x, x])\leqq\gamma$ and $d([x, y], y)=d([x, y], [y, y])\leqq\gamma$ for all $(x, y)\in Z_c\cap\Delta_\rho$. Therefore $(x, y)\in Z_c\cap\Delta_\rho$ implies $(x, y)\in Z_\varepsilon$.

(L.6) *Let $\mathcal{D}$ be a finite partition of X. Then every α-pseudo orbit with respect* *to $\mathcal{D}$ is β-traced if there is a strictly increasing sequence $\{M_n\}_{n\in N}$ of positive integers such that for every α-pseudo orbit $\{x_i\}_{i\in Z}$ with respect to $\mathcal{D}$, the following holds:*

$$\bigcap_{i=0}^{M_n} f^{-i}B_\beta(x_i) \neq \emptyset \qquad \text{for all} \quad n\in \boldsymbol{N}.$$

Proof. Let $\{x_i\}_{i\in Z}$ be an α-pseudo orbit with respect to $\mathcal{D}$ and $\{M_n\}_{n\in N}$ be a strictly increasing sequence of positive integers. For every $n\in \boldsymbol{N}$ define an α-pseudo orbit $\{y_i^n\}_{i\in Z}$ with respect to $\mathcal{D}$ by $y_i^n = x_{i-[M_n/2]}$ for all $i\in \boldsymbol{Z}$. ($[M]$ $(M\geqq 0)$ denotes the maximal integer which does not exeed M). By the assumption we have

$$E_n = f^{[M_n/2]}\bigcap_{i=0}^{M_n} f^{-i}B_\beta(y_i^n) \neq \emptyset$$

for all $n\in \boldsymbol{N}$, then $\bigcap_{i\in Z} f^{-i}B_\beta(x_i) = \bigcap_{n\in N} E_n \neq \emptyset$ since E_n are closed and decreasing.

The following (L.7) and (L.8) are general properties of topological spaces and we omit the proofs since they are easily checked.

(L.7) *If $A_1, \cdots, A_k \subset X$ are closed, then one has*

$$\bigcup_{i=1}^{k} \overline{\operatorname{int} A_i} = \overline{\operatorname{int}\left(\bigcup_{i=1}^{k} A_i\right)}$$

(L.8) *If $A_1, A_2 \subset X$ are closed and $A_1 \supset A_2$, then we have*

$$\overline{\operatorname{int} A_1} = \overline{\operatorname{int} A_2} \cup \overline{\operatorname{int}(A_1 \backslash A_2)}.$$

(L.9) *If (X, f) has a Markov partition, then (X, f) has Markov partitions with arbitrary small diameters.*

Proof. Let $\mathcal{P}$ be a Markov partition for (X, f). Define $\mathcal{P}^n = \{\overline{\bigcap_{i=-n}^{n} f^{-1}\operatorname{int} P_i} \mid P_i \in \mathcal{P} \text{ for } -n\leqq i\leqq n\}$ for $n\geqq 1$. Then the maximal diameter of the elements of $\mathcal{P}^n$ converges to 0 as $n\to\infty$ by condition 4) of the definition of Markov partitions. We also have that $\mathcal{P}^n$ $(n\geqq 1)$ is a Markov partition. Indeed, we have $\bigcup_{P\in\mathcal{P}^n} P = X$ by (L.7). Other properties for $\mathcal{P}^n$ to be a Markov partition for (X, f) is easily checked from the fact that $\mathcal{P}$ is a Markov partition.

3. Proof of Theorem

Theorem will be obtained in proving the following claims.

Claim 1 (I) $\Rightarrow$ (II).

Claim 2 (II) $\Rightarrow$ (III).

Claim 3 (III) $\Rightarrow$ (IV).

Claim 4 (IV) $\Rightarrow$ (I).

Proof of Claim 1 As in condition (I) let $0<c\leqq c^*/2$ and take $\eta=\eta(x)$ for $x\in X$. Then there are finite number of subsets $Y^1, \cdots, Y^n$ of $B_\eta(x)$ such that for every $y\in B_\eta(x)$ there exists $1\leqq i\leqq n$ such that $Y_c(y)\cap B_\eta(x)=Y^i$. Put $\hat{Y}^i=\{z\in Y^i \mid Y_c(z)\cap B_\eta(x)=Y^i\}$. Then $Z_c(y)\cap B_\eta(x)=Y_c(y)\cap \cup_{i\in\Lambda}\hat{Y}^i$ where $\Lambda=\{1\leqq i\leqq n \mid y\in Y^i\}$. Therefore $\{Z_c(y)\cap B_\eta(x) \mid y\in B_\eta(x)\}$ is finite.

Proof of Claim 2 The proof will be done along the following five steps.

Step 2.1. *Let c be as in condition (II). Then there exist $\delta_0>0$ and a finite partition $\mathcal{D}$ such that $x\underset{\mathcal{D}}{\sim}y$ implies $Z_c(x)\cap B_{\delta_0}(x)\subset Z_c(y)$.*

Proof. For $x\in X$ let $\delta=\delta(x)>0$ be as in condition (II). Since X is compact, we can find finite points $x_1, \cdots, x_k$ X such that

$$X=\cup_{i=1}^k \operatorname{int} B_{\delta(x_i)/2}(x_i)\,.$$

Put $R_i=B_{\delta(x_i)}(x_i)$ for $1\leqq i\leqq k$. By condition (II), for every $1\leqq i\leqq k$ there exist finite number of subsets $Z_1^i, \cdots, Z_{n_i}^i\subset R_i$ such that for every $y\in R_i$, $Z_c(y)\cap R_i=Z_j^i$ holds for some $1\leqq j\leqq n_i$. We can assume that Z_j^i's are different if j's are different. Denoting that $\hat{Z}_j^i=\{x\in R_i \mid Z_c(x)\cap R_i=Z_j^i\}$ for $1\leqq j\leqq n_i$ and $1\leqq i\leqq k$, we have $R_i=\cup_{j=1}^{n_i}\hat{Z}_j^i$ (disjoint union) for $1\leqq i\leqq k$. For every $x\in X$ and $1\leqq i\leqq k$, define $D_i(x)\subset X$ by

$$D_i(x)=\begin{cases}\hat{Z}_j^i \quad \text{with} \quad x\in\hat{Z}_j^i & \text{if} \quad x\in R_i\\ X\backslash R_i & \text{if} \quad x\notin R_i\,.\end{cases}$$

Put $D(x)=\cap_{i=1}^k D_i(x)$ for $x\in X$. Then $\mathcal{D}=\{D(x) \mid x\in X\}$ is a finite partition. For every $x\in X$ there exists $1\leqq i\leqq k$ such that $x\in B_{\delta(x_i)/2}(x_i)$ by the choice of $x_1, \cdots, x_k$, and then we have $B_{\delta_0}(x)\subset R_i$ where $\delta_0=\min_{1\leqq i\leqq k}\delta(x_i)/2$. Since $x\in\hat{Z}_j^i$ for some $1\leqq i\leqq n_i$, we have $y\in D(y)=D(x)\subset\hat{Z}_j^i$ for $y\in X$ with $x\underset{\mathcal{D}}{\sim}y$.

Let γ be a number such that $0<\gamma<\delta_0$ and

$$B_\gamma(x)\subset\cap_{i=-1}^1 f^{-i}B_c(f^ix)\,.$$

It is enough to show that for small $\beta>0$ there exists $\alpha>0$ such that every α-pseudo orbit with respect to $\mathcal{D}$ is β-traced.

Let $\beta>0$ be small enough. Then $\beta<\gamma$ and $Z_c\cap\Delta_\beta\subset Z_\gamma$ by (L.5). Similarly there is ε with $0<\varepsilon<\beta/6$ such that $Z_c\cap\Delta_{2\varepsilon}\subset Z_{\beta/3}$. By (L.2) we can find $M\in\boldsymbol{N}$ such that $f^MW_c^s(x)\subset W_{\varepsilon/2}^s(f^Mx)$ and $f^{-M}W_c^u(x)\subset W_{\varepsilon/2}^u(f^{-M}x)$ for all $x\in X$. Let $\alpha>0$ be a number such that $B_\alpha(x)\subset\cap_{i=0}^M f^{-i}B_{\varepsilon/2M}(f^ix)$ for all $x\in X$.

By (L.6), it is enough to show that for every α-pseudo orbit $\{x_i\}_{i\in\boldsymbol{Z}}$ with respect to $\mathcal{D}$ the following holds:

$$(*) \qquad \cap_{i=0}^{nM} f^{-i}B_\beta(x_i)\neq\emptyset \qquad \text{for all} \quad n\in\boldsymbol{N}\,.$$

To prove (*) we shall prepare steps 2.2, 3, 4, and 5. From now on, we fix any α-pseudo orbit $\{x_i\}_{i\in Z}$ with respect to $\mathcal{D}$.

Step 2.2. *For every $i\geqq 0$, $0\leqq j\leqq M$ and $0\leqq k\leqq M-j$,*

$$d(f^{j+k}x_i, f^k x_{i+j})\leqq \varepsilon/2\,.$$

Proof. Since $\{x_i\}_{i\in Z}$ is an α-pseudo orbit with respect to $\mathcal{D}$, by the choice of α we have

$$d(f^{j+k-l}(fx_{i+l-1}), f^{j+k-l}x_{i+l})\leqq \varepsilon/2M$$

for all j, k with $0\leqq j+k-l\leqq M$. Therefore

$$d(f^{j+k}x_i, f^k x_{i+j})\leqq \sum_{l=1}^{j} d(f^{j+k-l}(fx_{i+l-1}), f^{j+k-l}x_{i+l})\leqq \varepsilon/2$$

for all $0\leqq j\leqq M$ and $0\leqq k\leqq M-j$.

Step 2.3. *If $y\in W^s_{\beta/3}(x_i)$, then $(f^j y, x_{i+j})\in Z_\gamma$ for all $0\leqq j\leqq M$.*

Proof. Since $y\in W^s_{\beta/3}(x_i)$, we have $d(f^j y, f^j x_i)\leqq \beta/3$ for all $j\geqq 0$. By Step 2.2, we have $d(f^j x_i, x_{i+j})\leqq \varepsilon/2$ for all $0\leqq j\leqq M$. Thus $d(f^j y, x_{i+j})\leqq \beta/3+\varepsilon/2<\beta/2$. It is clear that $(y, x_i)\in Z_\gamma$. Assume that $(f^j y, x_{i+j})\in Z_\gamma$ for some $0\leqq j\leqq M-1$. By the choice of γ we have that $(f^{j+1}y, fx_{i+j})\in Z_c$. Since $fx_{i+j}\underset{\mathcal{D}}{\sim} x_{i+j+1}$, $(f^{j+1}y, x_{i+j+1})\in Z_c$ by Step 2.1. Thus the fact $d(f^{j+1}y, x_{i+j+1})<\beta$ implies $(f^{j+1}y, x_{i+j+1})\in Z_\gamma$.

Step 2.4. *For $n\geqq 0$ and $y\in W^s_{\beta/3}(x_{nM})$,*

$$(f^M y, x_{(n+1)M})\in Z_{\beta/3}\,.$$

Proof. By Step 2.3 we have $(f^M y, x_{(n+1)M})\in Z_\gamma$. By the choice of M, $f^M y\in W^s_{\varepsilon/2}(f^M x_{nM})$. Since $d(f^M x_{nM}, x_{(n+1)M})\leqq \varepsilon/2$ by Step 2.2, we have that $d(f^M y, x_{(n+1)M})\geq \varepsilon$ and so $(f^M y, x_{(n+1)M})\in Z_{\beta/3}$.

Put $y_0=x_0$ and $y_n=[x_{nM}, f^M y_{n-1}]$ for $n\geqq 1$ (Remark that $[x_{nM}, f^M y_{n-1}]$ is always defined by Step 2.4).

Step 2.5. *For every $v\geqq 0$ and $w\geqq 0$,*

$$f^{-wM}y_{v+w}\in W^u_{\varepsilon/2}(y_v)\,.$$

Proof. Take and fix any $v\geqq 0$ and $w\geqq 0$. Since $y_{v+w}\in W^u_{\beta/3}(f^M y_{v+w-1})$, it is easily checked that $f^{-M}y_{v+w}\in W^u_{\varepsilon/2}(y_{v+w-1})$. Thus the fact $y_{v+w-1}\in W^u_{\beta/3}(f^M y_{v+w-2})$ implies

$$f^{-M}y_{v+w}\in W^u_{\varepsilon/2+\beta/3}(f^M y_{v+w-2})\subset W^u_c(f^M y_{v+w-2})\,,$$

from which $f^{-2M}y_{v+w}\in W^u_{\varepsilon/2}(y_{v+w-2})$. Repeating this procedure, we get the conclusion.

Take and fix any $n\leqq 0$. Put $y=f^{-nM}y_n$. For $0\leqq i\leqq nM-1$ there exist $0\leqq j\leqq n-1$ and $0\leqq k\leqq M-1$ such that $i=jM+k$. By Step 2.5 we have $f^i y\in W^u_{\varepsilon/2}(f^{k-M}y_{j+1})$, and by the fact that $y_{j+1}\in W^u_{\beta/3}(f^M y_j)$, $f^k y_j\in W^u_{\beta/3}(f^{k-M}y_{j+1})$. Since $y_j\in W^s_{\beta/3}(x_{jM})$, we have $f^k y_j\in W^s_{\beta/3}(f^k x_{jM})$ by (1.1). Similarly $d(f^k x_{jM}, x_{jM+k})<\varepsilon/2$ by Step 2.2 and $f^i y\in B_{2\beta/3+\varepsilon/2}(x_i)\subset B_\beta(x_i)$. Therefore (*) is proved.

Proof of Claim 3. Let $\mathcal{D}$ be a finite partition in the definition of SPOTP and $c<0$ be a number such that $2c\leqq c^*$. Take $0<\beta<c/2$ so small that $(x, y)\in Z_c\cap\Delta_\beta$ implies $(fx, fy), (f^{-1}x, f^{-1}y)\in Z_c$ (such a β exists by (L.5)). Let $\alpha>0$ be a number such that every α-pseudo orbit with respect to $\mathcal{D}$ is $\beta/2$-traced. Take $0<\gamma<\alpha/2$ such that $d(x, y)<\gamma$ implies $d(fx, fy)<\alpha/2$. Since $\mathcal{D}$ is finite, we can find a finite set $T=\{t_1, \cdots, t_r\}\subset X$ such that for every $x\in X$ there exists $t_i\in T$ such that $x\underset{\mathcal{D}}{\sim}t_i$, $fx\underset{\mathcal{D}}{\sim}ft_i$, and $d(x, t_i)<\gamma$.

For every $v\in T^{\boldsymbol{Z}}$ we denote by v_i the i-th component of v. Let us put

$$\Sigma(T)=\{v\in T^{\boldsymbol{Z}}\mid \{v_i\}_{i\in\boldsymbol{Z}} \text{ is an } \alpha\text{-pseudo orbit with respect to } \mathcal{D}\}\ .$$

Since β is an expansive constant, we can define $\theta\colon\Sigma(T)\to X$ by

$$\theta(v)\in\bigcap_{i\in\boldsymbol{Z}}f^{-i}B_{\beta/2}(v_i)\qquad\text{for all}\quad v\in\Sigma(T)\ .$$

For every $x\in X$ and $n\in\boldsymbol{Z}$, take $v_n\in T$ such that $f^n x\underset{\mathcal{D}}{\sim}v_n$, $f^{n+1}x\underset{\mathcal{D}}{\sim}fv_n$ and $d(f^n x, v_n)<\gamma$. Then we have $v=(v_n)_{n\in\boldsymbol{Z}}\in\Sigma(T)$ and $\theta(v)=x$. Thus θ is surjective. It is easy to check that $f\circ\theta=\theta\circ\sigma$ where $\sigma\colon\Sigma(T)\to\Sigma(T)$ is the shift automorphism, i.e., $\sigma(v)_n=v_{n+1}$ for all $n\in\boldsymbol{Z}$ and $v\in\Sigma(T)$.

For $1\leqq i\leqq r$, put $\mathrm{cyl}(t_i)=\{v\in\Sigma(T)\mid v_0=t_i\}$ and $T_i=\theta(\mathrm{cyl}(t_i))$. Then $\operatorname{diam} T_i\leqq\beta$ and $\bigcup_{i=1}^r T_i=X$. Since f is expansive, θ is continuous by (L.1) and so T_i $(1\leqq i\leqq r)$ is closed. For every $v, w\in\Sigma(T)$ with $v_0=w_0$, we can define $[v, w]\in\Sigma(T)$ by $[v, w]_n=v_n$ for $n\geqq 0$ and $[v, w]_n=w_n$ for $n\leqq 0$. Then it is easy to check that $(\theta(v), \theta(w))\in Y_\beta$ and $\theta([v, w])=[\theta(v), \theta(w)]$.

To prove Claim 3, we shall prepare steps that will lead us to this end goal.

Step 3.1. *T_i's are closed rectangles.*

Proof. It was already shown that T_i's are closed. Thus it is only to show that T_i's are rectangles. Take any $1\leqq i\leqq r$ and any $x, y\in T_i$. Then there exist $v\in\theta^{-1}(x)$ and $w\in\theta^{-1}(y)$ with $v_0=w_0=t_i$. Since $\theta([v, w])=[\theta(v), \theta(w)]=[x, y]$ and $[v, w]_0=t_i$, we have $[x, y]\in T_i$ and hence T_i is a rectangle.

Step 3.2. *For $v\in\Sigma(T)$, we have the following;*

(1) *for every $y\in W^s_c(\theta(v))\cap\theta(\mathrm{cyl}(v_0))$ there exists $w\in\theta^{-1}(y)$ such that $w_n=v_n$ for all $n\geqq 0$,*

(2) *for every $y\in W^u_c(\theta(v))\cap\theta(\mathrm{cyl}(v_0))$ there exists $w\in\theta^{-1}(y)$ such that*

$w_n = v_n$ *for all* $n \leqq 0$.

Proof. For $y \in W^s_c(\theta(v)) \cap \theta(\mathrm{cyl}(v_0))$, there exists $v' \in \theta^{-1}(y)$ such that $v'_0 = v_0$. Therefore we can define $w = [v, v'] \in \Sigma(T)$ and $\theta(w) = \theta([v, v']) = [\theta(v), y] = y$. Then $w \in \theta^{-1}(y)$ and $w_n = v_n$ for all $n \geqq 0$. Thus (1) of Step 3.2 holds. Similarly we have (2).

Step 3.3. *For every* $x \in X$ *and* T_i $(1 \leqq i \leqq r)$ *with* $x \in T_i$ *we have*

$$W^\sigma_c(x) \cap T_i \subset W^\sigma_\beta(x) \qquad \textit{for} \quad \sigma = s, u\,.$$

Proof. For $y \in W^s_c(x) \cap T_i$ and $n \geqq 0$, there exists T_j $(1 \leqq j \leqq r)$ such that $f^n y, f^n x \in T_j$ by Step 3.2. Since $\mathrm{diam}\, T_j \leqq \beta$, we have $d(f^n x, f^n y) \leqq \beta$ for all $n \geqq 0$ and so $y \in W^s_\beta(x)$. Therefore $W^s_c(x) \cap T_i \subset W^s_\beta(x)$. Similarly we have $W^u_c(x) \cap T_i \subset W^u_\beta(x)$.

For $x \in X$, define subsets $\Lambda(x)$, $\Lambda^s(x)$, and $\Lambda^u(x)$ of $\{1, \cdots, r\}$ by

$$\Lambda(x) = \{1 \leqq i \leqq r \mid x \in T_i\}\,,$$
$$\Lambda^s(x) = \cup_{j \in \Lambda(x)} \{1 \leqq i \leqq r \mid T_i \cap T_j \cap W^s_c(x) \neq \emptyset\}\,,$$

and

$$\Lambda^u(x) = \cup_{i \in \Lambda(x)} \{1 \leqq i \leqq r \mid T_i \cap T_j \cap W^u_c(x) \neq \emptyset\}\,.$$

Step 3.4. *For every* $i \in \Lambda^\sigma(x)$ $(\sigma = s, u)$ *there exists* $j \in \Lambda(x)$ *such that* $T_i \cap T_j \cap W^\sigma_\beta(x) \neq \emptyset$.

Proof. We give the proof for $\sigma = s$. For $i \in \Lambda^s(x)$ there exists $j \in \Lambda(x)$ such that $T_j \cap T_j \cap W^s_c(x) \neq \emptyset$. Take $y \in T_i \cap T_j \cap W^s_c(x)$. By Step 3.3 we have $y \in W^s_\beta(x)$. Thus $T_i \cap T_j \cap W^s_\beta(x) \neq \emptyset$.

Step 3.5. $\Lambda^s(x) \cap \Lambda^u(x) = \Lambda(x)$ *for all* $x \in X$.

Proof. It is clear from definition that $\Lambda(x) \subset \Lambda^s(x) \cap \Lambda^u(x)$ for all $x \in X$. To prove that $\Lambda^s(x) \cap \Lambda^u(x) \subset \Lambda(x)$, we use Step 3.4. Indeed, for every $i \in \Lambda^s(x) \cup \Lambda^u(x)$ there exists $y \in T_i \subset W^s_\beta(x)$ and $z \in T_i \cap W^u_\beta(x)$. Since $y, z \in T_i$, we have $[y, z] \in T_i$. Therefore $[y, z] \in W^s_\beta(y) \cap W^u_\beta(z) \subset W^s_{2\beta}(x) \cap W^u_{2\beta}(x)$. Since $2\beta < c$ and $2c$ is an expansive constant, we have $x = [y, z] \in T_i$. Thus $i \in \Lambda(x)$.

For $x \in X$, put $L(x) = \{y \in X \mid \Lambda^\sigma(y) = \Lambda^\sigma(x)$ for $\sigma = s, u\}$. Clearly $x \in L(x)$ for all $x \in X$, and $y \in L(x)$ implies $L(y) = L(x)$. Put $\mathcal{L} = \{L(x) \mid x \in X\}$. Then $\mathcal{L}$ is a finite partition. For $L \in \mathcal{L}$, denote $\Lambda(L) = \Lambda(x)$, $\Lambda^s(L) = \Lambda^s(x)$, and $\Lambda^u(L) = \Lambda^u(x)$ for $x \in L$. Notice that $\Lambda(L)$, $\Lambda^s(L)$, and $\Lambda^u(L)$ are independent of the choice of $x \in L$.

Step 3.6. *Every* $L \in \mathcal{L}$ *is a rectangle.*

Proof. For $x, y \in L$, $[x, y]$ is defined and $[x, y] \in T_i$ for all $i \in \Lambda(L)$ since $x, y \in T_i$ for all $i \in \Lambda(L)$. Therefore we have $\Lambda([x, y]) \subset \Lambda(L)$. For every $i \in \Lambda^s(L)$ there exists $j \in \Lambda(x)$ such that $T_i \cap T_j \cap W^s_\beta(x) \neq \emptyset$ by Step 3.4. Since $\Lambda(x) = \Lambda(L) \subset \Lambda([x, y])$, we have $j \in \Lambda(([x, y])$. Since $x \in W^s_\beta([x, y])$, $T_i \cap T_j \cap W^s_c([x, y]) \neq \emptyset$ and so $i \in \Lambda^s([x, y])$. Thus $\Lambda^s([x, y]) \supset \Lambda^s(L)$. Conversely for every $i \in \Lambda^s([x, y])$ there exists $j \in \Lambda([x, y])$ such that $T_i \cap T_j \cap W^s_\beta([x, y]) \neq \emptyset$ For every $k \in \Lambda(x)$, we have $[x, y] \in T_k \cap T_j \cap W^s_\beta(x)$ since $\Lambda([x, y]) \supset \Lambda(x)$, and so $j \in \Lambda^s(x) = \Lambda^s(L)$. Similarly $j \in \Lambda^u(L)$. We obtain $j \in \Lambda(L)$ by Step 3.5. Since $[x, y] \in W^s_\beta(x)$, we have $T_i \cap T_j \cap W^s_c(x) \neq \emptyset$. Hence $i \in \Lambda^s(L)$. Therefore $\Lambda^s([x, y]) = \Lambda^s(L)$. Similarly we have $\Lambda^u([x, y]) = \Lambda^u(L)$.

Step 3.7. *For every $x \in X$,*

$$f(W^s_c(x) \cap L(x)) \subset W^s_c(fx) \cap L(fx)$$

and

$$f^{-1}(W^u_c(fx) \cap L(fx)) \subset W^u_c(x) \cap L(x)\,.$$

Proof. Take any $y \in W^s_c(x) \cap L(x)$. For any $i \in \Lambda^u(fx)$ there exist $j \in \Lambda(fx)$ and $z \in T_i \cap T_j \cap W^u_\beta(fx)$ by Step 3.4. Take $v \in \theta^{-1}(x)$ such that $v_1 = t_j$. Let $1 \leqq k \leqq r$ be the number such that $v_0 = t_k$. Since $z \in W^u_\beta(fx) \cap T_j$, we have $f^{-1}z \in T_k$ by Step 3.2 (2). It is clear that $y \in T_k \cap W^s_\beta(x)$, and $fy \in T_j$ by Step 3.2 (1). Since fy, $z \in T_j$, we have $[z, fy] \in T_j$. Since $z \in T_i$, there exists $w \in \theta^{-1}(z)$ such that $w_0 = t_i$. Let $1 \leqq l \leqq r$ be the number such that $w_{-1} = t_l$. Since $f^{-1}z \in T_l \cap T_k \cap W^u_\beta(x)$, we have $l \in \Lambda^u(x)$ and so $l \in \Lambda^s(y)$. By the fact that $l \in \Lambda^u(x) \cap \Lambda^u(y)$, we have $[f^{-1}z, y] \in T_l$. Thus $f[f^{-1}z, y] = [z, fy] \in T_i$ by Step 3.2 (1). Hence $[z, fy] \in T_i \cap T_j \cap W^u_\beta(fy)$, and so $i \in \Lambda^u(fy)$. Therefore $\Lambda^u(fy) \subset \Lambda^u(fx)$. Since $x \in W^s_c(y) \cap L(y)$, by symmetry $\Lambda^u(fy) \subset \Lambda^u(fx)$. Thus $\Lambda^u(fy) = \Lambda^u(fx)$, and similarly $\Lambda^s(fy) = \Lambda^s(fx)$. Since $fW^s_c(x) \subset W^s_c(fx)$, we have $f(W^s_c(x) \cap L(x)) \subset W^s_c(fx) \cap L(fx)$. Similarly we see that $f^{-1}(W^u_c(fx) \cap L(fx)) \subset W^u_c(x) \cap L(x)$.

Step 3.8. *For every $x \in X$ there exists a neighborhood U of x such that $\Lambda(x) \supset \Lambda(y)$ for all $y \in U$.*

Proof. For $x \in X$, put $U = X \setminus \cup_{i \notin \Lambda(x)} T_i$. Then U is open, $x \in U$ and $\Lambda(y) \subset \Lambda(x)$ for $y \in U$.

By Step 3.8, the following is clear.

Step 3.9. *Let $\sigma = s, u$ and $L \in \mathcal{L}$. For every $x \in X$ with $\Lambda^\sigma(x) \not\supset \Lambda^\sigma(L)$ there exists a neighborhood U of x such that $\Lambda^\sigma(y) \not\supset \Lambda^\sigma(L)$ for all $y \in U$.*

Step 3.10. $\cup_{L \in \mathcal{L}} \overline{\operatorname{int} L} = X$.

Proof. For $L\in\mathcal{L}$, define $\mathcal{L}(L)=\{M\in\mathcal{L}\mid\Lambda^{\sigma}(M)\supset\Lambda^{\sigma}(L)$ for $\sigma=s, u\}$ and $L^*=\bigcup_{M\in\mathcal{L}(L)} M$. Then L^* is closed by Step 3.9. Since $\bigcup_{L\in\mathcal{L}}L^*=\bigcup_{L\in\mathcal{L}}L=X$, we have $\bigcup_{L\in\mathcal{L}}\overline{\operatorname{int} L^*}=X$ by (L.7). Therefore it is sufficient to show that $\overline{\operatorname{int} M^*}\subset\bigcup_{L\in\mathcal{L}}\overline{\operatorname{int} L}$ for all $M\in\mathcal{L}$.

Put $\mathcal{L}_0=\{L\in\mathcal{L}\mid\mathcal{L}(L)=\{L\}\}$ and $\mathcal{L}_n=\{L\in\mathcal{L}\mid M\in\mathcal{L}_{n-1}$ for all $M\in\mathcal{L}(L)\setminus\{L\}\}$ $(n\geqq 1)$ inductively. If $M\in\mathcal{L}_0$, then it is trivial that $\overline{\operatorname{int} M^*}=\overline{\operatorname{int} M}\subset\bigcup_{L\in\mathcal{L}}\overline{\operatorname{int} L}$. Assume that $\overline{\operatorname{int} M^*}\subset\bigcup_{L\in\mathcal{L}}\overline{\operatorname{int} L}$ for all $M\in\mathcal{L}_{n-1}$. For $M\in\mathcal{L}_n$ put $A_1=M^*$ and $A_2=\bigcup_{N\in\mathcal{L}(M)\setminus\{M\}}N^*$. By (L.8), we have

$$\overline{\operatorname{int} M^*}=\overline{\operatorname{int} M}\cup\overline{\operatorname{int}\bigcup_{N\in\mathcal{L}(M)\setminus\{M\}}N^*}\,.$$

By (L.7),

$$\overline{\operatorname{int}\bigcup_{N\in\mathcal{L}(M)\setminus\{M\}}N^*}=\bigcup_{N\in\mathcal{L}(M)\setminus\{M\}}\overline{\operatorname{int} N^*}\,,$$

and so

$$\overline{\operatorname{int} M^*}=\overline{\operatorname{int} M}\cup\Big(\bigcup_{N\in\mathcal{L}(M)\setminus\{M\}}\overline{\operatorname{int} N^*}\Big)\,.$$

Since $\mathcal{L}(M)\setminus\{M\}\subset\mathcal{L}_{n-1}$, we have $\overline{\operatorname{int} M^*}\subset\bigcup_{L\in\mathcal{L}}\overline{\operatorname{int} L}$ for all $M\in\mathcal{L}_n$. By the fact that every $M\in\mathcal{L}$ satisfies $M\in\mathcal{L}_n$ for some $n\geqq 0$, we get the proof.

Step 3.11. *Let E be a rectangle. Then $x\in\operatorname{int} E$ if and only if there exists a neighborhood U of x such that $U\cup W_c^{\sigma}(x)\subset E$ for $\sigma=s, u$.*

Proof. Suppose that there is a neighborhood U of x such that $U\cap W_{\beta}^{c}(x)\subset E$ for $\sigma=s, u$. By Step 3.8, there is an neighborhood V of x such that $\Lambda(y)\subset\Lambda(x)$ for all $y\in V$. Then for $y\in V$ and $i\in\Lambda(y)$, we have $x, y\in T_i$, and so $(x, y)\in Z_c$. Since $x=[x, x]$ and $[\ \ ,\ \ ]$ is continuous, there exists a neighborhood $W\subset V$ such that $[x, y]\in U\cap W_c^s(x)$ and $[y, x]\in U\cap W_c^u(x)$ for all $y\in W$. Then we have $y=[[y, x], [x, y]]\in E$ for all $y\in W$. Therefore $x\in\operatorname{int} E$. The "only if" part is clear and so we omit the proof.

Step 3.12. *For every $L\in\mathcal{L}$, $\operatorname{int} L$ is a rectangle.*

Proof. For $x, y\in\operatorname{int} L$, it is clear that $[x, y]\in L$. By Step 3.8, there exists a neighborhood U of $[x, y]$ such that $\Lambda([x, y])\supset\Lambda(z)$ for all $z\in U$. Take $z\in U$ and $i\in\Lambda(z)$. Then $i\in\Lambda([x, y])=\Lambda(x)$ $(=\Lambda(L))$. Thus we have $x, z\in T_i$. Therefore we can define $[z, x]$ for all $z\in U$. Notice that $[\ \ ,\ \ ]$ is continuous. Since $x=[[x, y], x]$ and $x\in\operatorname{int} L$, there exists a neighborhood $V\subset U$ of $[x, y]$ such that $[z, x]\in L$ for all $z\in V$. Hence for $z\cap V\cap W_c^u([x, y])$ we have $z=[[z, x], [x, y]]\in L$. Similarly there exists a neighborhood V' of $[x, y]$ such that $z\in L$ for all $z\in V'\cap W_c^s([x, y])$. Thus $W\cap W_c^{\sigma}([x, y])\subset L$ $(\sigma=s, u)$ where $W=V\cap V'$. Therefore $[x, y]\in\operatorname{int} L$ by Step 3.11.

We prove that $\mathcal{P}=\{\overline{\operatorname{int} L}\mid\operatorname{int} L\neq\emptyset, L\in\mathcal{L}\}$ is a Markov partition. Obviously $\bigcup_{P\in\mathcal{P}}P=X$ (by Step 3.10) and $\overline{\operatorname{int} P}=P$ for all $P\in\mathcal{P}$. Since diam $P \leqq \beta$ for $P \in \mathcal{P}$, we have 4) in the definition of Markov partitions. Since $\operatorname{int} L$ ($L \in \mathcal{L}$) is a rectangle by Step 3.12, so is $\overline{\operatorname{int} L}$ by (L.4). Therefore $\mathcal{P}$ consists of closed rectangles. We remark that $\operatorname{int} L \cap \operatorname{int} L' = \emptyset$ for $L, L' \in \mathcal{L}$ with $L \neq L'$. Then for any $P \in \mathcal{P}$, there is a unique $L \in \mathcal{L}$ such that $P = \overline{\operatorname{int} L}$ and so we write $L(P)$ for such the L. For $P, Q \in \mathcal{P}$ with $P \neq Q$, we have $\operatorname{int} L(P) \cap \operatorname{int} L(Q) = \emptyset$ and so $P \cap Q$ has no interior. Therefore $\operatorname{int} P \cap \operatorname{int} Q = \emptyset$.

Step 3.13. *For every* $x \in \operatorname{int} L \cap f^{-1} \operatorname{int} M$ ($L, M \in \mathcal{L}$),

$$f(W_c^s(x) \cap \operatorname{int} L) \subset W_c^s(fx) \cap \operatorname{int} M$$

and

$$f^{-1}(W_c^u(fx) \cap \operatorname{int} M) \subset W_c^u(x) \cap \operatorname{int} L .$$

Proof. Take $y \in W_c^s(x) \cap \operatorname{int} L$. Then $fy \in M$ by Step 3.7 and there exists a neighborhood U of fy such that $\Lambda(z) \subset \Lambda(M)$ for all $z \in U$ by Step 3.8. For $z \in U$ and $i \in \Lambda(z)$, we have $fx, z \in T_i$, and so $(fx, z) \in Z_c$. Since $fx \in \operatorname{int} M$ and $fx = [fy, fx]$, there exists a neighborhood $V \subset U$ of fy such that $[z, fx] \in M$ and $[fx, z] \in M$ for all $z \in V \cap f(\operatorname{int} L)$. Thus $z = [[z, fx], [fx, z]] \in M$ for all $z \in V \cap f(\operatorname{int} L)$, and so $fy \in \operatorname{int} M$. Clearly $fy \in W_c^s(fx)$. Hence $f(W_c^s(x) \cap \operatorname{int} L) \subset W_c^s(fx) \cap \operatorname{int} M$. Similarly we have $f^{-1}(W_c^u(fx) \cap \operatorname{int} M) \subset W_c^u(x) \cap \operatorname{int} L$.

Step 3.14. $\overline{\operatorname{int} P \cap f^{-1} \operatorname{int} Q} = \overline{\operatorname{int} L(P) \cap f^{-1} \operatorname{int} L(Q)}$ *for all* $P, Q \in \mathcal{P}$.

Proof. We have $\operatorname{int} \bar{A} \cap \operatorname{int} \bar{B} = \operatorname{int}(\overline{A \cap B})$ for open sets A and B. Therefore $\overline{\operatorname{int} \bar{A} \cap \operatorname{int} \bar{B}} = \overline{A \cap B}$ for open sets A and B. Put $A = \operatorname{int} L(P)$ and $B = f^{-1} \operatorname{int} L(Q)$. Then

$$\overline{\operatorname{int} P \cap f^{-1} \operatorname{int} Q} = \overline{\operatorname{int} L(P) \cap f^{-1} \operatorname{int} L(Q)} .$$

Step 3.15. *For every* $x \in \overline{\operatorname{int} P \cap f^{-1} \operatorname{int} Q}$,

$$f(W_c^s(x) \cap P) \subset W_c^s(fx) \cap Q$$

and

$$f^{-1}(W_c^u(fx) \cap Q) \subset W_c^u(x) \cap P .$$

Proof. If $x \in \overline{\operatorname{int} P \cap f^{-1} \operatorname{int} Q}$, by Step 3.14 there exists a sequence $\{x_n\}_{n \in N}$ of points in $\operatorname{int} L(P) \cap f^{-1} \operatorname{int} L(Q)$ such that $x_n \to x$ as $n \to \infty$. For $y \in W_c^s(x) \subset P$, there exists a sequence $\{y_n\}_{n \in N}$ of the points in $\operatorname{int} L(P)$ such that $y_n \to y$ as $n \to \infty$. Since x_n and y_n are in a rectangle $\operatorname{int} L(P)$, we have that $[x_n, y_n] \in W_c^s(x_n) \cap \operatorname{int} L(P)$ for all $n \in \boldsymbol{N}$. Then by Step 3.13, we have that $f[x_n, y_n] \in W_c^s(fx_n) \cap \operatorname{int} L(Q)$ for all $n \in \boldsymbol{N}$. Since $[\ ,\ ]$ is continuous, $[x_n, y_n] \to [x, y] = y$ as $n \to \infty$. Thus $fy \in W_c^s(fx) \cap \overline{\operatorname{int} L(Q)} = W_c^s(fx) \cap Q$ and so $f(W_c^s(x) \cap P) \subset W_c^s(fx) \cap Q$. Similarly, we obtain that $f^{-1}(W_c^u(fx) \cap Q) \subset W_c^u(x) \cap P$.

Step 3.16. *For every* $x \in \operatorname{int} P \cap f^{-1} \operatorname{int} Q$,

$$f(W_c^s(x) \cap \operatorname{int} P) \subset W_c^s(fx) \cap \operatorname{int} Q$$

and

$$f^{-1}(W_c^u(fx) \cap \operatorname{int} Q) \subset W_c^u(x) \cap \operatorname{int} P .$$

Proof. For $y \in W_c^s(x) \cap \operatorname{int} P$, $\operatorname{int} P$ is a neighborhood of y and so $f(\operatorname{int} P)$ is a neighborhood of fy. By the choice of β, we have $f(W_c^s(x) \cap \operatorname{int} P) = f(W_c^s(y) \cap \operatorname{int} P) \supset W_\beta^s(fy) \cap f(\operatorname{int} P)$. By (L.3) there exists a neighborhood $V \subset f(\operatorname{int} P)$ of fy such that $V \cap W_c^s(fy) \subset W_\beta^s(fy)$. Thus $W_c^s(fy) \cap V \subset f(W_c^s(x) \cap \operatorname{int} P) \subset Q$ by Step 3.15. Since $fy \in Q$, there exists a neighborhood W of fy such that $\Lambda(z) \subset \Lambda(fy) = \Lambda(fx)$ $(=\Lambda(L(Q)))$ for all $z \in W$ by Step 3.8. For $z \in W$ and $i \in \Lambda(z)$, we have $z, fx \in T_i$ and so $(z, fx) \in Z_c$. Since $fx \in \operatorname{int} Q$ and $[fy, fx] = fx$, there exists a neighborhood $W' \subset W$ of fy such that $[z, fx] \in Q$. For $z \in W_c^u(fy) \cap W'$ we have $z = [[z, fx], fy] \in Q$. Put $U = V \cap W'$, then $W_c^\sigma(fy) \cap U \subset Q$ for $\sigma = s, u$ and so $fy \in \operatorname{int} Q$ by Step 3.11. Thus $f(W_c^s(x) \cap \operatorname{int} P) \subset W_c^s(fx) \cap \operatorname{int} Q$ by Step 3.15. Similarly we have $f^{-1}(W_c^u(fx) \cap \operatorname{int} Q) \subset W_c^u(x) \cap \operatorname{int} P$.

To show that $\mathcal{P}$ is a Markov partition, it remains only to show that $\mathcal{P}$ has 6) in the definition of Markov partitions.

For $P \in \mathcal{P}$, define

$$\partial^s P = \{x \in P \mid P \not\supset W_c^u(x) \cap U \quad \text{for all neighborhoods } U \text{ of } x\}$$

and

$$\partial^u P = \{x \in P \mid P \not\supset W_c^s(x) \cap U \quad \text{for all neighborhoods } U \text{ of } x\} .$$

Then we have $\partial P = \partial^s P \cup \partial^u P$ by Step 3.11. Put $B^\sigma = \bigcup_{P \in \mathcal{P}} \partial^\sigma P$ for $\sigma = s, u$. Then $\bigcup_{P \in \mathcal{P}} \partial P = B^s \cup B^u$. Thus the conclusion of Claim 3 is followed by the following Step.

Step 3.17. $fB^s \subset B^s$ *and* $f^{-1} B^u \subset B^u$.

Proof. For $x \in B^s$ there exists $P \in \mathcal{P}$ such that $x \in \partial^s P$. By (L.7) we have

$$\bigcup_{Q \in \mathcal{P}} \overline{\operatorname{int}(P \cap f^{-1} Q)} = \overline{\operatorname{int}(\bigcup_{Q \in \mathcal{P}} (P \cap f^{-1} Q))} .$$

Therefore $\bigcup_{Q \in \mathcal{P}} \overline{\operatorname{int} P \cap f^{-1} \operatorname{int} Q} = \overline{\operatorname{int} P} = P$, and so there exists $Q \in \mathcal{P}$ such that $x \in \overline{\operatorname{int} P \cap f^{-1} \operatorname{int} Q}$. Assume that $fx \in Q \backslash \partial^s Q$. Then there exists a neighborhood U of fx such that $U \cap W_c^u(fx) \subset Q$. Clearly $f^{-1} U$ is a neighborhood of x. By Step 3.15, $f^{-1} U \cap W_\beta^u(x) \subset f^{-1}(U \cap W_c^u(fx)) \subset P$. By (L.3) there exists a neighborhood V of x such that $W_c^u(x) \cap V \subset W_\beta^u(x)$. Thus $(V \cap f^{-1} U) \cap W_c^u(x) \subset P$ and so $x \in \partial^s P$. This is a contradiction. So we have $fx \in \partial^s Q \subset B^s$ and $fB^s \subset B^s$. Similarly we have $f^{-1} B^u \subset B^u$.

Proof of Claim 4. Let $c>0$ be a number such that $2c$ is an expansive constant for f and $\mathcal{P}$ be a Markov partition for (X, f). By (L.9) we can assume that $\max\{\operatorname{diam} P \mid P \in \mathcal{P}\} \leqq c/2$. It is easy to check that $W_c^\sigma(x) \cap P \subset W_{c/2}^\sigma(x)$ for all $x \in P\ (\in \mathcal{P})$ and for $\sigma=s, u$. Thus $W_{c/2}^s(x) \cap W_{c/2}^u(y) \neq \emptyset$ for all $x, y \in P$ $(\in \mathcal{P})$. Fix $x \in X$ and define

$$\Gamma^s(x) = \{P \in \mathcal{P} \mid P \cap W_{c/2}^s(x) \neq \emptyset\}$$

and

$$Y^*(x) = \{y \in X \mid W_{c/2}^u(y) \cap P \neq \emptyset \quad \text{for some } P \in \Gamma^s(x)\ .$$

It is easily checked that $Y_{c/2}(x) \subset Y^*(x) \subset Y_c(x)$ for all $x \in X$. Since $\mathcal{P}$ is finite, so is $\{\Gamma^s(x) \mid x \in X\}$. It is clear that $Y^*(x) = Y^*(y)$ whenever $\Gamma^s(x) = \Gamma^s(y)$. Therefore $\{Y^*(x) \mid x \in X\}$ is finite. By (L.5), there exists $\eta>0$ such that $\Delta_{2\eta} \cap Y_c \subset Y_{c/2}$. Then $Y_c(y) \cap B_\eta(x) = Y_{c/2}(y) \cap B_\eta(x)$ for all $x \in X$ and all $y \in B_\eta(x)$. Clearly we have $Y_c(y) \cap B_\eta(x) = Y_c^*(y) \cap B_\eta(x)$ for all $x \in X$ and all $y \in B_\eta(x)$. Therefore $\{Y_c(y) \cap B_\eta(x) \mid y \in B_\eta(x)\} = \{Y^*(y) \cap B_\eta(x) \mid y \in B_\eta(x)\}$ is finite for all $x \in X$.

4. Applications

This section contains some applications of our theorem.

Let S be a finite set, and σ be the shift automorphism of $S^{\boldsymbol{Z}}$. The usual product topology is given to $S^{\boldsymbol{Z}}$ and elements of $S^{\boldsymbol{Z}}$ will be written as $x=(x_n)_{n\in\boldsymbol{Z}}$. Let Σ be a σ-invariant closed subset of $S^{\boldsymbol{Z}}$. A system (Σ, σ) is said to be a *subshift*. A subshift (Σ, σ) is said to be of *finite type* is there exist $n \in \boldsymbol{N}$ and a subset B of S^n such that

$$\Sigma = \{x \in S^{\boldsymbol{Z}} \mid (x_i, x_{i+1}, \cdots, x_{i+n-1}) \in B \quad \text{for all } i \in \boldsymbol{Z}\}\ .$$

A subshift (Σ, σ) is said to be *sofic* if there exists a subshift (Σ', σ') of finite type such that (Σ, σ) is a *factor* of (Σ', σ'), i.e., there exists a continuous surjective map $\phi: \Sigma' \to \Sigma$ such that $\sigma \circ \phi = \phi \circ \sigma'$. Remark that a sofic subshift does not have POTP unless it is of finite type ([37]).

Application 1. *A subshift (Σ, σ) of $(S^{\boldsymbol{Z}}, \sigma)$ has Markov partitions if and only if it is sofic.*

Proof. If (Σ, σ) of $(S^{\boldsymbol{Z}}, \sigma)$ has Markov partitions, then (Σ, σ) is a factor of a subshift of finite type (c.f., see [15]), and so (Σ, σ) is sofic. Conversely, let (Σ, σ) be sofic and W be the set of words which occur in Σ. For $w \in W$, define $F(w) = \{w' \in W \mid ww' \in W\}$. Since (Σ, σ) is sofic, $\{F(w) \mid w \in W\}$ is finite, i.e., (Σ, σ) is F-finitary ([38]). For $x=(x_i)_{i\in\boldsymbol{Z}} \in \Sigma$, $F_n = F(x_{-n}, x_{-(n-1)}, \cdots, x_0)$ $(n \geqq 0)$ is a decreasing sequence of W. Since $\{F(w) \mid w \in W\}$ is finite, there exists $N \geqq 0$

such that $F_n=F_N$ for all $n\geqq N$. Put $\hat{F}(x)=F_N$. Clearly $\{\hat{F}(x)\,|\,x\in\Sigma\}\subset\{F(w)\,|\,w\in W\}$. Let $G(x)=\{y\in\Sigma\,|\,(y_1,\cdots,y_n)\in\hat{F}(x)$ for all $n\geqq 1\}$ for $x\in\Sigma$. For $x, y\in\Sigma$, define $z=(z_i)_{i\in Z}\in S^Z$ by $z_i=x_i$ for $i\leqq 0$ and $z_i=y_i$ for $i\geqq 1$. Then $z\in\Sigma$ if and only if $y\in G(x)$. It is clear that $\{G(x)\,|\,x\in\Sigma\}$ is finite. Define the metric on Σ by $d(x, y)=\max_{i\in Z}\delta(x_i, y_i)/2^{|i|}$ for $x, y\in\Sigma$ where $\delta(x_i, y_i)=0$ if $x_i=y_i$ and 1 if $x_i\neq y_i$. Put $c=1/3$. Then $2c$ is an expansive constant and for σ,

$$W^s_c(x)=\{y\in\Sigma\,|\,y_i=x_i \quad \text{for all} \quad i\geqq -1\}$$
$$W^u_c(x)=\{y\in\Sigma\,|\,y_i=x_i \quad \text{for all} \quad i\leqq 1\}\,.$$

Therefore we have that $Y_c(x)=\{y\in\Sigma\,|\,y\in G(x),\ y_0=x_0$ and $y_{-1}=x_{-1}\}$ for all $x\in\Sigma$. Since $\{G(x)\,|\,x\in\Sigma\}$ is finite and S is finite, we obtain that $\{Y_c(x)\,|\,x\in\Sigma\}$ is finite. This shows that (Σ, σ) satisfies condition (I) of our theorem and so (Σ, σ) has Markov partitions.

Let f be an expansive homeomorphism of a compact metric space X, and $c>0$ be a number such that $2c$ is an expansive constant for f. A point $x\in X$ is said to be a *singular point* if there exists a sequence $\{x_n\}_{n\in N}$ such that $x_n\to x$ as $n\to\infty$ and $x\notin\operatorname{int} Y_c(x_n)$ for all $n\in\boldsymbol{N}$. There are no singular points if (X, f) has POTP.

Application 2. *Pseudo-Anosov maps (for the definition, see* [18]*) have Markov partitions, but not have POTP.*

Proof. It is sketched in [17] that pseudo-Anosov maps have Markov partitions. But this is easily obtained by our theorem. For, from the definition pseudo-Anosov maps are expansive and have condition (I) of our theorem. Pseudo-Anosov maps do not have POTP since they have singular points by definition.

Let f be a hyperbolic automorphism of r-dimentional torus $\boldsymbol{T}^r$. It is known that $(\boldsymbol{T}^r, f)$ is expansive and has POTP. It is known also that for every fixed point $p\in\boldsymbol{T}^r$ of f, there is a point $p'(\neq p)$ $\boldsymbol{T}^r$ such that $f^n p'\to p$ as $n\to\pm\infty$.

Application 3. *Let f be a hyperbolic automorphism of $\boldsymbol{T}^r$ with fixed points p and q. Let X be the quotient space of $\boldsymbol{T}^r$ induced by identifying p with q, and g be the homeomorphism of X induced from f. Then (X, g) has Markov partitions, but it does not have POTP.*

Proof. Clearly (X, g) is expansive and has condition (I) of our theorem, and so it has Markov partitions. Since $p\,(=q)$ is singular point in X, (X, g) does not have POTP.

In Applications 2 and 3, the number of singular points are finite. However we can consider the case with infinite number of singular points as follows;

Application 4. *Let f be a hyperbolic automorphism of $\boldsymbol{T}^r$ with fixed points p and q, and p' $(\neq p)\in \boldsymbol{T}^r$ be a point such that $f^n p' \to p$ as $n\to\pm\infty$. Let X be the quotient space of $\boldsymbol{T}^r$ induced by identifying p with q and $f^n p'$ with $f^n p'+(q-p)$ $(n\in \boldsymbol{Z})$, and g be the homeomorphism of X induced from f. Then (X, g) has Markov partitions and has infinite number of singular points.*

Proof. It is easy to check that (X, g) is expansive. (X, g) has Markov partitions as in the proof of Application 3. Since $f^n p'$'s are different singular points, the conclusion is obtained.

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