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A NOTE ON TRANSITIVE PERMUTATION GROUPS OF DEGREE p

By

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Dedicated to Kenjiro Shoda on his sixtieth birthday

Let p and q be odd prime numbers such that $p=2q+1$. Let Ω be the set of symbols $1, \cdots, p$ and let $\mathfrak{G}$ be an insoluble transitive permutation group on Ω. Then by a famous theorem of Burnside $\mathfrak{G}$ is doubly transitive on Ω. In particular the order of $\mathfrak{G}$ is divisible by q. Let $\mathfrak{Q}$ and $Ns\mathfrak{Q}$ denote a Sylow q-subgroup of $\mathfrak{G}$ and its normalizer in $\mathfrak{G}$. Moreover let $\mathfrak{H}$ be the maximal subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes the symbol 1 and let $LF_2(n)$ denote the linear fractional group over the field of n elements.

Now the purpose of this note is (i) to give a proof for an unpublished result of Wielandt in 1955:

Theorem 1. *If $\mathfrak{H}$ is imprimitive on $\Omega-\{1\}$, then $\mathfrak{G}$ is isomorphic to $LF_2(7)$ with $p=7$,*
and (ii) to prove the following theorem:

Theorem 2. *If $Ns\mathfrak{Q}$ has order $2q$, then $\mathfrak{G}$ is isomorphic to either $LF_2(7)$ with $q=3$ or $LF_2(11)$ with $q=5$.*

§1. Proof of Theorem 1.

1. Let $\mathfrak{P}$ and $Ns\mathfrak{P}$ denote a Sylow p-subgroup of $\mathfrak{G}$ and its normalizer in $\mathfrak{G}$. We assume that $Ns\mathfrak{P}$ has order px. If $x=1$, then by the splitting theorem of Burnside $\mathfrak{G}$ contains a normal subgroup of index p. Hence $\mathfrak{H}$ is normal in $\mathfrak{G}$. Since $\mathfrak{G}$ is transitive on Ω, we have that $\mathfrak{H}=1$ and $\mathfrak{G}=\mathfrak{P}$. Then $\mathfrak{G}$ is soluble against our assumption. If $x=2$, let J be an involution in $Ns\mathfrak{P}$. Then the cycle structure of J consists of q transpositions. Since q is odd, J is an odd permutation. Let $\mathfrak{G}^*$ be the subgroup of $\mathfrak{G}$ consisting of all the even permutations of $\mathfrak{G}$. Then the index of $\mathfrak{G}^*$ in $\mathfrak{G}$ equals two. Since $\mathfrak{G}$ is insoluble, $\mathfrak{G}^*$ is also

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insoluble. But we have that $Ns\mathfrak{P}\cap\mathfrak{G}^*=\mathfrak{P}$. This is a contradiction as before. If $x=2q$, then by a theorem of Wielandt ([5], (27. 1)) $\mathfrak{G}$ is triply transitive on Ω. Hence $\mathfrak{H}$ is doubly transitive and necessarily primitive on $\Omega-\{1\}$. This is against our assumption. Hence we can assume that $x=q$.

2. $\mathfrak{G}$ is simple. Otherwise let $\mathfrak{N}$ be a proper normal subgroup ($\neq 1$) of $\mathfrak{G}$. Then since $\mathfrak{G}$ is doubly transitive on Ω, $\mathfrak{N}$ is transitive on Ω. Therefore $\mathfrak{N}$ contains $\mathfrak{P}$. Using Sylow's theorem we have that $\mathfrak{G}=Ns\mathfrak{P}.\mathfrak{N}$. Therefore we have that $\mathfrak{P}\subseteq Ns\mathfrak{P}\cap\mathfrak{N}\subsetneqq Ns\mathfrak{P}$. Since $Ns\mathfrak{P}:\mathfrak{P}=q$ is a prime number, we have that $Ns\mathfrak{P}\cap\mathfrak{N}=\mathfrak{P}$. This implies the solubility of $\mathfrak{N}$ and $\mathfrak{G}$ as before. This contradiction shows the simplicity of $\mathfrak{G}$.

3. The order of $\mathfrak{Q}$ is q and the cycle structure of every element ($\neq 1$) of $\mathfrak{Q}$ consists of two q-cycles. Otherwise $\mathfrak{Q}$ contains a q-cycle. Then by a classical theorem of Jordan $\mathfrak{G}$ must be the alternating group of degree p, which is obviously triply transitive on Ω. This is a contradiction as before.

4. Let $\mathfrak{K}$ be the subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes each of the symbols 1 and 2. Now since $\mathfrak{H}$ is imprimitive on $\Omega-\{1\}$, $\mathfrak{K}$ is not a maximal subgroup of $\mathfrak{H}$. Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{H}$ containing $\mathfrak{K}$. Since $\mathfrak{H}:\mathfrak{K}=2q$ two cases arise: (i) $\mathfrak{M}:\mathfrak{K}=q$ and $\mathfrak{H}:\mathfrak{M}=2$ and (ii) $\mathfrak{M}:\mathfrak{K}=2$ and $\mathfrak{H}:\mathfrak{M}=q$.

5. Case (i). Since $\mathfrak{M}$ has index two in $\mathfrak{H}$ and is intransitive on $\Omega-\{1\}$ $\Omega-\{1\}$ is divided into two domains of transitivity Ω_1 and Ω_2 of $\mathfrak{M}$ each of which has length q. Let $Cs\mathfrak{Q}$ denote the centralizer of $\mathfrak{Q}$ in $\mathfrak{G}$. Then we have that $Cs\mathfrak{Q}=\mathfrak{Q}$, because otherwise $Cs\mathfrak{Q}$ must contain a $2q$-cycle, which is an odd permutation against the simplicity of $\mathfrak{G}$. Now using Sylow's theorem we can assume that $\mathfrak{Q}$ is contained in $\mathfrak{H}$. Then $\mathfrak{Q}$ is contained in $\mathfrak{M}$ and we have that $\mathfrak{H}=Ns\mathfrak{Q}.\mathfrak{M}$, whence follows that $Ns\mathfrak{Q}:\mathfrak{M}\cap Ns\mathfrak{Q}=2$. Let $\mathfrak{T}$ be a Sylow 2-subgroup of $Ns\mathfrak{Q}$. Then $\mathfrak{T}$ is cyclic, because of $Cs\mathfrak{Q}=\mathfrak{Q}$. Anyway we have that $\mathfrak{T}\neq 1$. Let T be a generator of $\mathfrak{T}$. Then since $\mathfrak{T}$ is not contained in $\mathfrak{M}$, T must permute Ω_1 with Ω_2. Hence we have that $\alpha(T)=1$, where $\alpha(X)$ denotes the number of symbols of Ω which are fixed by a permutation X of $\mathfrak{G}$. If T is an involution, then the cycle structure of T consists of q transpositions and T must be an odd permutation, contradicting the simplicity of $\mathfrak{G}$. Hence the order of T, say 2^τ, is greater than two. Now we have that $\alpha(T^t)\leqq 3$ for $T^t\neq 1$ (t is an integer.), because otherwise T^t fixes at least two symbols of either Ω_1 or Ω_2. This means that T^t is commutative with the elements of $\mathfrak{Q}$, which is a contradiction since $Cs\mathfrak{Q}=\mathfrak{Q}$. Since $2q\not\equiv 0 \pmod 4$ the cycle structure of T must contain a transposition. Hence if t is even and $T^t\neq 1$ we have that $\alpha(T^t)=3$.

Therefore the cycle structure of T consists of one transposition and $(2q-2/2^{\tau})2^{\tau}$-cycles. Since T must be an even permutation, we have that $2q-2/2^{\tau}$ is odd. Anyway we obtain the following equality:

$$\text{(I)} \qquad 2q = 2+2^{\tau}(2q-2/2^{\tau}),$$

where $(2q-2/2^{\tau})$ is an odd number. On the other hand, T is contained in $Ns\mathfrak{Q}$ and $Cs\mathfrak{Q}=\mathfrak{Q}$. Therefore we obtain the following congruence:

$$\text{(II)} \qquad q \equiv 1 \pmod{2^{\tau}}.$$

(I) and (II) give us a contradiction. Hence the case (i) cannot occur.

6. Case (ii). Since $\mathfrak{M}:\mathfrak{K}=2$, the length of the domain Γ of transitivity of $\mathfrak{M}$ containing the symbol 2 of Ω must be two. Let us assume that Γ consists of two symbols 2 and 3 of Ω. Then since $\mathfrak{K}$ is normal in $\mathfrak{M}$, $\mathfrak{K}$ must fix also the symbol 3. Now let Φ denote the set of symbols of Ω which are fixed by $\mathfrak{K}$. Then the length f of Φ is at least three. By a theorem of Witt ([5], (9. 4)) the normalizer $Ns\mathfrak{K}$ of $\mathfrak{K}$ in $\mathfrak{G}$ is doubly transitive on Φ. In our case then $Ns\mathfrak{K}$ clearly has order $f(f-1)$. Since $f(f-1)$ must divide $2pq$ and f is smaller than p, we must have that $f=q$ and $f-1=2$. Thus we obtain that $q=3$ and $p=7$. Now it is easy to show that $\mathfrak{G}$ is isomorphic to $LF_2(7)$.

§ 2. Proof of Theorem 2.

If $Ns\mathfrak{Q}$ is cyclic, then $\mathfrak{G}$ contains by the splitting theorem of Burnside a normal subgroup $\mathfrak{N}$ of index q. Since $Ns\mathfrak{P}\cap\mathfrak{N}$ has order at most $2p$, $\mathfrak{N}$ and $\mathfrak{G}$ must be soluble as before in § 1.1 against our assumption. Hence $Ns\mathfrak{Q}$ must be a dihedral group of order $2q$. Therefore Theorem 2 is a special case of the following

Theorem 3. *Let n be an integer such that $n=2q+1$, where q is an odd prime number. Let Ω be the set of symbols $1,\cdots,n$ and let $\mathfrak{G}$ be an insoluble doubly transitive permutation group on Ω. Let $\mathfrak{Q}$ be a Sylow q-subgroup of $\mathfrak{G}$ and $Ns\mathfrak{Q}$ be the normalizer of $\mathfrak{Q}$ in $\mathfrak{G}$. If $Ns\mathfrak{Q}$ is a dihedral group of order $2q$, then $\mathfrak{G}$ is isomorphic to either $LF_2(7)$ with $q=3$ or $LF_2(11)$ with $q=5$.*

Proof of Theorem 3.

1. $\mathfrak{G}$ is simple. Otherwise let $\mathfrak{N}$ be a maximal normal subgroup $(\neq 1)$ of $\mathfrak{G}$. If $\mathfrak{N}$ contains $\mathfrak{Q}$, then we have that $\mathfrak{N}\cap Ns\mathfrak{Q}=\mathfrak{Q}$, since $Ns\mathfrak{Q}:\mathfrak{Q}=2$ and by Sylow's theorem $(Ns\mathfrak{Q})\mathfrak{N}=\mathfrak{G}$. Hence by the splitting theorem of Burnside $\mathfrak{N}$ contains a normal subgroup $\mathfrak{N}^*$ of index q. Since $Ns\mathfrak{Q}$ is a dihedral group of order $2q$, every element $(\neq 1)$ of $\mathfrak{N}^*$

is not commutative with any element ($\neq 1$) of $\mathfrak{Q}$. Therefore $\mathfrak{N}^*$ is nilpotent by a theorem of Thompson [4]. Then $\mathfrak{N}$ and $\mathfrak{G}$ become soluble against our assumption. If the order of $\mathfrak{N}$ is prime to q, then let us consider the subgroup $\mathfrak{N}\mathfrak{Q}$. Again by a theorem of Thompson $\mathfrak{N}$ becomes nilpotent. Let $\mathfrak{N}^*$ be a minimal normal subgroup of $\mathfrak{G}$ contained in $\mathfrak{N}$. Since $\mathfrak{G}$ is doubly transitive on Ω, every normal subgroup ($\neq 1$) of $\mathfrak{G}$ is transitive on Ω. Therefore $\mathfrak{N}^*$ must be an elementary abelian p-group for some prime number p and we have the following factorisation of $\mathfrak{G}: \mathfrak{G}=\mathfrak{N}^*\mathfrak{H}$, $\mathfrak{N}^* \cap \mathfrak{H}=1$, where $\mathfrak{H}$ denotes the maximal subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes the symbol 1. Since $\mathfrak{N}$ is nilpotent, $\mathfrak{N}^*$ is contained in the center of $\mathfrak{N}$. Since $\mathfrak{H}$ does not contain any normal subgroup ($\neq 1$) of $\mathfrak{G}$, we have that $\mathfrak{N} \cap \mathfrak{H}=1$ and $\mathfrak{N}=\mathfrak{N}^*$. On the other hand, since $\mathfrak{G}$ is insoluble, $\mathfrak{H}$ must be insoluble. Moreover since $\mathfrak{N}$ is also a maximal normal subgroup of $\mathfrak{G}$, $\mathfrak{H}$ is simple. Let p^ν be the order of $\mathfrak{N}$. Then we have the equality

$$n = 2q+1 = p^\nu .$$

Since $\mathfrak{H}$ is insoluble, we have that ν is greater than one. Hence we have that $p=3$ and $q=\frac{1}{2}(3^\nu-1)$. In particular we have that ν is greater than two. Then $\mathfrak{H}$ is isomorphic to a subgroup of the ν-dimensional special linear group $SL_\nu(3)$ over the field of three elements. But then $Ns\mathfrak{Q}$ has order $\nu q>2q$ [3]. This is a contradiction. Hence $\mathfrak{G}$ must be simple.

2. In the first place let us assume that $\mathfrak{H}$ is imprimitive on $\Omega-\{1\}$. Let $\mathfrak{K}$ be the subgroup of $\mathfrak{G}$ consisting of all the permutations each of which fixes each of the symbols 1 and 2 of Ω. Then $\mathfrak{K}$ is not a maximal subgroup of $\mathfrak{H}$. Let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{H}$ containing $\mathfrak{K}$. Since $\mathfrak{H}:\mathfrak{K}=2q$, we have two cases: (i) $\mathfrak{H}:\mathfrak{M}=2$, $\mathfrak{M}:\mathfrak{K}=q$ and (ii) $\mathfrak{H}:\mathfrak{M}=q$, $\mathfrak{M}:\mathfrak{K}=2$.

3. Case (i). Using Sylow's theorem we can assume that $\mathfrak{Q}$ is contained in $\mathfrak{H}$. Then $\mathfrak{Q}$ is contained in $\mathfrak{M}$. Hence by Sylow's theorem we have that $(Ns\mathfrak{Q})\mathfrak{M}=\mathfrak{H}$. Since $Ns\mathfrak{Q}:\mathfrak{Q}=2$, we have then that $Ns\mathfrak{Q}\cap\mathfrak{M}=\mathfrak{Q}$. By the splitting theorem of Burnside $\mathfrak{M}$ contains a normal subgroup of index q, which necessarily coincides with $\mathfrak{K}$. Since $\mathfrak{H}$ is transitive on $\Omega-\{1\}$, we have that $\mathfrak{K}=1$. Then it is easy to show the solubility of $\mathfrak{G}$ against our assumption. Thus Case (i) cannot occur.

4. Case (ii). If $\mathfrak{H}$ is simple, then by a previous result ([2], Theorem II) $\mathfrak{H}$ becomes primitive on $\Omega-\{1\}$. Hence in our case $\mathfrak{H}$ cannot be simple. Let $\mathfrak{N}$ be a maximal normal subgroup of $\mathfrak{H}$. If the order of $\mathfrak{N}$ is divisible by q, then $\mathfrak{N}$ has index two in $\mathfrak{H}\mathfrak{N}\cap Ns\mathfrak{Q}=\mathfrak{D}$, because $Ns\mathfrak{Q}$ is a dihedral group of order $2q$ and we have that $\mathfrak{H}=\mathfrak{N}Ns\mathfrak{Q}$ by Sylow's theorem. Now by the splitting theorem of Burnside $\mathfrak{N}$ contains a normal

subgroup $\mathfrak{N}^*$ of index q. Since $\mathfrak{H}$ is transitive on $\Omega-\{1\}$ $\mathfrak{N}^*$ is semi-transitive on $\Omega-\{1\}$ ([5], 11). Hence the length of domains of transitivity of $\mathfrak{N}^*$ from $\Omega-\{1\}$ equals two and $\mathfrak{N}^*$ is an elementary abelian 2-subgroup. Let us consider the subgroup $\mathfrak{Q}\mathfrak{N}^*$. Since $Ns\mathfrak{Q}$ is a dihedral group of order $2q$, every element $(\neq 1)$ of $\mathfrak{N}^*$ is not commutative with any element $(\neq 1)$ of $\mathfrak{Q}$. Hence we see in particular that the order of $\mathfrak{N}^*$ is congruent to 1 modulo q.

Since $\mathfrak{G}$ is simple and $Ns\mathfrak{Q}$ is a dihedral group of order $2q$, we see, using a method of Brauer-Fowler ([2], § 1. 3), that there is only class of conjugate involutions in $\mathfrak{G}$. Now let us consider the subgroup $Ns\mathfrak{Q}$. Every element of $Ns\mathfrak{Q}$ of order q has a cycle structure consisting of two q-cycles. Then it is easy to show that every involution in $Ns\mathfrak{Q}$ fixes just three symbols of Ω.

Since $\mathfrak{M}:\mathfrak{K}=2$, the length of the domain Γ of transitivity of $\mathfrak{M}$ containing the symbol 2 of Ω must be two. Let us assume that Γ consists of two symbols 2 and 3 of Ω. Then since $\mathfrak{K}$ is normal in $\mathfrak{M}$, $\mathfrak{K}$ must fix also the symbol 3. Let $\mathfrak{T}$ be a Sylow 2-subgroup of $\mathfrak{K}$. Then $\mathfrak{T}$ must be semi-regular on $\Omega-\{1, 2, 3\}$ ([5], § 4). Therefore the order of $\mathfrak{T}$ is a divisor of $2q-2$. Since the orders of $\mathfrak{T}$ and $\mathfrak{N}^*$ are same, we see that the order of $\mathfrak{T}$ equals $q+1$ and that $2q-2=q+1$. Thus we obtain that $q=3$. Now it is easy to show that $\mathfrak{G}$ is isomorphic to $LF_2(7)$.

If the order of $\mathfrak{N}$ is prime to q, then as before (see No. 1 of this proof) by a theorem of Thompson [4] $\mathfrak{N}$ itself becomes a nilpotent, therefore, an elementary abelian 2-group and the order of $\mathfrak{N}$ is congruent to 1 modulo q. If $\mathfrak{N}$ has index q in $\mathfrak{H}$, then we must have that $Ns\mathfrak{Q}=\mathfrak{Q}$ against our assumption. Since the factor group $\mathfrak{H}/\mathfrak{N}$ is simple, the order of $\mathfrak{H}/\mathfrak{N}$ is divisible by 4. Hence the order of $\mathfrak{N}$ is at most a half of that of $\mathfrak{T}$. Thus we obtain an absurd inequality $q-1\geqq q+1$.

5. So we can assume that $\mathfrak{H}$ is primitive on $\Omega-\{1\}$. Then $\mathfrak{H}$ is simple. Otherwise let $\mathfrak{N}$ be a proper normal subgroup $(\neq 1)$ of $\mathfrak{H}$. Since $\mathfrak{H}$ is primitive on $\Omega-\{1\}$, $\mathfrak{N}$ is transitive on $\Omega-\{1\}$. Hence the order of $\mathfrak{N}$ is divisible by $2q$. Since $Ns\mathfrak{Q}$ is a dihedral group of order $2q$ and $\mathfrak{H}=\mathfrak{N}Ns\mathfrak{Q}$ by Sylow's theorem, we must have that $\mathfrak{H}:\mathfrak{N}=2$ and $\mathfrak{N}\cap Ns\mathfrak{Q}=\mathfrak{Q}$. Then by the splitting theorem of Burnside $\mathfrak{N}$ contains a characteristic subgroup $\mathfrak{N}^*$ of index q. If $\mathfrak{N}^*\neq 1$, then the order of $\mathfrak{N}^*$ is divisible by $2q$. Therefore we obtain that $\mathfrak{N}^*=1$, which implies the solubility of $\mathfrak{G}$ against our assumption.

6. If $\mathfrak{H}$ is doubly transitive on $\Omega-\{1\}$, then by a previous result ([2], Theorem I), $\mathfrak{H}$ is isomorphic to $LF_2(r)$ with $2q=r+1$, and hence only the identity element of $\mathfrak{H}$ fixes at least three symbols of $\Omega-\{1\}$. Therefore $\mathfrak{G}$ is triply transitive on Ω and only the identity element fixes

at least four elements of Ω. Now using a theorem of Gorenstein-Hughes [1] we obtain that $2q+1=2^{\nu}+1$. This is a contradiction, since q is an odd prime number.

7. If $\mathfrak{H}$ is not doubly transitive on $\Omega-\{1\}$, then also by a previous result ([2], Theorem II) $\mathfrak{H}$ is isomorphic to the icosahedral group with $q=5$. Thus we have obtained that $n=11$ and the order of $\mathfrak{G}$ is 660. Now it is easy to show that $\mathfrak{G}$ is isomorphic to $LF_2(11)$.

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