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DADE’S CONJECTURE FOR 2-BLOCKS
OF SYMMETRIC GROUPS

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0. Introduction

Let $G$ be a finite group, $p$ a prime and $B$ a $p$-block of $G$. In [4] Dade conjectured that the number of ordinary irreducible characters of $B$ with a fixed defect can be expressed as an alternating sum of the numbers of ordinary irreducible characters of related defects in related blocks $B'$ of certain local $p$-subgroups of $G$. This (ordinary) conjecture has been proved by Olsson and Uno for the symmetric groups when $p$ is odd. In this paper, we prove the (ordinary) conjecture for the symmetric groups $G$ when $p = 2$.

In Section 1 we state the ordinary conjecture and fix some notation. In Section 2 we reduce the family of radical 2-chains $R(G)$ to a $G$-invariant subfamily $QR(G)$. In Section 3 we first give several more reductions, and then prove the conjecture for $p = 2$ using results of Olsson and Uno [6].

1. Dade’s ordinary conjecture

Throughout this paper we shall follow the notation of Dade [4]. Let $C$ be a $p$-subgroup chain of a finite group $G$,

\[(1.1) \quad C : P_0 < P_1 < \cdots < P_w.\]

Then $w = |C|$ is called the length of $C$,

\[(1.2) \quad N(C) = N_G(C) = N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_w)\]

is called the normalizer of $C$ in $G$, and

\[(1.3) \quad C_k : P_0 < P_1 < \cdots < P_k, \quad 0 \leq k \leq w\]

is called the $k$-th initial $p$-subchain of $C$. In addition, $C$ is called a radical $p$-chain if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for all $1 \leq k \leq w$.

Thus $P_{k+1}$ and $P_{k+1}/P_k$ are radical subgroups of $N(C_k)$ and $N(C_k)/P_k$, respectively for $0 \leq k \leq w - 1$, where a $p$-subgroup $R$ of $G$ is radical if $R = O_p(N_G(R))$. Let $R = R(G)$ be the set of all radical $p$-chains of $G$. 
Given $C \in \mathcal{R}$, $B$ a $p$-block of $G$ and $u$ a non-negative integer, let $k(N(C), B, u)$ be the number of characters of the set

\[(1.4) \quad \text{Irr}(N(C), B, u) = \{ \psi \in \text{Irr}(N(C)) : B(\psi)^G = B, \text{ and } d(\psi) = u \},\]

where $B(\psi)$ is the block of $N(C)$ containing $\psi$ and $d(\psi)$ is the $p$-defect of $\psi$ (see [4, (5.5)] for the definition). Then the following is Dade’s ordinary conjecture, [4, Conjecture 6.3].

**Dade’s ordinary conjecture.** If $O_p(G) = 1$ and $B$ is a $p$-block of $G$ with defect $d(B) > 0$, and if $u$ is a non-negative integer, then

\[(1.5) \quad \sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N(C), B, u) = 0,\]

where $\mathcal{R}/G$ is a set of representatives for the $G$-orbits in $\mathcal{R}$.

2. **The first reduction**

In this section we shall first define a $G$-invariant subfamily $\mathcal{Q}\mathcal{R}$ of radical 2-chains of a symmetric group and then reduce Dade’s conjecture to the family $\mathcal{Q}\mathcal{R}$. In the rest of the paper we always suppose $p = 2$.

We shall also follow the notation of Alperin and Fong [1]. Given a positive integer $n$, we denote by $S(n) = S(V)$ the symmetric group of degree $n$ acting on the set $V$ of cardinality $n$. For each non-negative integer $c$, let $A_c$ denote the elementary abelian group of order $2^c$ represented by its regular permutation representation. Thus $A_c$ is embedded uniquely up to conjugacy as a transitive subgroup of $S(2^c)$, $C_{S(2^c)}(A_c) = A_c$, and

\[N_{S(2^c)}(A_c) \simeq A_c \rtimes \text{GL}(c, 2).\]

For a sequence $c = (c_1, c_2, \ldots, c_\ell)$ of non-negative integers, let $|c| = c_1 + \ldots + c_\ell$ and let $A_c$ be the wreath product $A_{c_1} \wr A_{c_2} \wr \ldots \wr A_{c_\ell}$. Then $A_c$ is embedded uniquely up to conjugacy as a transitive subgroup of $S(2^{|c|})$. Moreover,

\[(2.1) \quad N_{S(2^{|c|})}(A_c) = N_{S(2^{c_1})}(A_{c_1}) \otimes N_{S(2^{c_2})}(A_{c_2}) \otimes \ldots \otimes N_{S(2^{c_\ell})}(A_{c_\ell}),\]

\[N_{S(2^{|c'|})}(A_{c'})/A_{c'} \simeq \text{GL}(c_1, 2) \times \text{GL}(c_2, 2) \times \ldots \times \text{GL}(c_\ell, 2),\]

where $c' = (c_2, \ldots, c_\ell)$ and $N_{S(2^{c_1})}(A_{c_1}) \otimes N_{S(2^{c_2})}(A_{c_2}) \otimes \ldots \otimes N_{S(2^{c_\ell})}(A_{c_\ell})$ is the tensor product of the normalizers $N_{S(2^{c_1})}(A_{c_1})$ and $N_{S(2^{c_2})}(A_{c_2})$. Suppose $R$ is a radical 2-subgroup of $G$. By Alperin and Fong [1, (2A)], there exists a corresponding decomposition

\[(2.2) \quad V = V_0 \cup V_1 \cup \ldots \cup V_\ell, \quad R = R_0 \times R_1 \times \ldots \times R_\ell\]
such that \( R_0 = \langle 1, v_0 \rangle \) and each \( R_i \) for \( i \geq 1 \) is conjugate to some \( A_c \) in \( S(V) \). Let \( A(R) \) be the subgroup generated by all normal abelian subgroups of \( R \), and let \( B(R) = C_{A(R)}([A(R), A(R)]) \), where \([A(R), A(R)]\) is the commutator subgroup of \( A(R) \). Then \( B(R) \) is a characteristic subgroup of \( R \) and \( N_G(R) \leq N_G(B(R)) \). By [2, (2A)],

\[
B(A_c) = \begin{cases} 
(A_c)^{2^{[e_i]}} & \text{if } c_1 \neq 1 \text{ or } c_2 \neq 1, \\
(D_8)^{2^{|w|}} & \text{if } c_1 = c_2 = 1,
\end{cases}
\]

where \( D_8 = A_1 \wr A_1 \) is a dihedral group of order 8 and \( w = (c_3, \ldots, c_t) \).

Let \( \Psi = \{ A_c : c = (1, c_2, c_3, \ldots, c_t), c_2 \geq 2 \} \), \( \Psi' = \{ A_c : c = (1, 1, c_3, \ldots, c_t) \} \) and \( \Psi^* = \{ A_c : c = (0) \text{ or } c = (1, 1, \ldots, 1) \} \). A 2-subgroup \( R \) with a decomposition (2.2) is radical in \( G \) if and only if \( m_R(P) \neq 2, 4 \) for all \( P \in \Psi^* \), where \( m_R(P) \) is the multiplicity of the components \( P \) in \( R \).

Let \( S_d \) be a Sylow 2-subgroup of \( N_{S(2^d)}(A_d) \). Then \( S_1 = A_1 \) and \( S_2 = D_8 \). Let \( \Delta(1) = \{ A_1 \wr A_2 \} \), \( \Delta(d) = \{ S_d \wr A_1, S_d \wr A_2, S_d \wr A_1 \wr A_1 \} \) for \( d \geq 2 \).

\[
\Delta^+ = \bigcup_{d \geq 1} \Delta(d) \quad \text{and} \quad \Delta = \Delta(1) \cup \Delta(2).
\]

Suppose \( R \) is a radical subgroup of \( G \) with a decomposition (2.2). Then \( m_R(D_8) \notin \{2, 4\} \), \( m_R(A_1) \neq 2 \) and \( m_R(D_8 \wr A_1) \neq 2 \). So \( B(R) = \prod_{i=1}^t B(R_i) \) is non-radical in \( G \) if and only if \( m_{B(R)}(A_1) = 4 \) or \( m_{B(R)}(D_8) \in \{2, 4\} \), which is equivalent to

(a) \( m_R(A_1 \wr A_2) = 1 \) but \( m_R(P) = 0 \) for \( P \in \Psi \setminus \{ A_1 \wr A_2 \} \), or
(b) \( \text{For } X \in \Delta(2), m_R(X) = 1 \) but \( m_R(P) = 0 \) for all \( P \in \Psi \setminus \{X\} \).

If \( B(R) \) is radical, then define \( K(R) = B(R) \). Suppose \( B(R) \) is non-radical. Define

\[
K(R) = \begin{cases} 
A_1 \wr A_2 \times \prod_{R_j \neq A_1 \wr A_2} B(R_j) & \text{if only case (a) occurs,} \\
X \times \prod_{R_j \neq X} B(R_j) & \text{if only case (b) occurs,} \\
A_1 \wr A_2 \times X \times \prod_{R_j \neq A_1 \wr A_2, X} B(R_j) & \text{if both cases (a) and (b) occur.}
\end{cases}
\]

Thus \( K(R) \leq R \), \( K(R) \) is a radical subgroup of \( G \) and

\[
N_G(R) \leq N_G(K(R)) \leq N_G(B(R)).
\]

In addition, if two radical subgroups \( R \) and \( W \) are \( G \)-conjugate, then \( K(R) \) and \( K(W) \) are \( G \)-conjugate, since \( B(R) \) and \( B(W) \) are \( G \)-conjugate. We also need the following lemma to define the chains in \( QR \).

\[ (2A). \text{ Given integer } d \geq 1, \text{ let } G = S(2^d) = S(V) \text{ and } N = N(A_d) = N_G(A_d). \]

(a) There exists a bijection between the classes of radical subgroups \( R \) of \( N \) and the compositions \( c = (c_1, c_2, \ldots, c_t) \) of \( d \) such that

\[
N_N(R)/R \simeq \text{GL}(c_1, 2) \times \text{GL}(c_2, 2) \times \cdots \times \text{GL}(c_t, 2).
\]
In particular, the subset \([V, R]\) of \(V\) consisting of all points moved by \(R\) is \(V\) itself.

(b) Let \(R\) be a radical subgroup of \(N\) and \(Q\) a radical subgroup of \(N_N(R)\). Then \(Q\) is radical in \(N\) and \(N_N(Q) \leq N_N(R)\).

Proof. (a) Since \(R\) is radical in \(N\), \(A_d \leq R\) and \(R/A_d\) is a radical subgroup of \(N/A_d \simeq \text{GL}(d, 2)\). Since \(N_N(R)/A_d \simeq N_N(R/A_d)\), it follows by Borel-Tits theorem \([3]\) that \(R/A_d\) is the unipotent radical of a parabolic subgroup of \(N/A_d\). The classes of parabolic subgroups of \(\text{GL}(d, 2)\) are labelled by compositions of \(d\), and so (a) follows easily.

(b) Suppose \(Q\) is a radical subgroup of \(N_N(R)\). Then \(R \leq Q\), and the proof of (b) is also straightforward by applying the Borel-Tits theorem to \(N_N(R)/A_d\).

Remark. Follow the notation of (2A). Then \(R\) is radical in \(G\) if and only if \(R = A_d\) except when \(d = 2\) and \(c_1 = c_2 = 1\), in which case either \(R = A_d\) or \(R = D_8\). Indeed, we may suppose \(d \geq 2\). Since \(A_d \leq R\), it follows that \(R\) acts transitively on \(V\), and \(R = A_w\) for some sequence \(w = (w_1, \ldots, w_\ell)\) of positive integers with \(|w| = d\). Note that \(A_d \leq A(R)\). If \(w_1 \geq 2\), then each \(A(R)\)-orbit in \(V\) has \(2^{w_1}\) elements, so that \(d = w_1\). If \(w_1 = 1\), then each \(A(R)\)-orbit in \(V\) is contained in some \(A_1 \wr A_w\)-orbit, so that \(w = (1, w_2)\) and \(d = 1 + w_2\). But \(|R/A_d| = 2^{w_2}\) and \(|A_1 \wr A_{w_2}| = 2^{w_2 + w_2}\), so \(2^{w_2} = d = w_2 + 1\) and \(w_2 = 0\) or \(1\). Thus \(w_2 = 1\) and \(R = D_8\).

The radical subgroup \(R\) of \(N_{S(2^d)}(A_d)\) determined by the composition \(c\) in (2A) (a) will be denoted by \(Q_c\) if \(R\) is not a radical subgroup of \(S(2^d)\). This holds in particular if \(d \geq 3\). We set \(B(Q_c) = A_d\). Now we can define the family \(QR\).

Let \(QR = Q(R)(G)\) be the \(G\)-invariant subfamily of \(R\) consisting of radical 2-chains

\[(2.3) \quad C : 1 < P_1 < \cdots < P_w\]

such that \(P_1 = K(P_1)\) and each \(P_i\) has a decomposition \(\prod_{j=1}^{t_i} Q_{i,j}\) with \(Q_{i,j} \in \Delta^+ \cup \{A_d, Q_c, D_8\}\) for all \(i, j\). Let \(M\) be the complement \(R \setminus QR\) of \(QR\) in \(R\), so that

\[R = QR \cup M \quad \text{(disjoint)}.\]

In the following we shall show that Dade's conjecture can be reduced to the family \(QR\). First of all, we consider the structure of the subgroup \(P_2\). By definition, \(P_2\) is a radical subgroup of \(N_G(P_1)\).

Let \(D\) be a radical subgroup of \(G\) such that \(D = K(D)\). Then

\[(2.4) \quad V = V^+ \cup V^* \quad \text{and} \quad D = D^+ \times D^* ,\]
where $D^+ = \prod_{X \in \Delta} (X)^{\alpha_X}$ with $\alpha_X \in \{0, 1\}$, $D^* = (D_8)^{m_2} \times \prod_{d \geq 0} (A_d)^{m_d}$, $V^+ = [V, D^+]$ and $V^* = V \setminus V^+$. Let $U_X$ be the underlying set of $X \in \Delta^+$ such that $U_X = [U_X, X]$, and $N_X = N_{S(U_X)}(X)$. Then

$$N(D) = N_G(D) = N(D)^+ \times N(D)^*,$$

where $N(D)^+ = \prod_{X \in \Delta} (N_X)^{\alpha_X}$ and $N(D)^* = D_8 \times \prod_{d \geq 0} N_{S(2^d)}(A_d) \times S(m_d)$. If $X = A_1 \cap A_2$, then $N_X = A_1 \cap S(4)$ and it has exactly two radical subgroups, $A_{(1,1,1)}$ and $A_1 \cap A_2$ up to conjugacy. Similarly, if $X = D_8 \cap A_2$, then $N_X = D_8 \cap S(4)$ and it has exactly two radical subgroups, $A_{(1,1,1,1)}$ and $D_8 \cap A_2$ up to conjugacy.

(2B). Let $D$ be a subgroup of $G = S(V)$ with a decomposition (2.2) such that $D = B(D)$ and $[V, D] = V$. In addition, let $R$ be a radical subgroup of $N = N(R)$. Suppose $D = D(1) = (A_1)^{m_1}$, $D(2) = (A_2)^{m_2}$ or $D(2)' = (D_8)^{m_2}$. Then $R$ is radical in $G$ and $K(R)$ is radical in $N$. If $L = N_N(K(R))$, then

$$L = \begin{cases} 
(A_1) \cap S(t_1) \times (D_8) \cap S(t_2') \times \prod_{X \in \Delta} (N_X)^{\beta_X} & \text{if } D = D(1), \\
N_{S(4)}(A_2) \cap S(t_2) \times (D_8) \cap S(t_2') \times \prod_{X \in \Delta(2)} (N_X)^{\beta_X} & \text{if } D = D(2), \\
(D_8) \cap S(t_2') \times \prod_{X \in \Delta(2)} (N_X)^{\beta_X} & \text{if } D = D(2)',
\end{cases}$$

where $t_1, t_2$ and $t_2'$ are some non-negative integers and $\beta_X = 0, 1$. Moreover, $N_N(R) \leq L$.

Proof. Suppose $D = D(1)$, so that $N = A_1 \cap S(m_1)$. It follows by [5, Proposition 4.7] or [6, Proposition 2.3 and the Remark 2.5] that $R = \prod_{i=1}^n R_i$, where $R_i = A_1 \cap R'_i$ with $R'_i \subseteq A_2$. Thus $R_i \subseteq \Psi \cup \Psi'$ and $B(R) = (A_1)^{\alpha} \times (D_8)^\beta$ for some integers $\alpha, \beta \geq 0$. Since $R/D = \prod_{i=1}^n R'_i$ is radical in $S(m_1)$, it follows that $m_{R/D}(A_c) \notin \{2, 4\}$, and hence $m_{R}(A_c) \notin \{2, 4\}$ for all $A_c \in \Psi^*$. Thus $R$ and then $K(R)$ are radical in $G$, $B(R) = (A_1)^{t_1} \times (D_8)^{t_2}$ with $t_1 + 2t_2 = m_1$ and $N(B(R)) = (A_1) \cap S(t_1) \times D_8 \cap S(t_2')$. Since $N(K(R)) \leq N(B(R)) \leq N$, $K(R)$ is radical in $N$. If $B(R)$ is radical in $G$, then $K(R) = B(R)$ and $N(K(R)) = N(B(R))$. Suppose $B(R)$ is non-radical in $G$. Then

$$K(R) = \begin{cases} 
A_1 \cap A_2 \times Y = R & \text{if } t_1 = 4 \text{ and } t'_2 \in \{2, 4\}, \\
(A_1)^{t_1} \times Y & \text{if } t_1 \notin \{2, 4\} \text{ and } t'_2 \in \{2, 4\}, \\
A_1 \cap A_2 \times (D_8)^{t_2'} & \text{if } t_1 = 4 \text{ and } t'_2 \notin \{2, 4\}
\end{cases}$$

for some $Y \in \Delta(2)$. Thus $N_N(K(R))$ is given by (2B). Since $N(R) \leq N(K(R))$, it follows that

$$N_N(R) = N(R) \cap N \leq N(K(R)) \cap N = N_N(K(R)) = L.$$
Suppose \( D = D(2) \), so that \( N = N_{S(4)}(A_2) \wr S(m_2) \). Since \( A_2 \) and \( D_8 \) are the only radical subgroups (up to conjugacy) in \( N_{S(4)}(A_2) \), it follows that \( R = \prod_{i=1}^{n} R_i \), where \( R_i = A_2 \wr R_i' \) or \( D_8 \wr R_i' \) with \( R_i' = A_2 \). Let \( B(R) = (A_2)^{t_2} \times (D_8)^{t_2} \), \( R(2) = \prod_i R_i \) and \( R(2)' = \prod_j R_j \), where \( i \) and \( j \) run over the indices such that \( R_i = A_2 \wr R_i' \) and \( R_j = D_8 \wr R_j' \), respectively. Then \( R(2)'/(A_2)^{t_2} \) is radical in \( GL(2,2) \wr S(t_2') \), since \( R/D \) is radical in \( GL(2,2) \wr S(m_2) \). Thus \( m_{R(2)'}/(D_8)^{t_2'}(A_2) \notin \{2,4\} \) and hence \( m_{R(2)'}(A_2) \notin \{2,4\} \) for each \( A_2 \in \Psi^* \). It follows that \( R \) is radical in \( G \). If \( B(R) \) is non-radical in \( G \), then \( t_2' \in \{2,4\} \) and \( R = R(2) \times Y \) for some \( Y \in \Delta(2) \), and \( K(R) = (A_2)^{t_2} \times Y \). Since \( N(B(R)) = N_{S(4)}(A_2) \wr S(t_2) \times (D_8) \wr S(t_2') \leq N \), it follows that \( N(K(R)) \) is given as (2B) and \( K(R) \) is radical in \( N \). A proof similar to above shows that \( N_{N}(R) \leq L \).

Suppose \( D = D(2)' \), so that \( N = D_8 \wr S(m_2') \). A proof similar to above shows that each component of \( R \) is an element of \( \Psi' \) and \( m_{R}(A_2) \notin \{2,4\} \) for all \( A_2 \in \Psi^* \). It follows that \( R \) is radical in \( G \) and \( B(R) = D \). If \( m_2' \notin \{2,4\} \), then \( K(R) = B(R) \). If \( m_2' = 2 \) or \( 4 \), then \( K(R) = R \in \Delta(2) \). This proves (2B).

Given sequences \( c = (c_1, \ldots, c_t) \) and \( z = (z_1, \ldots, z_t) \) of non-negative integers, let \( Q_{c,z} \) be the wreath product \( X \wr A_z \) in \( S(2|c|+|z|) \), where \( X = A_c \) or \( Q_c \). If \( X = A_c \), then \( Q_{c,z} = A_w \) and \( N_{S(2|w|)}(Q_{c,z})/Q_{c,z} \) is given by (2.1) with some obvious modifications, where \( w = (c_1, \ldots, c_t, z_1, \ldots, z_t) \). Suppose \( Q_{c,z} = X \wr A_z \) with \( X = Q_c \). Let \( d = |c| \) and \( M \) the underlying set of \( M \). Then we may suppose \( A_d \leq X \) and \( [M,X] = M \). Let \( X_1, X_2, \ldots, X_{2|z|} \) be copies of \( X \), and let \( U_1, U_2, \ldots, U_{2|z|} \) be disjoint underlying sets of \( X_1, X_2, \ldots, X_{2|z|} \). Then \( U = U_1 \cup U_2 \cup \cdots \cup U_{2|z|} \) can be taken as the underlying set of \( X \wr A_z \), and \( (\prod_{i=1}^{2|z|} X_i) \times A_z = X \wr A_z \). Let \( W_i \) be a normal subgroup of \( X_i \) isomorphic to \( A_d \). Then \( \{U_1, W_i\} = U_i \) and \( W = \prod_{i=1}^{2|z|} W_i \) is a normal abelian subgroup of \( X \wr A_z \), so that \( W \leq A(Q_{c,z}) \). If \( A \) is a normal abelian subgroup of \( X \), then \( A^{2|z|} \) is a normal abelian subgroup of \( Q_{c,z} \). It follows that \( (A)^{2|z|} \leq A(Q_{c,z}) \) and \( \prod_{i=1}^{2|z|} A(X_i) \leq A(Q_{c,z}) \). Since \( Q_c \) is nonabelian, it follows by \([2,2A]\) that each normal abelian subgroup of \( Q_{c,z} \) is a subgroup of \( \prod_{i=1}^{2|z|} X_i \). Thus \( A(Q_{c,z}) \leq \prod_{i=1}^{2|z|} A(X_i) \), so that \( A(Q_{c,z}) = \prod_{i=1}^{2|z|} A(X_i) \) and \( U_1, U_2, \ldots, U_{2|z|} \) are the orbits of \( A(Q_{c,z}) \) in \( U \). Since \( N_{S(2|c|+|z|)}(Q_{c,z}) \) normalizes \( A(Q_{c,z}) \), \( N_{S(2|c|+|z|)}(Q_{c,z}) \) permutes \( U_1, U_2, \ldots, U_{2|z|} \) among themselves, so that

\[
N_{S(2|c|+|z|)}(Q_{c,z}) = N_{S(2|c|)}(Q_c) \otimes N_{S(2|z|)}(A_z).
\]

In particular, \( N_{S(2|c|+|z|)}(Q_{c,z}) \) normalizes the subgroup \( \prod_{i=1}^{2|z|} X_i = (Q_c)^{2|z|} \) of \( Q_{c,z} \).

We claim that

\[
N_{N_{S(2|c|+|z|)}(W)}(Q_{c,z}) \simeq N_{S(2|c|)}(A_{|c|}) \otimes \otimes_{S(2|z|)}(A_z),
\]

where \( W = \prod_{i=1}^{2|z|} W_i \) is a normal abelian subgroup of \( Q_{c,z} \) such that each \( W_i \) is a
normal subgroup of \( X_i \) isomorphic to \( A_{|c|} \). Indeed, let

\[
N = N_{S(2^{|+|})}(Q_{c,x}), \quad H = N_{S(2^{|+|})}(A_{|c|}) \otimes N_{S(2^{|})}(A_x).
\]

If \( g \in N \), then \( g \) normalizes \( Q_{c,x} \), so that by (2.5) \( g = \text{diag} \{g_1, g_2, \cdots, g_2\} \sigma \), where \( g_i \in N_{S(M)}(Q_c) \) and \( \sigma \in N_{S(2^{|})}(A_x) \). Since \( W_i \leq X_i \) and \( g \) normalizes \( W \), it follows that \( g_i \) normalizes \( W_i \) and \( g \in H \). Conversely, if \( g \in H \), then \( g = \text{diag} \{g_1, g_2, \cdots, g_2\} \sigma \), where \( \sigma \in N_{S(2^{|})}(A_x) \) and \( g_i \in N_{S(2^{|})}(Q_c) \). Thus \( g \) normalizes \( Q_{c,x} \) and \( g \in N \), so that \( H = N \).

Let \( R = X \triangleleft A_x \) be a subgroup of \( S(2^{|+|}) \), where \( X = A_c \) or \( Q_c \). If \( R = A_c \triangleleft A_x \), then set \( QB(R) = B(R) \); if \( R = Q_c \triangleleft A_w \), then set \( QB(R) = (Q_c)^2 \) and \( B(R) = (A_{|c|})^2 \). By (2.5)

\[
N_{S(2^{|+|})}(Q_{c,x}) \leq N_{S(2^{|+|})}(QB(Q_{c,x})).
\]

(2C). Let \( G = S(n) = S(V) \), and let \( Q \) decompose as (2.2) with \( Q = B(Q) \) or \( Q = K(Q) \). Suppose \( R \) a radical subgroup of \( N(Q) \). Then there exists a corresponding decomposition

\[
V = M_0 \cup M_1 \cup \cdots \cup M_v, \\
R = R_0 \times R_1 \times \cdots \times R_v
\]

such that \( R_0 = \langle 1, M_0 \rangle \) and \( R_i = Q_{c,x} \leq S(M_i) \) for \( i \geq 1 \).

Proof. By (2B) and the remark before (2B), we may suppose \( Q = \prod_{d \geq 3} (A_d)^{m_d} \) and

\[
N = N(Q) = \prod_{d \geq 3} N_{S(2^d)}(A_d) \triangleleft S(m_d).
\]

By [6, Lemma (2.2)], \( R = \prod_{d \geq 3} R_d \), where \( R_d \) is a radical subgroup of \( N_{S(2^d)}(A_d) \) \( S(m_d) \) for all \( d \geq 3 \). By induction, we may suppose \( N \) acts transitively on \( V \), so that \( Q = (A_d)^{m_d} \). Thus \( R = Z_1 \times Z_2 \times \cdots \times Z_m \) and each \( Z_i = X \triangleleft Y \) for some subgroup \( Y = A_x \) of \( S(m_d) \) and a radical subgroup \( X \) of \( N_{S(2^d)}(A_d) \). By (2A) (a), \( X \in \{A_d, Q_c\} \), where \( c \) is a composition of \( d \). So \( Z_i = Q_{c,x} \) and this proves (2C). \( \square \)

Suppose \( R \) has a decomposition (2.7). Define \( QB(R) = R_0 \times \prod_{i=1}^v QB(R_i) \) and \( B(R) = R_0 \times \prod_{i=1}^v B(R_i) \).

(2D). Let \( R \) be a subgroup of \( G = S(n) = S(V) \) such that \( R \) decomposes as (2.7). Given sequences \( c = (c_1, c_2, \cdots, c_\ell) \), \( z = (z_1, z_2, \cdots, z_u) \), and
\(w = (w_1, w_2, \ldots, w_m)\) of positive integers, let \(M(c) = \bigcup_i M_i\), \(R(c) = \prod_i R_i\), \(M(w, z) = \bigcup_j M_j\), and \(R(w, z) = \prod_j R_j\), where \(i\) and \(j\) run over the indices such that \(R_i = A_c\) and \(R_j = Q_w \upharpoonright A_z\), respectively. Then

\[
N(R) = N_G(R) = S(M_0) \times \prod C(w, z)
\]

Moreover,

\[
S(M(c))(R(c)) \times \prod C(w, z)
\]

where \(M_c\) and \(M_{w, z}\) are the underlying sets of \(A_c\) and \(Q_w \upharpoonright A_z\), respectively, and \(t_c\), \(t_{w, z}\) are the numbers of components \(A_c\) and \(Q_w \upharpoonright A_z\) in \(R(c)\) and \(R(w, z)\), respectively. In particular, if \(D = QB(R)\), then \(N(R) \leq N(D)\).

Proof. Let \(D_i = QB(R_i)\), so that \(D = R_0 \times \prod_{i=1}^v D_i\), where \(v\) is given by the decomposition (2.7). If \(M\) is an \(R\)-orbit with \(|M| \geq 2\), then \(M = M_i\), for some \(i \geq 1\) and \(R_i = \{g \in R : gy = y\ \text{for all} \ y \in V \setminus M_i\}\). Thus \(N(R)\) acts as a permutation group on the set of pairs \((M_i, R_i)\). Suppose a component \(R_i\) is conjugate to a component \(R_j\), where \(1 \leq i, j \leq v\). Then \(|M_i| = |M_j|\), so that \(S(M_i)\) is conjugate to \(S(M_j)\) in \(G\). If \(R_i = A_c\), then \(R_i\) is radical in \(S(M_i)\), so is \(R_j\) in \(S(M_j)\). Thus \(R_j = A_c^e\) for some sequence \(e^c\) of non-negative integers. Since \(|M_i| = |M_j|\), it follows that \(|c| = |e^c|\) and so \(c = e^c\) as shown in the proof of [1, (2B)]. In particular, \(D_i\) is conjugate to \(D_j\). If \(R_i = Q_w \upharpoonright A_z\), then by the remark of (2A), \(R_i\) is non-radical in \(S(M_i)\), so is \(R_j\) in \(S(M_j)\). Thus \(R_j = Q_w \upharpoonright A_z\) for some sequences \(w^c\) and \(z^c\) of non-negative integers. Moreover, \(D_i = (Q_w)^{2|w|}\) and \(D_j = (Q_w)^{2|z|}\).

As shown in the proof of (2.5), an \(A(R_i^c)\)-orbit of \(M_i\) has \(2|w|\) elements and it is a underlying set of a factor \(Q_w\) of \(D_i\). Since \(A(R_i^c)\) is conjugate to \(A(R_j^c)\), it follows that \(|w| = |w^c|\), so that \(|z| = |z^c|\). Moreover, \(R_i\) induces a permutation group \(A_z\) on the set of \(A(R_i^c)\)-orbits and \(R_j\) induces a permutation group \(A_z^c\) on the set of \(A(R_j^c)\)-orbits. Thus \(z = z^c\) by [1, (2B)]. Let \(W = \prod_{k=1}^{2|w|} W_k\) be a normal subgroup of \(D_i\) such that \(W_k \simeq A_{|w|}\). Then \(W\) is a normal abelian subgroup of \(R_i\) and the underlying set \(U_k\) of \(W_k\) is an \(A(R_i^c)\)-orbit of \(M_i\). Suppose \(\sigma \in N(R)\) such that \(\sigma(M_i) = M_j\) and \(R_j^\sigma = R_j\). Then \(S(M_i)^\sigma = S(M_j)\) and \(A(R_i^c)^\sigma = A(R_j^c)\). Thus \(W^\sigma\) is a normal abelian subgroup of \(R_j\), so that \(W^\sigma \leq A(R_j^c)\). The image of an \(A(R_i)\)-orbit of \(M_i\) is an \(A(R_j)\)-orbit of \(M_j\). In particular, each \(\sigma(U_k)\) is an \(A(R_j)\)-orbit and it is the underlying set of a factor of \(D_j\). Thus \(W^\sigma = \prod_{k=1}^{2|w|} L_k\) is a normal subgroup of \(R_j\) such that \(L_k \simeq A_{|w|}\). So \(\sigma\) induces an isomorphism between \(N_{S(M_i^c)}(W)(R_i)/R_i\) and \(N_{S(M_j^c)}(W^\sigma)(R_j)/R_j\). By (2.6),

\[
N_{S(M_i^c)}(W)(R_i)/R_i \simeq N_{S(2|w|)}(A_{|w|})(Q_w)/Q_w \times N_{S(2|z|)}(A_z)/A_z
\]
It follows that \( w = w' \) as \( |w'| = |w| \). In particular, \( D_i \) is conjugate to \( D_j \). The remaining assertions of (2D) now follows easily.

Suppose \( R = \prod_{i=1}^{\ell} R_i \) is a subgroup of \( G \) with a decomposition (2.7). We define

\[
QK(R) = \prod_i QB(R_i) \times \prod_j R_j,
\]

where \( i \) runs over the indices such that either \( R_i \not\in \Delta^+ \) or \( R_i = S_{d_i} \triangleleft A_{c_i} \in \Delta^+ \) but \( m_{QB(R_i)}(S_{d_i}) \not\in \{2,4\} \), and \( j \) runs over the indices such that \( R_j = S_{d_j} \triangleleft A_{c_j} \in \Delta^+ \) and \( m_{QB(R_j)}(S_{d_j}) \in \{2,4\} \). If \( R \) and \( W \) are subgroups given by (2C) and they are \( G \)-conjugate, then \( QK(R) \) and \( QK(W) \) are also \( G \)-conjugate. Since \( P_2 \) is radical in \( N(P_1) \), it follows that \( P_2 = QK(P_2) \). Next, we study the structure of \( P_i \) for \( i \geq 3 \).

Let \( G = S(n) = S(V) \) and let

\[
(2.8) \quad H = \prod_{X \in \Delta^+} (N_X)^{\alpha_X} \times \prod_{c \in \Omega} N_{S_{|c|}(A_{|c|})}(X_c) \triangleleft S(t_c)
\]

be a subgroup of \( G \), where \( N_X = N_{S(H)}(X) \), \( \alpha_X \) and \( t_c \) are non-negative integers, \( X_c \in \{ A_{|c|}, Q_c, D_8 \} \) and \( \Omega = \Omega(H) = \{ w_1, w_2, \ldots, w_s \} \) is a subset of sequences \( w_i \) of non-negative integers. (It may happen that \( w_i = w_j \) for \( i \neq j \)). In addition, let \( H^+ = \prod_{X \in \Delta^+} (N_X)^{\alpha_X} \), \( H_c = N_{S_{|c|}(A_{|c|})}(X_c) \triangleleft S(t_c) \) and \( H^* = \prod_{c \in \Omega} H_c \).

(2E). Suppose \( W \) is a radical subgroup of \( H \). Then \( W = W^+ \times W^* \) such that \( W^+ = \prod_{Y \in \Delta^+} Y^{\beta_Y} \) and \( W^* = \prod_{c \in \Omega} W_c \), where \( Y \) and \( W_c \) are radical subgroups of \( N_X \) and \( H_c \), respectively and \( \beta_Y \) is a non-negative integer.

(a) Each \( W_c \) has a decomposition (2.7), and if \( |c| \in \{0,1,2\} \), then \( W_c \) is a radical subgroup of \( S(2^{|c|+t_c}) \). Thus \( W \) has a decomposition (2.7).

(b) Let \( QK_H(W) = W^+ \times \prod_{c \in \{0,1,2\}} K(W_c) \times \prod_{|c| \geq 3} QK(W_c) \), where \( c \) runs over \( \Omega \). In addition, let \( Q = QK_H(W) \) and \( L = N_H(Q) \). Then \( Q \) is radical in \( H \) and \( N_H(W) \leq N_H(Q) \). In particular, \( O_2(H) \leq Q \) and \( QK_H(Q) = Q \). Moreover, \( L = L^+ \times L^* \) such that \( L^+ = \prod_{Y \in \Delta^+} (N_Y)^{\delta_Y} \) and

\[
L^* = \prod_{w \in \Omega(L)} N_{S_{|w|}(A_{|w|})}(Y_w) \triangleleft S(t_w),
\]

where \( Y_w \in \{ A_{|w|}, Q_w, D_8 \} \), and \( \delta_Y \) and \( t_w \) are non-negative integers. In particular, \( L \) has a decomposition (2.8).

(c) Let \( R \) be a radical subgroup of \( L = N_H(Q) \). If \( QK_L(R) = Q \), then \( R \) is radical in \( H \) and \( N_H(R) = N_L(R) \).
Proof. The decomposition of \( W \) follows by [6, Lemma 2.2]. We now prove (a) and (b). If \( Y \) is a radical subgroup of \( N_X \) and \( X \in \Delta(d) \) for some \( d \geq 1 \), then \( N_X = N_Y \), or \( Y \in \{ X, S_d \} \), so \( Y \in \{ X, S_d \} \), and \( X \in \Delta(d) \). Thus \( W^+ = \prod_{Y \in \Delta(d)} (Y)^{\beta_Y} \) for some integers \( \beta_Y \).

If \( |c| \in \{ 0, 1, 2 \} \), then by (2B), \( W_c \) is radical in \( S(2|c|+t) \), so that \( K(W_c) \) is a radical subgroup of both \( S(2|c|+t) \) and \( H_c \). In particular, \( W_c \) has a decomposition (2.2). The normalizer \( N_{H_c}(K(W_c)) \) is given by [1, (2B)] or (2B). Suppose \( d = |c| > 3 \). Then \( W_c = W_1 \times \ldots \times W_m \) such that \( W^c = Z_{I_{A}}(A) \), where \( Z_{I_{A}}(A) \) is a radical subgroup of \( N_{S(2|c|+t)}(A) \). By (2B) (a), \( Z_{I_{A}}(A) = Z_{I_{A}}(A) \). Thus \( W^c = \Pi_v (Y^v) \), where \( Y^v \) is an integer and \( \beta^v = 0, 1 \).

If \( Z_{I_{A}}(A) \) is not a Sylow 2-subgroup of \( N_{S(2|c|+t)}(A) \), then \( Z_{I_{A}}(A) \) is not a self-normalizer. If \( Z_{I_{A}}(A) \) is a Sylow 2-subgroup of \( N_{S(2|c|+t)}(A) \), then \( \beta^v = 0, 1 \). It follows that \( Z_{I_{A}}(A) = Z_{I_{A}}(A) \), and so \( Z_{I_{A}}(A) \) is well-defined. By induction, we may suppose \( \Omega = \{ c \} \) and \( d = |c| \). Thus \( W = W_c \) and \( H = H_c \). Suppose \( m_Q(B(W))(S_d) \in \{ 2,4 \} \). Since \( V_F \) is radical in \( H \), it follows that \( m_Q(B(W))(S_d) \in \{ 2,4 \} \). If \( m_Q(B(W))(S_d) = 2 \), then \( \pi_2(W) = 1 \). If \( m_Q(B(W))(S_d) = 4 \), then \( \pi_2(W) = 1 \) for one \( E \in \{ S_d, M_2, S_d, M_2 \} \). It follows that \( \pi_2(W) = 1 \) for one \( E \in \{ S_d, M_2, S_d, M_2 \} \). Thus \( m_2(W) = 1 \).

(c) In the notation above, \( Q = \prod_{E \in \Delta} \prod_{c \in \Omega} Q(E) \), where \( Y \) and \( Q(E) \) are radical subgroups of \( N_X \) and \( H_c \), respectively. In addition, \( Q(E) = \prod_{Z \in \Delta} (Z)^{\gamma_Z} \prod_{w \in W} Z \) for some \( \gamma_Z = 0, 1 \). Since \( R \) is radical in \( L \), it follows that \( R = \prod_{E \in \Delta} (E)^{\epsilon_E} \prod_{c \in \Omega} R_c \), where \( E \) and \( R_c \) are radical subgroups of \( N_Y \) and \( L_c = N_{H_c}(Q(E)) \), respectively. But \( K(R)(R) = Q \), so \( E = Y \) and \( \epsilon_E = \beta_Y \). By induction, we may suppose \( \Omega = \{ c \} \) and \( Q = Q(E) \).

If \( |c| = 0 \), then \( H = S(t_c) \), \( Q = Q^* \times Q^* \) with \( Q = Q(K) \) and \( R \) is given by (2C), where \( Q^+ = \prod_{Z \in \Delta} Z^{\gamma_Z} \) with \( \gamma_Z = 0, 1 \) and \( Q^* = (D_8)^{m_2} \times \prod_{d \geq 0} (A_d)^{m_2} \). Thus \( L = L^* \times L^* \) and \( R = R^* \times R^* \), so \( L^* = D_8 \times S(m_2) \times \prod_{d \geq 0} N_{S(2d)}(A_d) \), \( R^* = D_8 \times \prod_{d \geq 0} R_d \). So \( E \) is a radical subgroup of \( N_Z \), \( R^2 \) is a radical subgroup of \( D_8 \times S(m_2) \) and \( R_d \) is a radical subgroup of \( N_{S(2d)}(A_d) \). Since \( Q(K_L)(R) = Q \), it follows that \( E = Z \) and \( \epsilon_E = \gamma_Z \), so that \( R^+ = Q^+ \). By (2B), \( R_d \) and \( R^2 \) are radical in \( S(2d+m^2) \) and \( S(2d+m^2) \), respectively, where \( d = 1, 2 \). By definition,

\[
Q = Q(K_L)(R) = Q^+ \times Q(K(R_d^2)) \times \prod_{d = 0, 1, 2} K(R_d) \times \prod_{d \geq 3} Q(K(R_d)).
\]

Thus \( R_0 = (A_0)^{m_0} \), \( K(R_1) = (A_1)^{m_1} \), \( K(R_2) = (D_8)^{m_2} \), \( K(R_2) = (A_2)^{m_2} \), and \( m_1, m_2 \notin \{ 2, 4 \} \), since \( Q \) is radical in \( H \). By definition, \( K(R_d^2) = B(R_d^2) \) and \( K(R_d) = B(R_d) \) for \( d = 1, 2 \). Similarly, since \( Q(K(R_d)) = (A_d)^{m_4} \), it follows that
$R_d$ has a decomposition (2.2) and $\text{QK}(R_d) = \text{QB}(R_d) = \text{B}(R_d)$ for $d \geq 3$. If $\gamma_Z = 1$ for some $Z \in \Delta$, then $Z = A_1 \updownarrow A_2$ or $Z \in \Delta(2)$. In the former case $m_{B(Q)}(A_1) = 4$, since $Q = \text{QK}_H(Q) = K(Q)$. So $m_1 = 0$ and $R_1 = 1$. In the latter cases $m_{B(Q)}(D_8) \in \{2, 4\}$, so that $m_q' = 0$ and $R_q' = 1$. In particular, $m_{R^*}(Z) = 0$. Since $R$ is radical in $L$, it follows that $R$ is radical in $G$ and $\text{QK}_L(R) = K(R) = Q$. Thus $N_H(R) \leq N_H(Q) = L$ and $N_L(R) = N_H(R)$.

If $|c| = 1$, then $H = A_1 \updownarrow S(t_c)$ and $Q = Q^+ \times Q^*$, where $Q^+ = \prod_{Z \in \Delta(d)}(Z)^{\gamma_Z}$ with $\gamma_Z = 0, 1$ and $Q^* = (A_d)^{t_1} \times (D_8)^{t_2}$. If $|c| = 2$, then $H = \text{NS}_4(X_c) \updownarrow S(t_c)$ and $Q = Z^* \times Q^*$, where $Z \in \Delta(2)$, $\gamma_Z = 0, 1$ and $Q^* = (A_2)^{t_2} \times (D_8)^{t_2}$ or $(D_8)^{t_2}$ according as $X_c = A_2$ or $D_8$. The same proof as above shows that $K(R) = Q$, $N_H(R) = N_L(R)$ and $R$ is radical in $H$.

\textbf{Remark.} In the notation of (2E), suppose $O_2(H) = \prod_{X \in \Delta^+}X^{\alpha_X} \times \prod_{c \in \Omega}(X_c)^{t_c}$. Then $N_X$, $\alpha_X$, $H_c$ and $t_c$ are determined uniquely by $H$. In particular, $\text{QK}_H(R)$ is independent of the choice of decompositions of $H$. Indeed, the underlying set $U$ of $H_c$ and $U_X$ are $H$-orbits of $V$, and $N_X = \{y \in H : gy = y \}$ for all $y \in V \setminus U_X$ and $H_c = \{y \in H : gy = y \}$ for all $y \in V \setminus U$. Thus $H_c$ and $N_X$ are determined by $H$. In addition, $X = O_2(N_X) \in \Delta^+$, $(X_c)^{t_c} = O_2(H_c) \notin \Delta^+$ and $\alpha_X = m_{Q^*}(S_d)$.

Let $G = S(n) = S(V)$ and let $C \in \mathcal{Q}_R$ be the chain given by (2.3) with $w \geq 1$. In addition, let $R$ be a radical subgroup of $N(C)$. Then $R$ has a decomposition (2.7). If $D = \text{QK}_N(C)(R)$, then $D$ is radical in $N(C)$, $P_w \leq D$, and $N_{N(C)}(R) = N_{N(C)}(D)$. In addition, if $P_w \neq D$ and $C' = P_0 < P_1 < \cdots < P_w < D$,
then \( C' \in QR \), \( R \) is radical in \( N(C') \), and \( N_{N(C')}(R) = N_{N(C)}(R) \). If \( P_w = D \), then \( R \) is radical in \( N(C_{w-1}) \) and \( N_{N(C_{w-1})}(R) = N_{N(C)}(R) \).

Proof. Since \( P_1 \) is radical in \( G \) and \( QK_G(P_1) = K(P_1) = P_1 \), it follows that \( N(C_1) = N(P_1) \) has a decomposition (2.8) and \( P_2 \) is radical in \( N(C_1) \) with \( QK_{N(C_1)}(P_2) = P_2 \). By (2E) (b), \( N(C_2) = N_{N(C_1)}(P_2) \) has a decomposition (2.8) and by induction, \( N(C) \) has a decomposition (2.8). Thus \( R \) decomposes as (2.7) and \( D = QK_{N(C)}(R) \) is well-defined. By (2E) (b) again, \( D \) is radical in \( N(C) \) and moreover, \( N_{N(C)}(R) \leq N_{N(C)}(D) \). Thus \( N_{N(C)}(R) \leq N(C) \) and \( N_{N(C)}(R) = N_{N(C)}(R) \). If \( P_w = D \), then apply (2E) (c) to \( H = N(C_{w-1}) \) and \( Q = P_w \). Thus \( R \) is radical in \( N(C_{w-1}) \) and \( N_{N(C_{w-1})}(R) = N_{N(C)}(R) \). This proves (2F).

We can now prove the main result of this section.

(2G). Let \( G = S(n) = S(V) \) with \( O_2(G) = \{1_V\} \), and let \( B \) be a positive defect 2-block of \( G \) and \( u \) an integer. Then

\[
\sum_{C \in QR/G} (-1)^{|C|} k(N(C), B, u) = \sum_{C \in QR/G} (-1)^{|C|} k(N(C), B, u)
\]

where \( QR/G \) is a set of representatives for the \( G \)-orbits in \( QR \).

Proof. It suffices to show that

(2.9) \[
\sum_{C \in \mathcal{M}/G} (-1)^{|C|} k(N(C), B, u) = 0,
\]

where \( \mathcal{M} = R \setminus QR \). Suppose \( C \in \mathcal{M} \) is given by (1.1). Then \( C_0 \in QR \) and \( C = C_w \not\in QR \), so that there must be some minimal \( m = m(C) \in \{0, 1, \ldots, w - 1\} \) such that \( C_m \in QR \) and \( C_{m+1} \not\in QR \). Since \( P_{m+1} \) is radical in \( N(C_m) \), \( P_{m+1} \) has a decomposition (2.7). We can apply (2F) to \( C_m \). If \( D = QK_{N(C_m)}(P_{m+1}) \), then \( D \neq P_{m+1} \), \( D \) is radical in \( N(C_m) \) and \( N_{N(C_m)}(P_{m+1}) \leq N_{N(C_m)}(D) \), so that \( P_m \leq D \). Moreover, if \( P_m = D \), then \( P_{m+1} \) is radical in \( N(C_{m-1}) \) and \( N_{N(C_{m-1})}(P_{m+1}) = N_{N(C_m)}(P_{m+1}) \).

Define

\[
\varphi(C) : \begin{cases} 
1 < P_1 < \ldots < P_{m-1} < P_m < P_{m+1} < \ldots < P_w & \text{if } P_m = D, \\
1 < P_1 < \ldots < P_m < D < P_{m+1} < \ldots < P_w & \text{if } P_m < D.
\end{cases}
\]

Then \( \varphi(C) \in \mathcal{M} \) and \( N(C) = N(\varphi(C)) \). Moreover, \( \varphi(\varphi(C)) = C \) and \( |\varphi(C)| = |C| \pm 1 \). Thus \( \varphi \) is a bijection from \( \mathcal{M} \) to itself. This implies (2.9).

\( \square \)
3. More reductions and the proof of the conjecture

In this section we shall follow the notation of Sections 1 and 2. Let $QR^0$ be the $G$-invariant subfamily of $QR$ consisting of chains $C$ given by (2.3) such that $m_{P_t}(S_d \cdot D_8) = 0$ for all $d \geq 1$ except when $d = 1$, in which case if $m_{P_t}(A_1 \cdot D_8) \neq 0$, then $(A_2)^2$ is a component of some $P_k$ for $k < i$, and $[V, (A_2)^2] = [V, D_8 \cdot A_1]$ and $(A_2)^2 \leq D_8 \cdot A_1 = A_1 \cdot D_8$. If $QR^1 = QR \setminus QR^0$, then

$$QR = QR^0 \cup QR^1 \quad \text{(disjoint)}.$$ 

We shall first reduce Dade’s conjecture to the family $QR^0$.

Fix integer $d \geq 1$. Let $X \in \{S_d \cdot A_2, S_d \cdot D_8\}$, and let $X \times Q$ be a subgroup of $G = S(n) = S(V)$ with a decomposition (2.7). If $U_X = [V, X]$ and $U_Q = V \setminus U_X$, then $V = U_X \cup U_Q$. Suppose $C(0) \in QR^0$ is a fixed radical chain with $|C(0)| = s$. Let $QR(C(0), X \times Q)$ be the subfamily of $QR$ consisting of all chains $C$ given by (2.3) such that its $s$-th subchain $C_s$ is $C(0)$ and its $(s + 1)$-st subgroup $P_{s+1}$ is $X \times Q$ up to conjugacy in $G$. Since $X \in \Delta^+$ and $N(C_{s+1})$ has a decomposition (2.8), it follows that $N(C_{s+1}) = N_X \times N(s+1)$, where $N_X = N_{G(U_X)}(X)$ and $N(s+1) \leq S(U_Q)$. Let $P_t$ be the $t$-th subgroup of $C$ with $t \geq s + 1$. Then $P_t = Y(t) \times Z(t)$, where $Y(t) \in \{X, S_d \cdot A_1 \cdot A_1\}$ and $Z(t) \leq N(s+1)$. Note that $QR(C(0), S_d \cdot D_8 \times Q) \subseteq QR^1$ whenever $d \geq 2$.

Let $M = M(C(0), S_d \cdot A_2 \times Q)$ be the subset of $QR(C(0), S_d \cdot A_2 \times Q)$ consisting of all chains $C$ such that $Y(t) = S_d \cdot D_8$, that is, $P_t = S_d \cdot D_8 \times Z(t)$ (up to conjugacy) for some $t \geq s + 2$. In particular, $M(C(0), S_d \cdot A_2 \times Q) \subseteq QR^1$ and

$$QR^1 = \bigcup_{C(0), S_d \cdot A_2 \times Q} S(C(0), S_d \cdot A_2 \times Q) \quad \text{(disjoint),}$$

where $S(C(0), S_d \cdot A_2 \times Q) = M(C(0), S_d \cdot A_2 \times Q) \cup (QR(C(0), S_d \cdot D_8 \times Q) \cap QR^1)$, $C(0)$ runs over $QR^0$ and $S_d \cdot A_2 \times Q$ runs over subgroup of $G$ with a decomposition (2.7).

For $C \in M$, denote by $m = m(C)$ the smallest integer such that $P_m = S_d \cdot D_8 \times Z(m)$, so that $Q \leq Z(m)$. Let $M_0$ and $M_+$ be the subsets of $M$ consisting of all chains $C$ such that $Z(m) = Q$ and $Z(m) \neq Q$, respectively.

(3A). In the notation above, suppose $S = S(C(0), S_d \cdot A_2 \times Q)$. Then

$$\sum_{C \in S/G} (-1)^{|C|} k(N(C), B, u) = 0$$

for all 2-blocks $B$ and integers $u \geq 0$.

Proof. Set $X = S_d \cdot A_2$. Suppose $C \in M_+$ is given by (2.3). Then $m = m(C) \geq s + 2$ and $P_{m-1} = X \times Z(m - 1)$. So $Z(m - 1) \leq Z(m)$ and $N(C_t) = N_X \times N(t)$.
for \(s + 1 \leq t \leq m - 1\). In particular, \(Z(m - 1)\) is a radical subgroup of \(N(m - 2)\) and moreover, if \(m = s + 2\), then \(Q = Z(m - 1) < Z(m)\). Define a map \(\varphi\) such that

\[
\varphi(C) : \begin{cases} 
1 < P_1 < \ldots < P_{m-2} < P_m < \ldots < P_w & \text{if } Z(m-1) = Z(m), \\
1 < P_1 < \ldots < P_{m-1} < X \times Z(m) < P_m < \ldots < P_w & \text{if } Z(m-1) < Z(m).
\end{cases}
\]

Then \(\varphi(C) \in M_+\), \(N(C) = N(\varphi(C))\), \(\varphi(\varphi(C)) = C\) and \(|\varphi(C)| = |C| \pm 1\). Thus

\[
\sum_{C \in (M_+)/G} (-1)^{|C|} k(N(C), B, u) = 0.
\]

Suppose \(C \in M_0\) is given by (2.3). Since \(X\) and \(S_d \wr D_8\) are the only two radical subgroups of \(N_X = S_d \wr (4)\) up to conjugacy containing \((S_d)^4\), it follows that \(m(C) = s + 2\), that is, \(P_{s+2} = S_d \wr D_8 \times Q\). Thus

\[
g(C) : 1 < P_1 < \ldots < P_s < P_{s+2} < \ldots < P_w
\]

is a chain of \(Q\mathcal{R}(C(0), S_d \wr D_8 \times Q) \cap Q\mathcal{R}^1\) and \(N(C) = N(g(C))\). Conversely, suppose

\[
C' : 1 < P'_1 < \ldots < P'_s < P'_{s+1} < \ldots < P'_{w'}
\]

is a chain of \(Q\mathcal{R}(C(0), S_d \wr D_8 \times Q) \cap Q\mathcal{R}^1\), then \(P'_{s+1} = S_d \wr D_8 \times Q\) and

\[
h(C') : 1 < P'_1 < \ldots < P'_s < X \times Q < P'_{s+1} < \ldots < P'_{w'}
\]

is a chain of \(M_0\). It is clear that \(g(h(C')) = C'\), \(h(g(C)) = C\) and \(|g(C)| = |C| - 1\). Thus

\[
\sum_{C \in (M_0 \cup (Q\mathcal{R}(C(0), S_d \wr D_8 \times Q) \cap Q\mathcal{R}^1))}/G (-1)^{|C|} k(N(C), B, u) = 0.
\]

This proves (3A).

\[\square\]

It follows by (3A) that Dade's conjecture can be reduced to the family \(Q\mathcal{R}^0\). Let \(Z = (A_1)^{m_1}\) be a radical subgroup of \(S(2^{m_1}) = S(U_Z)\), and \(W \neq Z\) a radical subgroup of \(N_Z = N_{S(U_Z)}(Z)\) such that \(K(W) = W\). As shown in the proof of (2B)

\[
W \in \Phi = \{D_8 \wr A_2 \times A_1 \wr A_2, D_8 \wr A_2 \times (A_1)^{t_1}, A_1 \wr A_2 \times (D_8)^{t_2}, (A_1)^{t_1} \times (D_8)^{t_2}\},
\]

where \(t_1 \notin \{2, 4\}\). If \(W = A_1 \wr A_2 \times (D_8)^{t_2}\), then \(t_2 \neq 0\), since \(Z\) is radical in \(S(U_Z)\). Similarly, if \(W = (A_1)^{t_1} \times (D_8)^{t_2}\) and \(t_1 = 0\), then \(t_2 \neq 1\). Thus \(N_{N_{S(U_Z)}(Z)}(W) = \)
Let $\mathcal{Q}(C(0), Z \times Q) = \mathcal{Q}(C(0), Z \times Q) \cap \mathcal{Q}(C(0), Z \times Q, W)$ be the subset of $\mathcal{Q}(C(0), Z \times Q)$ consisting of chains $C$ given by (2.3) such that $P_m = W \times Z(m)$ (up to conjugacy) for some $m \geq s + 2$ and $P_t = Z \times Z(t)$ for $s + 1 \leq t \leq m$, where $\mathcal{Q}(C(0), Z \times Q)$ is defined as in (3A) and $Z(t) \leq S(U_Q)$.

(3B). In the notation above, let $S = \mathcal{M}(C(0), Z \times Q, W) \cup \mathcal{Q}(C(0), W \times Q)$, where $W \in \Phi$. Then (3.1) holds for $S$.

Proof. Replacing $X = S_d \cap A_2$ by $Z$, $S_d \cap D_8$ by $W$ and some obvious modifications in the proof of (3A), we have (3B).

Let $\mathcal{Q}^+\hat{\mathcal{R}}$ be the complement of $\bigcup_{C(0), Z, W, Q}(\mathcal{M}(C(0), Z \times Q, W) \cup \mathcal{Q}(C(0), W \times Q))$ in $\mathcal{Q}$, where $C(0)$ runs over $\mathcal{Q}(0)$, $Z = (A_1)^{m_1}$ with $m_1 \notin \{2, 4\}$, $W$ runs over $\Phi$, and $Q$ runs over subgroups of $S(U_Q)$ with a decomposition (2.7). It follows by (3A) and (3B) that

$$\sum_{C \in \mathcal{Q}^+/G} (-1)^{|C|} k(N(C), B, u) = \sum_{C \in \mathcal{Q}/G} (-1)^{|C|} k(N(C), B, u).$$

Let $D = P_1$ be the first non-trivial subgroup of $C \in \mathcal{Q}^+\hat{\mathcal{R}}$. Then $D = K(D)$ and $D = D^+ \times D^\ast$ decomposes as (2.4). Now

$$\Delta = \{A_1 \mid A_2, D_8 \mid A_1, D_8 \mid A_2, D_8 \mid D_8\}.$$

By (3A), $m_D(A_1 \mid A_2) = 0$. Since $D_8 \mid A_2 \in \Phi$, it follows by (3B) that $m_D(D_8 \mid A_2) = 0$. Similarly, $m_D(D_8) = 0$ and $m_Q(A_1 \mid A_2) = 0, 1$. If $m_D(D_8 \mid A_1) \neq 0$, then $D$ is not the first non-trivial subgroup of any chain in $\mathcal{Q}$. Suppose $m_D(A_1 \mid A_2) \neq 0$. Since $A_1 \mid A_2 \times (D_8)^{t_2} \in \Phi$ for $t_2 \geq 1$, it follows by (3B) that $m_D(D_8) = 0$. But $K(D) = D$, so $m_B(A_1) = 4$ and $m_B(D^+) = 1$. Similarly, if $m_D(D_8) \neq 0$, then $m_D(A_1 \mid A_2) = m_D(A_1) = 0$. Thus

$$(3.2) \quad D = D(0) \times X \times \prod_{d \geq 2} D(d),$$

where $D(d) = (A_d)^{m_d}$ for $d \neq 1$ and $X \leq S(2^{m_1})$ such that

$$X = \begin{cases} D_8 & \text{if } m_1 = 2, \\ A_1 \mid A_2 & \text{if } m_1 = 4, \\ (A_1)^{m_1} & \text{if } m_1 \notin \{2, 4\}. \end{cases}$$

For simplicity, we denote by $D(1)$ the subgroup $X$. Thus $N(D) = \prod_{d \geq 0} N(D_d)$ such that $N(D_d) = N_{S(2^d)}(A_d) \mid S(m_d)$. 

Suppose \( Q = P_2 \) is the second subgroup of \( C \). Then \( Q \) is a radical subgroup of \( N(D) \), so that \( Q = \prod_{d \geq 0} Q_d \), where \( Q_d \) is a radical subgroup of \( N(D)_d \). Thus \( Q_0 \) is of form (3.2). It follows by (3B) that \( Q_1 = D(1) \). In general, if \( W = P_i \) is the \( i \)-th subgroup of \( C \) for \( i \geq 1 \), then \( W = \prod_{d \geq 0} W_d \) with \( W_1 = D(1) \) and \( W_d \leq N(D)_d \) for all \( d \geq 1 \). By (3B) again, if \( m_W(D(\ell A_2)) = 0 \), then there is some \( 1 \leq k \leq i - 1 \) such that \( (A_2)^4 \) is a component of \( P_k \), \( [V, (A_2)^4] = [V, (D(8) \ell A_2)] \) and \( (A_2)^4 \leq (D(8) \ell A_2) \).

Let \( \Delta' = \{D_8, A_1 \ell A_2\} \) and let

\[
P = \prod_{X \in \Delta'} (X)^{\alpha_X} \times \prod_{d=0}^s (A_d)^{m_d},
\]

be a subgroup of \( G \), where \( \alpha_X \) and \( m_d \) are non-negative integers. Set \( P^+ = \prod_{X \in \Delta'} (X)^{\alpha_X} \) and \( P^* = \prod_{d=0}^s (A_d)^{m_d} \). Let \( U_X \) be the underlying set of \( X \in \Delta' \) such that \( U_X = [U_X, X] \), and \( N_X = N_{S(U_X)}(X) \).

Suppose \( C \in \mathcal{QR}^+ \) is given by (2.3). Denote by \( C_V(C) \) the fixed-point set \( C_V(P_w) \) of the final subgroup \( P_w \) of \( C \). Let \( \ell = \ell(C) \) be the largest integer such that \( P_\ell \) has a decomposition (3.3), and let \( \mathcal{QR}^+(P) \) be the subset of \( \mathcal{QR}^+ \) consisting of all chains \( C \) given by (2.3) such that \( P_\ell = P \). Then \( \mathcal{QR}^+ = \bigcup_P \mathcal{QR}^+(P) \) (disjoint),

where \( P \) runs over subgroups of \( G \) with a decomposition (3.3). Thus

\[
(3.4) \quad N(C_\ell) \simeq S(V(0)) \times \prod_{X \in \Delta'} (N_X)^{\alpha_X} \times \prod_{d=1}^s \left( \prod_{j=1}^{h_d} N_{S(2^d)}(A_d) \ell S(\lambda_{d,j}) \right),
\]

where \( (\lambda_{d,1}, \ldots, \lambda_{d,h_d}) \) is a partition of \( m_d \) and \( V(0) = C_V(P) \).

Fix partitions \( \lambda_d = (\lambda_{d,1}, \ldots, \lambda_{d,h_d}) \) of \( m_d \), and set \( \lambda = (\lambda_1, \ldots, \lambda_s) \). Let \( \mathcal{QR}^+(P, \lambda) \) be the subset of \( \mathcal{QR}^+(P) \) consisting of all chains \( C \) such that \( N(C_\ell) \) is given by (3.4). Then \( \mathcal{QR}^+(P) = \bigcup_\lambda \mathcal{QR}^+(P, \lambda) \) (disjoint),

where \( \lambda \) runs over all \( s \)-tuple partitions \( \lambda_d \) of \( m_d \).

Suppose \( W \) is a \( G \)-conjugate of \( P \). Then \( W^g = P \) for some \( g \in G \), and \( C^g \in \mathcal{QR}^+(P) \) for each \( C \in \mathcal{QR}^+(W) \). Thus a set of representatives for the \( N(P) \)-conjugacy classes of \( \mathcal{QR}^+(P) \) can be regarded as a set of representatives for the \( G \)-conjugacy classes of the \( G \)-orbit containing \( \mathcal{QR}^+(P) \). It is clear that \( \mathcal{QR}^+(P) \) and \( \mathcal{QR}^+(P, \lambda) \) both are \( N(P) \)-invariant.

Let \( \mathcal{QR}'(P, \lambda) = \{C \in \mathcal{QR}^+(P, \lambda) : C_V(C) = C_V(P)\} \), and let \( \mathcal{QR}''(P, \lambda) \) be the complement of \( \mathcal{QR}'(P, \lambda) \) in \( \mathcal{QR}^+(P, \lambda) \).
(3C). In the notation above,

$$\sum_{C \in \mathcal{QR}^+(P, \lambda)/N(P)} (-1)^{|C|} k(N(C), B, u) = 0$$

for all 2-blocks $B$ and integers $u \geq 0$.

Proof. Let $C : 1 < P_1 < \ldots < P_\ell = P < P_{\ell+1} < \ldots < P_w$ be a chain of $M = \mathcal{QR}''(P, \lambda)$. Then $C_V(P_w) \neq C_V(P)$. Let $m = m(C)$ be the smallest integer such that $C_V(P_m) \neq C_V(P) = V(0)$. Then $\ell + 1 \leq m \leq w$.

Let $V(+) = \{V, P\}$ and $P(+) = P^+ \times \prod_{d \geq 1} (A_d)^{m_d},$ where $P^+$ is defined after (3.3). Then $P = P(0) \times P(\cdot)$ and $N(P) = S(V(0)) \times N(P)(\cdot)$, where $P(0) = \langle 1_{V(0)} \rangle$ and $N(P)(\cdot) = N_S(V(\cdot))(P(\cdot))$. Thus $N(C_{m-1}) = S(V(0)) \times N(C_{m-1})(\cdot)$, where $N(C_{m-1})(\cdot) \leq S(V(\cdot))$. So $W = P_m$ decomposes as $W = W_0 \times W_+$, where $W_0$ is a radical subgroup of $S(V(0))$ and $W_+ \leq S(V(\cdot))$. In particular, $W_0$ is a non-trivial subgroup with a decomposition (3.3). By definition, $P_m$ has no decompositions as that of (3.3), so that $m_{W_+}(Z) \neq 0$ for some $Z \in \{Q_\varepsilon, S_d \cdot A_1, S_d \cdot A_2\}$, where $d \geq 2$ and $\varepsilon$ is a sequence of positive integers. Let $D = \langle 1_{V(0)} \rangle \times W_+$ and $R = P_{m-1}$. Then $D < P_m$, $R = R(0) \times R(\cdot) \leq D$, and $D$ is radical in $N(C_{m-1})$, where $R(0) = \langle 1_{V(0)} \rangle$ and $R(\cdot) = O_2(N(C_{m-1})(\cdot))$. If $R(\cdot) = W_+$, then $m \geq \ell + 2$ and $P_m = W_0 \times W_+$ is radical in $N(C_{m-2})$. Let $\varphi(C) \in \mathcal{QR}^+$ such that

$$\varphi(C) : \begin{cases} 1 < P_1 < \ldots < P_{m-2} < P_m < \ldots < P_w & \text{if } P_{m-1} = D, \\ 1 < P_1 < \ldots < P_{m-1} < D < P_m < \ldots < P_w & \text{if } P_{m-1} < D. \end{cases}$$

Then $\varphi(C) \in M$, $N(C) = N(\varphi(C))$, $|\varphi(C)| = |C| \pm 1$ and $\varphi(\varphi(C)) = C$. Thus $\varphi$ is a permutation of $M$ and preserves $N(P)$-classes in $M$. This implies (3C).

Let $\mathcal{QR}'_1(P, \lambda)$ be the subset of $\mathcal{QR}'(P, \lambda)$ consisting of all the chains whose final subgroup is $P$. For any $C(0) \in \mathcal{QR}'_1(P, \lambda)$ with length $|C(0)| = \ell$, let $\mathcal{QR}'(C(0), \lambda)$ denote the subset of $\mathcal{QR}'(P, \lambda)$ consisting of all the chains $C$ such that $C_\ell = C(0)$. Thus

$$\mathcal{QR}'(P, \lambda) = \bigcup_{C(0) \in \mathcal{QR}'_1(P, \lambda)} \mathcal{QR}'(C(0), \lambda) \quad \text{(disjoint)}.$$ 

In addition, two chains $C(0)$ and $C(0)'$ of $\mathcal{QR}'_1(P, \lambda)$ are $N(P)$-conjugate if and only if $\mathcal{QR}'(C(0), \lambda)$ and $\mathcal{QR}'(C(0)', \lambda)$ are $N(P)$-conjugate.

Now we can prove the main result of this paper.

(3D). Dade's ordinary conjecture holds for any positive defect 2-block of the symmetric groups $S(n)$ with $O_2(S(n)) = 1$. 

Proof. (1) First of all, we show that if \( m_P(A_d) \neq 0 \) for some \( d \geq 2 \), then

\[
\sum_{C \in \mathcal{QR}'(C(0), \lambda)/N(C(0))} (-1)^{|C|} k(N(C), B, u) = 0
\]

for all 2-blocks \( B \) and integers \( u \geq 0 \).

Let \( K = \prod_{d=2}^s \prod_{j=1}^{h_d} \text{GL}(d, 2)^{\lambda_{d,j}} \) and let \( \mathcal{R}(K) \) be the set of all radical 2-chains of \( K \). In addition, let \( \mathcal{S} = S(C(0), \lambda) \) be the set of all chains

\[
C : 1 < P_1 < \ldots < P_\ell = P < P_{\ell+1} < \ldots < P_w
\]

of \( G \) such that \( C_\ell = C(0) \) and \( C/P : P_\ell/P < P_{\ell+1}/P < \ldots < P_w/P \) is a chain of \( \mathcal{R}(K) \). The map \( \varphi : \mathcal{S}(C(0), \lambda) \to \mathcal{R}(K) \) given by \( \varphi(C) = C/P \) is a bijection (see [6, (5.7)]).

The same proof as that after (5.7) of [6] shows that

\[
\sum_{C \in \mathcal{S}(C(0), \lambda)/N(C(0))} (-1)^{|C|} k(N(C), B, u) = 0.
\]

It suffices to show that there exists a bijective map \( \psi \) from \( \mathcal{QR}'(C(0), \lambda) \) to \( S(C(0), \lambda) \) such that \( N(C) = N(\psi(C)) \).

Let \( C \) be a chain of \( \mathcal{QR}'(C(0), \lambda) \) given by (2.3) and let \( N_i = N(C_i) \) for \( 0 \leq i \leq w \). If \( D = P_t \) is the \( t \)-th subgroup of \( C \), then

\[
D = D(0) \times D(1) \times \prod_{d \geq 2} [(S_d \cap A_1)^{\alpha_d} \times (S_d \cap A_2)^{\beta_d} \times D(d)],
\]

such that \( D(0) = (A_0)^{m_0} \), \( D(1) = (D_8)^{\alpha_1} \times (A_1 \cap A_2)^{\beta_1} \times (A_1)^{t_1} \) and \( D(d) = \prod_{c} (X_c)^{t_c} \) with \( |c| = d \), where \( X_c \in \{A_d, S_2 = D_8, Q_c\} \), and \( \alpha_i, \beta_i, t_1 \) and \( t_c \) are non-negative integers. Let

\[
\psi(D) = D(0) \times D(1) \times \prod_{d \geq 2} [(S_d)^{2\alpha_d} \times (S_d)^{4\beta_d} \times D(d)].
\]

Equivalently, if \( D = \prod_i D_i \) such that \( D_i \in \Delta^+ \) or \( D_i \in \{A_d, D_8, Q_c\} \), then \( \psi(D) = \prod_k QB(D_k) \times (A_1 \cap A_2)^{\beta_1} \), where \( \beta_1 = m_D(A_1 \cap A_2) \) and \( k \) runs over the indices such that \( D_k \neq A_1 \cap A_2 \). Define

\[
\psi(C) : 1 < \psi(P_1) < \psi(P_2) < \ldots < \psi(P_w).
\]

Then \( \psi(C_\ell) = \psi(C) \) for \( 1 \leq t \leq w \). We shall show that \( \psi(C) \in \mathcal{S} \) and \( \psi \) is a bijection satisfying \( N(C) = N(\psi(C)) \). If \( \alpha_d = \beta_d = 0 \) for \( d \geq 2 \), then \( \psi(D) = D \). In particular, \( \psi(P_t) = P_t, \psi(C_\ell) = C_\ell = C(0) \) and \( N_G(C_t) = N_G(\psi(C_t)) \) for \( 1 \leq t \leq \ell \).
Suppose $\psi(C_t) \in S$ for some $t \geq \ell$. Then $N_t$ is of the form (2.8), and moreover, if $V(0) = C_V(P_t) = C_V(P)$, then

$$(3.7) \quad N_t = S(V(0)) \times N_t(1) \times \prod_{d \geq 2} \left[ (S_d \wr A_1)^{\alpha_d} \times (S_d \wr S(4))^{\beta_d} \times N_t(d) \right],$$

where $N_t(1) = (D_8)^{\alpha_1} \times (A_1 \wr S(4))^{\beta_1} \times A_1 \wr S(t_1)$ and

$$N_t(d) = \prod_{|c| = d} N_{N_{B(2d)}(A_d)}(X_c) \wr S(t_c).$$

Since $P_{t+1}$ is radical subgroup of $N_t$ and $C \in \mathcal{QR}'(C(0), \lambda)$, it follows that

$$P_{t+1} = D(0) \times D(1) \times \prod_{d \geq 2} \left[ (S_d \wr A_1)^{\alpha_d} \times (S_d \wr A_2)^{\beta_d} \times W(d) \right],$$

where $W(d) = \prod_{|c| = d} W_c$ such that $W_c$ is radical in $H_c = N_{N_{B(2d)}(A_d)}(X_c) \wr S(t_c)$. As shown in the proof of (2E) (c) $W_c = \prod_w (Y_w)^{\eta w} \times (S_d)^{\gamma w}$, where the $w$'s are sequences of positive integers such that $|w| = |c| = d$, $Z \in \{S_d \wr A_1, S_d \wr A_2\}$, $Y_w \in \{A_d, D_8, Q_w\}$ and $\gamma Z = 0, 1$. Moreover, $X_c \leq Y_w$ and $(Z)^{\gamma w} = S_d \wr A_1 \text{ or } S_d \wr A_2$ according as $m_{QB(W_c)}(S_d) = 2 \text{ or } 4$. So $QB(W_c) = \prod_w (Y_w)^{\eta w} \times (S_d)^{\gamma w}$ and $(X_c)^{\eta w} \leq QB(W_c)$, where $\eta = 2 \text{ or } 4$ according as $(Z)^{\gamma w} = S_d \wr A_1 \text{ or } S_d \wr A_2$. In particular, $N_{H_c}(W_c) = N_{H_c}(QB(W_c))$, and $QB(W_c)/(A_d)^{\eta w} = \prod_w (Y_w/A_d)^{\eta w} \times (S_d/A_d)^{\gamma w}$ is a radical subgroup of $GL(d,2)^{\eta w}$. By definition,

$$\psi(P_{t+1}) = D(0) \times D(1) \times \prod_{d \geq 2} \left[ (S_d)^{2\alpha_d} \times (S_d)^{4\beta_d} \times QB(W(d)) \right],$$

so that $N_{N_t}(\psi(P_{t+1})) = N_{t+1}$. By (3.6), $\psi(P_t) \subseteq \psi(P_{t+1})$ and $\psi(P_{t+1})/P$ is a radical subgroup of $K$. Thus $\psi(C_t)/P \in \mathcal{R}(K)$, and by induction, $\psi(C)/P \in \mathcal{R}(K)$, so that $\psi(C) \in S$. Since $N_t = N(C_t) = N(\psi(C_t))$ for $t \geq 1$, it follows that $P_t = O_{2}(N(\psi(C_t)))$, so that $C$ is determined uniquely by $\psi(C)$. Thus $\psi$ is a bijection if and only if it is onto.

Let $C' : 1 < P'_1 < \ldots < P'_w$ be a chain in $S$, and let $C$ be the chain of length $w$ such that its $t$-th non-trivial subgroup $P_t$ is $O_2(N(C'_t))$. Since $C'_t = C(0)$ is radical, it follows that $C_t = C(0)$, and so $\psi(C_t) = C'_t$ for $0 \leq t \leq \ell$. Suppose $\psi(C_t) = C'_t$ and $C_t \in \mathcal{QR}'(C(0), \lambda)$ for some $\ell \leq t \leq w$. Then $N_t = N(C_t) = N(C'_t)$ is given by (3.7). Since $C_t$ is a radical chain and $P_t = O_2(N_t)$, it follows that $P_t = D$ is given by (3.5) and $P'_t = \psi(D)$ is given by (3.6). Since $P'_t \leq P'_{t+1} \leq N_t$ and $P'_{t+1}/P$ is radical in $K$, it follows that

$$P'_{t+1} = D(0) \times D(1) \times \prod_{d \geq 2} \left[ (S_d)^{2\alpha_d} \times (S_d)^{4\beta_d} \times T(d) \right]$$

such that $T(d) = \prod_{|c| = d} T_c$, where $T_c/(A_d)^{\lambda c}$ is a radical subgroup of $GL(d,2)^{\lambda c}$. 

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By (2A) (b), \( T_c = \prod_w (Y_w)^{m_w} \), where \(|w| = |c| = d\) and \( Y_w \in \{ A_d, D_8, Q_w \} \). Thus

\[
N_{H_c}(T_c) = \prod_w N_{N_{S(2d)}(A_d)}(Y_w) \lhd S(m_w).
\]

Since \( Y_w \) is self-normalizing if and only if \( Y_w = S_d \), it follows that \( O_2(N_{H_c}(T_c)) = T_c \) except when \( m_{T_c}(S_d) \in \{2, 4\} \), in which case \( O_2(N_{H_c}(T_c)) = \prod_{Y_w \neq S_d} (Y_w)^{m_w} \times Z \), where \( Z = S_d \lhd A_1 \) or \( S_d \lhd A_2 \) according as \( m_{T_c}(S_d) = 2 \) or \( 4 \). Thus \( \psi(O_2(N_{H_c}(T_c))) = T_c \) and \( \psi(P_{t+1}) = P_{t+1}' \). By induction, \( \psi(C) = C' \) and \( \psi \) is onto. Thus \( \psi \) is a bijection.

(2) In order to complete the proof, it suffices to consider chains \( C \in Q\mathcal{R}'(P, \lambda) \) such that

\[
(3.8) \quad P = (A_0)^{m_0} \times (D_8)^{\alpha} \times (A_1 \lhd A_2)^{\beta} \times (A_1)^{m_1},
\]

where \( \alpha, \beta, m_0 \) and \( m_1 \) are non-negative integers. It follows by (3B) and (3C) that \( P \) is the final subgroup for each chain \( C \in Q\mathcal{R}'(P, \lambda) \). Let \( Q\mathcal{R}^*(G) = \cup_{P, \lambda} Q\mathcal{R}'(P, \lambda) \), where \( P \) runs over subgroups of form (3.8) and \( \lambda \) runs over partitions of \( m_1 \). It suffices to show that

\[
(3.9) \quad \sum_{C \in Q\mathcal{R}^*(G)/G} (-1)^{|C|} k(N(C), B, u) = 0
\]

for all positive defect 2-blocks \( B \) and integers \( u \geq 0 \).

Now each subgroup of a chain \( C \in Q\mathcal{R}^*(G) \) is of the form (3.8). Let \( \phi(P) = (A_0)^{m_0 \times (A_1)^{2\alpha} \times (A_1)^{4\beta} \times (A_1)^{m_1}} \) and let

\[
\phi(C) : 1 < \phi(P_1) < \phi(P_2) < \ldots < \phi(P_w)
\]

for chain \( C \in Q\mathcal{R}^*(G) \) given by (2.3). In addition, let \( S(G) = \{ \phi(C) : C \in Q\mathcal{R}^*(G) \} \). A proof similar to that of (1) above shows that \( \phi \) is a bijection between \( Q\mathcal{R}^*(G) \) and \( S(G) \), and \( N(C) = N(\phi(C)) \).

The same proof as that of [6, Proposition (6.1)] shows that

\[
\sum_{C \in S(G)/G} (-1)^{|C|} k(N(C), B, u) = 0,
\]

which implies (3.9). This completes the proof.

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References


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