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THE FIRST COHOMOLOGY GROUP OF A COMMUTATIVE RING

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Given a commutative ring K, let $\mathcal{U}(K)$ be the group of units of K. If G is an arbitrary group, represented by ring automorphisms of K, let $D(K, G)$ be the trivial crossed product of K with G, and k be the subring of K consisting of all elements left fixed by G.

This note is devoted to perform a description of the first cohomology group $H^1(G, \mathcal{U}(K))$ as the group of ring automorphisms of $D(K, G)$ leaving K pointwise fixed, modulo inner ones. This result holds under the assumption that K is its own centralizer in $D(K, G)$. Traducing this condition in terms of the G-action on K, the description of $H^1(G, \mathcal{U}(K))$ mentioned above provides, in the Galois case, the first four terms of the exact sequence associated to a Galois extension [2, 5.5, p. 31].

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Throughout this paper, all rings will be assumed to have units, and all modules will be unitary. A subring will contain the unit element of the ring, and a morphism of rings will preserve unit elements. The general notations are given below.

Let k be a commutative ring, and K be any k-algebra. A (left) K-module A, together with a operation $(x, y) \mapsto x.y$, which is associative, with unit element, and $k.1$-bilinear, will be called, for shortness, a *k-algebra over K*. The definition of *morphism of K-algebra over K* is obvious.

If G is a group and K is a commutative ring, given a representation of G by ring automorphisms of K, let $D(K, G)=D$ denote the *trivial crossed product* of K with G. This means that D is the free K-module $K^{(G)}$ generated by G, with multiplication defined by the formula

$$(aw_s)(bw_t) = as(b)w_{st},\ a,\ b\in K \quad \text{and} \quad s,\ t\in G,$$

where $(w_s)_{s\in G}$ is the canonical basis of $K^{(G)}$. If k is the subring ${}^GK$ of K consisting of all elements left fixed by G, and $w : G\to D$ is the map $s\mapsto w_s$, then D is a k-algebra over K and w is a multiplicative morphism. The pair (D, w)

is characterized by the following universal mapping property: if A is any k-algebra over K and $m: G \rightarrow A$ is any multiplicative morphism, there exists a unique morphism of k-algebras over K $\varphi: D \rightarrow A$ such that $\varphi w = m$ (φ is defined as the K-module morphism $w_s \mapsto m(s)$, $s \in G$). Note that the ring structure of D has w_1 for its identity element, and that the map $x \mapsto x w_1$ imbeds K as a subring of D.

Let k be a commutative ring, and A be a k-algebra. If M is a two-sided A-module, ${}^A M$ will denote the k-submodule $\{x \in M;\ ax = xa,\ a \in A\}$. Note that if A is a subalgebra of a k-algebra B, then ${}^A B$ is just the centralizer of A in B; in particular, ${}^A A$ is the center of A. If α and β are k-algebra endomor phisms of A, the two-sided A-module with additive group equal to the additive group of M and actions $axb = \alpha(a) x \beta(b)$, a, $b \in A$ and $x \in M$, will be denoted ${}_\alpha M_\beta$. Finally, set $\mathcal{J}_\alpha(M) = {}^A{}_\alpha M_1$.

Henceforth, unless otherwise specified, K is a commutative ring, G is a group represented by ring automorphisms of K, and $k = {}^G K$.

The following result will be needed for the anounced description of the first cohomology group.

(1) **Proposition.** *K is its own centralizer in D* (i. e. ${}^K D = K$) *if and only if $\mathcal{J}_s(K) = 0$, for $s \in G$ and $s \neq 1$.*

Proof. Explicitly, $\mathcal{J}_s(K) = \{a \in K;\ as(x) = xa,\ x \in K\}$.

Sufficiency. Let $a \in D$, $a = \sum_{s \in G} a_s w_s$, commute with all elements of K. Since $(w_s)_{s \in G}$ is a basis of D, as a K-module, and $(s(x) w_1) w_s = s(x) w_s = w_s (x w_1)$, for $s \in G$ and $x \in K$, from the relation

$$a(x w_1) = (x w_1) a,\ x \in K,$$

it follows that

$$a_s s(x) = x a_s,\ s \in G \quad \text{and} \quad x \in K.$$

Therefore, $a_s \in \mathcal{J}_s(K)$, for all $s \in G$, and so, $a = a_1 w_1$.

Necessity. Given $s \in G$, if $a \in \mathcal{J}_s(K)$, then

$$(a w_s)(x w_1) = (x w_1)(a w_s),\ x \in K,$$

and so, $a w_s \in {}^K D = K$. Hence, if $s \neq 1$, $a = 0$.

In the rest of the paper, it will be assumed that $\mathcal{J}_s(K) = 0$, for $s \in G$ and $s \neq 1$. Then,

(2) **Corollary.** *D is a central k-algebra.*

Proof. If a is in the center of D, then $a = c w_1$, for some $c \in K$, because K is its own centralizer in D. But,

$$w_s(cw_1) = (cw_1)w_s,\ s\in G,$$

and so,

$$s(c) = c,\ s\in G.$$

Since c is left fixed by all elements of G, $c\in k$.

If k is a commutative ring, K is a k-algebra and A is a k-algebra over K, $\mathcal{O}_K(A)$ will denote the group of (k-algebra over K) automorphisms of A, and $\mathcal{I}_K(A)$ will denote the subgroup of inner automorphisms of A. If u is a unit of A, which defines an inner automorphism σ of A, an element $x\in A$ is left fixed by σ if and only if u commutes with x. Thus, $u\in {}^K A$, and it is proved the following

(3) Corollary. *If σ_u is the inner automorphism of D defined by a unit u of K, $x\mapsto uxu^{-1}$, then $\mathcal{I}_K(D)=\{\sigma_u;\ u\in\mathcal{U}(K)\}$.*

Passing to the cohomological context, if $f\in Z^1(G,\ \mathcal{U}(K))$, let $\rho(f)$ be the endomorphism of the K-module structure of D defined by

$$\rho(f)(w_s) = f(s)w_s,\ s\in G.$$

Since $(f(s)w_s)_{s\in G}$ is a basis of D, because $f(s)\in\mathcal{U}(K)$, for all $s\in G$, $\rho(f)$ is a K-module automorphism of D. The fact that f is a cocycle translates into the formula

$$\rho(f)(w_s w_t) = \rho(f)(w_s)\rho(f)(w_t),\ s,\ t\in G,$$

and so, $\rho(f)\in\mathcal{O}_K(D)$. An easy computation, using the basis $(w_s)_{s\in G}$, shows that the map $\rho : Z^1(G,\ \mathcal{U}(K))\to\mathcal{O}_K(D)$ is an injective group morphism. Moreover, given $\alpha\in\mathcal{O}_K(D)$, from the relation

$$w_s(xw_1) = (s(x)w_1)w_s,\ s\in G \text{ and } x\in K,$$

it follows that

$$(\alpha(w_s)w_s^{-1})(xw_1) = (xw_1)(\alpha(w_s)w_s^{-1}),\ s\in G \text{ and } x\in K.$$

Hence, $\alpha(w_s)w_s^{-1}$ commutes with K (elementwise), for all $s\in G$. Now, applying (1), for each $s\in G$ there is an element $f(s)\in K$, uniquely determined, such that $\alpha(w_s)w_s^{-1}=f(s)w_1$; but $f(s)\in\mathcal{U}(K)$, since $\alpha(w_s)w_s^{-1}\in\mathcal{U}(D)$. A simple calculation shows that f is a cocycle, and so, $\rho(f)=\alpha$. Therefore, ρ is a surjective map. Finally, given $u\in\mathcal{U}(K)$, if f_u is the coboundary defined by u (i. e. $f_u(s)=us(u)^{-1}$, $s\in G$), then $\rho(f_u)=\sigma_u$. Hence, applying (3),

$$\rho(B^1(G,\ \mathcal{U}(K))) = \mathcal{I}_K(D),$$

and it is proved the following

(4) Theorem. *The map* $\rho : Z^1(G, \mathcal{U}(K)) \to \mathcal{O}_K(D)$ *is a group isomorphism, and* $\rho(B^1(G, \mathcal{U}(K))) = \mathcal{I}_K(D)$.

(5) Corollary. $\mathcal{O}_K(D)$ *is an abelian group, and* $H^1(G, \mathcal{U}(K)) \cong \mathcal{O}_K(D)/\mathcal{I}_K(D)$.

(6) Corollary. *If* $\mathcal{U}(K) \to \mathcal{O}_K(D)$ *and* $\mathcal{U}(K) \to B^1(G, \mathcal{U}(K))$ *are the maps* $u \mapsto \sigma_u$ *and* $u \mapsto f_u$, *respectively, then the following diagram is commutative and exact:*

$$\begin{tikzcd}
 & 1 \arrow[d] & & \\
 & \mathcal{U}(k) \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{U}(k) \arrow[r] & \mathcal{U}(K) \arrow[r] \arrow[d] & \mathcal{O}_K(D) \arrow[r] \arrow[d, "\rho^{-1}"] & \mathcal{O}_K(D)/\mathcal{I}_K(D) \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & B^1(G, \mathcal{U}(K)) \arrow[r] \arrow[d] & Z^1(G, \mathcal{U}(K)) \arrow[r] \arrow[d] & H^1(G, \mathcal{U}(K)) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 & 1 &
\end{tikzcd}$$

Hence, the following diagram is commutative and exact:

$$\begin{tikzcd}
 & & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{U}(k) \arrow[r] \arrow[d, equal] & \mathcal{U}(K) \arrow[r] \arrow[d, equal] & Z^1(G, \mathcal{U}(K)) \arrow[r] \arrow[d, "\rho"] & H^1(G, \mathcal{U}(K)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{U}(k) \arrow[r] & \mathcal{U}(K) \arrow[r] & \mathcal{O}_K(D) \arrow[r] \arrow[d] & \mathcal{O}_K(D)/\mathcal{I}_K(D) \arrow[r] \arrow[d] & 1 \\
 & & & 1 & 1 &
\end{tikzcd}$$

Finally, it will be shown that the description of $H^1(G, \mathcal{U}(K))$ given in (5) yields, in the Galois case, the first four terms of the exact sequence of a Galois extension [2, 5.5, p. 31]. The key result is the Rosenberg-Zelinsky generalization of the Skolem-Noether theorem [4, 3.7, p. 1112]:

Let k *be a commutative ring, and* $\mathcal{P}(k)$ *be the projective class group of* k. *If* A *is an Azumaya* (i. e. *central, separable*) k*-algebra, the map* $\Lambda : \mathcal{O}_k(A) \to \mathcal{P}(k)$, $\alpha \mapsto \mathrm{cls}(\mathcal{J}_\alpha(A))$, *is a group morphism,* $\mathrm{Ker}\ \Lambda = \mathcal{I}_k(A)$, *and* $\mathrm{Im}\ \Lambda = \{\mathrm{cls}(P);\ A \otimes_k P \cong A$, as left A*-modules*$\}$.

Since $\mathcal{O}_K(D) \cap \mathcal{I}_k(D) = \mathcal{I}_K(D)$, the inclusion of $\mathcal{O}_K(D)$ in $\mathcal{O}_k(D)$ induces a group monomorphism $\mathcal{O}_K(D)/\mathcal{I}_K(D) \to \mathcal{O}_k(D)/\mathcal{I}_k(D)$, and so, (5) provides a group monomorphism $H^1(G, \mathcal{U}(K)) \to \mathcal{O}_k(D)/\mathcal{I}_k(D)$.

Now, it will be assumed that G is a finite group, and that K is a Galois extension of k (relative to G), in the sense of Auslander-Goldman [1, Appendix, p. 396]. The last requirement holds if and only if K is a separable k-algebra, and $\mathcal{J}_s(K)=0$, for $s\in G$ and $s\neq 1$ (that is, the general hypothesis of this paper) [3, prop. 2, p. 344]. The assumption that K is a Galois extension of k guarantees that D is an Azumaya k-algebra (cf. (2) and [1, 5.1, p. 380]), and so, the Rosenberg-Zelinsky theorem yields a group monomorphism $\lambda : H^1(G, \mathcal{U}(K)) \to \mathcal{P}(k), f \mapsto \mathrm{cls}\,(\mathcal{J}_{\rho(f)}(D))$. If $\varepsilon : \mathcal{P}(k) \to H^0(G, \mathcal{P}(K))$ is the group morphism $\mathrm{cls}\,(P) \mapsto \mathrm{cls}\,(K\otimes_k P)$, then $\mathrm{Im}\lambda = \mathrm{Ker}\,\varepsilon = \{\mathrm{cls}(P);\ K\otimes_k P \simeq K$, as K-modules$\}$.

$\subseteq$. Given $\alpha\in\mathcal{O}_k(D)$, a general result [5, 3.4, pp. 1111–2] guarantees that the map $\mu : D\otimes_k \mathcal{J}_\alpha(D) \to {}_\alpha D_1$, $x\otimes a \mapsto a.x$, is an isomorphism of two-sided D-modules. Supposing that $\alpha\in\mathcal{O}_K(D)$, it follows that $\mathcal{J}_\alpha(D)\subseteq {}^K D = K$. Thus, if $\iota : K\otimes_k \mathcal{J}_\alpha(D) \to D\otimes_k \mathcal{J}_\alpha(D)$ is the K-module morphism deduced from the imbedding of K into D, $\mu\iota$ induces a K-module morphism $\nu : K\otimes_k \mathcal{J}_\alpha(D) \to K$. It is clear that ν is a monomorphism, because μ and ι are monomorphisms. Moreover, given $x\in K$, it can be written in the form $x=\sum_{1\leqslant i\leqslant n} a_i x_i$, with $a_i\in\mathcal{J}_\alpha(D)$ and $x_i\in D$, $1\leqslant i\leqslant n$, since μ is an epimorphism (that is, $D=\mathcal{J}_\alpha(D).\,D$). Hence, if $x_i=\sum_{s\in G} x_{is}w_s$, with $x_{is}\in K$, $1\leqslant i\leqslant n$ and $s\in G$, then $x=\sum_{1\leqslant i\leqslant n} a_i x_{i1}\in\mathcal{J}_\alpha(D).\,K$; and so, ν is an epimorphism.

$\supseteq$. Let P be a finitely generated, projective k-module, of rank one, such that $K\otimes_k P\cong K$, as K-modules. If f is such an isomorphism, the map $\varphi : D\otimes_k P \to D$, defined by $\varphi(w_s\otimes x)=f(1\otimes x)w_s$, $s\in G$ and $x\in P$, is a K-module isomorphism (note that $w_s\otimes x$, with $s\in G$ and $x\in P$, are generators of $D\otimes_k P$, as a K-module). Moreover, since $\varphi((w_s\otimes x)w_t)=f(1\otimes x)w_{st}=\varphi(w_s\otimes x)w_t$, s, $t\in G$ and $x\in P$, φ is an isomorphism of right D-modules. Therefore, the argument of [4, 3.7 (proof), p. 1113] provides a k-algebra automorphism of D, say α, as follows: If g is a free generator of $D\otimes_k P$, as a right D-module, for each $x\in D$ there is an element $\alpha(x)\in D$, uniquely determined, such that $xg=g\alpha(x)$. It can be assumed that $g\in\iota(K\otimes_k P)$; for example, take $g=\varphi^{-1}(w_1)$, and apply the commutativity of the diagram

$$\begin{tikzcd} K\otimes_k P \arrow[r, "f"] \arrow[d, "\iota"'] & K \arrow[d] \\ D\otimes_k P \arrow[r, "\varphi"'] & D \end{tikzcd}$$

Thus, $xg=gx$, for all $x\in K$, since K is contained in the centralizer of $\iota(K\otimes_k P)$ in $D\otimes_k P$. Hence, $\alpha\in\mathcal{O}_K(D)$, and it is proved the following

(7) Proposition. *Let G be a finite group. If K is a separable k-algebra (equivalently, K is a Galois extension of k, relative to G), then the sequence*

$$1 \longrightarrow H^1(G, \mathcal{U}(K)) \xrightarrow{\lambda} \mathcal{P}(k) \xrightarrow{\varepsilon} H^0(G, \mathcal{P}(K))$$

is exact.

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References

[1] M. Auslander and O. Goldman: *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[2] S.U. Chase, D.K. Harrison and A. Rosenberg: *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. No. 52 (1965), 15–33.

[3] M. Harada: *Supplementary results on Galois extensions*, Osaka J. Math. **2** (1965), 343–350.

[4] A. Rosenberg and D. Zelinsky: *Automorphisms of separable algebras*, Pacific J. Math. **11** (1961), 1109–1117.