

Title	The first cohomology group of a commutative ring
Author(s)	Martínez, Juan José
Citation	Osaka Journal of Mathematics. 1972, 9(3), p. 415-420
Version Type	VoR
URL	<a href="https://doi.org/10.18910/8381">https://doi.org/10.18910/8381</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

Martinez, J. J.  
Osaka J. Math.  
9 (1972), 415-420

## THE FIRST COHOMOLOGY GROUP OF A COMMUTATIVE RING

JUAN JOSE MARTINEZ

(Received October 1, 1971)

Given a commutative ring  $K$ , let  $\mathcal{U}(K)$  be the group of units of  $K$ . If  $G$  is an arbitrary group, represented by ring automorphisms of  $K$ , let  $D(K, G)$  be the trivial crossed product of  $K$  with  $G$ , and  $k$  be the subring of  $K$  consisting of all elements left fixed by  $G$ .

This note is devoted to perform a description of the first cohomology group  $H^1(G, \mathcal{U}(K))$  as the group of ring automorphisms of  $D(K, G)$  leaving  $K$  pointwise fixed, modulo inner ones. This result holds under the assumption that  $K$  is its own centralizer in  $D(K, G)$ . Traducing this condition in terms of the  $G$ -action on  $K$ , the description of  $H^1(G, \mathcal{U}(K))$  mentioned above provides, in the Galois case, the first four terms of the exact sequence associated to a Galois extension [2, 5.5, p. 31].

The author would like to thank Professor O. E. Villamayor for many useful conversations about the subject.

Throughout this paper, all rings will be assumed to have units, and all modules will be unitary. A subring will contain the unit element of the ring, and a morphism of rings will preserve unit elements. The general notations are given below.

Let  $k$  be a commutative ring, and  $K$  be any  $k$ -algebra. A (left)  $K$ -module  $A$ , together with a operation  $(x, y) \mapsto x.y$ , which is associative, with unit element, and  $k$ -1-bilinear, will be called, for shortness, a  $k$ -algebra over  $K$ . The definition of *morphism of  $K$ -algebra over  $K$*  is obvious.

If  $G$  is a group and  $K$  is a commutative ring, given a representation of  $G$  by ring automorphisms of  $K$ , let  $D(K, G) = D$  denote the *trivial crossed product* of  $K$  with  $G$ . This means that  $D$  is the free  $K$ -module  $K^{(G)}$  generated by  $G$ , with multiplication defined by the formula

$$(aw_s)(bw_t) = as(b)w_{st}, \quad a, b \in K \quad \text{and} \quad s, t \in G,$$

where  $(w_s)_{s \in G}$  is the canonical basis of  $K^{(G)}$ . If  $k$  is the subring  ${}^G K$  of  $K$  consisting of all elements left fixed by  $G$ , and  $w : G \rightarrow D$  is the map  $s \mapsto w_s$ , then  $D$  is a  $k$ -algebra over  $K$  and  $w$  is a multiplicative morphism. The pair  $(D, w)$

is characterized by the following universal mapping property: if  $A$  is any  $k$ -algebra over  $K$  and  $m : G \rightarrow A$  is any multiplicative morphism, there exists a unique morphism of  $k$ -algebras over  $K$   $\varphi : D \rightarrow A$  such that  $\varphi w = m$  ( $\varphi$  is defined as the  $K$ -module morphism  $w_s \mapsto m(s)$ ,  $s \in G$ ). Note that the ring structure of  $D$  has  $w_1$  for its identity element, and that the map  $x \mapsto xw_1$  imbeds  $K$  as a subring of  $D$ .

Let  $k$  be a commutative ring, and  $A$  be a  $k$ -algebra. If  $M$  is a two-sided  $A$ -module,  ${}^A M$  will denote the  $k$ -submodule  $\{x \in M; ax = xa, a \in A\}$ . Note that if  $A$  is a subalgebra of a  $k$ -algebra  $B$ , then  ${}^A B$  is just the centralizer of  $A$  in  $B$ ; in particular,  ${}^A A$  is the center of  $A$ . If  $\alpha$  and  $\beta$  are  $k$ -algebra endomorphisms of  $A$ , the two-sided  $A$ -module with additive group equal to the additive group of  $M$  and actions  $axb = \alpha(a)x\beta(b)$ ,  $a, b \in A$  and  $x \in M$ , will be denoted  ${}_{\alpha}M_{\beta}$ . Finally, set  $\mathcal{J}_{\alpha}(M) = {}^A M_1$ .

Henceforth, unless otherwise specified,  $K$  is a commutative ring,  $G$  is a group represented by ring automorphisms of  $K$ , and  $k = {}^G K$ .

The following result will be needed for the announced description of the first cohomology group.

**(1) Proposition.**  $K$  is its own centralizer in  $D$  (i. e.  ${}^K D = K$ ) if and only if  $\mathcal{J}_s(K) = 0$ , for  $s \in G$  and  $s \neq 1$ .

Proof. Explicitly,  $\mathcal{J}_s(K) = \{a \in K; as(x) = xa, x \in K\}$ .

Sufficiency. Let  $a \in D$ ,  $a = \sum_{s \in G} a_s w_s$ , commute with all elements of  $K$ .

Since  $(w_s)_{s \in G}$  is a basis of  $D$ , as a  $K$ -module, and  $(s(x)w_1)w_s = s(x)w_s = w_s(xw_1)$ , for  $s \in G$  and  $x \in K$ , from the relation

$$a(xw_1) = (xw_1)a, x \in K,$$

it follows that

$$a_s s(x) = xa_s, s \in G \text{ and } x \in K.$$

Therefore,  $a_s \in \mathcal{J}_s(K)$ , for all  $s \in G$ , and so,  $a = a_1 w_1$ .

Necessity. Given  $s \in G$ , if  $a \in \mathcal{J}_s(K)$ , then

$$(aw_s)(xw_1) = (xw_1)(aw_s), x \in K,$$

and so,  $aw_s \in {}^K D = K$ . Hence, if  $s \neq 1$ ,  $a = 0$ .

In the rest of the paper, it will be assumed that  $\mathcal{J}_s(K) = 0$ , for  $s \in G$  and  $s \neq 1$ . Then,

**(2) Corollary.**  $D$  is a central  $k$ -algebra.

Proof. If  $a$  is in the center of  $D$ , then  $a = cw_1$ , for some  $c \in K$ , because  $K$  is its own centralizer in  $D$ . But,

$$w_s(cw_1) = (cw_1)w_s, s \in G,$$

and so,

$$s(c) = c, s \in G.$$

Since  $c$  is left fixed by all elements of  $G$ ,  $c \in k$ .

If  $k$  is a commutative ring,  $K$  is a  $k$ -algebra and  $A$  is a  $k$ -algebra over  $K$ ,  $\mathcal{O}_K(A)$  will denote the group of ( $k$ -algebra over  $K$ ) automorphisms of  $A$ , and  $\mathcal{I}_K(A)$  will denote the subgroup of inner automorphisms of  $A$ . If  $u$  is a unit of  $A$ , which defines an inner automorphism  $\sigma$  of  $A$ , an element  $x \in A$  is left fixed by  $\sigma$  if and only if  $u$  commutes with  $x$ . Thus,  $u \in {}^K A$ , and it is proved the following

**(3) Corollary.** *If  $\sigma_u$  is the inner automorphism of  $D$  defined by a unit  $u$  of  $K$ ,  $x \mapsto uxu^{-1}$ , then  $\mathcal{I}_K(D) = \{\sigma_u; u \in \mathcal{U}(K)\}$ .*

Passing to the cohomological context, if  $f \in Z^1(G, \mathcal{U}(K))$ , let  $\rho(f)$  be the endomorphism of the  $K$ -module structure of  $D$  defined by

$$\rho(f)(w_s) = f(s)w_s, s \in G.$$

Since  $(f(s)w_s)_{s \in G}$  is a basis of  $D$ , because  $f(s) \in \mathcal{U}(K)$ , for all  $s \in G$ ,  $\rho(f)$  is a  $K$ -module automorphism of  $D$ . The fact that  $f$  is a cocycle translates into the formula

$$\rho(f)(w_s w_t) = \rho(f)(w_s) \rho(f)(w_t), s, t \in G,$$

and so,  $\rho(f) \in \mathcal{O}_K(D)$ . An easy computation, using the basis  $(w_s)_{s \in G}$ , shows that the map  $\rho : Z^1(G, \mathcal{U}(K)) \rightarrow \mathcal{O}_K(D)$  is an injective group morphism. Moreover, given  $\alpha \in \mathcal{O}_K(D)$ , from the relation

$$w_s(xw_1) = (s(x)w_1)w_s, s \in G \text{ and } x \in K,$$

it follows that

$$(\alpha(w_s)w_s^{-1})(xw_1) = (xw_1)(\alpha(w_s)w_s^{-1}), s \in G \text{ and } x \in K.$$

Hence,  $\alpha(w_s)w_s^{-1}$  commutes with  $K$  (elementwise), for all  $s \in G$ . Now, applying (1), for each  $s \in G$  there is an element  $f(s) \in K$ , uniquely determined, such that  $\alpha(w_s)w_s^{-1} = f(s)w_1$ ; but  $f(s) \in \mathcal{U}(K)$ , since  $\alpha(w_s)w_s^{-1} \in \mathcal{U}(D)$ . A simple calculation shows that  $f$  is a cocycle, and so,  $\rho(f) = \alpha$ . Therefore,  $\rho$  is a surjective map. Finally, given  $u \in \mathcal{U}(K)$ , if  $f_u$  is the coboundary defined by  $u$  (i. e.  $f_u(s) = us(u)^{-1}$ ,  $s \in G$ ), then  $\rho(f_u) = \sigma_u$ . Hence, applying (3),

$$\rho(B^1(G, \mathcal{U}(K))) = \mathcal{I}_K(D),$$



group monomorphism  $H^1(G, \mathcal{U}(K)) \rightarrow \mathcal{O}_k(D) / \mathcal{I}_k(D)$ .

Now, it will be assumed that  $G$  is a finite group, and that  $K$  is a Galois extension of  $k$  (relative to  $G$ ), in the sense of Auslander-Goldman [1, Appendix, p. 396]. The last requirement holds if and only if  $K$  is a separable  $k$ -algebra, and  $\mathcal{J}_s(K) = 0$ , for  $s \in G$  and  $s \neq 1$  (that is, the general hypothesis of this paper) [3, prop. 2, p. 344]. The assumption that  $K$  is a Galois extension of  $k$  guarantees that  $D$  is an Azumaya  $k$ -algebra (cf. (2) and [1, 5.1, p. 380]), and so, the Rosenberg-Zelinsky theorem yields a group monomorphism  $\lambda : H^1(G, \mathcal{U}(K)) \rightarrow \mathcal{P}(k)$ ,  $f \mapsto \text{cls}(\mathcal{J}_{\rho(f)}(D))$ . If  $\varepsilon : \mathcal{P}(k) \rightarrow H^0(G, \mathcal{P}(K))$  is the group morphism  $\text{cls}(P) \mapsto \text{cls}(K \otimes_k P)$ , then  $\text{Im} \lambda = \text{Ker} \varepsilon = \{\text{cls}(P); K \otimes_k P \simeq K, \text{ as } K\text{-modules}\}$ .

$\subseteq$ . Given  $\alpha \in \mathcal{O}_k(D)$ , a general result [5, 3.4, pp. 1111-2] guarantees that the map  $\mu : D \otimes_k \mathcal{J}_\alpha(D) \rightarrow {}_\alpha D$ ,  $x \otimes a \mapsto a \cdot x$ , is an isomorphism of two-sided  $D$ -modules. Supposing that  $\alpha \in \mathcal{O}_K(D)$ , it follows that  $\mathcal{J}_\alpha(D) \subseteq {}^K D = K$ . Thus, if  $\iota : K \otimes_k \mathcal{J}_\alpha(D) \rightarrow D \otimes_k \mathcal{J}_\alpha(D)$  is the  $K$ -module morphism deduced from the imbedding of  $K$  into  $D$ ,  $\mu \iota$  induces a  $K$ -module morphism  $\nu : K \otimes_k \mathcal{J}_\alpha(D) \rightarrow K$ . It is clear that  $\nu$  is a monomorphism, because  $\mu$  and  $\iota$  are monomorphisms. Moreover, given  $x \in K$ , it can be written in the form  $x = \sum_{1 \leq i \leq n} a_i x_i$ , with  $a_i \in \mathcal{J}_\alpha(D)$  and  $x_i \in D$ ,  $1 \leq i \leq n$ , since  $\mu$  is an epimorphism (that is,  $D = \mathcal{J}_\alpha(D) \cdot D$ ). Hence, if  $x_i = \sum_{s \in G} x_{i,s} w_s$ , with  $x_{i,s} \in K$ ,  $1 \leq i \leq n$  and  $s \in G$ , then  $x = \sum_{1 \leq i \leq n} a_i x_{i1} \in \mathcal{J}_\alpha(D)$ .  $K$ ; and so,  $\nu$  is an epimorphism.

$\supseteq$ . Let  $P$  be a finitely generated, projective  $k$ -module, of rank one, such that  $K \otimes_k P \simeq K$ , as  $K$ -modules. If  $f$  is such an isomorphism, the map  $\varphi : D \otimes_k P \rightarrow D$ , defined by  $\varphi(w_s \otimes x) = f(1 \otimes x)w_s$ ,  $s \in G$  and  $x \in P$ , is a  $K$ -module isomorphism (note that  $w_s \otimes x$ , with  $s \in G$  and  $x \in P$ , are generators of  $D \otimes_k P$ , as a  $K$ -module). Moreover, since  $\varphi((w_s \otimes x)w_t) = f(1 \otimes x)w_{st} = \varphi(w_s \otimes x)w_t$ ,  $s, t \in G$  and  $x \in P$ ,  $\varphi$  is an isomorphism of right  $D$ -modules. Therefore, the argument of [4, 3.7 (proof), p. 1113] provides a  $k$ -algebra automorphism of  $D$ , say  $\alpha$ , as follows: If  $g$  is a free generator of  $D \otimes_k P$ , as a right  $D$ -module, for each  $x \in D$  there is an element  $\alpha(x) \in D$ , uniquely determined, such that  $xg = g\alpha(x)$ . It can be assumed that  $g \in \iota(K \otimes_k P)$ ; for example, take  $g = \varphi^{-1}(w_1)$ , and apply the commutativity of the diagram

$$\begin{array}{ccc}
 K \otimes_k P & \xrightarrow{f} & K \\
 \downarrow \iota & & \downarrow \\
 D \otimes_k P & \xrightarrow{\varphi} & D
 \end{array}$$

Thus,  $xg = gx$ , for all  $x \in K$ , since  $K$  is contained in the centralizer of  $\iota(K \otimes_k P)$  in  $D \otimes_k P$ . Hence,  $\alpha \in \mathcal{O}_K(D)$ , and it is proved the following

**(7) Proposition.** *Let  $G$  be a finite group. If  $K$  is a separable  $k$ -algebra (equivalently,  $K$  is a Galois extension of  $k$ , relative to  $G$ ), then the sequence*

$$1 \longrightarrow H^1(G, \mathcal{U}(K)) \xrightarrow{\lambda} \mathcal{P}(k) \xrightarrow{\varepsilon} H^0(G, \mathcal{P}(K))$$

*is exact.*

UNIVERSIDAD DE BUENOS AIRES

---

### References

- [1] M. Auslander and O. Goldman: *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.
- [2] S.U. Chase, D.K. Harrison and A. Rosenberg: *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. No. 52 (1965), 15–33.
- [3] M. Harada: *Supplementary results on Galois extensions*, Osaka J. Math. **2** (1965), 343–350.
- [4] A. Rosenberg and D. Zelinsky: *Automorphisms of separable algebras*, Pacific J. Math. **11** (1961), 1109–1117.