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ON MEASURE-COMPACTNESS AND BOREL MEASURE-COMPACTNESS

SUSUMU OKADA AND YOSHIAKI OKAZAKI

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1. Introduction

Moran has introduced "measure-compactness' and "strongly measure-compactness' in [7], [8]. For measure-compactness, Kirk has independently presented in [4]. For Borel measures, "Borel measure-compactness' and "property B" are introduced by Gardner [1].

We study, defining "a weak Radon space", the relation among these properties at the last of §2. Furthermore we investigate the stability of countable unions in §3 and the hereditariness in §4.

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2. Preliminaries and fundamental results

All spaces considered in this paper are completely regular and Hausdorff. Let $C^b(X)$ be the space of all bounded real-valued continuous functions on a topological space X. By $Z(X)$ we mean the family of *zero sets* $\{f^{-1}(0); f\in C^b(X)\}$ and $U(X)$ the family of *cozero sets* $\{Z^c; Z\in Z(X)\}$. We denote by $\mathcal{B}_a(X)$ the *Baire field*, that is, the smallest σ-algebra containing $Z(X)$ and by $\mathcal{B}(X)$ the σ-algebra generated by all open subsets of X, which is called the *Borel field*.

A *Baire measure* [resp. *Borel measure*] is a totally finite, non-negative countably additive set function defined on $\mathcal{B}_a(X)$ [resp. $\mathcal{B}(X)$]. For a Baire measure μ, the support of μ is

$$\text{supp}\,\mu=\{x\in X;\ \mu(U)>0 \quad \text{for every } U\in U(X) \text{ containing } x\}\ .$$

Similarly the support of a Borel measure μ is

$$\text{supp}\,\mu=\{x\in X;\ \mu(O)>0 \quad \text{for every open } O \text{ containing } x\}\ .$$

The following is clear.

Lemma 2.1. *Let μ be a Borel measure on a topological space X. If we denote by ν the restriction of μ to the Baire field, then we have*

$$\operatorname{supp} \mu = \operatorname{supp} \nu .$$

A Baire measure μ is *τ-smooth* if, whenever a net $\{Z_\alpha\}$ of zero sets decreases to the empty set $\emptyset$, $\lim_\alpha \mu(Z_\alpha)=0$. A Baire measure μ is *tight* if for every $\varepsilon>0$, there is a compact set K in X such that $\mu^*(K)>\mu(X)-\varepsilon$, where $\mu^*(K)=\inf\{\mu(U); K\subset U\in U(X)\}$. A topological space is said to be *measure-compact* [resp. *strongly measure-compact*] if every Baire measure is τ-smooth [resp. tight]. For measure-compactness and strongly measure-compactness, the followings are given by Moran:

(I) ([7; Th. 2.1]) A topological space X is measure-compact if and only if every non-zero Baire measure on X has a non-empty support;

(II) ([8; Prop. 4.4]) A topological space X is strongly measure-compact if and only if for every non-zero Baire measure μ there is a compact set K in X such that $\mu^*(K)>0$.

A Borel measure μ is *weakly τ-smooth* if, whenever a net $\{F_\alpha\}$ of closed sets decreases to the empty set $\emptyset$, $\lim_\alpha \mu(F_\alpha)=0$. We call a Borel measure μ is *τ-smooth* if, whenever a net $\{F_\alpha\}$ of closed sets decreases to a closed set F, $\lim_\alpha \mu(F_\alpha)=\mu(F)$. A Borel measure μ is *regular* if for every B in $\mathscr{B}(X)$,

$$\mu(B) = \sup\{\mu(F); F\subset B \quad \text{and} \quad F \text{ is closed}\}.$$

A Borel measure μ is a *Radon measure* if for every Borel set A in $\mathscr{B}(X)$,

$$\mu(A) = \sup\{\mu(K); K\subset A \quad \text{and} \quad K \text{ is compact}\}.$$

We say a topological space X has the *property B* if every Borel measure μ is weakly τ-smooth. A topological space X is called an *HB-space* if every subspace of X has the property B. A topological space X is *Borel measure-compact* if every regular Borel measure on X is τ-smooth. Gardner has given the following three results:

(III) ([1; Th. 4.1]) A topological space X has the property B if and only if every non-zero Borel measure on X has non-empty support;

(IV) ([1; Th. 5.2]) A topological space X is an HB-space if and only if every Borel measure on X is τ-smooth;

(V) ([1; Th. 3.1]) A topological space X is Borel measure-compact if and only if every non-zero regular Borel measure on X has a non-empty support.

A topological space is said to be a *Borel-regular space* if every Borel measure is regular. We call a topological space X is a *weak Radon space* [resp. *Radon space*] if every regular Borel measure [resp. Borel measure] is a Radon measure.

For the properties defined above, we offer the relation among them.

$$
\begin{tikzcd}
\sigma\text{-compact space} \arrow[r, Rightarrow, "(1)"] & \text{strongly measure-compact space} \arrow[r, Rightarrow, "(2)"] \arrow[d, Rightarrow, "(3)"] & \text{measure-compact space} \arrow[dd, Rightarrow, "(5)"] \\
\text{weak Radon space} \arrow[r, Rightarrow, "(4)"'] & \text{Borel measure-compact space} & \\
\text{Radon space} \arrow[u, Rightarrow, "(6)"] \arrow[r, Rightarrow, "(8)"'] & \text{HB space} \arrow[r, Rightarrow, "(9)"'] \arrow[d, Rightarrow, "(10)"'] & \text{property B} \arrow[ul, Rightarrow, "(7)"'] \\
& \text{Borel regular space} &
\end{tikzcd}
$$

(1), (2), (4), (6), (8) and (9) are obvious from the definitions. (7) is evident from (III) and (V). Since X is a regular space, each point in X has a neighborhood base consisting of closed sets, which means (10) by (IV) (Gardner [1; Th. 6.1]).

Theorem 2.2. (i) *A strongly measure-compact space X is a weak Radon space.*

(ii) *A measure-compact space X has the property B.*

Proof. (i) Let μ be a regular Borel measure on X and ν be the restriction of μ to the Baire field. Since X is strongly measure-compact, ν is a tight measure. By Knowles [6; Th. 2.5] ν is uniquely extended to a Radon measure $\tilde{\nu}$. Then μ equals $\tilde{\nu}$ by Kirk [5; Cor. 1.15].

(ii) Let μ be a Borel measure on X and ν be the restriction of μ to the Baire field. By (I) supp ν is non-empty. Since supp μ is equal to supp ν, supp μ is non-empty. By (III) X has the property B. This completes the proof.

Let Ω be the first uncountable ordinal. Then $[0, \Omega]$ is an example that the converses of (6) and (9) do not hold and it implies that the property B does not necessarily imply Borel-regularity. Let A be a non-Lebesgue-measurable subset of $[0, 1]$ such that $\mu^*(A)>0$, where μ is the Lebesgue measure on $[0, 1]$ and μ^* denotes the outer measure derived from μ. Then this space A is an example that the converses of (4), (7) and (8) do not hold. Furthermore the countable product $\boldsymbol{R}^\infty$ of real lines gives an example that the converse of (1) does not hold.

3. Countable union

Let X be a topological space. Gardner [1; Th. 3.7, Th. 7.4] has shown that if X is the union of a sequence $\{X_n\}$ of Borel measure-compact [resp. C*-embedded, measure-compact] subsets, then X is Borel measure-compact [resp. measure-compact]. We examine whether other spaces defined in the preceding section have the same property. We recall that a subset A of X is *Baire-embedded* in X if $\mathscr{B}_a(A)=A\cap\mathscr{B}_a(X)$. It is clear that if A is C*-embedded, then A is Baire-embedded.

The following lemma is derived from the same argument as in Lemma 3.8 of Gardner [1].

Lemma 3.1. *Let μ be a totally finite measure on a measurable space $(X, \mathscr{B})$. If a subset A of X satisfies $\mu^*(A)>0$, then μ^* is a totally finite measure on $(A, A\cap\mathscr{B})$, where $\mu^*(B)=\inf\{\mu(C); B\subset C\in\mathscr{B}\}$ for a subset B of X.*

We denote by μ_A the above $\mu^*|_{A\cap\mathscr{B}}$, which we call the *restriction* of μ to A.

Theorem 3.2. *Let X be the countable union of a sequence $\{X_n\}$.*

(i) *If every X_n is Baire-embedded measure-compact, then X is measure-compact.*

(ii) *If every X_n is Baire-embedded strongly measure-compact, then X is strongly measure-compact.*

Proof. (i) It follows by the way similar to Gardner [1; Th. 3.7].

(ii) Let μ be a Baire measure on X, then there exists an n such that $\mu^*(X_n)>0$. Since X_n is Baire-embedded in X, the restriction μ_{X_n} of μ to X_n is non-zero Baire measure on X_n by Lemma 3.1. By (II) in §2, there exists a compact subset K of X_n such that $\mu^*_{X_n}(K)>0$. For every Baire set B in $\mathscr{B}_a(X)$ containing K, we have

$$\mu(B)\geqq\mu_{X_n}(B\cap X_n)\geqq\mu^*_{X_n}(K)>0\,,$$

which implies $\mu^*(K)>0$. The theorem is proved.

Theorem 3.3. *Suppose that a space X is the countable union of $\{X_n\}$.*

(i) *If every X_n has the property B, then X also has the property B.*

(ii) *If every X_n is an HB-space, then so is X.*

(iii) *If every X_n is a weak Radon space, then so is X.*

(iv) *If every X_n is a Radon space, then so is X.*

Proof. Let μ be a Borel measure on X. We put $\mu^*(A)=\inf\{\mu(B); A\subset B\in\mathscr{B}(X)\}$ for every subset A.

(i) There is an n such that $\mu^*(X_n)>0$. Then the restriction μ_{X_n} is a non-zero Borel measure on X_n. Since X_n has the property B, supp μ_{X_n} is non-empty, that is, supp μ_{X_n} contains an element x. For every open neighborhood U of x, it follows that

$$\mu(U)\geqq\mu_{X_n}(U\cap X_n)>0\,.$$

Therefore x is contained in supp μ. By (III) in §2 X has the property B.

(ii) Let A be any subset of X. We shall show A has the property B.

Since every X_n is an HB-space, $A \cap X_n$ has the property B. Then by (i) $A = \bigcup_{n=1}^{\infty} (A \cap X_n)$ has the property B.

(iii) Suppose μ is regular, then for every n the measure μ_{X_n} is regular on X_n. Since μ_{X_n} is a Radon measure, there exists a compact subset K_n of X_n such that $\mu_{X_n}(X_n - K_n) < \varepsilon/2^n$. Then we have

$$\mu(X - \bigcup_{n=1}^{\infty} K_n) \leqq \sum_{n=1}^{\infty} \mu_{X_n}(X_n - K_n) < \varepsilon .$$

It follows $\mu(X) = \sup \{\mu(K); K \text{ is compact}\}$. By the regularity of μ, μ is a Radon measure on X.

(iv) By the argument of (iii), we have $\mu(B) = \sup \{\mu(K); K \subset B$ and K is compact$\}$ for every B in $\mathscr{B}(X)$.

4. Hereditariness

Every Baire set of a measure-compact space X is measure-compact by Moran [8; Prop. 5.1] and for closed subsets, by Kirk [4; Th. 3.5]. We show the following:

Theorem 4.1. (i) *Let X be a measure-compact space and A be a subset of X. Suppose that for every closed subset F which is disjoint from A, there exists a Baire set B such that $B \supset A$ and $B \cap F = \emptyset$. Then A is measure-compact.*

(ii) *Let X be a normal and measure-compact space. Then every F_σ-subset is measure-compact.*

Proof. (i) It follows by the same way as in the proof of Moran [8; Prop. 5.1].

(ii) Since a closed subset of a normal space is Baire-embedded in X, every F_σ-subset is measure-compact by Theorem 3.2.

For a strongly measure-compact space X, Baire sets are strongly measure-compact by Moran [8; Prop. 4.5] and closed sets are strongly measure-compact by Mosiman and Wheeler [9; Prop. 3.2]. If X is a normal, strongly measure-compact space, then each F_σ-set is strongly measure-compact by Theorem 3.2.

Next we consider subspaces of a space having the property B.

Theorem 4.2. *Suppose a topological space X has the property B.*

(i) *Every closed subset F has the property B.*

(ii) *If a subset A of X satisfies the condition that for every closed subset F which is disjoint from A, there is a Baire set B such that $B \supset A$ and $B \subset F = \emptyset$, then A has the property B. In particular, every Baire set has the property B.*

Proof. (i) Let μ be a Borel measure on F. We define a Borel measure on X by $\bar{\mu}(B) = \mu(B \cap F)$ for B in $\mathscr{B}(X)$. Then $\bar{\mu}$ is weakly τ-smooth. For a net $\{F_\alpha\}$ of closed sets in F decreasing to $\emptyset$, we have

$$\lim_\alpha \mu(F_\alpha) = \lim_\alpha \bar{\mu}(F_\alpha) = 0\,,$$

which implies F has the property B.

(ii) For a Borel measure μ on A, $\bar{\mu}$ is the same one on X as in (i). Since $\bar{\mu}$ is weakly τ-smooth, supp $\bar{\mu}$ is non-empty and inf $\{\bar{\mu}(U);\ U \supset \text{supp}\ \bar{\mu}$ and U is open$\} = \bar{\mu}(X) = \mu(A)$. If supp $\bar{\mu} \cap A = \emptyset$, then there is a Baire set B such that $B \supset A$ and $B \cap \text{supp}\ \bar{\mu} = \emptyset$. It follows

$$\begin{aligned} 0 &= \mu(B^c) = \inf \{\bar{\mu}(V);\ B^c \subset V \in U(X)\} \\ &\geqq \inf \{\bar{\mu}(U);\ U \supset \text{supp}\ \bar{\mu} \quad \text{and} \quad \text{U is open}\} \\ &= \mu(A)\,, \end{aligned}$$

which is a contradiction. Therefore there is an x in supp $\bar{\mu} \cap A$. Let U be an open subset of X containing x, then we have

$$\mu(U \cap A) = \bar{\mu}(U) > 0\,,$$

which shows x belongs to supp μ. By (III) in §2 A has the property B. The proof of Theorem 4.2 is complete.

We give a necessary and sufficient condition under which a subset of a Borel measure-compact space is Borel measure-compact.

Theorem 4.3. *Let X be a Borel measure-compact space. A subset A of X is Borel measure-compact if and only if for every regular Borel measure μ on A, $\iota(\mu)$ is a regular measure on X, where ι is the natural injection of A into X.*

Proof. The necessity of the condition is derived from Gardner [1; Th. 5.4]. Conversely suppose $\iota(\mu)$ is regular. Then $\iota(\mu)$ is τ-smooth. For a net $\{F_\alpha\}$ of closed subsets of A decreasing to $\emptyset$, we have

$$\begin{aligned} \inf_\alpha \mu(F_\alpha) &= \inf_\alpha \mu(\bar{F}_\alpha \cap A) \\ &= \inf_\alpha \iota(\mu)(\bar{F}_\alpha) \\ &= \iota(\mu)(\bigcap_\alpha \bar{F}_\alpha) = \mu((\bigcap_\alpha \bar{F}_\alpha) \cap A) \\ &= \mu(\bigcap_\alpha F_\alpha) = \mu(\emptyset) = 0\,, \end{aligned}$$

which implies μ is a τ-smooth measure on A. We have proved the theorem.

The following lemma is easy to prove, so the proof is omitted.

Lemma 4.4. *Let X and Y are two topological spaces and f be a continuous closed map of X to Y. Provided μ is a regular Borel measure, then the image* *measure $f(\mu)$ defined by $f(\mu)(B)=\mu(f^{-1}(B))$ for B in $\mathscr{B}(Y)$ is a regular Borel measure on Y.*

Proposition 4.5. *Every F_σ-subset of a Borel measure-compact space is Borel measrue-compact.*

Proof. Every closed subset is Borel measure-compact by Theorem 4.3 and Lemma 4.4. By Gardner [1; Th. 3.7] every F_σ-set is Borel measure-compact.

For a weak Radon space, we have the following proposition by Theorem 3.3 and Lemma 4.4.

Proposition 4.6. *Every F_σ-subset of a weak Radon space is a weak Radon space.*

REMARK 4.7. Let μ be the Dieudonné measure on $[0, \Omega]$ (see Halmos [2; p. 231, (10)]) and ν be the restriction of μ to $[0, \Omega)=[0, \Omega]-\{\Omega\}$. Then ν is a regular Borel measure on $[0, \Omega)$ but supp ν is empty. Hence $[0, \Omega)$ is not Borel measure-compact. This implies that an open subset of a Borel measure-compact [resp. strongly measure-compact, measure-compact, weak Radon] space is not necessarily a Borel measure-compact [resp. strongly measure-compact, measure-compact, weak Radon] space. And even if X has the property B, an open subset has not necessarily the property B.

As for a Borel-regular space, the following proposition is easily derived.

Proposition 4.8. *Every subset of a Borel-regular space is also a Borel-regular space.*

Appendix

We study a condition for a locally compact space to be measure-compact. Let X be a locally compact space and μ be a Baire measure on X. As well-known, there exists a Radon measure $\nu(\mu)$ on X such that

$$\int_X f d\mu = \int_X f d\nu(\mu)$$

for every f in $C_0(X)$, where $C_0(X)$ is the space of continuous functions vanishing at infinity.

Lemma. *μ is τ-smooth if and only if $\nu(\mu)(X)$ is identical to $\mu(X)$.*

Proof. For a cozero set U in $U(X)$, we have

$$\begin{aligned}\nu(\mu)(U) &= \sup \{\int_X f d\mu;\ \chi_U \geqq f \in C_0(X)\} \\ &\leqq \sup \{\int_X f d\mu;\ \chi_U \geqq f \in C^b(X)\} \\ &= \mu(U),\end{aligned}$$

where χ_U is the characteristic function of U. Then it holds

$$\nu(\mu) \leqq \mu \quad \text{on} \quad \mathcal{B}_a(X).$$

Assume $\nu(\mu)(X)=\mu(X)$. Then we have $\nu(\mu)=\mu$ on $\mathcal{B}_a(X)$, which shows μ is τ-smooth.

Suppose μ is τ-smooth. Let $\{f_\alpha\}$ be a net in $C_0(X)$ increasing to 1 on X. Since μ is τ-smooth, it follows

$$\begin{aligned}\nu(\mu)(X) &= \lim_\alpha \int_X f_\alpha d\nu(\mu) \\ &= \lim_\alpha \int_X f_\alpha d\mu = \mu(X),\end{aligned}$$

which proves the lemma.

Theorem. *Let X be a locally compact space. For X to be measure-compact, it is necessary and sufficient that for every Baire measure μ, $\nu(\mu)(X)$ equals $\mu(X)$.*

OSAKA CITY UNIVERSITY (Current address; Australian National University)
KYUSHU UNIVERSITY

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