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A GROUP ALGEBRA OF A p-SOLVABLE GROUP

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1. Introduction

This paper is a sequel to our earlier one [6] and we are concerned also with the radical of a group algebra of a finite group, especially of a p-solvable group. Let G be a finite group of order $|G|=p^n g'$, where p is a fixed prime number, n is an integer $\geqq 0$ and $(p, g')=1$. Let S_p be a Sylow p-group of G and k a field of characteristic p. We denote by $\mathfrak{N}$ the radical of the group algebra kG (These notations will be fixed throughout this paper). Let B be a block of defect d in kG. Then $\mathfrak{N}B$ is the radical of B. First we shall show $(\mathfrak{N}B)^{p^d}=0$, when G is solvable or a p-solvable group with an abelian Sylow p-group. In §3, we assume S_p is abelian. Let H be a normal subgroup of G and $\mathfrak{R}$ the radical of kH. It follows from Clifford's Theorem that $\mathfrak{R}\subset\mathfrak{N}$, hence $\mathfrak{L}=kG\cdot\mathfrak{R}=\mathfrak{R}\cdot kG$ is a two sided ideal contained in $\mathfrak{N}$. If $[G: H]$ is prime to p, we have $\mathfrak{L}=\mathfrak{N}$ (Proposition 1 [6]). In another extreme, suppose $[G:H]=p$. Then we can show there exists a central element c in $\mathfrak{N}$ such that $\mathfrak{N}=\mathfrak{L}+(kG)c$. Hence if G is p-solvable, $\mathfrak{N}$ can be constructed somewhat explicitly using a special type of a normal sequence of G (Theorem 2). If S_p is normal in G, then $\mathfrak{N}$ is generated over kG by the radical of kS_p ([7] or Proposition 1 [6]). Hence Theorem 2 may be considered as a generalization of the above fact to the case that S_p is abelian. In the special case that S_p is cyclic, our main results will be improved in the final section.

Besides the notation introduced above we use the following; H will always denote a normal subgroup of G, $\mathfrak{R}$ the radical of kH and $\mathfrak{L}=kG\cdot\mathfrak{R}$. For a subset T in G, $N_G(T)$ and $C_G(T)$ are the normalizer and the centralizer of T in G. For an element x in G, $[x]$ denotes the sum of the elements in the conjugate class contaning x. Finally, we assume k is a splitting field for every subgroup of G.

2. Radical of a block

We begin with some considerations on the central idempotents. Let $\mathfrak{A}=\{\eta_i\}$ be the set of the block idempotents in kH. G induces a permutation group on $\mathfrak{A}$ by $\eta_i\to g^{-1}\eta_i g$, $g\in G$. Let $\mathfrak{F}_1\cdots\mathfrak{F}_s$, be the set of transitivity. We use the same letter $\tilde{\mathfrak{J}}_i$ to denote the set of the blocks whose block idempotents are in $\tilde{\mathfrak{J}}_i$. Consider the sum $\varepsilon_i = \sum \eta_i$ taken over the idempotents in $\tilde{\mathfrak{J}}_i$. ε_i is a central idempotent in kG, hence it is the sum of certain block idempotents in kG, say $\varepsilon_i = \sum \delta_k$. Let $\mathfrak{J}_i$ be the set of the blocks of kG whose block idempotents appear in the summation above. The different $\mathfrak{J}_i$ are disjoint, since $\varepsilon_i \varepsilon_j = 0$ for $i \neq j$, and there is a 1–1 correspondence

$$\mathfrak{J}_i \leftrightarrow \tilde{\mathfrak{J}}_i .$$

The following lemma is obvious.

Lemma 2.1. *Let M be a principal indecomposable (irreducible resp.) module belonging to a block in $\mathfrak{J}_i$. Then every principal indecomposable (irreducible resp.) kH-direct summand of M_H belongs to a block in $\tilde{\mathfrak{J}}_i$* [1)]. *Conversely if N is a principal indecomposable (irreducible resp.) kH-module belonging to a block in $\tilde{\mathfrak{J}}_i$, then every principal indecomposable (irreducible resp.) kG-direct summand (kG-composition factor module resp.) of the induced module $N^G = kG \otimes_{kH} N$ belongs to a block in $\mathfrak{J}_i$.*

The following result is completely due to Fong [3].

Lemma 2.2. *Suppose $[G:H] = q$ is a prime number. Then we have*

(1) ((1E), (3J) *in* [3]) *Every block of kG in $\mathfrak{J}_i$ has the same defect group. We denote it by D.*

(2) ((1F) *in* [3]) *If $q \neq p$, then D is a defect group of some block in $\tilde{\mathfrak{J}}_i$. In particular, every block in $\mathfrak{J}_i$ or in $\tilde{\mathfrak{J}}_i$ has the same defect.*

Here we recall some of the results in [6]. Let $kH = \oplus \sum (kH)e_i$ be a direct sum of principal indecomposable modules, where e_i is a primitive idempotent of kH. We assume the first $\{(kH)e_1, \cdots, (kH)e_r\}$ is the set of the non-isomorphic ones. From the natural exact sequence, $0 \to \mathfrak{N} \to kH \to kH/\mathfrak{N} \to 0$, we have the following commutative diagram and natural isomorphisms,

$$\begin{tikzcd}
0 \arrow[r] & kG\otimes\mathfrak{N} \arrow[r] \arrow[d, "\wr"] & kG\otimes kH \arrow[r] \arrow[d, "\wr"] & kG\otimes kH/\mathfrak{N} \arrow[r] \arrow[d, "\wr"] & 0 & \text{(exact)} \\
0 \arrow[r] & \mathfrak{L} \arrow[r] & kG \arrow[r] & kG/\mathfrak{L} \arrow[r] & 0 & \text{(exact)},
\end{tikzcd}$$

where $\otimes = \otimes_{kH}$.

Naturally we may regard $kH/\mathfrak{N} \subset kG/\mathfrak{L} = A$. The above isomorphisms induce an isomorphism $kG \otimes (kH/\mathfrak{N})\bar{e}_i \simeq A\bar{e}_i$, where $\bar{e}_i$ indicates the class of e_i in $kH/\mathfrak{N}$. For an irreducible kH-module V, the inertia group is the subgroup $H^*(V) = \{x \in G \mid x \otimes V \simeq V$ *as kH-modules*$\}$.

Now we assume $[G:H] = p$. $kH/\mathfrak{N}$ is arranged in the following form,

1) M_H is the kH-module obtained by restricting the operators to kH.

$kH/\mathfrak{R}=\sum_{i=1}^{m}u_i(kH/\mathfrak{R})\bar{e}_i\oplus\sum_{i=m+1}^{r}u_i(kH/\mathfrak{R})\bar{e}_i$, where $u_i(kH/\mathfrak{R})\bar{e}_i$ denotes a direct sum of u_i modules isomorphic to $(kH/\mathfrak{R})\bar{e}_i$ and $u_i=dim_k(kH/\mathfrak{R})\bar{e}_i$. We assume $H^*((kH/\mathfrak{R})\bar{e}_i)=G\,(1\leqslant i\leqslant m)$ and $H^*((kH/\mathfrak{R})\bar{e}_i)=H\,(m<i\leqslant r)$. Thus $A=\oplus\sum_{1\leqslant i\leqslant m}u_iA\bar{e}_i\oplus\sum_{m<i\leqslant r}u_iA\bar{e}_i$.

In [6] we proved;

(1) The composition factor modules of $A\bar{e}_i$ are all isomorphic. We denote it by M_i. For $i<m$, $A\bar{e}_i$ is irreducible and $\oplus\sum_{m<i\leqslant r}u_iA\bar{e}_i$ is a semisimple algebra over k. For $1\leqslant i\leqslant m$, the composition length of $A\bar{e}_i$ is p and $C_i=u_iA\bar{e}_i$ is a block of A. Furthermore we have $(M_i)_H\simeq(kH/\mathfrak{R})\bar{e}_i$.

(2) $\mathfrak{R}^p\subset\mathfrak{L}$.

Lemma 2.3. *$A\bar{e}_i$ is indecomposable.*

Proof. It suffices to show this only for $i\leqslant m$. From the first part of (2), $A\bar{e}_i$ is indecomposable or completely reducible (Proposition 2 [6]). Suppose it is completely reducible. Then $C_i=u_iA\bar{e}_i$ is a simple algebra over k and $A\bar{e}_i\simeq p\cdot M_i$. Thus we have $dim_k C_i=p\cdot u_i^2$. However since C_i is a simple algebra over a splitting field, we have $dim_k C_i=(dim_k M_i)^2=u_i^2$. This is a contradiction.

Corollary 2.4. *$(kG)e_i$ is indecomposable.*

REMARK 1. It follows from this corollary that the representatives of primitive idempotents of kG can be taken from kH. This is a key point for the later arguments.

Lemma 2.5. *$A\bar{e}_i$ is irreducible if and only if M_i is (G, H)-projective.*

Proof. If $A\bar{e}_i$ is irreducible, then $M_i=A\bar{e}_i=kG\otimes(kH/\mathfrak{R})\bar{e}_i$. Thus M_i is (G, H)-projective. Conversely, suppose $A\bar{e}_i$ is not irreducible and M_i is (G, H)-projective. Then $A\bar{e}_i\simeq kG\otimes(M_i)_H$ and M_i is a direct summand of $kG\otimes(M_i)_H$, which contradicts the indecomposability of $A\bar{e}_i$. This completes the proof.

In [4], Green proved the following; *Let B be a block and D its defect group. Then every irreducible module M belonging to B is (G, D)-projective. Moreover if M is of height* 0, *then D is the vertex of M.*

Lemma 2.6. *Let H be a normal subgroup of index p. Let B be a block of kG and D the defect group. If $D\subset H$, then we have $\mathfrak{R}B=\mathfrak{L}B$.*

Proof. It suffices to show that $\mathfrak{R}e_i=\mathfrak{L}e_i$ for certain primitive idempotents e_i such that $\sum e_i=\delta$, where δ is the block idempotent of B. We may assume each e_i is in kH by Remark 1. Since $A\bar{e}_i=(kG/\mathfrak{L})\bar{e}_i\simeq kGe_i/\mathfrak{L}e_i$, M_i belongs to B. Hence M_i is (G, D)-projective. However, since H contains D by the assumption, we know M_i is (G, H)-projective. Thus $A\bar{e}_i$ is irreducible by Lemma 2.4, which means $\mathfrak{N}e_i=\mathfrak{L}e_i$ since $(\mathfrak{N}/\mathfrak{L})\bar{e}_i$ is a maximal submodule of $A\bar{e}_i$. This completes the poof.

Theorem 1. *Suppose G is a solvable group, or a p-solvable group with an abelian Sylow p-group. Let B be a block of defect d. Then we have $(\mathfrak{N}B)^{p^d}=0$.*

Proof. We proceed by induction on the order of G. We may assume there exists a proper normal subgroup H of index p or prime to p.

Case 1. $[G:H]=p$. Let D be the defect group of B and δ the block idempotent. Since H contains all the p-regular elements, δ is actually in kH. Hence we have $\delta=\sum\eta_i$ and $B=kG\cdot\sum\tilde{B}_i$, where η_i is a block idempotent in kH and $\tilde{B}_i$ is the corresponding block of kH of defect d_i. Let ψ_i' be the linear character which defines the block $\tilde{B}_i$. Then we have $\psi_i'(\delta)=\sum_i\psi_i'(\eta_i)=1$. Hence $D\cap H$ contains the defect group of $\tilde{B}_i$, in particular $d\geqq d_i$. If $D\subset H$, we have $\mathfrak{N}B=\mathfrak{L}B$ by Lemma 2.5. Thus $(\mathfrak{N}B)^{p^d}=kG\cdot\sum_i(\mathfrak{N}\tilde{B}_i)^{p^{d_i}}=0$, since $(\mathfrak{N}\tilde{B}_i)^{p^{d_i}}=0$ by the induction hypothesis. If $D\not\subset H$, then we have $d<d_i$ and thus $p^d\geqq p\cdot p^{d_i}$. Since $(\mathfrak{N}B)^p\subset\mathfrak{L}B$, we have $(\mathfrak{N}B)^{p^d}\subset(\mathfrak{L}B)^{p^{d_i}}=kG\cdot\sum_i(\mathfrak{N}\tilde{B}_i)^{p^{d_i}}=0$.

Case 2. $[G:H]$ *is prime to* p.

(α) Suppose G is solvable. We may assume $[G:H]$ is a prime number. Let f be a primitive idempotent in B. Since $(kG)f$ is a projective kG-module, it is a also projective as a kH-module. Hence $(kG)f$ is isomorphic to a direct sum of principal indecomposable modules of kH, say $((kG)f)_H\simeq\sum_i(kH)e_i$. By Lemma 2.2, each $(kH)e_i$ belongs to a block of defect d in kH. Thus $\mathfrak{N}^{p^d}f=\mathfrak{N}^{p^d}(kG)f\simeq\sum_i\mathfrak{N}^{p^d}e_i=0$ by the hypothesis. Since f is an arbitrary idempotent in B, we have $(\mathfrak{N}B)^{p^d}=0$.

(β) Suppose G is a p-solvable and S_p is abelian. We cannot assume $[G:H]$ is a prime number in general. However, from the proof of the (α) part, it is sufficient to show that (2) in Lemma 2.2 holds also in this case.

We recall that the defect groups of the blocks in $\tilde{\mathfrak{J}}_i$ are conjugate in G. Let $\tilde{D}$ be one of them. Using the same notation as that of the beginning of this section, we have

Lemma 2.7. *Suppose G is p-solvable, S_p is abelian and $[G:H]$ is prime to p. Let D be the defect group of some block B in $\mathfrak{J}_i$. Then D is conjugate to $\tilde{D}$ in G. (In this case we write $D\underset{G}{=}\tilde{D}$).*

Proof. Let M be any irreducible kG-module belonging to B. The height of M is 0 by Thoerem (3F) [3]. Hence we have $v_G(M)=D$ by Green's Theorem refered above, where $v_G(M)$ is the vertex of M in G. Since H is normal, M_H

is a direct sum of irreducible kH-modules belonging to a block in $\tilde{\mathfrak{F}}_i$: $M_H = \oplus\sum N_i$. We have also $v_H(N_i) \underset{G}{=} \tilde{D}$. Since $[G:H]$ is prime to p, M is (G, H)-projective. Therefore there exists some N_i such that $v_G(M) \underset{G}{=} v_H(N_i)$. Thus we have $D = v_G(M) \underset{G}{=} v_H(N_i) \underset{G}{=} \tilde{D}$. This completes the proofs of Lemma 2.7 and Theorem 1.

3. Generators of the radical

In this section we assume S_p is abelian. Furthermore we assume the field k is the residue class field $\mathfrak{o}/\mathfrak{p}\mathfrak{o}$, where $\mathfrak{p}$ is a fixed prime divisor of p in a algebraic number field containing the $|G|$-th roots of unity and $\mathfrak{o}$ is the ring of $\mathfrak{p}$-integral elements. For $\sigma \in \mathfrak{o}$, σ^* indicates the image of σ by the natural map $\mathfrak{o} \to \mathfrak{o}/\mathfrak{p}\mathfrak{o}$. First we shall determine a geneator of $\mathfrak{N}/\mathfrak{L}$ over kG. If $[G:H]$ is prime to p, then $\mathfrak{N} = \mathfrak{L}$. If $[G:H] = p$ and the defect group of a block B is contained in H, then we have $\mathfrak{N}B = \mathfrak{L}B$. Hence we may consider only those blocks whose defect groups are not in H.

Lemma 3.1. *Suppose $[G:H]=p$. Let B be a block, D its defect group and let ψ be the linear character which defines the block B. If $D \not\subset H$, then there exists an element x in G but not in H such that $\psi([x]) \neq 0$.*

Proof. Let y be a p-regular element such that D is a defect group of y and $\psi([y]) \neq 0$. Since $[G:H] = p$, y is contained in H. Let ξ be an irreducible character of height 0 in B. Then $\psi([y]) = \left(\frac{|G|}{n(y)} \frac{\xi(y)}{z}\right)^* = \left(\frac{|G|}{n(y) \cdot z}\right)^* \xi(y)^* \neq 0$, where $n(y)$ is the order of the centralizer of y in G and z is the degree of ξ. Since $D \not\subset H$, there exists an element $a \in D$ and $a \notin H$. Then we have $N_G(ay) = N_G(a) \cap N_G(y) \supset D$, since D is abelian. Hence D is a defect group of ay. Thus $\frac{|G|}{n(ay) \cdot z}$ is also a $\mathfrak{p}$-integral element and $\left(\frac{|G|}{n(ay) \cdot z}\right)^* \neq 0$. On the other hand, since $ay = ya$ and a is a p-element, we have $\xi(ay)^* = \xi(y)^* \neq 0$. Thus $\psi([ay]) = \left(\frac{|G|}{n(ay) \cdot z}\right)^* \xi(ay)^* \neq 0$. This completes the proof.

Let $B_1, \cdots, B_s$ be the blocks of kG and $\delta_1, \cdots, \delta_s$ the block idempotents respectively. Let ψ_i be the linear character which defines the block B_i. Then $\{\psi_1 \cdots \psi_s\}$ is the set of the linear characters on the center of kG. Since the center is a commutative k-algebra, its radical is the intersection of the kernels of ψ_i's. In particular, for any element z of the center, $(z - \psi_i(z))\delta_i$ is an element in $\mathfrak{N}$.

Proposition 3.2. *Suppose $[G:H]=p$ and the defect group of the block B_i is not contained in H. Let x be any element in G such that $x \notin H$ and $\psi_i([x]) \neq 0$. Then we have $\mathfrak{N}B = \mathfrak{L}B + kG \cdot ([x] - \psi_i([x]))\delta_i$.*

Proof. we put $\delta=\delta_i$ and $\psi=\psi_i$ for convenience Let $\delta=\sum e_j$ be a decomposition into the sum of primitive idempotents. We may assume each e_j is in kH by Remark 1. Let $e=e_j$ be arbitrary and fixed. Since x is not in H, we may put $x=av$, where $a^{p-1}\notin H$ and $v\in H$. Then we have $([x]-\psi([x]))^{p-1}\delta e =a^{p-1}z_1+a^{p-2}z_2+\cdots+az_{p-1}+\psi([x])^{p-1}e$, where $z_i\in kH$. The right hand is not contained in $\mathfrak{L}e=a^{p-1}\mathfrak{N}e+\oplus a^{p-2}\mathfrak{N}e\oplus\cdots\oplus\mathfrak{N}e$, since $\psi([x])\neq 0$. Hence we have a sequence

$$A\bar{e}\supsetneqq([x]-\psi([x]))A\bar{e}\supsetneqq([x]-\psi([x]))^2A\bar{e}\supsetneqq\cdots\supsetneqq([x]-\psi([x]))^{p-1}A\bar{e}\supsetneqq 0\,.$$

However, since $A\bar{e}$ has p composition factors, $([x]-\psi([x]))\ A\bar{e}$ must be maximal, that is $([x]-\psi([x]))A\bar{e}=(\mathfrak{N}/\mathfrak{L})\bar{e}$. Therefore we have $kG\cdot([x]-\psi([x]))e +\mathfrak{L}e=\mathfrak{N}e$. and thus $\mathfrak{N}B=\mathfrak{L}B+kG([x]-\psi([x]))\delta$, since e is arbitrary. This completes the proof.

Corollary 3.3. *We put $c=\sum([x_i]-\psi_i([x_i]))\delta_i$, where δ_i ranges over all the block idempotents of the blocks whose defect groups are not is H and x_i is any element of G such that $x_i\notin H$ and $\psi_i([x_i])\neq 0$. Then we have $\mathfrak{N}B=\mathfrak{L}B+(kG)c$.*

From the above Corollary we have the following Theorem.

Theorem 2. *Suppose G is p-solvable and S_p is abelian. Consider a normal sequence,*

$$G=H_0\supset G_1\supset H_1\supset G_2\supset H_2\supset\ \cdots\ \supset G_n\supset H_n\supset G_{n+1}=\{1\}\,,$$

where G_{i+1} is the minimal normal subgroup of H_i such that $[H_i\colon G_{i+1}]$ is prime to p and H_i is a normal subgroup of G_i of index p (possibly $H_i=G_{i+1}$). Then there exists a central element c_i in kG_i such that $\{c_i\}_{i=1}^n$ generate $\mathfrak{N}$ over kG. In particular $\{\mathfrak{S}_i\}_{i=1}^n$ generates $\mathfrak{N}$ over kG, where $\mathfrak{S}_i$ is the radical of the center of kG_i.

4. The case where S_p is cyclic.

In this section we assume S_p is cyclic and we shall improve the main results of the preceeding sections. Let θ be a generator of S_p and $U=N_G(S_p)/C_G(S_p)$.

Lemma 4.1. *U is a cyclic group. Let t be the order of U and σ in $N_G(S_p)$ correspond to a generating element of U. Then t divides $p-1$ and $\sigma^{-1}\theta\sigma=\theta^l$. The conjugate class containing θ in $N_G(S_p)$ consists of θ, θ^l, $\cdots$, $\theta^{l^{t-1}}$. Furthermore, let ϕ be the Brauer homomorphism of the center of kG into the center of $kN_G(S_p)$. Then we have $\phi([\theta])=\theta+\theta^l+\cdots+\theta^{l^{t-1}}$.*

Proof. The first half is well known. We omit the proofs. Since the defect group of θ is S_p, we know $\phi([\theta])$ is the sum of the elements in the conjugate class containing θ. Thus we have $\phi([\theta])=\theta+\theta^l+\cdots+\theta^{l^{t-1}}$.

REMARK 2. Though the proof is easy, the following fact is worth while remarking. By the definition t is the order of l *mod* p^n. However, since t is prime to p, t is also the order of l *mod* p.

Lemma 4.2. *If G has a normal subgroup of index p, then G has a normal p-Sylow complement.*

Proof. By Burnside's Theorem, it suffices to show that $N_G(S_p)=C_G(S_p)$. We use the same notation as that of Lemma 4.1. The transfer map $G\to S_p$ induces an isomorphism $G/T\simeq Z\cap S_p$, where Z is the center of $N_G(S_p)$ and T is the minimal normal subgroup of G such that G/T is abelian p-group ([8]). We have $G/T\neq\{1\}$ by the assumption, hence there exists θ^k in S_p, $0<k<p^n$ and θ^k commutes with σ. Since $\sigma^{-1}\theta\sigma=\theta^l$, we have $\sigma^{-1}\theta^k\sigma=\theta^k=\theta^{lk}$. It follows that p^n divides $(l-1)k$. Since $p^n\not| k$, $(l-1)$ is divisible by a suitable power p^{n_0} $(n_0>0)$. Thus we have $l\equiv 1$ *mod* p. Hence we have $t=1$ by Remark 2. This completes the proof.

Lemma 4.3. *Let l and t be integers such that t is the order of l mod p. We assume l is greater than p. Let $F(X)=X+X^l+X^{l^2}+\cdots+X^{l^{t-1}}-t$ be a polynomial over k. Then we have $F(X)=(X-1)^tG(X)$, where $G(X)$ is a polynomial over k and $G(1)\neq 0$.*

Proof. It suffices to show that $F(1)=F'(1)=\cdots=F^{(t-1)}(1)=0$ and $F^{(t)}(1)\neq 0$, since $1\leqslant t<p$ (the characteristic of k). It follows directly that $F(1)=0$ and $F^{(v)}(1)=\sum_{i=1}^{t-1}l^i(l^i-1)\cdots(l^i-v+1)$. We put $Y(Y-1)\cdots(Y-v+1)=\sum_{j=1}^{v}a_jY^j$, then we have $\sum a_j=0$ and $F^{(v)}(1)=\sum_{j=1}^{v}a_j(\sum_{m=1}^{t-1}l^{mj})$. If $j\leqslant v<t$, then $\sum_{m=1}^{t-1}l^{mj}=\dfrac{l^j(l^{j(t-1)}-1)}{l^j-1}=-1$. Thus $F^{(v)}(1)=-\sum a_j=0$. For $v=t$, we have $F^{(t)}(1)=\sum_{j=1}^{t-1}(-a_j)+(t-1)=t\neq 0$. This completes the proof.

Now let $\delta_1\cdots\delta_r$ be the block idempotents of the blocks of full defect. It is clear that $\psi_i([\theta])=h$ in k, where h is the number of the elements in the conjugate class containing θ in G. In particular, we have $\psi_i([\theta])\neq 0$.

Proposition 4.4. *Let t be the order of U and $f=\dfrac{p^n-1}{t}$. Then for some i $(1\leqslant i\leqslant r)$, we have $([\theta]-h)^f\delta_i\neq 0$. In particular, we have $\mathfrak{N}^f\neq 0$.*

Proof. Since $[G:N_G(S_p)]\equiv 1$ *mod* p, we have $h=[G:N_G(S_p)]$ $[N_G(S_p):C_G(S_p)]\equiv t$ *mod* p. Hence $\phi(([\theta]-h)^f\delta_i)=(\theta+\theta^l+\cdots+\theta^{t-1}-t)^f\phi(\delta_i)$. As is well known, $\phi(\delta_i)$ is not zero and a block idempotent in $kN_G(S_p)$ and furthermore $\sum\phi(\delta_i)=1$. Hence it is sufficient to show that $(\theta+\theta^l+\cdots+\theta^{l^{i-1}}-t)^f\neq 0$. By Remark 2, t is also the order of l *mod* p. We use Lemma 4.3 replacing l by $l+p^n$ if necessary and we get $F(\theta)=\theta+\theta^l+\cdots\theta^{l^{t-1}}-t=(\theta-1)^tG(\theta)$. Furthermore $G(1)\neq 0$ means that the sum of the coefficients of $G(X)$ is not zero. Hence

$G(\theta)$ is a unit in kS_p (see [5] or pp. 189 [2]) Thus we have $F(\theta)^f=(\theta-1)^{p^n-1}G(\theta)^f\neq 0$.

Corollary 4.5. *If S_p has a normal complement in G, we have $([\theta]-h)^{p^n-1}\delta_i\neq 0$, for all i $(i\leqslant i\leqslant r)$.*

Proof. It follows from the assumption that $t=1$ and $f=p^n-1$. Hence we need to show only that $F(\theta)^{p^n-1}\phi(\delta_i)\neq 0$ for all i $(l\leqslant i\leqslant r)$. Now suppose $F(\theta)^{p^n-1}\delta_i'=0$ for some i, where $\delta_i'=\phi(\delta_i)$. Then we have $(\theta-1)^{p^n-1}\delta_i'=0$, since $G(\theta)$ is a unit. From this it follows that $\theta^{p^n-1}\delta_i'+a_1\theta^{p^n-2}\delta_i'+\cdots+a_*\theta\delta_i'=-\delta_i'$, where $a_i\in k$. However this is a contradiction, since all the elements of G which appear in the summation in the left hand side are p-irregular and the right hand side is a sum of p-regular elements. This completes the proof.

Lemma 4.6. *Let $\mathfrak{S}$ be the radical of the center of kG. If S_p has a normal complement in G, we have $\mathfrak{R}=kG\cdot\mathfrak{S}$.*

Proof. There exists a normal subgroup H of index p. Since S_p has only one subgroup of order p^v for $0\leqslant v\leqslant n$, all the defect groups of the blocks of defect smaller than n are contained in H. Hence by Corollary 3.3, we have $\mathfrak{R}=\mathfrak{L}+kG\cdot([\theta]-h)\rho$, where ρ is the sum of the block idempotents of the blocks of full defect. Let T be the normal complement. There exists a normal sequence,

$$G=G_0\supset G_1\supset G_2\supset\cdots\supset G_{n-1}\supset G_n=T\,,$$

where G_{k+1} is the normal subgroup of G_k of index p. G_k is unique and even normal in G. It is clear that θ^{p^k} generates a Sylow p-subgroup of G_k and the conjugate class containing θ^{p^k} in G_k is also the conjugate class in G. We denote by h_k the number of the elements in the class. Also it is clear that the sum, say ρ_k, of all the block idempotents of the blocks of full defect in kG_k is central in kG. Now, replacing G and H by G_k and G_{k+1} respectively, we have $\mathfrak{R}_k=\mathfrak{L}_k+kG([\theta^{p^k}]-h_k)\rho_k$, where $\mathfrak{R}_k$ is the radical of kG_k, $\mathfrak{L}_k=kG\cdot\mathfrak{R}_{k+1}$ and $\mathfrak{R}_{k+1}$ is the radical of kG_{k+1}. Thus $\{([\theta^{p^k}]-h_k)\rho_k\}_{k=0}^{n-1}$ generate $\mathfrak{R}$ over kG and they are central. This completes the proof.

Theorem 3. *Let G be a p-solvable group with a cyclic Sylow p-group. Then we have*

(1) *$\mathfrak{R}=kG\cdot\mathfrak{S}_T$, where $\mathfrak{S}_T$ is the radical of the center of kT and T is the minimal normal subgroup such that $[G:T]$ is prime to p.*

(2) *Let d be the defect of a certain block of kG. Then there exists a block of defect d, say B such that p^d is the smallest integer for which $(\mathfrak{R}B)^{p^d}=0$. This holds for any block of defect d, if G has a normal p-Sylow complement.*

Proof.

(1) Let $\mathfrak{R}$ be the radical of kT. Since $[G\colon T]$ is prime to p, we have $\mathfrak{N}=\mathfrak{L}=kG\cdot\mathfrak{R}$. Since G is p-solvable, T has a normal subgroup of index p. Then T has a normal p-Sylow complement by Lemma 4.2. Thus we have $\mathfrak{N}=kG\cdot\mathfrak{R}=kG(kT\cdot\mathfrak{S}_T)=kG\cdot\mathfrak{S}_T$ by Lemma 4.6.

(2) We prove by induction on the order of G. First, we prove the second statement. We have only to show $(\mathfrak{N}B)^{p^d-1}\neq 0$ for any block B of defect d. If $d=n$, we have already proved this in Corollary 4.5. Hence we may assume $d<n$. Let H be a normal subgroup of index p. H also has a normal p-Sylow complement. Let δ be the block idempotent of B and $\delta=\sum_{i=1}^{m}\eta_i$, where η_i is a block idempotent in kH. Since $d<n$, the defect group of B is contained in H. Therefore we have $\mathfrak{N}B=\mathfrak{L}B=\mathfrak{R}B$ and $d=d_i$ for all i $(1\leqslant i\leqslant m)$, d_i being the defect of the block corresponding to η_i in kH. Thus we have $\mathfrak{N}^{p^d-1}\delta=kG\cdot\oplus\sum_{i=1}^{m}\mathfrak{R}^{p^d-1}\eta_i\neq 0$ by the induction hypothesis. Now we prove the first part. If G has a normal subgroup of index p, our statement is obvious by Lemma 4.2 and the second part just proved. Thus we may assume there exists a proper normal subgroup of index prime to p. From the 1–1 correspondence $\mathfrak{J}_i\leftrightarrow\tilde{\mathfrak{J}}_i$ and Lemma 2.7, it follows that there exists a block of defect d in kH. Let $\tilde{\mathfrak{J}}_i$ be the set which contains a block $\tilde{B}$ such that $(\mathfrak{R}\tilde{B})^{p^d-1}\neq 0$. Then there exists a primitive idempotent e in $\tilde{B}$ such that $\mathfrak{R}^{p^d-1}e\neq 0$. Let $(kG)e=\oplus\sum_j(kG)f_j$ be a sum of principal indecomposable modules of kG. Each $(kG)f_j$ belogs to some block in $\mathfrak{J}_i$. We have $\oplus\sum_j\mathfrak{N}^{p^d-1}f_j=\mathfrak{N}^{p^d-1}e=kG\cdot\mathfrak{R}^{p^d-1}e\neq 0$. Hence there exists some f_j such that $\mathfrak{N}^{p^d-1}f_j\neq 0$. This completes the proof.

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