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Osaka University
OVERRINGS OF KRULL ORDERS

HIDETOSHI MARUBAYASHI AND KENJI NISHIDA

(Received September 13, 1978)

Introduction. Recently, one of the authors introduced a Krull order \( R \) in a simple artinian ring \( Q \)[6], that is, \( R \) is called Krull if the following conditions hold:

(K1) \( R = \bigcap_{i \in I} R_i \cap S(R) \), where \( R_i \) and \( S(R) \) are essential overrings of \( R \)(see [6] for the definition), and \( S(R) \) is the Asano overring of \( R \);

(K2) each \( R_i \) is a noetherian, local, Asano order in \( Q \), and \( S(R) \) is a noetherian, simple ring;

(K3) if \( c \) is any regular element of \( R \), then \( cR_i \neq R_i \) for only finitely many \( i \) in \( I \) and \( R_i c \neq R_i \) for only finitely many \( k \) in \( I \).

The fundamental properties of Krull orders were studied in [6]. Let \( P \) be the set of all prime \( \nu \)-ideals of \( R \) and \( P_0 \) any subset of \( P \). Then, in §1, we shall show that an order \( T = \bigcap_{P \in P_0} R_P \cap S(R) \) is also Krull and, in particular, \( T \) is an \( RL \)-order in the sense of Cozzens and Sandomierski [1], if we take \( P_0 \) to be the set of all invertible prime ideals of \( R \). In §2 we apply the results of §1 to the case where \( R \) is a \( D \)-order in a central simple algebra, where \( D \) is a unique factorization domain. §3 is devoted to state an example of a maximal order which has the noninvertible prime \( \nu \)-ideals.

1. Overrings of Krull orders. Let \( R \) be an order in a simple artinian ring \( Q \). A right \( R \)-submodule \( X \) of \( Q \) is called a right \( R \)-ideal, if \( aR \supset X \supset bR \) for units \( a,b \) in \( Q \). A left \( R \)-ideal and a two-sided \( R \)-ideal are defined by the similar way. An \( \nu \)-ideal in \( R \) is simply called an ideal. For a one-sided \( R \)-ideal \( X \) of \( R \), put \( O_r(X) = \{ x \in Q; Xx \subseteq X \} \), \( O_l(X) = \{ x \in Q; xX \subseteq X \} \), \( X^{-1} = \{ x \in Q; XxX \subseteq X \} = \{ x \in Q; Xx \subseteq O_r(X) \} \), \( X^{-1} = \{ x \in Q; xX \subseteq O_l(X) \} \), and \( X^* = X^{-1} \). \( X \) is called a \( \nu \)-ideal (invertible ideal), if \( X = X^*(R = XX^{-1} = X^{-1}X) \).

We state some results in [6] concerning Krull orders. Let \( R \) be a Krull order in a simple artinian ring \( Q \). \( R \) is a maximal order [6, Proposition 2.1]. Let \( P_i \) be a unique maximal ideal of \( R_i \). Then \( P_i = P_i \cap R \) is a prime \( \nu \)-ideal of \( R \)(cf. [4, Lemma 1.5]), \( P_i = R_iP_i \) [3, Proposition 1.1], and \( R_i = R_{P_i} \), where \( R_{P_i} \) is the localization of \( R \) at \( P_i \), that is, \( R_{P_i} = \{ xy^{-1} \in Q; x \in R, y \in C(P_i) \} \) with \( C(P_i) = \{ y \in R; y + P_i \) is a regular element of \( R/P_i \} \).

Let \( P = \{ P_i; i \in I \} \) be the set of all prime \( \nu \)-ideals of \( R \)(cf. [4, Proposition...
1.7), $P_i$ any subset of $P$, and $T = \cap_{P_i \in P} R_{P_i} \cap S(R)$. In order to show that $T$ is Krull, we prepare some definitions and lemmas. We sometimes drop the index $i$ of $P_i \in P$ for abbreviation.

Throughout this section, $R$ is a Krull order in a simple artinian ring $Q$.

Let $M$ be a subset of $Q$. Then $M$ is called a right $R$-set, if $M$ is a right $R$-module and contains a regular element of $Q$. We put $\overline{M} = \cup I^*$, where $I$ ranges over the set of all right $R$-ideals which are contained in $M$. $M$ is called closed if $\overline{M} = M$. By the similar way we define a left $R$-set and a closed left $R$-set. Let $A$ be a right $v$-ideal and $B$ a left $v$-ideal. Then we define $A \circ B = (AB)^*$ (cf. [5]).

Lemma 1.1. The following statements hold for right $R$-sets $M, N$.

(i) $\overline{M}$ is a right $R$-set which contains $M$.

(ii) If $N \subset M$, then $\overline{N} \subset \overline{M}$.

(iii) $(\overline{M})^* = \overline{M}$.

(iv) If $I$ is a right $R$-ideal, then $I^* = I^*$.

(v) $(\overline{M} \cap \overline{N})^* = \overline{M} \cap \overline{N}$.

(vi) If $M$ is closed, then $M = \cap_{P \in p\text{MR} \cap MS(R)}$.

In the following, we assume $M, N$ to be closed and $A, B v$-ideals. Let $M \circ A = \cup X \circ A$, where $X$ ranges over the set of all right $v$-ideals which are contained in $M$.

(vii) $M \circ A = (\overline{MA})^*$.

(viii) $(M \circ A) \circ B = M \circ (A \circ B)$.

(ix) $M \circ (A \cap B) = M \circ A \cap M \circ B$.

Proof. (i) and (ii) are easily proved. (iii); It holds that $x \in (\overline{M})^*$ if and only if there is a right $R$-ideal $I \subset \overline{M}$ such that $x \in I^*$. Since we can take $I$ to be finitely generated by [6, proposition 2.1], $I = a_1 R + \cdots + a_k R$ with $a_i \in \overline{M}$. Thus, there are right $R$-ideals $I_i \subset M$ such that $a_i \in I_i^* (i = 1, \ldots, k)$. We have $(I_1 + \cdots + I_k)^* = (I^* + \cdots + I_k^*) \subset I^*$ by [5, Lemma 2] and $I_1 + \cdots + I_k \subset M$. Hence $x \in \overline{M}$, that is, $(\overline{M})^* = \overline{M}$. (iv) is easily proved. (v); I. holds that $(\overline{M} \cap \overline{N})^* \supset x$ if and only if there is a right $R$-ideal $I$ such that $I \subset \overline{M} \cap \overline{N}$, and $x \in I^*$. We have $I^* = I_i \subset (\overline{M})^*$, $\overline{M}$, and, similarly, $I^* \subset \overline{N}$. Hence $x \in I^* \subset \overline{M} \cap \overline{N}$. (vi); Let $x \in \cap_{P \in p\text{MR} \cap MS(R)}$ be a regular element, Then we have $xc \in M$ for $c \in C(P)$ and $xB \in M$ for a nonzero ideal $B$. If we put $X = \Sigma cP R + B$, then $R = \cap X R \cap XS(R) \subset X^*$ by [6, Proposition 2.1 and 3, Corollary 4.2]. Thus $R = X^*$ and $x \in xX^* = (xx)^* = xX \subset \overline{M} = M$. Hence $\cap MR \cap MS(R) = M$ by [3, Lemma 2.2]. (vii); Let $X$ be a right $v$-ideal in $M$. Then $X \circ A = (XA)^* = (X^A) \subset (MA)^*$. Conversely, if $x \in (MA)^*$, then $x \in I^*$ for a finitely generated right $R$-ideal $I \subset MA$. By the same way as (iii) we have $I^* \subset X \circ A$ for a right $v$-ideal $X$ in $M$. Hence $x \in M \circ A$. (viii); It holds that $(X \circ A) \circ B = ((X \circ A)B)^* = (X(AB))^* = X \circ (AB)^* = X \circ (A \circ B)$ for a right $v$-ideal $X$. Hence $(M \circ A) \circ B$
(ix); By [5, Lemma 3] the lattice anti-isomorphism between the set of right \( v \)-ideals and one of left \( v \)-ideals yields the proof of (ix).

Let \( P_0 \) be an arbitrary subset of \( P \), \( T = \bigcap_{P \in P_0} R_P \cap S(R) \), \( F = \{ X \subseteq R; X \) is a right \( R \)-ideal such that \( XR_p = R_p \) for every \( P \in P_0 \) and \( XS(R) = S(R) \}, \) and \( F_1 = \{ Y \subseteq R; Y \) is a left \( R \)-ideal such that \( R_P Y = R_p \) for every \( P \in P_0 \) and \( S(R) Y = S(R) \}. \) In the following, the notation provided above will be preserved.

**Lemma 1.2.** \( F \) is a right additive topology on \( R \) and \( R \) \( F \) \( = T \). Similarly, \( F_1 \) is a left additive topology on \( R \) and \( T = R \) \( F_1 \).

Proof. To show that \( F \) is a right additive topology, we shall prove the followings [10]:

(i) If \( X \in F \) and \( r \in R \), then \( r^{-1}X = \{ x \in R; rx \in X \} \in F \).
(ii) If \( Y \) is a right ideal and \( X \in F \) such that \( x^{-1}Y \in F \) for all \( x \in X \), then \( Y \in F \).

(i); Considering the fact \( XR_p = R_p \Rightarrow C(P) \cap X \neq \phi \) and \( XS(R) = S(R) \overset{r}{\Rightarrow} X \) contains a nonzero ideal we can show \( r^{-1}X \in F \) by the right Ore condition of \( C(P) \). (ii); We have \( R_p \supset YR_p \supset \Sigma_{x \in X} (x^{-1}Y)R_p = \Sigma_{x \in X} xR_p = XR_p = R_p \), and then \( R_p = YR_p \). Similarly, \( S(R) = YS(R) \) which implies \( Y \in F \). If \( X \in F \), then \( X^{-1}T \subset \bigcap X^{-1}R_p \cap X^{-1}S(R) = \bigcap R_p \cap S(R) = T \). Thus \( R_p \subset T \). Conversely, if \( t \in T \), then there are \( c_r \in C(P) \) and a nonzero ideal \( B \) such that \( tc_r \in R \) and \( tB \subset R \). Putting \( X = \Sigma c_r R + B \) we have that \( X \in F \) and \( tX \subset R \). Hence \( t \in R \). This completes the proof.

**Lemma 1.3.** We have \( \overline{T} = \overline{T} = \overline{S(R)} = \overline{S(R)} \) and \( (\overline{R}_p) = \overline{R}_p \).

Proof. In general, if \( G \) is a right additive topology which consists of a family of essential right ideals, then \( (\overline{R}_c) = R_c \). For, it holds that \( x \in (\overline{R}_c) \) if and only if \( x \in I^* \) for a finitely generated left \( R \)-ideal \( I \subset R_c \). Then we can take \( X \in G \) such that \( I \subset X^{-1} \) by the same way as the proof of Lemma 1.1 (i); Hence \( x \in I^* \subset (X^{-1})^* = X^{-1} \). This completes the proof.

**Lemma 1.4.** If \( X \in F \) and \( Y \in F \), then \( Y^{-1}X^{-1} \subset T \).

Proof. We can assume \( X \) to be a right \( v \)-ideal and \( Y \) a left \( v \)-ideal. Then \( Y^{-1}X^{-1} = (X^{-1})^{-1} = (XY)^{-1} \) by [5, Lemma 3]. If \( c \in (XY)^{-1} \), then \( cX \subset Y^{-1} \subset T = R_f \). Thus \( c \in \text{Hom}_{\mathbb{R}}(X, R_f) \) which implies \( c \in R_f \) by [10, Proposition 7.8].

Let \( I_f = \bigcup I \cdot X^{-1} \), where \( X \) ranges over \( F \), for a right \( v \)-ideal \( I \).

**Lemma 1.5.** The following statements hold for a right \( v \)-ideal \( I \).

(i) \( I_f \) is a right \( T \)-ideal and a \( v \)-ideal as a right \( T \)-ideal.
(ii) If \( I \subset R \), then \( I_f = T \) if and only if \( I \in F \).
(iii) If $X \in F$, then $(X^{-1})_{F_1} = T$.

Proof. (i); If $X, Y \in F$, then $I_0 X^{-1} + I_0 Y^{-1} \subseteq (I_0 X^{-1} + I_0 Y^{-1})^* = I_0(X^{-1} + Y^{-1})^* = I_0(X \cap Y)^{-1}$ by [5, Lemma 3]. Thus $I_0 X^{-1} + I_0 Y^{-1} \subseteq I_1$. Let $t \in T$, $Y \in F$ such that $t \in Y^{-1}$, $x \in I_0 X^{-1}$ for $X \in F$, and $Z = \{r \in R ; t \in X \} \in F$. Then we have $xX \subseteq I$ and $xY \subseteq xX \subseteq I$, which implies $(xT + I_0 Z^{-1})Z \subseteq I$ and $xT + I_0 Z^{-1}$ to be a right $S$-ideal with $S = O_1(Z)$. Since we can take $X$ to be a right $x$-ideal, $Z$ is also a right $x$-ideal by [5, Lemma 3]. Thus $I_0 X + I_0 Y \subseteq I$. Let $t \in T$, $F \in F$ such that $t \in Y^{-1}$, $x \in I_0 X^{-1}$ for $X \in F$, and $Z = \{r \in R ; t \in x \} \in F$. Then we have $I \subseteq I_0 X + I_0 Y \subseteq I$. Thus $I_0 X + I_0 Y \subseteq I$.

Hence $I_0 F \subseteq I$ is a right $I$-ideal. To prove that $I_0 F$ is a $I$-ideal as a right $T$-ideal, it suffices to show $(I_0 F)^{-1} = (I^{-1})_{I_1} = (I^{-1})_I$. Then we have $I_0 F = (I^{-1})_{I_1} = (I^{-1})_I$. Hence again by [3, Lemma 2.2] $(I^{-1})_{I_1} \subseteq (I^{-1})_I$. This completes the proof. (ii); If $I \subseteq R$, then $I_0 X^{-1} \subseteq R \subseteq X^{-1} \subseteq T$ for any $X \in F$, that is, $I_0$ is an integral right $T$-ideal. It holds that $I_0 F = T \subseteq I \subseteq I_1$. For any $X \in F$, we have $y(I_0 X^{-1}) = yI_0 X^{-1} \subseteq Y^{-1} \subseteq T$ by Lemma 1.4. Thus $yI_0 \subseteq T$, that is, $y \in (I^{-1})_I$. Hence again by [3, Lemma 2.2] $(I^{-1})_{I_1} \subseteq (I^{-1})_I$. This completes the proof. (iii); We have $(X^{-1})_{I_1} = (X^{-1})_F = (X^{-1})_{I_1}$.

Lemma 1.6. If $P \subseteq P$ and $A$ is a nonzero ideal of $R$, then $AR_P = (AR_P)_{I_1} = (R_P A) = R_P A$.

Proof. If $AR_P = R_P$, then $A \cap C(P) \subseteq \phi$ which implies $R_P A = R_P = AR_P$. Thus we assume $R_P \neq R_P$, that is, $A \subset P$. If $AP^{-m}R_P \neq R_P$ for every integer $m \geq 1$, then $AP^{-m} \subseteq P$. For $AP^{-m} \subseteq P$ implies, $AP^{-m} \subseteq P \subseteq R$. If $AP^{-m} \subseteq P$, then $AP^{-m} \cap C(P) \subseteq \phi$, that is, $AP^{-m} \cap C(P) \subseteq \phi$. Thus $AP^{-m} \subseteq P$ and the assertion holds by induction. There is an increasing chain of proper right ideals of $R_P$;

$AR_P \subseteq AP^{-1}R_P \subseteq AP^{-2}R_P \subseteq \cdots \subseteq AP^{-m}R_P \subseteq \cdots$

which must stabilize since $R_P$ is noetherian. Therefore there is an integer $n$ such that $AR_P \subseteq AP^{-n}R_P \neq AP^{-m}R_P$. We have $R_P A R_P = R_P A R_P \neq R_P$ and $R_P \subseteq O_1(R_P A) \subseteq R_P$ which is a contradiction. Thus $AP^{-m} \subseteq R_P$ for some integer $m \geq 1$. Let $m$ be the smallest such integer. We conclude that $AP^{-m} \subseteq P$ and then $AP^{-m} \subseteq R$ which implies $AP^{-m} \cap C(P) = \phi$. Thus $AP^{-m} \subseteq R_P \neq R_P A P^{-m}$, and then $R_P A \supseteq R_P A P^{-m} \neq AP^{-m} R_P R^m = AR_P$. By the similar way we have $AR_P \supseteq R_P A$. Hence $AR_P = R_P A$. Now, it suffices to show $AR_P = (AR_P)_{I_1}$. Let $a$ be a regular element in $A$ such that $AR_P = a R_P = R_P a$ and $x \in (AR_P)_{I_1}$. Then
there exists \( J = Ra_1 + \cdots + Ra_n \subseteq AR_p \) with \( x \in J^* \). Put \( a_i = r_i c^{-1} a \) for \( c \in C(P) \) \((i = 1, \ldots, n)\). We have \( x \in J^* = (\sum_{i=1}^{n} ra_i c^{-1} a) = (\Sigma r_i) c^{-1} a \subseteq R c^{-1} a \subseteq R_p a = AR_p \).

Hence \((AR_p) \subseteq AR_p\). This completes the proof.

**Lemma 1.7.** Let \( A \) be an \( R \)-ideal in \( R \). Then \( A^* = \bigcap_{p \in P} AR_p \cap S(R) \).

Proof. Since \((A^{-1}A)^* = R\), we have \( A^* A^{-1} \subseteq P \) and \( A^{-1}A \subseteq P \), that is, \( R_p = R_p A A^{-1} = A^{-1}AR_p \) for all \( P \in \mathcal{P} \). Thus \( A^* R_p = A^* (A^{-1}A) R_p = R_p (A^* A^{-1} A) = R_p A = AR_p \) by Lemma 1.6 and \( A^* S(R) = AS(R) = S(R) \). Hence we have \( A^* = \bigcap_{p \in P} A^* R_p \cap A^* S(R) = \bigcap_{p \in P} AR_p \cap AS(R) \) by [3, Corollary 4.2].

Let \( P \in \mathcal{P}_0 \) be the corresponding unique maximal ideal of \( R_p \), and \( P'' = P' \cap T \) the minimal prime \( v \)-ideal of \( T \). Then \( P'' R_p = P'' P R_p \), since \( T \) is Krull in the sense of [4]. It is noted that the same proof as Lemma 1.7 yields \( B^* = \bigcap_{p \in P_0} B R_p \cap S(R) \) for every ideal \( B \) of \( T \). We shall write \( I_T = I \circ R_P = I \circ T \) for a right \( v \)-ideal \( I \).

**Lemma 1.8.** We have \((P^n)^* o T = T o (P^n)^* = (P''^n)^* = P'' R_p \cap T\) for every \( P \in \mathcal{P}_o \) and every natural number \( n \).

Proof. Since \((P^n)^* o T \) is a right \( v \)-ideal by Lemma 1.5, we have \((P^n)^* o T = \bigcap_{p \in P_0} ((P^n)^* o T) R_p \cap S(R) \supseteq \bigcap_{p \in P_0} P^n R_p \cap S(R) = P^n R_p \cap T = P''^n R_p \cap T = (P''^n)^* \). On the other hand, \((P^n)^* o T = (P^n)^* T \) \( \subseteq \bigcap_{p \in P_0} P^n R_p \cap T \) by Lemmas 1.1(ii), 1.6, and 1.7. Hence \((P^n)^* o T = P'' R_p \cap T = (P''^n)^* \). By the similar way \( T \cap P'' R_p = T o (P^n)^* \). This completes the proof.

**Lemma 1.9.** (i) If \( A \) is a \( v \)-ideal of \( R \), then \( A o T = T o A \) is a \( v \)-ideal of \( T \).

(ii) If \( A'' \) is a \( v \)-ideal of \( T \), then \( A = A'' \cap R \) is a \( v \)-ideal of \( R \) and \( A'' = A o T \).

Proof. (i); Let \( A = (P_i^n) \) with \( P_i \in \mathcal{P}_0 \) \((i = 1, \ldots, l)\) and \( P_j \in \mathcal{P} - \mathcal{P}_0 \) \((j = l+1, \ldots, k)\). Since \( A o T \) is a right \( v \)-ideal, \( A o T = \bigcap_{p \in P} (A o T) R_p \cap S(R) \supseteq \bigcap_{p \in P} A R_p \cap S(R) = \bigcap_{p \in P} P^n R_p \cap T = (P^n)^* \cap \cdots \cap (P^n)^* \). On the other hand, \( A o T = (A T) \subseteq \bigcap_{p \in P_0} A R_p \cap S(R) = \bigcap_{p \in P_0} P^n R_p \cap T = (P''^n)^* \cap \cdots \cap (P''^n)^* \). Thus \( A o T = (P''^n)^* \cap \cdots \cap (P''^n)^* = T o A \) by the similar way. (ii); It holds that \( \{P''; P \in P_0\} \) is the set of all prime \( v \)-ideals of \( T \) (cf. [4, Proposition 1.7]). Thus we have \( A'' = (P''^n)^* \cap \cdots \cap (P''^n)^* = P'' R_p \cap T \cap \cdots \cap P'' R_p \cap T \) by Lemma 1.8. Hence \( A = (P'' R_p \cap R) \cap \cdots \cap (P'' R_p \cap R) = (P''^n)^* \cap \cdots \cap (P''^n)^* \) is a \( v \)-ideal of \( R \). Moreover, we have \( A o T = ((P''^n)^* \cap \cdots \cap (P''^n)^*) o T = (P''^n)^* o T \cap \cdots \cap (P''^n)^* o T = (P''^n)^* \cap \cdots \cap (P''^n)^* = A'' \) by Lemmas 1.1(ix) and 1.8. This completes the proof.

**Theorem 1.10.** \( T \) is a Krull order in \( Q \).

Proof. In order to prove \( T \) to be Krull, it suffices to show \( S(R) = S(T) \), because \( T \) is Krull in the sense of [4] (cf. [4, Proposition 1.2]). Let \( A \) be a
v-ideal of R. Then $A''=A\circ T$ is also a v-ideal of $T$ and $A^{-1}\cap T\circ A^{-1}=(A\circ T)^{-1}\subset S(T)$ by the proof of Lemma 1.5 (i). Thus $S(R)\subset S(T)$. Conversely, let $A''$ be a v-ideal of $T$ and $A=A''\cap R$ a v-ideal of $R$. Then $(A'')^{-1}=T\circ A^{-1}=(TA^{-1})^{-1}\subset (S(R))^{-1}=S(R)$. Hence $S(R)=S(T)$ and thus $T$ is Krull.

Corollary 1.11. If we choose $P_0$ such that $P-P_0$ is a finite set, then $R=T\cap T'$, where $T=\cap_{P\in P_0} R_P \cap S(R)$ is a Krull order and $T'=\cap_{P\in P_0} R_P$ is a bounded Dedekind prime ring, and is a right and left principal ideal ring.

Proof. This follows from Theorem 1.10 and [3, Lemma 3.3].

Now, we specially choose $P_0$ to be the set of all invertible prime ideals. Then $T$ is an $RI$-order in the sense of Cozzens and Sandomierski [1], here an order in a simple artinian ring is said to be an $RI$-order, its two-sided v-ideals form a group under the ordinary multiplication.

Theorem 1.12. If $P_0$ is the set of all invertible prime ideals of $R$, then $T=\cap_{P\in P_0} R_P \cap S(R)$ is an $RI$-order.

Proof. It holds that \{P''; $P''=P'\cap T$ and $P'$ is a unique maximal ideal of $R_P$, $P\in P_0$\} is the set of all minimal prime ideals of $T$. Thus by [1, Proposition 2.4] we only show that each $P''$ is invertible. Let $x\in P''$ and $X\in F$ with $x\in X^{-1}$. Then $xX\subset R\cap P'=P$ and $xR_P=xXR_P\subset PR_P$. We have $x\in PR_P \cap P\cap S(R)=\cap_{P\in P_0} PR_P \cap S(R)=PT$ by the invertibility of $P$ and [6, Proposition 2.1]. Thus $P''\subset PT$. The converse inclusion is clear, so that $P''=PT$. By the same way we have $P''=TP$. Hence $P''$ is invertible.

We shall give the structure of the integral v-ideals of $R$.

Theorem 1.13. Let $R$ be a Krull order in $Q$ and $A$ a v-ideal of $R$. Then $A=P_1^*\cdots P_k^*B=P_1^*\cdots P_k^*B$, where each $P_i$ is invertible ($i=1,\ldots,k$) and $B$ is a v-ideal such that $B\subset P$ for every invertible prime ideal $P$.

Proof. Let $A=(P_1^*\cdots P_k^*P_{k+2}^*\cdots P_l^*)^*$, where each $P_i$ is an invertible prime ideal ($i=1,\ldots,k$) and each $P_j$ is a noninvertible prime v-ideal ($j=k+1,\ldots,l$). Then by [1, Lemma 2.1] we have that $(P_1^*\cdots P_k^*)^{-1}A=((P_1^*\cdots P_k^*)^{-1}A)^*=(P_1^*\cdots P_k^*)^{-1}(P_1^*\cdots P_k^*)^*=B$ is a v-ideal. It is clear that $B$ satisfies the condition of the theorem. Hence $A=P_1^*\cdots P_k^*B$. Starting with $A=(P_1^*\cdots P_k^*P_{k+1}^*\cdots P_l^*)^*$ we get $A=BP_1^*\cdots P_k^*$. This completes the proof.

2. Maximal orders over unique factorization domains

Throughout this section, let $R$ be a unique factorization domain and $\Lambda$ a maximal $R$-order in the sense of Fossum [2], that is, $\Lambda$ satisfies the following
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Let $\mathcal{P}$ be the set of all minimal prime ideals of $R$, $\mathcal{P}_1=\{p\in\mathcal{P};$ the minimal prime ideal $P$ of $\Lambda$ with $P\cap R=p$ is invertible in $R\},$ $\mathcal{P}_2=\mathcal{P}-\mathcal{P}_1$, $\Lambda_1=\cap p\in\mathcal{P}_2\Lambda_p$, and $\Lambda_2=\cap q\in\mathcal{P}_2\Lambda_q$. It is well-known that there is a unique minimal prime ideal $P$ of $\Lambda$ with $p=P\cap R$ for every $p\in\mathcal{P}$. Let $\delta$ be the different of $\Lambda$, that is, $\delta=C(\Lambda)^{-1}$, where $C(\Lambda)$ is the complementary ideal, $C(\Lambda)=\{x\in\Sigma; \text{tr}(x\Lambda)\subset R\}$, in which tr denotes the usual trace function in a simple algebra [8]. Let $P_1,\ldots,P_n$ be all minimal primes of $\Lambda$ which contain $\delta$. We shall classify the minimal primes of $\Lambda$ with respect to the invertibility.

Lemma 2.1. If $P$ is a minimal prime of $\Lambda$ with $P\neq P_i(i=1,\ldots,n)$, then $P=p\Lambda(p=P\cap R)$ is invertible. If $P$ is one of $P_i$'s, then;

(i) There is an integer $t>1$ such that $P^t=p\Lambda(p=P\cap R)$ is invertible.
(ii) $P^t\neq p\Lambda(p=P\cap R)$ for any integer $t\neq P$ is not invertible.

Proof. By [9, §5] we only prove the second statement. (i); $\Rightarrow$: Since $p\Lambda$ is invertible by hypothesis, $P$ is, too. $\Leftarrow$: There is an integer $t>1$ such that $P_i^t=p\Lambda_x$. If $q\in\mathcal{P}$ with $q\neq p$, then $P_i^t=p\Lambda_q=\Lambda_q$. Thus $P^t=\cap q\in\mathcal{P}_i^t\cap q\in\mathcal{P}\Lambda_q$ $=p\Lambda$, since $P^t$ and $p\Lambda$ are $v$-ideals. Now, (ii) holds at once.

Theorem 2.2. Let $\Lambda$, $\Lambda_1$, and $\Lambda_2$ are the same as in the first paragraph of this section. Then $\Lambda=\Lambda_1\cap\Lambda_2$, where $\Lambda_1$ is an RI-order and $\Lambda_2$ is a bounded Dedekind prime ring, and is a right and left principal ideal ring.

Proof. This follows from Lemma 2.1, Corollary 1.11, and Theorem 1.12.

Applying Theorem 1.13 to a $v$-ideal of $\Lambda$ we get the following.

Proposition 2.3. If $A$ is a $v$-ideal of $\Lambda$, then $A\Lambda_1$ is also a $v$-ideal of $\Lambda_1$.

Proof. Let $A=P_1^t\cdots P_n^tB$, where $P_i$'s are minimal primes with $P_i\cap R\in\mathcal{P}_1$ and $B$ is a $v$-ideal of $\Lambda$ such that $B\subseteq P$ for every minimal prime $P$ with $P\cap R\in\mathcal{P}_1$. Let $Q$ be a minimal prime ideal of $\Lambda$ with $q=Q\cap R\in\mathcal{P}_2$ and $R_i=\cap p\in\mathcal{P}_2 R_p$. Then $q=\epsilon R$ for $\epsilon\in R$, since $R$ is a unique factorization domain. We have $qR_1=\epsilon R_1=\cap p\in\mathcal{P}_2 \epsilon R_p=R_1$, and then $q\Lambda_1=\Lambda_1$ which implies $Q\Lambda_1=\Lambda_1$. Therefore, $B\Lambda_1=\Lambda_1$. Hence $A\Lambda_1=P_1^t\cdots P_n^tA_1$ is a $v$-ideal of $\Lambda_1$ (in fact, an invertible ideal).

3. Example. In this section, we study an example of a maximal $R$-order which has a noninvertible prime $v$-ideal.

Remark. It was shown in [1, §2] that an arbitrary maximal $R$-order,
where $R$ is a noetherian integrally closed domain, is an $R$-$I$-order. However, the following example turns out to be the counter example of this statement.

Now, our example is originated by Ramras [7]. Let $k$ be a perfect field of characteristic $\neq 2$, $R=k[[X, Y]]$ with $X$ and $Y$ transcendental over $k$, and $K=k((X, Y))$. Let $\Lambda$ be the quaternion algebra $K[\ell, a, \beta, \alpha \beta]$ with $\alpha^2=X$, $\beta^2=Y(Y-X)(Y+X)$, and $\alpha \beta=-\alpha \beta$. Then the $R$-free order $R[\ell, a, \beta, \alpha \beta]=\Lambda$ is a maximal $R$-order in $\Sigma$ by [7]. We shall compute the different $\delta$ of $\Lambda$, and for the reader's convenience we state the process of it. Let $L=K(\beta)$ be the cyclic extension of $K$ and $\Sigma=L \otimes L$. We put $\alpha=k-\beta l$ for any $a=k+\beta l \in L(k, l \in K)$. $L$ is the splitting field of $\Sigma$ and the ring isomorphism $L \otimes K \Sigma \cong M_2(L)$ is given by

$$1 \otimes a \rightarrow \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ and } 1 \otimes ab \rightarrow \begin{pmatrix} 0 & Xb \\ b & 0 \end{pmatrix}$$

for $a, b \in L$ (cf. [8, Example 9.4]). Thus $tr(xy)=a+a$ for $x=a+\alpha \beta \in \Sigma$. Let $x, y \in \Sigma$ and $x=k_1+\beta l_1+\alpha(k_2+\beta l_2)$, $y=m_1+\beta n_1+\alpha(m_2+\beta n_2)(k_i, l_i, m_i, n_i \in K)$. Then:

$$xy = k_1m_1+Xk_2m_2+\psi_1l_1n_1-X\psi_1m_2+\beta(k_1n_1+Xk_2n_2+l_1m_1-Xl_2m_2)$$

$$+\alpha(k_2m_2+k_2m_1-\psi_1l_2n_2+\psi_1m_1+\beta(k_2n_2-l_1m_2+k_2n_1+l_1m_1)),$$

where $\psi=Y(Y-X)(Y+X)$. Now, we get $tr(xy)=2(k_1m_1+Xk_2m_2+\psi_1l_1n_1-X\psi_1m_2)$, $C(\Lambda)=\{x=k_1+\beta l_1+\alpha(k_2+\beta l_2) \in \Sigma; k_i \in R, k_2 \in (1/X)R, l_1 \in (1/\beta)R, l_2 \in (1/X)R\}$, and then $\delta=C(\Lambda)^{-1}=\{y=m_1+\beta n_1+\alpha(m_2+\beta n_2) \in \Lambda; m_i \in \psi(X), m_2 \in \psi(Y), n_i \in (X), n_2 \in (Y)\}$. Let $P_0=\{y=m_1+\beta n_1+\alpha(m_2+\beta n_2) \in \Lambda; m_1, n_1 \in (X), m_2, n_2 \in (Y)\}=\alpha \Lambda$, $P_1=\{y=m_1+\beta n_1+\alpha(m_2+\beta n_2) \in \Lambda; m_1, m_2 \in \psi(X), n_1, n_2 \in (X)\}$, where $\psi_1=Y, \psi_2=Y-X, \psi_3=Y+X(i=1, 2, 3)$. Thus $P_0 \supset \delta(i=0, 1, 2, 3), P_0$ is invertible, each $P_i$ is a $v$-ideal, since $P_i^{-1}=\{x=k_1+\beta l_1+\alpha(k_2+\beta l_2) \in \Sigma; k_i, l_2 \in (1/\psi)R, k_2 \in (X)R\} (i=1, 2, 3)$. Since $\alpha \Lambda(\psi)=\text{Rad}(\Lambda(\psi)), \beta \Lambda(\psi)=\text{Rad}(\Lambda(\psi))$, each $P_i(i=0, 1, 2, 3)$ is a prime ideal by the equations $\alpha \Lambda(\psi) \cap \Lambda=P_0, \beta \Lambda(\psi) \cap \Lambda=P_i(i=1, 2, 3)$. If $P_i(i=1, 2, 3)$ is invertible, there exist $k_{ij}, l_{ij}, m_{ij}, n_{ij} \in R(i=1, 2, 3)$ such that $\psi_1 \Sigma_1 k_{ij} l_{ij}+X \psi_1 \Sigma_1 k_{ij} m_{ij}+(\psi_1 \psi_1) \Sigma_1 l_{ij} n_{ij}+X(\psi_1 \psi_1) \Sigma_1 l_{ij} n_{ij}+\psi_1 l_{ij} n_{ij}=1$. However, the left hand side of this equation is contained in $(X, Y)$ which is a contradiction. Thus each $P_i$ is not invertible $(i=1, 2, 3)$. It holds that $P_0^2=X \Lambda, P_i=\beta \Lambda+\psi_i \Lambda$, and no power of $P_i$ equals $\psi_i \Lambda(i=1, 2, 3)$. If $P$ is a minimal prime ideal which contains $\delta$, then $P \cap R \supset (X \psi)$. Thus $P \cap R \supset (X \psi)$ or $(\psi_i)$ which implies that $P$ equals one of $P_i(i=0, 1, 2, 3)$. Summarizing the above results we get the followings. $\Lambda$ is a maximal $R$-order which has the prime $v$-ideals $P_i(i=0, 1, 2, 3), P_0$ is invertible, $P_0 \not \subset X \Lambda$, and $P_0^2=X \Lambda$ and $P_i$ is noninvertible, $P_i \not \subset \psi_i \Lambda$ for any positive integer $t$, and $P_i=\beta \Lambda+\psi_i \Lambda(i=1, 2, 3)$. Every prime $v$-ideal $P \not \subset P_i(i=0, 1, 2, 3)$ is invertible and $P=\beta \Lambda(\beta=P \cap R)$. 

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References


