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Author(s)	Tsushima, Yukio
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## A GROUP ALGEBRA OF A $p$ -SOLVABLE GROUP

YUKIO TSUSHIMA

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### 1. Introduction

This paper is a sequel to our earlier one [6] and we are concerned also with the radical of a group algebra of a finite group, especially of a  $p$ -solvable group. Let  $G$  be a finite group of order  $|G| = p^n g'$ , where  $p$  is a fixed prime number,  $n$  is an integer  $\geq 0$  and  $(p, g') = 1$ . Let  $S_p$  be a Sylow  $p$ -group of  $G$  and  $k$  a field of characteristic  $p$ . We denote by  $\mathfrak{R}$  the radical of the group algebra  $kG$  (These notations will be fixed throughout this paper). Let  $B$  be a block of defect  $d$  in  $kG$ . Then  $\mathfrak{R}B$  is the radical of  $B$ . First we shall show  $(\mathfrak{R}B)^{p^d} = 0$ , when  $G$  is solvable or a  $p$ -solvable group with an abelian Sylow  $p$ -group. In §3, we assume  $S_p$  is abelian. Let  $H$  be a normal subgroup of  $G$  and  $\mathfrak{R}$  the radical of  $kH$ . It follows from Clifford's Theorem that  $\mathfrak{R} \subset \mathfrak{R}$ , hence  $\mathfrak{Z} = kG \cdot \mathfrak{R} = \mathfrak{R} \cdot kG$  is a two sided ideal contained in  $\mathfrak{R}$ . If  $[G:H]$  is prime to  $p$ , we have  $\mathfrak{Z} = \mathfrak{R}$  (Proposition 1 [6]). In another extreme, suppose  $[G:H] = p$ . Then we can show there exists a central element  $c$  in  $\mathfrak{R}$  such that  $\mathfrak{R} = \mathfrak{Z} + (kG)c$ . Hence if  $G$  is  $p$ -solvable,  $\mathfrak{R}$  can be constructed somewhat explicitly using a special type of a normal sequence of  $G$  (Theorem 2). If  $S_p$  is normal in  $G$ , then  $\mathfrak{R}$  is generated over  $kG$  by the radical of  $kS_p$  ([7] or Proposition 1 [6]). Hence Theorem 2 may be considered as a generalization of the above fact to the case that  $S_p$  is abelian. In the special case that  $S_p$  is cyclic, our main results will be improved in the final section.

Besides the notation introduced above we use the following;  $H$  will always denote a normal subgroup of  $G$ ,  $\mathfrak{R}$  the radical of  $kH$  and  $\mathfrak{Z} = kG \cdot \mathfrak{R}$ . For a subset  $T$  in  $G$ ,  $N_G(T)$  and  $C_G(T)$  are the normalizer and the centralizer of  $T$  in  $G$ . For an element  $x$  in  $G$ ,  $[x]$  denotes the sum of the elements in the conjugate class containing  $x$ . Finally, we assume  $k$  is a splitting field for every subgroup of  $G$ .

### 2. Radical of a block

We begin with some considerations on the central idempotents. Let  $\mathfrak{A} = \{\eta_i\}$  be the set of the block idempotents in  $kH$ .  $G$  induces a permutation group on  $\mathfrak{A}$  by  $\eta_i \rightarrow g^{-1}\eta_i g$ ,  $g \in G$ . Let  $\tilde{\mathfrak{S}}_1 \cdots \tilde{\mathfrak{S}}_s$  be the set of transitivity. We use the

same letter  $\tilde{\mathfrak{S}}_i$  to denote the set of the blocks whose block idempotents are in  $\tilde{\mathfrak{S}}_i$ . Consider the sum  $\varepsilon_i = \sum \eta_i$  taken over the idempotents in  $\tilde{\mathfrak{S}}_i$ .  $\varepsilon_i$  is a central idempotent in  $kG$ , hence it is the sum of certain block idempotents in  $kG$ , say  $\varepsilon_i = \sum \delta_k$ . Let  $\mathfrak{S}_i$  be the set of the blocks of  $kG$  whose block idempotents appear in the summation above. The different  $\mathfrak{S}_i$  are disjoint, since  $\varepsilon_i \varepsilon_j = 0$  for  $i \neq j$ , and there is a 1-1 correspondence

$$\mathfrak{S}_i \leftrightarrow \tilde{\mathfrak{S}}_i.$$

The following lemma is obvious.

**Lemma 2.1.** *Let  $M$  be a principal indecomposable (irreducible resp.) module belonging to a block in  $\mathfrak{S}_i$ . Then every principal indecomposable (irreducible resp.)  $kH$ -direct summand of  $M_H$  belongs to a block in  $\tilde{\mathfrak{S}}_i$ <sup>1)</sup>. Conversely if  $N$  is a principal indecomposable (irreducible resp.)  $kH$ -module belonging to a block in  $\tilde{\mathfrak{S}}_i$ , then every principal indecomposable (irreducible resp.)  $kG$ -direct summand ( $kG$ -composition factor module resp.) of the induced module  $N^G = kG \otimes_{kH} N$  belongs to a block in  $\mathfrak{S}_i$ .*

The following result is completely due to Fong [3].

**Lemma 2.2.** *Suppose  $[G:H] = q$  is a prime number. Then we have*

(1) ((1E), (3J) in [3]) *Every block of  $kG$  in  $\mathfrak{S}_i$  has the same defect group. We denote it by  $D$ .*

(2) ((1F) in [3]) *If  $q \neq p$ , then  $D$  is a defect group of some block in  $\tilde{\mathfrak{S}}_i$ . In particular, every block in  $\mathfrak{S}_i$  or in  $\tilde{\mathfrak{S}}_i$  has the same defect.*

Here we recall some of the results in [6]. Let  $kH = \bigoplus \sum (kH)e_i$  be a direct sum of principal indecomposable modules, where  $e_i$  is a primitive idempotent of  $kH$ . We assume the first  $\{(kH)e_i, \dots, (kH)e_r\}$  is the set of the non-isomorphic ones. From the natural exact sequence,  $0 \rightarrow \mathfrak{R} \rightarrow kH \rightarrow kH/\mathfrak{R} \rightarrow 0$ , we have the following commutative diagram and natural isomorphisms,

$$\begin{array}{ccccccc} 0 & \rightarrow & kG \otimes \mathfrak{R} & \rightarrow & kG \otimes kH & \rightarrow & kG \otimes kH/\mathfrak{R} \rightarrow 0 \quad (\text{exact}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{E} & \longrightarrow & kG & \longrightarrow & kG/\mathfrak{E} \longrightarrow 0 \quad (\text{exact}), \end{array}$$

where  $\otimes = \otimes_{kH}$ .

Naturally we may regard  $kH/\mathfrak{R} \subset kG/\mathfrak{E} = A$ . The above isomorphisms induce an isomorphism  $kG \otimes (kH/\mathfrak{R})\bar{e}_i \cong A\bar{e}_i$ , where  $\bar{e}_i$  indicates the class of  $e_i$  in  $kH/\mathfrak{R}$ . For an irreducible  $kH$ -module  $V$ , the inertia group is the subgroup  $H^*(V) = \{x \in G \mid x \otimes V \simeq V \text{ as } kH\text{-modules}\}$ .

Now we assume  $[G:H] = p$ .  $kH/\mathfrak{R}$  is arranged in the following form,

1)  $M_H$  is the  $kH$ -module obtained by restricting the operators to  $kH$ .

$kH/\mathfrak{R} = \sum_{i=1}^m u_i(kH/\mathfrak{R})\bar{e}_i \oplus \sum_{i=m+1}^r u_i(kH/\mathfrak{R})\bar{e}_i$ , where  $u_i(kH/\mathfrak{R})\bar{e}_i$  denotes a direct sum of  $u_i$  modules isomorphic to  $(kH/\mathfrak{R})\bar{e}_i$  and  $u_i = \dim_k(kH/\mathfrak{R})\bar{e}_i$ . We assume  $H^*((kH/\mathfrak{R})\bar{e}_i) = G$  ( $1 \leq i \leq m$ ) and  $H^*((kH/\mathfrak{R})\bar{e}_i) = H$  ( $m < i \leq r$ ). Thus  $A = \bigoplus_{1 \leq i \leq m} u_i A\bar{e}_i \oplus \bigoplus_{m < i \leq r} u_i A\bar{e}_i$ .

In [6] we proved;

(1) The composition factor modules of  $A\bar{e}_i$  are all isomorphic. We denote it by  $M_i$ . For  $i < m$ ,  $A\bar{e}_i$  is irreducible and  $\bigoplus_{m < i \leq r} u_i A\bar{e}_i$  is a semisimple algebra over  $k$ . For  $1 \leq i \leq m$ , the composition length of  $A\bar{e}_i$  is  $p$  and  $C_i = u_i A\bar{e}_i$  is a block of  $A$ . Furthermore we have  $(M_i)_H \simeq (kH/\mathfrak{R})\bar{e}_i$ .

(2)  $\mathfrak{R}^p \subset \mathfrak{R}$ .

**Lemma 2.3.**  $A\bar{e}_i$  is indecomposable.

Proof. It suffices to show this only for  $i \leq m$ . From the first part of (2),  $A\bar{e}_i$  is indecomposable or completely reducible (Proposition 2 [6]). Suppose it is completely reducible. Then  $C_i = u_i A\bar{e}_i$  is a simple algebra over  $k$  and  $A\bar{e}_i \simeq p \cdot M_i$ . Thus we have  $\dim_k C_i = p \cdot u_i^2$ . However since  $C_i$  is a simple algebra over a splitting field, we have  $\dim_k C_i = (\dim_k M_i)^2 = u_i^2$ . This is a contradiction.

**Corollary 2.4.**  $(kG)e_i$  is indecomposable.

REMARK 1. It follows from this corollary that the representatives of primitive idempotents of  $kG$  can be taken from  $kH$ . This is a key point for the later arguments.

**Lemma 2.5.**  $A\bar{e}_i$  is irreducible if and only if  $M_i$  is  $(G, H)$ -projective.

Proof. If  $A\bar{e}_i$  is irreducible, then  $M_i = A\bar{e}_i = kG \otimes (kH/\mathfrak{R})\bar{e}_i$ . Thus  $M_i$  is  $(G, H)$ -projective. Conversely, suppose  $A\bar{e}_i$  is not irreducible and  $M_i$  is  $(G, H)$ -projective. Then  $A\bar{e}_i \simeq kG \otimes (M_i)_H$  and  $M_i$  is a direct summand of  $kG \otimes (M_i)_H$ , which contradicts the indecomposability of  $A\bar{e}_i$ . This completes the proof.

In [4], Green proved the following; Let  $B$  be a block and  $D$  its defect group. Then every irreducible module  $M$  belonging to  $B$  is  $(G, D)$ -projective. Moreover if  $M$  is of height 0, then  $D$  is the vertex of  $M$ .

**Lemma 2.6.** Let  $H$  be a normal subgroup of index  $p$ . Let  $B$  be a block of  $kG$  and  $D$  the defect group. If  $D \subset H$ , then we have  $\mathfrak{R}B = \mathfrak{B}B$ .

Proof. It suffices to show that  $\mathfrak{R}e_i = \mathfrak{B}e_i$  for certain primitive idempotents  $e_i$  such that  $\sum e_i = \delta$ , where  $\delta$  is the block idempotent of  $B$ . We may assume each  $e_i$  is in  $kH$  by Remark 1. Since  $A\bar{e}_i = (kG/\mathfrak{R})\bar{e}_i \simeq kGe_i/\mathfrak{B}e_i$ ,  $M_i$  belongs to  $B$ . Hence  $M_i$  is  $(G, D)$ -projective. However, since  $H$  contains  $D$  by the

assumption, we know  $M_i$  is  $(G, H)$ -projective. Thus  $A\bar{e}_i$  is irreducible by Lemma 2.4, which means  $\Re e_i = \mathfrak{L}e_i$  since  $(\Re/\mathfrak{L})\bar{e}_i$  is a maximal submodule of  $A\bar{e}_i$ . This completes the poof.

**Theorem 1.** *Suppose  $G$  is a solvable group, or a  $p$ -solvable group with an abelian Sylow  $p$ -group. Let  $B$  be a block of defect  $d$ . Then we have  $(\Re B)^{p^d} = 0$ .*

*Proof.* We proceed by induction on the order of  $G$ . We may assume there exists a proper normal subgroup  $H$  of index  $p$  or prime to  $p$ .

*Case 1.*  $[G:H] = p$ . Let  $D$  be the defect group of  $B$  and  $\delta$  the block idempotent. Since  $H$  contains all the  $p$ -regular elements,  $\delta$  is actually in  $kH$ . Hence we have  $\delta = \sum \eta_i$  and  $B = kG \cdot \sum \tilde{B}_i$ , where  $\eta_i$  is a block idempotent in  $kH$  and  $\tilde{B}_i$  is the corresponding block of  $kH$  of defect  $d_i$ . Let  $\psi_i'$  be the linear character which defines the block  $\tilde{B}_i$ . Then we have  $\psi_i'(\delta) = \sum \psi_i'(\eta_i) = 1$ . Hence  $D \cap H$  contains the defect group of  $\tilde{B}_i$ , in particular  $d \geq d_i$ . If  $D \subset H$ , we have  $\Re B = \mathfrak{L}B$  by Lemma 2.5. Thus  $(\Re B)^{p^d} = kG \cdot \sum (\Re \tilde{B}_i)^{p^{d_i}} = 0$ , since  $(\Re \tilde{B}_i)^{p^{d_i}} = 0$  by the induction hypothesis. If  $D \not\subset H$ , then we have  $d < d_i$  and thus  $p^d \geq p \cdot p^{d_i}$ . Since  $(\Re B)^p \subset \mathfrak{L}B$ , we have  $(\Re B)^{p^d} \subset (\mathfrak{L}B)^{p^{d_i}} = kG \cdot \sum (\Re \tilde{B}_i)^{p^{d_i}} = 0$ .

*Case 2.*  $[G:H]$  is prime to  $p$ .

( $\alpha$ ) Suppose  $G$  is solvable. We may assume  $[G:H]$  is a prime number. Let  $f$  be a primitive idempotent in  $B$ . Since  $(kG)f$  is a projective  $kG$ -module, it is also projective as a  $kH$ -module. Hence  $(kG)f$  is isomorphic to a direct sum of principal indecomposable modules of  $kH$ , say  $((kG)f)_H \cong \sum_i (kH)e_i$ . By Lemma 2.2, each  $(kH)e_i$  belongs to a block of defect  $d$  in  $kH$ . Thus  $\Re^{p^d} f = \Re^{p^d} (kG)f \cong \sum_i \Re^{p^d} e_i = 0$  by the hypothesis. Since  $f$  is an arbitrary idempotent in  $B$ , we have  $(\Re B)^{p^d} = 0$ .

( $\beta$ ) Suppose  $G$  is a  $p$ -solvable and  $S_p$  is abelian. We cannot assume  $[G:H]$  is a prime number in general. However, from the proof of the ( $\alpha$ ) part, it is sufficient to show that (2) in Lemma 2.2 holds also in this case.

We recall that the defect groups of the blocks in  $\tilde{\mathfrak{S}}_i$  are conjugate in  $G$ . Let  $\tilde{D}$  be one of them. Using the same notation as that of the beginning of this section, we have

**Lemma 2.7.** *Suppose  $G$  is  $p$ -solvable,  $S_p$  is abelian and  $[G:H]$  is prime to  $p$ . Let  $D$  be the defect group of some block  $B$  in  $\mathfrak{S}_i$ . Then  $D$  is conjugate to  $\tilde{D}$  in  $G$ . (In this case we write  $D = \tilde{D}$ ).*

*Proof.* Let  $M$  be any irreducible  $kG$ -module belonging to  $B$ . The height of  $M$  is 0 by Theorem (3F) [3]. Hence we have  $v_G(M) = D$  by Green's Theorem referred above, where  $v_G(M)$  is the vertex of  $M$  in  $G$ . Since  $H$  is normal,  $M_H$

is a direct sum of irreducible  $kH$ -modules belonging to a block in  $\tilde{\mathfrak{N}}_i$ :  $M_H = \bigoplus \sum N_i$ . We have also  $v_H(N_i) = \tilde{D}$ . Since  $[G:H]$  is prime to  $p$ ,  $M$  is  $(G, H)$ -projective. Therefore there exists some  $N_i$  such that  $v_G(M) = v_H(N_i)$ . Thus we have  $D = v_G(M) = v_H(N_i) = \tilde{D}$ . This completes the proofs of Lemma 2.7 and Theorem 1.

### 3. Generators of the radical

In this section we assume  $S_p$  is abelian. Furthermore we assume the field  $k$  is the residue class field  $\mathfrak{o}/\mathfrak{p}\mathfrak{o}$ , where  $\mathfrak{p}$  is a fixed prime divisor of  $p$  in an algebraic number field containing the  $|G|$ -th roots of unity and  $\mathfrak{o}$  is the ring of  $\mathfrak{p}$ -integral elements. For  $\sigma \in \mathfrak{o}$ ,  $\sigma^*$  indicates the image of  $\sigma$  by the natural map  $\mathfrak{o} \rightarrow \mathfrak{o}/\mathfrak{p}\mathfrak{o}$ . First we shall determine a generator of  $\mathfrak{R}/\mathfrak{I}$  over  $kG$ . If  $[G:H]$  is prime to  $p$ , then  $\mathfrak{R} = \mathfrak{I}$ . If  $[G:H] = p$  and the defect group of a block  $B$  is contained in  $H$ , then we have  $\mathfrak{R}B = \mathfrak{I}B$ . Hence we may consider only those blocks whose defect groups are not in  $H$ .

**Lemma 3.1.** *Suppose  $[G:H] = p$ . Let  $B$  be a block,  $D$  its defect group and let  $\psi$  be the linear character which defines the block  $B$ . If  $D \not\subseteq H$ , then there exists an element  $x$  in  $G$  but not in  $H$  such that  $\psi([x]) \neq 0$ .*

*Proof.* Let  $y$  be a  $p$ -regular element such that  $D$  is a defect group of  $y$  and  $\psi([y]) \neq 0$ . Since  $[G:H] = p$ ,  $y$  is contained in  $H$ . Let  $\xi$  be an irreducible character of height 0 in  $B$ . Then  $\psi([y]) = \left( \frac{|G|}{n(y)} \frac{\xi(y)}{z} \right)^* = \left( \frac{|G|}{n(y) \cdot z} \right)^* \xi(y)^* \neq 0$ , where  $n(y)$  is the order of the centralizer of  $y$  in  $G$  and  $z$  is the degree of  $\xi$ . Since  $D \not\subseteq H$ , there exists an element  $a \in D$  and  $a \notin H$ . Then we have  $N_G(ay) = N_G(a) \cap N_G(y) \supset D$ , since  $D$  is abelian. Hence  $D$  is a defect group of  $ay$ . Thus  $\frac{|G|}{n(ay) \cdot z}$  is also a  $\mathfrak{p}$ -integral element and  $\left( \frac{|G|}{n(ay) \cdot z} \right)^* \neq 0$ . On the other hand, since  $ay = ya$  and  $a$  is a  $p$ -element, we have  $\xi(ay)^* = \xi(y)^* \neq 0$ . Thus  $\psi([ay]) = \left( \frac{|G|}{n(ay) \cdot z} \right)^* \xi(ay)^* \neq 0$ . This completes the proof.

Let  $B_1, \dots, B_s$  be the blocks of  $kG$  and  $\delta_1, \dots, \delta_s$  the block idempotents respectively. Let  $\psi_i$  be the linear character which defines the block  $B_i$ . Then  $\{\psi_1 \cdots \psi_s\}$  is the set of the linear characters on the center of  $kG$ . Since the center is a commutative  $k$ -algebra, its radical is the intersection of the kernels of  $\psi_i$ 's. In particular, for any element  $z$  of the center,  $(z - \psi_i(z))\delta_i$  is an element in  $\mathfrak{R}$ .

**Proposition 3.2.** *Suppose  $[G:H] = p$  and the defect group of the block  $B_i$  is not contained in  $H$ . Let  $x$  be any element in  $G$  such that  $x \notin H$  and  $\psi_i([x]) \neq 0$ . Then we have  $\mathfrak{R}B = \mathfrak{I}B + kG \cdot ([x] - \psi_i([x]))\delta_i$ .*

**Proof.** we put  $\delta = \delta_i$  and  $\psi = \psi_i$  for convenience. Let  $\delta = \sum e_j$  be a decomposition into the sum of primitive idempotents. We may assume each  $e_j$  is in  $kH$  by Remark 1. Let  $e = e_j$  be arbitrary and fixed. Since  $x$  is not in  $H$ , we may put  $x = av$ , where  $a^{p-1} \notin H$  and  $v \in H$ . Then we have  $([x] - \psi([x]))^{p-1} \delta e = a^{p-1} z_1 + a^{p-2} z_2 + \cdots + a z_{p-1} + \psi([x])^{p-1} e$ , where  $z_i \in kH$ . The right hand is not contained in  $\mathfrak{V}e = a^{p-1} \mathfrak{R}e + \oplus a^{p-2} \mathfrak{R}e \oplus \cdots \oplus \mathfrak{R}e$ , since  $\psi([x]) \neq 0$ . Hence we have a sequence

$$A\bar{e} \supseteq ([x] - \psi([x])) A\bar{e} \supseteq ([x] - \psi([x]))^2 A\bar{e} \supseteq \cdots \supseteq ([x] - \psi([x]))^{p-1} A\bar{e} \supseteq 0.$$

However, since  $A\bar{e}$  has  $p$  composition factors,  $([x] - \psi([x])) A\bar{e}$  must be maximal, that is  $([x] - \psi([x])) A\bar{e} = (\mathfrak{R}/\mathfrak{V})\bar{e}$ . Therefore we have  $kG \cdot ([x] - \psi([x]))e + \mathfrak{V}e = \mathfrak{R}e$ . and thus  $\mathfrak{R}B = \mathfrak{V}B + kG([x] - \psi([x]))\delta$ , since  $e$  is arbitrary. This completes the proof.

**Corollary 3.3.** *We put  $c = \sum ([x_i] - \psi_i([x_i]))\delta_i$ , where  $\delta_i$  ranges over all the block idempotents of the blocks whose defect groups are not is  $H$  and  $x_i$  is any element of  $G$  such that  $x_i \notin H$  and  $\psi_i([x_i]) \neq 0$ . Then we have  $\mathfrak{R}B = \mathfrak{V}B + (kG)c$ .*

From the above Corollary we have the following Theorem.

**Theorem 2.** *Suppose  $G$  is  $p$ -solvable and  $S_p$  is abelian. Consider a normal sequence,*

$$G = H_0 \supset G_1 \supset H_1 \supset G_2 \supset H_2 \supset \cdots \supset G_n \supset H_n \supset G_{n+1} = \{1\},$$

where  $G_{i+1}$  is the minimal normal subgroup of  $H_i$  such that  $[H_i : G_{i+1}]$  is prime to  $p$  and  $H_i$  is a normal subgroup of  $G_i$  of index  $p$  (possibly  $H_i = G_{i+1}$ ). Then there exists a central element  $c_i$  in  $kG_i$  such that  $\{c_i\}_{i=1}^n$  generate  $\mathfrak{R}$  over  $kG$ . In particular  $\{\mathfrak{S}_i\}_{i=1}^n$  generates  $\mathfrak{R}$  over  $kG$ , where  $\mathfrak{S}_i$  is the radical of the center of  $kG_i$ .

#### 4. The case where $S_p$ is cyclic.

In this section we assume  $S_p$  is cyclic and we shall improve the main results of the preceeding sections. Let  $\theta$  be a generator of  $S_p$  and  $U = N_G(S_p)/C_G(S_p)$ .

**Lemma 4.1.**  *$U$  is a cyclic group. Let  $t$  be the order of  $U$  and  $\sigma$  in  $N_G(S_p)$  correspond to a generating element of  $U$ . Then  $t$  divides  $p-1$  and  $\sigma^{-1}\theta\sigma = \theta^t$ . The conjugate class containing  $\theta$  in  $N_G(S_p)$  consists of  $\theta, \theta^t, \dots, \theta^{t^{t-1}}$ . Furthermore, let  $\phi$  be the Brauer homomorphism of the center of  $kG$  into the center of  $kN_G(S_p)$ . Then we have  $\phi([\theta]) = \theta + \theta^t + \cdots + \theta^{t^{t-1}}$ .*

**Proof.** The first half is well known. We omit the proofs. Since the defect group of  $\theta$  is  $S_p$ , we know  $\phi([\theta])$  is the sum of the elements in the conjugate class containing  $\theta$ . Thus we have  $\phi([\theta]) = \theta + \theta^t + \cdots + \theta^{t^{t-1}}$ .

**REMARK 2.** Though the proof is easy, the following fact is worth while

remarking. By the definition  $t$  is the order of  $l \bmod p^n$ . However, since  $t$  is prime to  $p$ ,  $t$  is also the order of  $l \bmod p$ .

**Lemma 4.2.** *If  $G$  has a normal subgroup of index  $p$ , then  $G$  has a normal  $p$ -Sylow complement.*

Proof. By Burnside's Theorem, it suffices to show that  $N_G(S_p) = C_G(S_p)$ . We use the same notation as that of Lemma 4.1. The transfer map  $G \rightarrow S_p$  induces an isomorphism  $G/T \simeq Z \cap S_p$ , where  $Z$  is the center of  $N_G(S_p)$  and  $T$  is the minimal normal subgroup of  $G$  such that  $G/T$  is abelian  $p$ -group ([8]). We have  $G/T \neq \{1\}$  by the assumption, hence there exists  $\theta^k$  in  $S_p$ ,  $0 < k < p^n$  and  $\theta^k$  commutes with  $\sigma$ . Since  $\sigma^{-1}\theta\sigma = \theta^l$ , we have  $\sigma^{-1}\theta^k\sigma = \theta^{lk} = \theta^{tk}$ . It follows that  $p^n$  divides  $(l-1)k$ . Since  $p^n \nmid k$ ,  $(l-1)$  is divisible by a suitable power  $p^{n_0}$  ( $n_0 > 0$ ). Thus we have  $l \equiv 1 \bmod p$ . Hence we have  $t=1$  by Remark 2. This completes the proof.

**Lemma 4.3.** *Let  $l$  and  $t$  be integers such that  $t$  is the order of  $l \bmod p$ . We assume  $l$  is greater than  $p$ . Let  $F(X) = X + X^l + X^{l^2} + \dots + X^{l^{t-1}} - t$  be a polynomial over  $k$ . Then we have  $F(X) = (X-1)^t G(X)$ , where  $G(X)$  is a polynomial over  $k$  and  $G(1) \neq 0$ .*

Proof. It suffices to show that  $F(1) = F'(1) = \dots = F^{(t-1)}(1) = 0$  and  $F^{(t)}(1) \neq 0$ , since  $1 \leq t < p$  (the characteristic of  $k$ ). It follows directly that  $F(1) = 0$  and  $F^{(v)}(1) = \sum_{i=1}^{t-1} l^i(l^i-1)\dots(l^i-v+1)$ . We put  $Y(Y-1)\dots(Y-v+1) = \sum_{j=1}^v a_j Y^j$ , then we have  $\sum a_j = 0$  and  $F^{(v)}(1) = \sum_{j=1}^v a_j (\sum_{m=1}^{t-1} l^{mj})$ . If  $j \leq v < t$ , then  $\sum_{m=1}^{t-1} l^{mj} = \frac{l^j(l^{j(t-1)}-1)}{l^j-1} = -1$ . Thus  $F^{(v)}(1) = -\sum a_j = 0$ . For  $v=t$ , we have  $F^{(t)}(1) = \sum_{j=1}^{t-1} (-a_j) + (t-1) = t \neq 0$ . This completes the proof.

Now let  $\delta_1 \dots \delta_r$  be the block idempotents of the blocks of full defect. It is clear that  $\phi_i([\theta]) = h$  in  $k$ , where  $h$  is the number of the elements in the conjugate class containing  $\theta$  in  $G$ . In particular, we have  $\phi_i([\theta]) \neq 0$ .

**Proposition 4.4.** *Let  $t$  be the order of  $U$  and  $f = \frac{p^n-1}{t}$ . Then for some  $i$  ( $1 \leq i \leq r$ ), we have  $([\theta] - h)^f \delta_i \neq 0$ . In particular, we have  $\Re^f \neq 0$ .*

Proof. Since  $[G:N_G(S_p)] \equiv 1 \bmod p$ , we have  $h = [G:N_G(S_p)] [N_G(S_p):C_G(S_p)] \equiv t \bmod p$ . Hence  $\phi(([\theta] - h)^f \delta_i) = (\theta + \theta^l + \dots + \theta^{l^{t-1}} - t)^f \phi(\delta_i)$ . As is well known,  $\phi(\delta_i)$  is not zero and a block idempotent in  $kN_G(S_p)$  and furthermore  $\sum \phi(\delta_i) = 1$ . Hence it is sufficient to show that  $(\theta + \theta^l + \dots + \theta^{l^{t-1}} - t)^f \neq 0$ . By Remark 2,  $t$  is also the order of  $l \bmod p$ . We use Lemma 4.3 replacing  $l$  by  $l+p^n$  if necessary and we get  $F(\theta) = \theta + \theta^l + \dots + \theta^{l^{t-1}} - t = (\theta-1)^t G(\theta)$ . Furthermore  $G(1) \neq 0$  means that the sum of the coefficients of  $G(X)$  is not zero. Hence



$G(\theta)$  is a unit in  $kS_p$  (see [5] or pp. 189 [2]) Thus we have  $F(\theta)^f = (\theta - 1)^{p^n - 1}$   
 $G(\theta)^f \neq 0$ .

**Corollary 4.5.** *If  $S_p$  has a normal complement in  $G$ , we have  $([\theta] - h)^{p^n - 1} \delta_i \neq 0$ , for all  $i$  ( $i \leq i \leq r$ ).*

*Proof.* It follows from the assumption that  $t=1$  and  $f=p^n-1$ . Hence we need to show only that  $F(\theta)^{p^n-1} \phi(\delta_i) \neq 0$  for all  $i$  ( $1 \leq i \leq r$ ). Now suppose  $F(\theta)^{p^n-1} \delta_i' = 0$  for some  $i$ , where  $\delta_i' = \phi(\delta_i)$ . Then we have  $(\theta - 1)^{p^n-1} \delta_i' = 0$ , since  $G(\theta)$  is a unit. From this it follows that  $\theta^{p^n-1} \delta_i' + a_1 \theta^{p^n-2} \delta_i' + \dots + a_{p-1} \theta \delta_i' = -\delta_i'$ , where  $a_i \in k$ . However this is a contradiction, since all the elements of  $G$  which appear in the summation in the left hand side are  $p$ -irregular and the right hand side is a sum of  $p$ -regular elements. This completes the proof.

**Lemma 4.6.** *Let  $\mathfrak{S}$  be the radical of the center of  $kG$ . If  $S_p$  has a normal complement in  $G$ , we have  $\mathfrak{R} = kG \cdot \mathfrak{S}$ .*

*Proof.* There exists a normal subgroup  $H$  of index  $p$ . Since  $S_p$  has only one subgroup of order  $p^v$  for  $0 \leq v \leq n$ , all the defect groups of the blocks of defect smaller than  $n$  are contained in  $H$ . Hence by Corollary 3.3, we have  $\mathfrak{R} = \mathfrak{S} + kG \cdot ([\theta] - h)\rho$ , where  $\rho$  is the sum of the block idempotents of the blocks of full defect. Let  $T$  be the normal complement. There exists a normal sequence,

$$G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_{n-1} \supset G_n = T,$$

where  $G_{k+1}$  is the normal subgroup of  $G_k$  of index  $p$ .  $G_k$  is unique and even normal in  $G$ . It is clear that  $\theta^{p^k}$  generates a Sylow  $p$ -subgroup of  $G_k$  and the conjugate class containing  $\theta^{p^k}$  in  $G_k$  is also the conjugate class in  $G$ . We denote by  $h_k$  the number of the elements in the class. Also it is clear that the sum, say  $\rho_k$ , of all the block idempotents of the blocks of full defect in  $kG_k$  is central in  $kG$ . Now, replacing  $G$  and  $H$  by  $G_k$  and  $G_{k+1}$  respectively, we have  $\mathfrak{R}_k = \mathfrak{S}_k + kG ([\theta^{p^k}] - h_k)\rho_k$ , where  $\mathfrak{R}_k$  is the radical of  $kG_k$ ,  $\mathfrak{S}_k = kG \cdot \mathfrak{R}_{k+1}$  and  $\mathfrak{R}_{k+1}$  is the radical of  $kG_{k+1}$ . Thus  $\{([\theta^{p^k}] - h_k)\rho_k\}_{k=0}^{n-1}$  generate  $\mathfrak{R}$  over  $kG$  and they are central. This completes the proof.

**Theorem 3.** *Let  $G$  be a  $p$ -solvable group with a cyclic Sylow  $p$ -group. Then we have*

(1)  $\mathfrak{R} = kG \cdot \mathfrak{S}_T$ , where  $\mathfrak{S}_T$  is the radical of the center of  $kT$  and  $T$  is the minimal normal subgroup such that  $[G:T]$  is prime to  $p$ .

(2) Let  $d$  be the defect of a certain block of  $kG$ . Then there exists a block of defect  $d$ , say  $B$  such that  $p^d$  is the smallest integer for which  $(\mathfrak{R}B)^{p^d} = 0$ . This holds for any block of defect  $d$ , if  $G$  has a normal  $p$ -Sylow complement.

Proof.

(1) Let  $\mathfrak{R}$  be the radical of  $kT$ . Since  $[G:T]$  is prime to  $p$ , we have  $\mathfrak{R} = \mathfrak{L} = kG \cdot \mathfrak{R}$ . Since  $G$  is  $p$ -solvable,  $T$  has a normal subgroup of index  $p$ . Then  $T$  has a normal  $p$ -Sylow complement by Lemma 4.2. Thus we have  $\mathfrak{R} = kG \cdot \mathfrak{R} = kG(kT \cdot \mathfrak{S}_T) = kG \cdot \mathfrak{S}_T$  by Lemma 4.6.

(2) We prove by induction on the order of  $G$ . First, we prove the second statement. We have only to show  $(\mathfrak{R}B)^{p^{d-1}} \neq 0$  for any block  $B$  of defect  $d$ . If  $d=n$ , we have already proved this in Corollary 4.5. Hence we may assume  $d < n$ . Let  $H$  be a normal subgroup of index  $p$ .  $H$  also has a normal  $p$ -Sylow complement. Let  $\delta$  be the block idempotent of  $B$  and  $\delta = \sum_{i=1}^m \eta_i$ , where  $\eta_i$  is a block idempotent in  $kH$ . Since  $d < n$ , the defect group of  $B$  is contained in  $H$ . Therefore we have  $\mathfrak{R}B = \mathfrak{L}B = \mathfrak{R}B$  and  $d = d_i$  for all  $i$  ( $1 \leq i \leq m$ ),  $d_i$  being the defect of the block corresponding to  $\eta_i$  in  $kH$ . Thus we have  $\mathfrak{R}^{p^{d-1}}\delta = kG \cdot \bigoplus_{i=1}^m \mathfrak{R}^{p^{d-1}}\eta_i \neq 0$  by the induction hypothesis. Now we prove the first part. If  $G$  has a normal subgroup of index  $p$ , our statement is obvious by Lemma 4.2 and the second part just proved. Thus we may assume there exists a proper normal subgroup of index prime to  $p$ . From the 1-1 correspondence  $\mathfrak{F}_i \leftrightarrow \tilde{\mathfrak{F}}_i$  and Lemma 2.7, it follows that there exists a block of defect  $d$  in  $kH$ . Let  $\tilde{\mathfrak{F}}_i$  be the set which contains a block  $\tilde{B}$  such that  $(\mathfrak{R}\tilde{B})^{p^{d-1}} \neq 0$ . Then there exists a primitive idempotent  $e$  in  $\tilde{B}$  such that  $\mathfrak{R}^{p^{d-1}}e \neq 0$ . Let  $(kG)e = \bigoplus_j (kG)f_j$  be a sum of principal indecomposable modules of  $kG$ . Each  $(kG)f_j$  belongs to some block in  $\mathfrak{F}_i$ . We have  $\bigoplus_j \mathfrak{R}^{p^{d-1}}f_j = \mathfrak{R}^{p^{d-1}}e = kG \cdot \mathfrak{R}^{p^{d-1}}e \neq 0$ . Hence there exists some  $f_j$  such that  $\mathfrak{R}^{p^{d-1}}f_j \neq 0$ . This completes the proof.

OSAKA CITY UNIVERSITY

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