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THE FIRST COHOMOLOGY GROUP
OF A COMMUTATIVE RING

JUAN JOSE MARTINEZ

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Given a commutative ring $K$, let $\mathcal{U}(K)$ be the group of units of $K$. If $G$ is an arbitrary group, represented by ring automorphisms of $K$, let $D(K, G)$ be the trivial crossed product of $K$ with $G$, and $k$ be the subring of $K$ consisting of all elements left fixed by $G$.

This note is devoted to perform a description of the first cohomology group $H^1(G, \mathcal{U}(K))$ as the group of ring automorphisms of $D(K, G)$ leaving $K$ pointwise fixed, modulo inner ones. This result holds under the assumption that $K$ is its own centralizer in $D(K, G)$. Traducing this condition in terms of the $G$-action on $K$, the description of $H^1(G, \mathcal{U}(K))$ mentioned above provides, in the Galois case, the first four terms of the exact sequence associated to a Galois extension [2, 5.5, p. 31].

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Throughout this paper, all rings will be assumed to have units, and all modules will be unitary. A subring will contain the unit element of the ring, and a morphism of rings will preserve unit elements. The general notations are given below.

Let $k$ be a commutative ring, and $K$ be any $k$-algebra. A (left) $K$-module $A$, together with a operation $(x, y) \mapsto x.y$, which is associative, with unit element, and $k$-1-bilinear, will be called, for shortness, a $k$-algebra over $K$. The definition of morphism of $K$-algebra over $K$ is obvious.

If $G$ is a group and $K$ is a commutative ring, given a representation of $G$ by ring automorphisms of $K$, let $D(K, G) = D$ denote the trivial crossed product of $K$ with $G$. This means that $D$ is the free $K$-module $K^{(G)}$ generated by $G$, with multiplication defined by the formula

$$(aw)(bw) = as(b)w,$$  

where $(w_s)_{s \in G}$ is the canonical basis of $K^{(G)}$. If $k$ is the subring $^GK$ of $K$ consisting of all elements left fixed by $G$, and $w : G \rightarrow D$ is the map $s \mapsto w_s$, then $D$ is a $k$-algebra over $K$ and $w$ is a multiplicative morphism. The pair $(D, w)$
is characterized by the following universal mapping property: if $A$ is any $k$-algebra over $K$ and $m : G \to A$ is any multiplicative morphism, there exists a unique morphism of $k$-algebras over $K \varphi : D \to A$ such that $\varphi w = m(\varphi)$ (where $\varphi$ is defined as the $K$-module morphism $w_s \mapsto m(s)$, $s \in G$). Note that the ring structure of $D$ has $w_1$ for its identity element, and that the map $x \mapsto xw_1$ imbeds $K$ as a subring of $D$.

Let $k$ be a commutative ring, and $A$ be a $k$-algebra. If $M$ is a two-sided $A$-module, $^A M$ will denote the $k$-submodule \{ $x \in M$; $ax = xa$, $a \in A$ \}. Note that if $A$ is a subalgebra of a $k$-algebra $B$, then $^A B$ is just the centralizer of $A$ in $B$; in particular, $^A A$ is the center of $A$. If $\alpha$ and $\beta$ are $k$-algebra endomorphisms of $A$, the two-sided $A$-module with additive group equal to the additive group of $M$ and actions $axb = \alpha(a)x\beta(b)$, $a, b \in A$ and $x \in M$, will be denoted $^M A B$. Finally, set $^A S(M) = A \otimes _k M$.

Henceforth, unless otherwise specified, $K$ is a commutative ring, $G$ is a group represented by ring automorphisms of $K$, and $k = k^G K$.

The following result will be needed for the announced description of the first cohomology group.

(1) **Proposition.** $K$ is its own centralizer in $D$ (i. e. $^KD = K$) if and only if $^A S_s (K) = 0$, for $s \in G$ and $s \neq 1$.

Proof. Explicitly, $^A S_s (K) = \{ a \in K; as(x) = xa, x \in K \}$.

Sufficiency. Let $a \in D$, $a = \sum s a_s w_s$, commute with all elements of $K$. Since $(w_s)_{s \in G}$ is a basis of $D$, as a $K$-module, and $(s(x)w_1)w_s = s(x)w_s = w_s(xw_1)$, for $s \in G$ and $x \in K$, from the relation

$$ a(xw_1) = (xw_1)a, x \in K, $$

it follows that

$$ a_s(x) = xa_s, s \in G \text{ and } x \in K. $$

Therefore, $a_s \in ^A S_s (K)$, for all $s \in G$, and so, $a = a w_1$.

Necessity. Given $s \in G$, if $a \in ^A S_s (K)$, then

$$ (aw_s)(xw_1) = (xw_1)(aw_s), x \in K, $$

and so, $aw_s \in K^D = K$. Hence, if $s \neq 1$, $a = 0$.

In the rest of the paper, it will be assumed that $^A S_s (K) = 0$, for $s \in G$ and $s \neq 1$.

Then,

(2) **Corollary.** $D$ is a central $k$-algebra.

Proof. If $a$ is in the center of $D$, then $a = cw_1$, for some $c \in K$, because $K$ is its own centralizer in $D$. But,
and so,
\[ s(c) = c, \quad s \in G. \]

Since \( c \) is left fixed by all elements of \( G \), \( c \in k \).

If \( k \) is a commutative ring, \( K \) is a \( k \)-algebra and \( A \) is a \( k \)-algebra over \( K \), \( \mathcal{O}_K(A) \) will denote the group of \( (k \)-algebra over \( K \) \) automorphisms of \( A \), and \( \mathcal{L}_K(A) \) will denote the subgroup of inner automorphisms of \( A \). If \( u \) is a unit of \( A \), which defines an inner automorphism \( \sigma \) of \( A \), an element \( x \in A \) is left fixed by \( \sigma \) if and only if \( u \) commutes with \( x \). Thus, \( u \in K^*A \), and it is proved the following

**Corollary.** If \( \sigma_u \) is the inner automorphism of \( D \) defined by a unit \( u \) of \( K \), \( \sigma_u : \mathcal{U}(K) \rightarrow \mathcal{U}(K) \), then \( \mathcal{L}_K(D) = \{ \sigma_u ; u \in \mathcal{U}(K) \} \).

Passing to the cohomological context, if \( f \in Z^1(G, \mathcal{U}(K)) \), let \( \rho(f) \) be the endomorphism of the \( K \)-module structure of \( D \) defined by
\[ \rho(f)(w_s) = f(s)w_s, \quad s \in G. \]
Since \( (f(s)w_s)_{s \in G} \) is a basis of \( D \), because \( f(s) \in \mathcal{U}(K) \), for all \( s \in G \), \( \rho(f) \) is a \( K \)-module automorphism of \( D \). The fact that \( f \) is a cocycle translates into the formula
\[ \rho(f)(w_s w_t) = \rho(f)(w_s) \rho(f)(w_t), \quad s, t \in G, \]
and so, \( \rho(f) \in \mathcal{O}_K(D) \). An easy computation, using the basis \((w_s)_{s \in G}\), shows that the map \( \rho : Z^1(G, \mathcal{U}(K)) \rightarrow \mathcal{O}_K(D) \) is an injective group morphism. Moreover, given \( \alpha \in \mathcal{O}_K(D) \), from the relation
\[ w_s(xw_t) = (s(x)w_t)w_s, \quad s \in G \quad \text{and} \quad x \in K, \]
it follows that
\[ (\alpha(w_s)w_t^{-1})(xw_t) = (xw_t)(\alpha(w_s)w_t^{-1}), \quad s \in G \quad \text{and} \quad x \in K. \]
Hence, \( \alpha(w_s)w_t^{-1} \) commutes with \( K \) (elementwise), for all \( s \in G \). Now, applying (1), for each \( s \in G \) there is an element \( f(s) \in K \), uniquely determined, such that \( \alpha(w_s)w_t^{-1} = f(s)w_t \); but \( f(s) \in \mathcal{U}(K) \), since \( \alpha(w_s)w_t^{-1} \in \mathcal{U}(D) \). A simple calculation shows that \( f \) is a cocycle, and so, \( \rho(f) = \alpha \). Therefore, \( \rho \) is a surjective map. Finally, given \( u \in \mathcal{U}(K) \), if \( f_u \) is the coboundary defined by \( u \) (i.e. \( f_u(s) = us(u)^{-1}, \quad s \in G \)), then \( \rho(f_u) = \sigma_u \). Hence, applying (3),
\[ \rho(B^1(G, \mathcal{U}(K))) = \mathcal{L}_K(D), \]
and it is proved the following:

(4) Theorem. The map \( \rho : Z'(G, \mathcal{U}(K)) \to \mathcal{O}_K(D) \) is a group isomorphism, and \( \rho(B'(G, \mathcal{U}(K))) = \mathcal{I}_K(D) \).

(5) Corollary. \( \mathcal{O}_K(D) \) is an abelian group, and \( H'(G, \mathcal{U}(K)) \cong \mathcal{O}_K(D)/\mathcal{I}_K(D) \).

(6) Corollary. If \( \mathcal{U}(K) \to \mathcal{O}_K(D) \) and \( \mathcal{U}(K) \to B'(G, \mathcal{U}(K)) \) are the maps \( u \mapsto \sigma_u \) and \( u \mapsto f_u \), respectively, then the following diagram is commutative and exact:

\[
\begin{array}{c}
1 \\
\downarrow \\
\downarrow \\
1 \to \mathcal{U}(k) \to \mathcal{U}(K) \to \mathcal{O}_K(D) \to \mathcal{O}_K(D)/\mathcal{I}_K(D) \to 1 \\
\downarrow \rho^{-1} \\
1 \to B'(G, \mathcal{U}(K)) \to Z'(G, \mathcal{U}(K)) \to H'(G, \mathcal{U}(K)) \to 1 \\
\end{array}
\]

Hence, the following diagram is commutative and exact:

\[
\begin{array}{c}
1 \\
\downarrow \\
\downarrow \\
1 \to \mathcal{U}(k) \to \mathcal{U}(K) \to Z'(G, \mathcal{U}(K)) \to H'(G, \mathcal{U}(K)) \to 1 \\
\downarrow \rho \\
1 \to \mathcal{U}(k) \to \mathcal{U}(K) \to \mathcal{O}_K(D) \to \mathcal{O}_K(D)/\mathcal{I}_K(D) \to 1 \\
\end{array}
\]

Finally, it will be shown that the description of \( H'(G, \mathcal{U}(K)) \) given in (5) yields, in the Galois case, the first four terms of the exact sequence of a Galois extension [2, 5.5, p. 31]. The key result is the Rosenberg-Zelinsky generalization of the Skolem-Noether theorem [4, 3.7, p. 1112]:

Let \( k \) be a commutative ring, and \( \mathcal{P}(k) \) be the projective class group of \( k \). If \( A \) is an Azumaya (i.e. central, separable) \( k \)-algebra, the map \( \Lambda : \mathcal{O}_k(A) \to \mathcal{P}(k), \alpha \mapsto \text{cls}(\mathcal{I}_k(A)) \), is a group morphism, Ker \( \Lambda = \mathcal{I}_k(A) \), and Im \( \Lambda = \{ \text{cls}(P); A \otimes_k P \cong A, \text{as left } A\text{-modules} \} \).

Since \( \mathcal{O}_K(D) \cap \mathcal{I}_k(D) = \mathcal{I}_K(D) \), the inclusion of \( \mathcal{O}_K(D) \) in \( \mathcal{O}_k(D) \) induces a group monomorphism \( \mathcal{O}_K(D)/\mathcal{I}_K(D) \to \mathcal{O}_k(D)/\mathcal{I}_k(D) \), and so, (5) provides a
group monomorphism $H^1(G, \mathcal{U}(k)) \rightarrow \mathcal{O}_k(D)/\mathcal{J}_k(D)$.

Now, it will be assumed that $G$ is a finite group, and that $K$ is a Galois extension of $k$ (relative to $G$), in the sense of Auslander-Goldman [1, Appendix, p. 396]. The last requirement holds if and only if $K$ is a separable $k$-algebra, and $\mathcal{J}_a(K) = 0$, for $s \in G$ and $s \neq 1$ (that is, the general hypothesis of this paper) [3, prop. 2, p. 344]. The assumption that $K$ is a Galois extension of $k$ guarantees that $D$ is an Azumaya $k$-algebra (cf. (2) and [1, 5.1, p. 380]), and so, the Rosenberg-Zelinsky theorem yields a group monomorphism $\lambda : H^1(G, \mathcal{U}(k)) \rightarrow \mathcal{O}(k), f \mapsto \text{cls}(\mathcal{J}_kF(D))$. If $\varepsilon : \mathcal{O}(k) \rightarrow H^0(G, \mathcal{P}(k))$ is the group morphism $\varepsilon : (P) \mapsto \{\text{cls}(P); K \otimes_k P \simeq K\}$, then $\text{Im} \lambda = \text{Ker} \varepsilon = \{\text{cls}(P); K \otimes_k P \simeq K\}$, as $K$-modules.

Given $\alpha \in \mathcal{O}_K(D)$, a general result [5, 3.4, pp. 1111–2] guarantees that the map $\mu : D \otimes_k \mathcal{J}_a(D) \rightarrow_a D$, $x \otimes a \mapsto a.x$, is an isomorphism of two-sided $D$-modules. Supposing that $\alpha \in \mathcal{O}_K(D)$, it follows that $\mathcal{J}_a(D) \subseteq D = K$. Thus, if $\iota : K \otimes_k \mathcal{J}_a(D) \rightarrow D \otimes_k \mathcal{J}_a(D)$ is the $K$-module morphism deduced from the imbedding of $K$ into $D$, $\mu \iota$ induces a $K$-module morphism $\nu : K \otimes_k \mathcal{J}_a(D) \rightarrow K$. It is clear that $\nu$ is a monomorphism, because $\mu$ and $\iota$ are monomorphisms. Moreover, given $x \in K$, it can be written in the form $x = \sum_{i \leq i \leq n} a_i x_i$, with $a_i \in \mathcal{J}_a(D)$ and $x_i \in D$, $1 \leq i \leq n$, since $\mu$ is an epimorphism (that is, $D = \mathcal{J}_a(D)$). Hence, if $x_i = \sum_{t \leq i \leq n} x_{is} w_s$, with $x_{is} \in K$, $1 \leq i \leq n$ and $s \in G$, then $x = \sum_{i \leq i \leq n} a_i x_{is} \in \mathcal{J}_a(D)$. It and so, $\nu$ is an epimorphism.

Let $P$ be a finitely generated, projective $k$-module, of rank one, such that $K \otimes_k P \simeq K$, as $K$-modules. If $f$ is such an isomorphism, the map $\varphi : D \otimes_k P \rightarrow D$, defined by $\varphi(w_s \otimes x) = f(1 \otimes x)w_s$, $s \in G$ and $x \in P$, is a $K$-module isomorphism (note that $w_s \otimes x$, with $s \in G$ and $x \in P$, are generators of $D \otimes_k P$, as a $K$-module). Moreover, since $\varphi((w_s \otimes x)w_t) = f(1 \otimes x)w_{st} = \varphi(w_s \otimes x)w_t$, $s, t \in G$ and $x \in P$, $\varphi$ is an isomorphism of right $D$-modules. Therefore, the argument of [4, 3.7 (proof), p. 1113] provides a $k$-algebra automorphism of $D$, say $\alpha$, as follows: If $g$ is a free generator of $D \otimes_k P$, as a right $D$-module, for each $x \in D$ there is an element $\alpha(x) \in D$, uniquely determined, such that $xg = g\alpha(x)$. It can be assumed that $g \in \iota(K \otimes_k P)$; for example, take $g = \varphi^{-1}(w_1)$, and apply the commutativity of the diagram

$$
\begin{array}{ccc}
K \otimes_k P & \xrightarrow{f} & K \\
\phantom{f} \downarrow \iota & \phantom{f} & \phantom{f} \\
D \otimes_k P & \xrightarrow{\varphi} & D
\end{array}
$$

Thus, $xg = gx$, for all $x \in K$, since $K$ is contained in the centralizer of $\iota(K \otimes_k P)$ in $D \otimes_k P$. Hence, $\alpha \in \mathcal{O}_K(D)$, and it is proved the following.
(7) **Proposition.** Let $G$ be a finite group. If $K$ is a separable $k$-algebra (equivalently, $K$ is a Galois extension of $k$, relative to $G$), then the sequence

$$1 \rightarrow H^1(G, \mathcal{U}(K)) \rightarrow \mathcal{D}(k) \rightarrow H^0(G, \mathcal{D}(K))$$

is exact.

**References**


