THE AFFINE STRUCTURES ON THE REAL TWO-TORUS (I)

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Introduction

In these two papers we intend to study the space of all affine structures on the real 2-dimensional torus T^2, a problem suggested by C. Ehresmann in 1936, or more specifically by S. S. Chern in one of his lectures and attacked by N. H. Kuiper [6] among others. An affine structure on a mainfold is a maximal atlas whose coordinate transformations belong to the affine transformation group $A(n)$ on the affine space.

Our main purpose is to describe the set $\{\Gamma\}$ of all affine structures on T^2 module the group $\text{Diff}[T^2]_e$; here $\text{Diff}[T^2]_e$ is the group of all diffeomorphisms of T^2 which induce the identity on the fundamental group $\pi_1(T^2)$. The space $\{\Gamma\}/\text{Diff}[T^2]_e$, equipped with an appropriate topology, is regarded as an affine version of the Teichmüller space.

In the usual case the holonomy group H of an affine structure on a mainfold is defined as a subgroup of the affine transformation group $A(n)$ up to the conjugate class. In this work, however, we construct a modified holomony group H^* for an affine structure so that in the case of 2-dimensional affine torus the group H^* is a subgroup of $\widetilde{A(2)}_e$, the universal covering group of the identity component of $A(2)$. We do this in such a way that the modified holonomy group H^* is mapped onto the usual holonomy group by the projection mapping.

With this modification of the holonomy group the first main result in the paper could be summarized as follows (Theorem 3.3 and 4.15): the affine structures on T^2 are completely determined by their modified holonomy groups H^*.

Carrying out the determination of holonomy groups H^*, we describe the space $\{\Gamma_h\}/\text{Diff}[T^2]_e$ of the homogeneous affine structures on T^2. As Y. Matsushima [7] discusses for complex tori in a somewhat different way, we show the following (Theorem 3.10): the space $\{\Gamma_h\}/\text{Diff}[T^2]_e$ is an affine algebraic or, more precisely a 4-dimensional quadratic cone in R^6 without singularities variety, except at the vertex, the vertex itself corresponding to the natural affine

structure. We shall be concerned with the whole space $\{\Gamma\}/\mathrm{Diff}[T^2]_e$ in the second paper where it will be shown that the space is 4-dimensional and connected. That it is finite dimensional is not surprising in view of the Kodaira-Spencer theory of deformations or rather the general theory of van Quê [12], since the infinitesimal affine transformations satisfy a certain elliptic differential equation.

Technically, the great difficulty lies in establishing the fact that the developing map from the universal covering space of an affine torus into the affine plane is a covering map. The first section, besides explaining basic concepts and their properties, is largely devoted to this question. Actually the object of our concern in this section is not just the affine structure but a more general one which includes Ehresmann's locally homogenerous structure. The reason for this generalization is to separate general facts from ones which are particularly related to the affine sturcture.

Since the holonomy group H in $A(2)$ is a homomorphism image of a fundamental group $\pi_1(T^2)$, it is an abelian subgroup with at most 2 generators. In the second section, in order to obtain a better understanding of the holonomy groups, we take a methodical approach to the algebraic study of the maximal abelian subgroup G of $A(2)$ which in effect allows some results to remiain valid in higher dimensional cases. The main part is purely algebraic with an added survey of the action of G, from which one could obtain conclusions on some geometric properties (e.g., completeness, convexity) of an indivisual or the entire affine structures. The third section, concerned with homogeneous case, is a straightforward application of the preceding sections. In the forth section, the study of the inhomogeneous case is concerned in part with topological dynamics. It will also be proved that the inhomgeneous affine structure is completely determined by the number of the closed H-invariant geodesics together with the usual holonomy group H.

To conclude the introduction, we wish to touch on the existence of affine structures in a more general setting, since we shall have no occasion to do so in the text. The torus is the only closed orientable surface which admits affine structures (Benzecri [1] or Milnor [8] for a more general treatment). Diagonally opposite to it, every open surface admits affine structures and indeed many of them. A more complete result was obtained for open Riemann surface R as the solution to the Riemann-Hilbert problem (Hilbert 21 st problem) by Röhrl [14], which in our context roughly reads that for any preassigned homomorphism: $\pi_1(R)\rightarrow GL(1;C)$ there exists a complex affine structure on R whose linear holonomy group is the image of the homomorphism. As to the real affine structures, it is easy to see that we have the analogue on the domain $D=R^2-\{p,q\}\,(p\neq q)$, for any homomorphism: $\pi_1(D)\rightarrow A(2)$. For that matter the surfaces with boundaries are more or less the same as open surfaces. However, it seems extremely difficult to extend to the whole surface an affine structure given on a neighborhood of the

boundary (cf. Poenaru [11]). Still open is the question of whether a compact affine mainfold has the Euler number=0 or not; we affirm this in special cases (e.g. Corollary 1.24). Some results in this paper were briefly announced in [18].

1. Basic concepts and properties

We will discuss some rudiments of "locally flat" structures which are more general than affine structures. But we confine ourselves to establishing the facts to be used in later sections rather than developing a general theory of what we call an A-structure.

Now let A be a connected (real) analytic mainfold, and A be an effective group of analytic transfomrations of A. A is given the discrete topology. An A-structure or an (A, A)-structure on a mainfold M is by definition a maximal atlas $\mathcal{A}=\{(\alpha, U_\omega)\}$ for M such that (U_α), is an open covering of M, each α is a homeomorphism of an open set U_α into A and each coordinate transformation $\beta \circ \alpha^{-1}$ is the restriction to $\alpha(U_\alpha \cap U_\beta)$ of a member g in A. A member (α, U_α) of an A-sturcture is called an A-chart. Note that g is defined globally on A and g unique (unless $U_\alpha \cap U_\beta$ is empty) for $\alpha, \beta \in \mathcal{A}$. These properties of $\beta \circ \alpha^{-1}$ make the A-sturcture differ from Chern's G-structure and Ehresmann's pseudogroup structure which are much more general. A mainfold M with an A-sturcture is called an A-*manifold*. M is then an analytic mainfold. For instance, an A-structure is a real n-dimensional affine structure when A is the affine group $A(n)$ of the real-dimensional affine space A^n. Given an A-mainfold M and an equidimensional (viz. dim N=dim M) immersion $f: N \rightarrow M$, we define an A-structure of N by the atlas $\{\alpha \circ f \mid \alpha \in \mathcal{A}\}$. Given two A-mainfolds M, N, we say an immersion $f: N \rightarrow M$ is an A-map when the A-structure of N is the one induced by f. We thus have the category of the A-manifolds and the A-maps, called the category (A, A). The related terms like A-isomorphisms will hardly need definitions. To avoid an unnecessary difficulty we always assume the connectedness for every mainfold in this paper. The following fundamental proposition is practically known.

Proposition 1.1. *There exists a covariant "functor" $\mathcal{H}$ of the category (A, A) into itself (modulo certain equivalence relation) having these properties*: (i) $\mathcal{H}H$ *is a principal bundle over M*, (ii) *the structure group of $\mathcal{H}M$ is a subgroup H of A*, (iii) *the projection $p: \mathcal{H}M \rightarrow M$ is an A-map*, (iv) *the A-map $\mathcal{H}f: \mathcal{H}N \rightarrow \mathcal{H}M$ commutes with the projections for each A-map $f: N \rightarrow M$*, and (v) *there exists a A-map $d: \mathcal{H}M \rightarrow A$.*

Proof. We will constructure $\mathcal{H}M$ and $\mathcal{H}f$. Consider the disjoint union $\cup_{\alpha \in \mathcal{A}}\{\alpha\} \times U_\alpha \times A$, where $\mathcal{A}$ is the A-structure of M and the other notations are as before. A acts on this space to the right, $g \in A$ carrying (α, x, a) into (α, x, ag). We pass to the quotient space $B=B_M$ with equivalence realtion: $(\alpha, x, a) \sim (\beta, y, b) \Leftrightarrow x=y$ and $\beta \alpha^{-1}=ba^{-1}$. The group A acts on B too. B is

then a principal A-bundle over M. The structure group A permutes the connected components of B among themselves. Each component is a covering space of M and is given the A-manifold-structure induced by the projection of B onto M. The componentes are A-isomorphic by the members of A. Let $\boldsymbol{M}$ be one of the components and let $p\colon \boldsymbol{M}\to M$ be the projection. The structure group $H=H_{\boldsymbol{M}}$ of $\boldsymbol{M}$ is carried into $g^{-1}Hg$ when $\boldsymbol{M}$ is carried into another component $\boldsymbol{M}g$ by $g\in A$ For an A-map $f\colon N\to M$, B_N is the induced bundle from B_M by f, and $\boldsymbol{N}$ is A-isomorphic with a subbundle of the pullback of $\boldsymbol{M}$. Hence the inclusion map gives the lift $\mathcal{H}f\colon \mathcal{H}N\to \mathcal{H}M$ that is, $\mathcal{H}f$ commutes with the projections and H_N-equivariant (meaning that, $H_N \subset H_M$ acting on $\boldsymbol{M}$ and $\boldsymbol{N}$, $\mathcal{H}f$ commutes with the members of H_N). Finally the map $\boldsymbol{d}\colon \boldsymbol{M}\to A$ is obtained as the restriction of $\boldsymbol{M}$ of the map: $B\to A$ which in turn is induced from the map: $(\alpha, x, a)\to a^{-1}\alpha(x)$ defined on the disjoint union above. $\boldsymbol{d}$ is obviously an A-map. One has only to put $\mathcal{H}M=\boldsymbol{M}$.   Q.E.D.

REMARK. All the conditions except (ii) will be satisfied if $\mathcal{H}M$ is taken as the universal covering space of M. The functor constructed above is characterized by the minimum condition:

(1.2)   Let $p'\colon M'\to M$ be a covering map of an A-manifold M. If there exists an A map $d'\colon M'\to A$ then there exists an A-map $q\colon M'\to \mathcal{H}M$ which factors $d'=\boldsymbol{d}\circ q$ mod A.

REMARK and DEFINITIONS. We will use $\boldsymbol{M}$ as $\mathcal{H}M$, though the condition (1.2) will not be used substantially. The group H which is determined up to the conjugate class in A will be called *the holonomy group* of the A-manifold M. Keeping in mind the obvious ambiguity, we call $\boldsymbol{M}$ and $\boldsymbol{d}$ *the holonomy covering space* and *the developing map*.

REMARK. In general the developing mapping $\boldsymbol{d}\colon \boldsymbol{M}\to A$ is not a covering map onto its image.

We list a few corollaries:

(1.3a) H acts on $\boldsymbol{M}$ as an A-automorphism group.

(1.3b)   The developing map $\boldsymbol{d}\colon \boldsymbol{M}\to A$ is H-equivariant.

(1.3c)   There is a natural epimorphism: $\pi_1(M)\to H$ where $\pi_1(M)$ is the fundamental group of M.

(1.3d) $\boldsymbol{d}(\boldsymbol{M})$ is an open connected submainfold of A.

(1.3e) M is naturally A-isomorphic with the quotient A-manifold $\boldsymbol{M}/H$.

(1.3f)   The developing map $\boldsymbol{d}\colon \boldsymbol{M}\to A$ induces a homomorphism $d_*\colon \mathrm{Aut}(\boldsymbol{M})\to \mathrm{Aut}(A)$ of the A-automorphism group and $\boldsymbol{d}$ is $\mathrm{Aut}(\boldsymbol{M})$-equivariant with respect to d_*. The image of d_* leaves $\boldsymbol{d}(\boldsymbol{M})$ invariant. Moreover if an automorphism $g\in \mathrm{Aut}(\boldsymbol{M})$ centralizes H, then so does $d_*(g)$.

Proof.   Since $d \circ g$, $g \in \mathrm{Aut}\,(M)$, is another developing map, we have some $g' \in \mathrm{Aut}\,(A)$ with $d \circ g = g' \circ d$. g' is unique for the given g and globally defined on A.   Put $d_*(g) = g'$.   The rest should be obvious.   Q.E.D.

By Proposition 1.1, an A-manifold M gives rise to two surjective, equidimensional and H-equivariant immersions $p \colon \overline{M} \to M$ and $d \colon \overline{M} \to d(\overline{M}) \subset A$ with $M = \overline{M}/H$, where H acts on M as the identity.   This situation allows us to transplant certain geometric objects into $d(\overline{M})$ and vice versa.   Generally, we call those two processes *the development,* denoted by $\mathscr{D}$, and *the envelopment,* denoted by $\mathscr{E}$.   Although this may not be envelopment in the usual sense of the word, we find these symbls convenient for our purpose.   Of course we have to give a clear definition for them each time we have a type of "geometric objects", as follows.

(1.4$\mathscr{D}$)   Given a continous map $f \colon X \to M$ from a 1-connected (meaning connected and simply connected) space X into the A-manifold M, there is a continuous map $\mathscr{D}f \colon X \to d(\overline{M})$ with $\mathscr{D}f = d \circ F$ for some $F \colon X \to \overline{M}$ satisfying $f = p \circ F$. $\mathscr{D}f$ is unique up to the composite with a member of H.   If X is a mainfold and f is smooth, an immersion, etc., then so is $\mathscr{D}f$ repectively.

For instance, any curve in M, regarded as a map of an interval into M, is developed into a curve in $d(\overline{M}) \subset A$.

(1.4a$\mathscr{D}$)   The above conclusion follows when X is not 1-connected but f is homotopic to a constant mapping.

To construct another example, take a subset U of M which is either 1-connected or contractible in M to a point, then, we can imbed U into $d(\overline{M})$ through the development $\mathscr{D}i$ of the inclusion $i \colon U \to M$.   Therefore we have

(1.4b$\mathscr{D}$) M has an open covering $\{U\}$ such that $\{\mathscr{D}i_U\}$ is an atlas of M which defines an (H, A)-structure on M, where i_U is the inclusion map of U.

(1.4$\mathscr{E}$)   Let Y be a space on which H acts (trivially or not).   If $g \colon d(\overline{M}) \to Y$ is an H-equivariant map, then there is a unique map $\mathscr{E}g \colon M \to Y/H$ with $\mathscr{E}g \circ p = \pi \circ g \circ d$ where π is the projection: $Y \to Y/H$ onto the orit space Y/H.

This assertion has the following two applications.

(1.5)   If A is not compact but M is, then H is an infinite group.

Proof.   Let g be the identity map of $d(\overline{M})$.   Then $d(\overline{M})/H$ is compact, while $d(\overline{M})$ is open in A.

(1.5a)   Let $R \subset A \times A$ be an equivalence relation for points in A such that (1) the projection: $A \to A/R$ is open and (2) A/R is a noncompact Hausdorff space.   Then H cannot fix each equivalence class for a compacgt A-manifold M.

Perhaps a more concrete lemma should be stated along this line:

(1.5b)   If an A-manifold M is compact, then H cannot fix each leaf of a folia-

tion Φ on A such that A/Φ is a non-compact manifold in the natural way (i.e. the space A/Φ obtained by smashing each leaf to a point is a mainfold in such a way that the projection: $A \to A/\Phi$ is a submersion. See Palais [9]).

Here the Hausdorff property of A/Φ is not quite relevant; for instance, if A is diffeomorphic with R^2 then H cannot fix each leaf of any flow on A by virtue of W. Kaplan's theorem [5]. For an application of (1.5b) see (3.0).

(1.6$\mathcal{D}$)   If B is a nonempty subset of M then $\mathcal{D}B = d \circ p^{-1}(B)$ is a nonempty H-invariant subset of A. And if B is open, an immersed mainfold, k-dimensional, etc. then so is $\mathcal{D}B$ respectively.

With this $\mathcal{D}$, $d(M)$ can be written as $\mathcal{D}M$ and called *the development of M*.

(1.6$\mathcal{E}$)   If C is a nonempty H-invariant subset of $\mathcal{D}M = d(M)$, then $\mathcal{E}C = p \circ d^{-1}(C)$ is a nonempty subset and we have $\mathcal{D}\mathcal{E}C = C$. If C is closed, open, a submanifold, etc. in $\mathcal{D}M$. then so is $\mathcal{E}C$ respectively.

(1.7)   Let F be a fibre bundle over $\mathcal{D}M$ on which H acts as an automorphism group.   Then a unique fibre bundle $\mathcal{E}F$ is defined over M in such a way that its pullback by p is that of F by d. Moreover if v is an H-invariant section of F then a section $\mathcal{E}v$ of $\mathcal{E}F$ is defined in the obvious way.

This is particularly important when F is given by a functor. Here are two examples.

(1.7a)   If A and M are orientable, then H preserves the orientation.

Proof.   Obvious if we take as F the orientation bundle (= the orientable double covering) of $\mathcal{D}M$.   Q.E.D.

(1.7b)   An H-invariant vector field v on $\mathcal{D}M$ gives rise to a vector field $\mathcal{E}v$ on M. $\mathcal{E}v$ is carried locally into v by the A-chart. If v is nonvanishing then so is $\mathcal{E}v$.

The next lemma is an important application of (1.7b):

Lemma 1.8.   *Let G be a connceted Lie transformation group of A. Assume that G centralizes H, i.e. every member of G commutes with every member of H. If M is a compact A-manifold, then a connected Lie group $\mathcal{E}G$ acts on M (and hence its covering group G acts on M) in such a way that both d and p induce locally injective Lie group homomorphisms.*

Proof.   The vector fields in the Lie algebra of G are "enveloped" into vector fields on M by (1.7b). This is a Lie algebra isomorphism also by (1.7b). Since M is compact, the Lie algebra generates a Lie transformation group $\mathcal{E}G$. The rest is obvious.   Q.E.D.

We have a few corollaries to the lemma.

(1.8a)   Under the above assumptions, suppose that $\mathcal{D}M$ meets an open G-

orbit. Then $\mathcal{D}M$ contains the orbit and $\mathcal{E}G$ has open orbits ($\neq\emptyset$) in M. In particular, if $\mathcal{D}M$ does not meet G-orbits of lower dimensions ($< \dim A$) under the assumption of (1.8), then $\mathcal{D}M$ is a G-orbit, and $\mathcal{E}G$ is transitive on M.

(1.8b)   Still under the same assumption, G leaves $\mathcal{D}M$ invariant.

(1.8c)   Under the same assumption, if B is the union of k-dimensional $\mathcal{E}G$-orbits in M, then $\mathcal{D}B$ is that of k-dimensional G-orbits in $\mathcal{D}M$.

Convention 1.9.   We shall denote $\mathcal{E}G$ and G by the common symbol G sometimes for simplicity since this will cause no confusion in the sequel.

Lemma 1.10.   *Under the assumptions of* (1.8), *let C be a G-obit$\subset A$. Assume that C is an H-invariant closed submainfold of $\mathcal{D}M$. Then each G-orbit $B\subset\mathcal{E}C$ is a closed* (*and hence compact*) *submanifold of $\mathcal{E}C$. Moreover the G-orbits in $\mathcal{E}C$ are finite in number.*   Here "a closed sumbanifold" means a closed subset C which is locally defined by x^{k+1}=const., $\cdots$, x^n=const. in terms of some coordinates, k=dim C, and $n-k$=codim C.

Proof of (1.10)   In the notation of (1.4b$\mathcal{D}$), we choose a neighborhood U of an arbitrary point in B. We may assume that the image V of $\mathcal{D}i_U$ is contained in the neighborhood mentioned above and $C\cap V$ is connected. Then $B\cap U$ is exactly $(\mathcal{D}i_U)^{-1}(C\cap V)$, since C is H-invariant; recall that the set $\{\mathcal{D}i_U\}$ gives an H-structure. Thus B is a closed sumbanifold. B is compact. We see that B has a neighborhood in which B is the only G-orbit contained in $\mathcal{E}C$. Since $\mathcal{E}C$ is compact by (1.6$\mathcal{E}$) the number of the G-orbits$\subset\mathcal{E}C$ is finite.

<div align="right">Q.E.D.</div>

One of the most technically important questions about an A-structure is whether the developing map d is a covering map or not. For the most part, the rest of the section will be devoted to anserwing to this question. Before stating the next proposition which tells us what follows if d is covering, we introduce some notation.   Given an A-manifold M, let A^* denote the universal covering of $\mathcal{D}M\subset A$. The projection $\pi: A^*\to\mathcal{D}M$ pulls back the A-structure to A^*. Let A^* denote the automorphism group of the A-manifold A^*. Then M becomes an A^*-manifold in the obvious fashion. Finally let H^* denote the holonomy group of the A^*-manifold M.

Proposition 1.11.   *If the developing map d is covering, then M is A-isomorphic with the A-manifold A^*/H^* in the above notation; the action of H^* is free and properly discontinuous.*

Proof. M is an A^*-manifold too. The map d is an A^*-map. Let d^*: $M^*\to A^*$ be the developing map for the A^*-manifold M. The map d^* is also covering, since d is covering as well as $\pi: A^*\to A$ and the projection $p^*: M^*\to M$.

Thus d^* is an A^*-isomorphism, since A^* is 1-connected.   The holonomy covering space M^* of the A^*-manifold M is that of the A^*-manifold M too by (1.2), since A^* is 1-connected.   The proposition follows now from (1.3e)   Q.E.D.

Corollary 1.12. $H^* \cong \pi_1(M)$ *naturally in* (1.11).

Corollary 1.12a.   (Uniqueness) *Assume that M has two A-structures and their developing maps are both covering.   Then the two A-structures are isomorphic if and only if the correspondings A^*'s are A-isomorphic and, thereby identifying them, the corresponding holonomy groups H^*'s are conjugate in A^*.*

We are about to prove several sufficient conditions for d to be covering, since we do not know a powerful theory in this regard except for fragmentary results like this:

(1.13)   An equidimensional immersion $f: M \to N$ is covering if M is compact, where M, N connected manifolds.

Problem.   Thus compacteness of M implies that d is covering, but we do not know if that of M does.

(1.14)   The developing map d is covering if the A-manifold M is compact and the action of H on $\mathcal{D}M$ is proper.

Proof.   Omitted. (Compare Lemma 1.17).

(1.15) $d: M \to \mathcal{D}$ is covering if dim $A = 1$ and M is compact.

Proof.   We may assume that M is not compact and hence diffeomorphic with the real line R, by (1.13).   Then H is an infinite cyclic group of translations with respect to the pullback of a Riemannian metric on M.   Suppose d is not covering. Then $\mathcal{D}M$ is compact by a theorem in Calculus.   And either some interval $[a, \infty)$ or $(-\infty, a]$ is carried bijectively onto $\mathcal{D}M$ by d.   Obviously the point $d(a)$ is left fixed by H acting on $\mathcal{D}M$.   But H nas a member h which carries a to $h(a) \in (a, \infty)$ or $h(a) \in (-\infty, a)$ according to the case.   Certainly $dh(a) = hd(a) = d(a)$, contrary to the injectively of d restricted to the interval.   Q.E.D.

Remark.   We do not know if (1.15) is ture in the case dim $A = 2$.   To illustrate the difficulty, we point out that there exists a surjective immersion: $R^2 \to R^2$ which is not covering.

Proposition 1.16.   *Assume that either A or M admits a connected Lie transformation group G of A-automorphisms and that G centralizes H; H is supposed to act on M trivially.   Consider the following conditions (i)—(v):*
(i)   *G acts transitively on A;*
(ii)   *G acts on A and $\mathcal{D}M$ does not meet lower dmensional G-orbits;*
(iii)   *G acts on M transitively;*

(iv) *G acts on **A** and $\mathcal{D}M$ is a G-orbit; and*

(v) ***d** is covering.*

Then (iii) *implies* (iv) *and* (v). *Trivially* (i) *implies* (ii), *and* (iv) *implies* (ii). *And, if M is compact, then* (ii), (iii) *and* (iv) *are equivalent to one another and follow from* (i).

Proof. We have been using Convention (1.9). Assume (iii). Then G acts on M transitively and centralizes H there by (1.3e). Further G acts on A through the homomorphism d_* in (1.3f). And G centralizes H by (1.3f). Also $\mathcal{D}M$ is a G-orbit again by (1.3f), since the connected group G is transitive on M. Thus we have (iv). We see (v) follows immediately from the fact that $d: M \to \mathcal{D}M$ is a G-equivariant map between these homogeneous manifolds of the connected Lie group G. Finally, if M is compact, then (ii) implies (iii) by Lemma 1.8.

<div align="right">Q.E.D.</div>

Lemma 1.17. *Let G be a topological transformation group of manifolds M and N. Then an equidimenisonal G-equivariant immersion $f: M \to N$ is covering under the following assumptions* (a) *and* (b): (a) *A sequence (g_n) from G contains a convergent subsequence if there is a convergent dsequence (z_n) from N with $g_n(z_n) = z_n$ for each n, and* (b) *the induced map $f/G; M/G \to N/G$ is covering.*

REMARK 1.18. The condition (a) is satisfied if the action of G is *propoer* on N, i.e. for any two points $x, y \in G$, distinct or not, there exist neighborhoods U, V of x, y respectively such that the subset $\{g \in G \mid U \cap gV \neq \emptyset\}$ is relatively compact in G. The action is proper, for instance, if it is properly discontinuous and G is discrete.

Proof of Lemma 1.17. Consider the curves $c: [0,1] \to N$ and $C: [0,1) \to M$ with $c = f \circ C$ on $[0, 1)$. We have to extend C to $[0, 1)$. The curves induce c/G and C/G on N/G and M/G respectively. The curve C/G extends to $[0, 1]$ by (b). The orbit $\subset M$ corresponding to the point $(C/G)(1)$ in M/G is mapped onto the orbit $G(c(1)) \subset N$ by f. Hence there is a point $x \in M$ with $f(x) = c(1)$. Take a neighborhood U of x on which f is a diffeomorphism onto $f(U)$. The acr $c([s, 1])$ is contained in $f(U)$ for some $s < 1$. There is a unique curve $\Gamma: [s, 1] \to U$ with $f \circ \Gamma = c$ on $[s, 1]$. We have $C/G = \Gamma/G$ on $[s, 1]$, since we have $(C/G)(1) = (\Gamma/G)(1)$ and f is covering. Therefore there exists some $g(t) \in G$ for each $t \in [s, 1)$ such we have $C(t) = g(t)(\Gamma(t))$. (The map $g: [s, 1) \to G$ may not be continuous.) Since both $C(t)$ and $\Gamma(t)$ are carried into $c(t)$ by the G-equivariant map f, it follows from (a) that there is a sequence $(t_n) \uparrow 1$ such that the sequence $(g(t_n))$ converges to some $g \in G$. We have $f \circ g(\Gamma(1)) = c(1)$. Thus we can extend C to $[0, 1]$ by putting $C(1) = g(\Gamma(1))$, since the map f, restricted to $g(U)$, is a homeomorphism and $g(U)$ contains $C(t_n)$.

<div align="right">Q.E.D.</div>

Corollary 1.19. *An equidimensional G-equivariant immersion $f: M \to N$ is*

covering if G is both transitive and proper on N.

Proof. Simply becuse N/G is a single point. Note that M/G is discrete.

Q.E.D.

Corollary 1.19a. *An equidimensional G-equivariant immersion $f: M{\to}N$ is covering if G is proper on N, the orbit space M/G is compact, and both M/G N/G are manifolds in the natural way.*

Proof. Immediate from (1.13), (1.18) and (1.17).   Q.E.D.

The next proposition is crucial in the fourth section.

Proposition 1.20. *Assume on A that a connected Lie transformation group G acts on A as analytic transformations and satisfies these conditions: (1) the orbit space A/G is a one-dimensional manifold in the natural way, and (2) A admits a G-invariant Riemannian metric. (This is the case when the action is proper; See Palais [10]). Let M be a compact (A, A)-manifold. Then the developing map $d: M{\to}\mathscr{D}M$ is covering if the holonomy group H centralizes G and if no compact G-orbit in M is homologous to zero.*

Proof. Our plan is to use (1.17) and (1.15). But first we will show that M/G is a 1-dimensional manifold in the natural way; G acts on M by (1.8). Since d is a G-equivariant local diffeomorphism and the G-orbits $\subset A$ are closed submanifolds by (1), so are those in M. Besides G is an isometry group of the pullback by d of the Riemannian metic on A mentioned in (2). We define the distance between G-orbits as usual by $d(G(x), G(y)){=}\mathrm{Min}\ \{d(g(x), g(y))\,|\,g{\in}G\}$, $x, y{\in}M$, by means of the distance function d given by the G-invariant Riemannian metric. This induces a metric on M/G, which is compatible with the quotient topology. The last point will be seen as follows. Since every orbit is closed, there exists a point $y_0{\in}G(y)$ such that we have $d(x, y_0){=}\mathrm{Min}\ \{d(x, g(y))\,|\,g{\in}G\}$. Thus we have $d(G(x), G(y_0)){=}g(x, y_0)$ by the invariance of d under G. We conclude that M/G is a Hausdorff space. Hence it follows from Palais [9] that M/G is a manifold; the smooth structure is uniquely determined if we require that every (maximal) curve which is normal to the G-orbit at every point is immersed into M/G by the projection: $M{\to}M/G$, since G permutes such curves among themsalves.

Unfortunately (1.15) does not directly apply to the situation. But a closer look at its proof will reveal that we have only to show that H has no fixed point in M/G, in order to conclude that d/G is covering. We may assume that M/G is diffeomorphic with R, as before. Suppose H has a fixed point. Then H leaves invariant the corresponding G-orbit, say $G(x)$, which is closed. Therefore its image under p is compact. Call it $G(x)$. $G(x)$ divides M into two domains, since $G(x)$ does, which in turn follows from $M/G{=}R$; $R{-}\{0\}$ has two connected components. Therefore $G(x)$ is a bounding cycle, contrary to the assumption.

Thus d/G is covering. Now we want to use (1.17) to complete the proof. G does not quite satisfy the assumption in (1.17). For the remedy we take the closure $\bar{G}$ in the isometry group of A with the compact-open topology. It is easy to see that $\bar{G}$ satisfies the condition in the proposition; in particular $\bar{G}$ has the same orbits in A as G. $\bar{G}$ is proper and (1.17) applies to $\bar{G}$.   Q.E.D.

The next proposition makes it possible to use (1.20) in a later section and gives some information about the shape of $\mathscr{D}M$.

Proposition 1.21. *Assume that the standard space A has a vector field v satisfying the following conditions* (See Remark 1.22 below): (1) *the vanishing point set $V_v=V=v^{-1}(0)$ of v is a submanifold with $\dim V < \dim A$,* (2) *the integral curve of v through each point $a \in A$ has the $\lim_{t \to -\infty} e^{tv}(a)$, denoted by $\lambda(a)$, and* (3) *the map $\lambda\colon A \to V \subset A$ is continuous. Further assume that V is an orbit of some connected Lie transformation group G. Now if M is a compact A-manifold and if the holonomy group H centralizes both G and v, then the development $\mathscr{D}M$ does not meet V; in particular the Euler number $\chi(M)=0$.*

REMARK 1.22.   A vector field v satisfies the conditions (1), (2) and (3) above if A has a vector bundle structure with positive fibre dimension and v generates the group of positive scalar multiplications.

REMARK 1.23.   The condition on G is void when V is a single point.

Proof of (1.21).   Suppose $\mathscr{D}M$ meets V at a point 0. Since G leaves $\mathscr{D}M$ invariant by (1.8), the orbit $G(0)=V$ must be entirely contained in $\mathscr{D}M$. Thus λ carries $\mathscr{D}M$ into itself. Let w denote the vector field v defined in (1.7b). Since M is compact, the integral curve $e^{tw}(x)$ is defined for all $(t,x) \in \mathbf{R} \times M$. Its development has a limit in $V \subset \mathscr{D}M$ when t tends to $-\infty$. Therefore the curve $e^{tw}(x)$ has the $\lim_{t \to -\infty} e^{tw}(x)$, denoted by μ. Developing a 1-connected neighborhood of the curve by (1.4a$\mathscr{D}$), one immediately sees that μ is continuous. Therefore we have a continuous map $\mu\colon M \to V_w$, where V_w is the vanishing point set of w. Hence V_w is a deformation retract of M by the integral curves of w. But V_w is a manifold with $\dim V_w < \dim M$. This is absurd, since every n-dimensional compact connected manifold has the homology group $H_n(M; \mathbf{Z}_2) \cong \mathbf{Z}_2$. Therefore $\mathscr{D}M$ does not meet V. The vanishing of $\chi(M)$ follows from the fact that V_w is empty and w vanishes nowhere.   Q.E.D.

Corollary 1.24.   *A compact affine manifold M has $\chi(M)=0$ if the holonomy group H fixes a point 0 in the affine space A^n, $n=\dim M$.*

Proof. A^n is then a vector bundle over the single point 0. Then the proposition applies in view of Remark 1.22.   Q.E.D.

Corollary 1.25.   *The holonomy group of a compact affine manifold M does*

not fix a point in the development $\mathcal{D}M$.

2. The maximal abelian subgroups of $A(2)$

The main prupose of this section is to classify and examine the maximal abelian subgroups of the affine group $A(2)$ of the real affine plane $\boldsymbol{A}^2$. We prefer a more systematic method to those of elementary linear algebra, using some rudiments of the module theory (c.f. Bourbaki [2]: §6), becuase, beside the obvious merits of a general theory, our method takes care of the transition to the conjugate classes automatically.

Let U be an m-dimensional vector space over $\boldsymbol{R}$. To begin with, we study the maximal commutative nilpotent subalgebras of End U, the algebra of all endomorphisms of U, where "nilpotent" means that the members are nilpotent operators $\in$ End U. Let N be one. Put $N^{+1}=\boldsymbol{R}1\oplus N$, where 1 is the identity $\in$ End U. N^{+1} is a commuttive subalgebra with the unit 1 of End U. Its radical, $\mathcal{R}(N^{+1})$, is N. We regard U as an N^{+1}-module. Then the following facts (2.1), (2.2) and (2.3) are well known.

$$(2.1) \qquad\qquad \mathcal{R}(U) \neq U,$$

where $\mathcal{R}(U)$ is the radical of U, i.e. the intersection of all the proper maximal submodules.

$$(2.2) \qquad NU = \mathcal{R}(N^{+1})U \subset \mathcal{R}(U),$$

where NU is the image of the multiplication: $N \otimes U \rightarrow U$.

(2.3) *A submodule* $P = U$ *if* $P + \mathcal{R}(U) = U$.

(2.3a) *The* N^{+1}-*modules* U *is indecomposable.*

Proof. Suppose that U is the direct sum of nonzero N^{+1}-modules V and W. Fix $v \in V$ which is not in $\mathcal{R}(U)$, (2.1). Take $w \in W$ which lies in the intersection of the kernels of the endomorphisms in N, or from $N^k U = N(N^{k-1}U)$, where k is the largest integer such that $N^k U \neq 0$. Define a linear map $f \in$ End U by these conditions: (1) $f(v) = w$, (2) the restriction $f \,|\, V$ has rank $= 1$, and (3) $f = 0$ on W. Then certainly f centralizes N and is nilpotent without belonging to N, contrary to the maximality of N.   Q.E.D.

We will scrutinize the two extreme cases of codim $\mathcal{R}(U) = 1$ and of dim $\mathcal{R}(U) = 1$.

Lemma 2.4. $NU = \mathcal{R}(U)$ *if* codim $\mathcal{R}(U) = 1$.

Proof. Take any $z \in U$ which is not in $\mathcal{R}(U)$. Then the vector subspace $N^{+1} \cdot z + NU = \boldsymbol{R} \cdot z + NU$ is a submodule over N^{+1}. We have $(\boldsymbol{R} \cdot z + NU) + \mathcal{R}(U) = U$ by the assumption. This gives $\boldsymbol{R} \cdot z + NU = U$ by (2.3). Hence

NU is maximal in U. Thus we have $NU=\mathcal{R}(U)$ by (2.2) and (2.1).

<div style="text-align:right">Q.E.D.</div>

Lemma 2.5. *If* dim $\mathcal{R}(U)=1$, *then* $NU=\mathcal{R}(U)$ *and* N *is characterized as the totality of the linear maps of* U *into a fixed* 1-*dimensional vector subspace* NU *of* U(*on which they are* 0).

Proof. Then $m=$dim $U>1$. Hence dim $NU\geq 1$ by the maximality. Thus we have $NU=\mathcal{R}(U)$ by (2.2) and the assumption. Every member f of N is a linear map: $U\to NU$. Its restriction to NU is 0, since f is nilpotent and dim $NU=1$. Conversely, given a 1-dimensional vector subspace V of U, let N' be the space of the linear maps $f\colon U\to V$ with $f|V=0$. Then N is a commutative nilpotent subalgebra of End U. To prove the maximality of N', let g denote a nilpotent endomorphism of U which centralizes N'. Then g levaes V invariant; $gV=gNU=NgU\subseteq NU=V$. Hence $gV=0$ by dim $V=1$. Thus we have $NgU=gNU=gV=0$. This gives $gU\subseteq V$, since V is the intersection of the kernels of the members of $N'\subset$ End U. Hence we have $g\in N'$ and so the maximality of N'.   Q.E.D.

Back to the case of codim $\mathcal{R}(U)=1$, we will obtain the clssification (2.6). Define the successive radicals $\mathcal{R}^i U$ inductively by $\mathcal{R}^0 U=U$ and $\mathcal{R}^{i+1}U=\mathcal{R}(\mathcal{R}^i U)$ for the given N.

Lemma 2.6. *One has* $\mathcal{R}^{m-1}(U)\neq 0$ *if and only if th algebra* N *is generated over* $\boldsymbol{R}$ *by a member* n *with* $n^{m-1}\neq 0$ (*as an endomorphism of* U).

Proof. The "if-part" is obvious from $N^{m-1}U\subseteq\mathcal{R}^{m-1}U$ which is a consequence of (2.2). Now the converse. We have dim $\mathcal{R}^i U/\mathcal{R}^{i+1}U=1$ for $0\leq i<m$ by (2.1) under the assumption. Hence we obtain $\mathcal{R}^i U=N^i U$ by Lemma 2.4 applied to $\mathcal{R}^i U$ successively. In particular, we see $N^{m-1}U=\mathcal{R}^{m-1}U\neq 0$. Thus the product $n_1 n_2\cdots n_{m-1}\neq 0$ for some $n_1,\cdots,n_{m-1}\in N$. Hence there must exist some linear combination $n=\sum_{i=1}^{m-1}t_i n_i$, $t_i\in\boldsymbol{R}$, for which we hve $n^{m-1}\neq 0$. In fact, if we regard n as a function of $t_1,\cdots,t_{m-1}$, we have the partial derivative $(\partial^{m-1}/\partial t_1\partial t_2\cdots\partial t_{m-1})n^{m-1}=$(const.) $t_1\cdots t_{m-1}\neq 0$ by the commutativity. It follows from $n^{m-1}\neq 0$ that we have a basis $(n^{m-1}z,\cdots,nz,1)$ of U for any member z of U which is not in $\mathcal{R}(U)$. Now let p be any member of N. Then pz is a unique linear combination $\sum_{i=1}^{m-1}a_i n^i z$. And we have $pn^k z=n^k pz=\sum a_i n^i(n^k z)$ for $1\leq k<m$. This shows that p itself is the polynomial $\sum a_i n^i$ in n over $\boldsymbol{R}$.

<div style="text-align:right">Q.E.D.</div>

The last two lemmas 2.5 and 2.6 are enough to classify N in the case of dim $U\leq 3$, which we will need later. Note that there is only one N up to the automorphism of the vector space U for each of the cases referred to in those lemmas. First assume (I): dim $U\leq 1$. Then $N=0$ trivially. Next assume (II): dim $U=2$. Then we see dim $N=1$ by either one of (2.5) and (2.6). Finally assume (III): dim $U=3$. When codim $\mathcal{R}(U)=1$, there are two cases to be

distinguished; i.e. the classes (III, 1) of N for dim $\mathcal{R}^2 U = 1$ and (III, 2) of N for dim $\mathcal{R}^2 U = 0$. When codim $\mathcal{R}(U) = 2$, we have only one classify (2.6). We record the result in the next lemma for the future reference.

Lemma 2.7. *Let N be a maximal commutative nilpotent subalgebra of End U of a vector space U of dim $U \leq 3$. Then N is, with respect a suitable basis for U, exactly one of the algebras consisting of the matrices*: (I) 0, $n=1$; (II) $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $n=2$; (III, 1) $\begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$, (III, 2) $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$, or (III, 3) $\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $n=3$, *where a and b are arbitrary real numbers.*

This result will be used to study the maximal abelian subgroups G of the affine group $A(2)$. We fix the usual representation: $A(n) \rightarrow GL(n+1, \boldsymbol{R})$. We have

(2.8)   *There exists a vector subspace T of $\boldsymbol{R}^{n+1}$ with codim $T = 1$ such that a linear transformation $g \in GL(n+1, R)$ belongs to $A(n)$ if and only if g leaves T invariant and acts trivially on the quotient vector space $\boldsymbol{R}^{n+1}/T$.*

(2.8a)   *The rotation part of $g \in A(n)$ is its restriction to T by definition.*

With this convention, the affine space A^n is interpreted as the hyperplane in $\boldsymbol{R}^{n+1}$ which is the inverse of a fixed nonzero vector under the projction: $\boldsymbol{R}^{n+1} \rightarrow \boldsymbol{R}^{n+1}/T$. Throughout this section, G will denote a maximal abelian subgroup of $A(n)$. We aim at the structure theorem (2.11) below. A well known theorem on linear trnasformations (e.g. Chevalley [3]; Theorem 18, p.184) asserts in our context that

(2.9)   *The maximal abelian subgroup G of $A(n)$ is isomorphic with the direct product $S \times (1+N)$ of its sugbroups S and $1+N$, where S is the totality of the semisimple ($=$completely reducible) members of G and $1+N$ is that of unipotent ones ($=1+$ [nilpotent linear endomorhsim]).*

Proof. G is abelian, and $A(n)$ is an algebraic subgroup of $GL(n+1, \boldsymbol{R})$.
                                                                                    Q.E.D.

The following is also more or less knwon:

(2.10)   *The G-module $\boldsymbol{R}^{n+1}$ admits a unique (up to the order) direct sum decomposition: $\boldsymbol{R}^{n+1} = \sum_{\alpha=0}^{p} U_\alpha \otimes V_\alpha$ such that, in the notations of (2.9), S acts irreducibly on $V_1, \cdots, V_p$ but trivially on $V_0 = \boldsymbol{R}$, $U_0, \cdots, U_p$ which $1+N$ acts trivially on V_0, $\cdots, V_p$ but as a maximal ableian unipotent (viz. N nilpotent) linear transformation group on $U_0, \cdots, U_p$.*

Proof.   Since S is abelian and consists of semisimple operators, $\boldsymbol{R}^{n+1}$ is decomposed into the unique direct sum of minimal S-modules W_i, $1 \leq i \leq q$, and

the subspace W_0 of S-invariant vectors as an S-module. We partition the collection $\{W_0, W_1, \cdots, W_q\}$ into the equivalence classes by the S-module isomorphism. Choose a representative V_α, from each class. Then the sum of all S-modules in the class of V_α is G-invariant by (2.9) and is isomorphic with the tensor product $U_\alpha \otimes V_\alpha$ as a G-modules, where U_α is an N-module and G acts on $U_\alpha \otimes V_\alpha$ through the decomposition of (2.9). The uniqueness follows from the fact that $U_\alpha \otimes V_\alpha$ is then decomposable by (2.3a) and a well known uniqueness theorem (See Theorem 1 in §2 of [2], for instance).   Q.E.D.

A few remarks are due about the decomposition above.

(2.10a) $$U_0 \otimes V_0 \cong U_0 \quad \text{is not zero.}$$

Proof. $A(n) \supset G$ is trivial on R^{n+1}/T. And S is a commutative set of semisimple operators.   Q.E.D.

(2.10b) dim V_α, $1 \leq \alpha \leq p$, *is either 1 or 2. S acts on V_α as the general linear group $GL(1, R)$ or $GL(1, C)$ accordingly.*

Proof. Obvious from the maximality of G and hence of S.   Q.E.D.

(2.10c) *Any two distinct spaces $U_\alpha \otimes V_\alpha$, $U_\beta \otimes V_\beta$ are not G-isomorphic with each other, and U_α, U_β are not S-isomorphic either.*

Proposition 2.11. *A maximal abelian subgroup G of $A(n)$, or its conjugate, is characterized as a linear group.*

(2.12) $\prod_{\alpha=0}^{p}(1+N_\alpha) \times S_\alpha$ (group direct rproduct) *acting on R^{n+1} in harmony with the decomposition in (2.10): $R^{n+1} = \sum_{\alpha=0}^{p} U_\alpha \otimes V_\alpha$; $1+N_\alpha$ acts on U_α as a maximal abelian unipotent linear group, S_α, $\alpha \neq 0$, acts on V_α as $GL(1, R)$ or $GL(1, C)$ according as dim $V_\alpha = 1$ or 2, S_0 is $\{1\}$ on $V_0 = R$, and $U_0 \neq \{0\}$.*

Proof. We have proved that G is a group of type (2.12). Conversely let G be the group (2.12). Then G is abelian. G is maximal because of the uniqueness of the decomposition (2.12) and of the maximal properties in the statement above. Let W be an N_0-invariant hyperplane $\ni 0$ in $U_0 \cong U_0 \otimes V_0$. Let f be a linear automorphism of R^{n+1} which carries the hyperplane $W + \sum_{\alpha=1}^{p} U_\alpha \otimes V_\alpha$ onto T. Then $f\,G\,f^{-1}$ is a subgroup of $A(n)$ by (2.8).   O.E.D.

In the following corollaries (2.13)-(2.14d) we use the notations of (2.11) freely.

Corollary 2.13. *The connected component G_e containing 1 of G is $\prod_{\alpha=0}^{p}(1+N_\alpha) \times S_{\alpha,e}$ where $S_{\alpha,e}$ is the identity component $\cong \{x \in R \mid x < 0\}$ of S_α when $S_\alpha \cong GL(1, R)$ and $S_{\alpha,e} = S_\alpha$ otherwise.*

Corollary 2.14. *In the action of G on A^n, every G-orbit is a finite union of G_e-orbits of the same dimension and G/G_e permutes these orbits transitively.*

Corollary 2.14a. *The exponential map of the Lie algebra into G_e is surjective. And the map is a diffeomorphism if no V_α is C.*

Proof. The exponential map for $1+N_\alpha$ is a diffeomorphism: $N_\alpha \to 1+N_\alpha$.

$$\text{Q.E.D.}$$

Corollary 2.14b. *Always G fixes each leaf of some foliation in A^n whose leaves are parallel affine subspaces of codimension$=\dim(U_0/\mathcal{R}U_0)-1$, where $\mathcal{R}U_0$ is the radical of the module U_0 over $N_0^{+1}=\boldsymbol{R}1+N_0$. And smilarly if $\mathcal{R}U_0$ is replaced by NU_0.*

Proof. Consider the special case where U_0 is the whole space $\boldsymbol{R}^{n+1}$, to begin with. We then have $G=1+N_0$. Let $G(x)$ be an arbitrary G-orbit, $x \in A^n \subset \boldsymbol{R}^{n+1}$. Then the points of $G(x)$ are $e^m x = x + \sum_{k=1}^n m^k x/k!$, $m \in N=N_0$. Thus $G(x)$ is contained in the affine subspace $x+NU_0$, viz. the space with the tangent space parallel to NU_0 at x. Since $NU_0 \subseteq \mathcal{R}U_0$, we also conclude that $G(x)$ is contained in the affine subspace which has the tangenet space $\mathcal{R}U_0$ at x (and hence verywhere). And codim $G(x)=n-\dim \mathcal{R}U_0=\dim(U_0/\mathcal{R}U_0)-1$. In the general case, we observe that the affine space A^n can be thought of as the cartesian product of the affine space in U_0 defined by $U_0 \cap T$, T as in (2.8), and of the vector space $\sum_{i=1}^p U_i \otimes V_i$. Every G-orbit $G(x)$ in A^n is therefore contained in the cartesian product of the orbit $(1+N_0)(x)$ in U_0 and the space $\sum U_i \otimes V_i$. Q.E.D.

Corollary 2.14c. *G is transitive on A^n if and only if $\dim \mathcal{R}U_0=n$.*

Corollary 2.14d. *G has a fixed point if and only if $\dim U_0=1$.*

Now we proceed to the classification of the maximal abelian groups G of $A(2)$. We employ the decomposition (2.10) along with its interpretation (2.11). Recall that $\dim U_0 \geqq 1$. First consider the case (I): $\dim U_0=1$. (See (2.14d) for the geometric meaning). The lemma 2.7 allows us to enumerate the three possibilities: (I, 1) $\dim U_1=2$, $p=1$; (I, 2) $\dim U_1=\dim U_2=1$, $p=2$, and (I, 3) $\dim V_1=2, p=1$. Turning to the case (II): $\dim U_0=2$, we find only one possibility: $\dim U_1 \otimes V_1=1$, $p=1$. The lemma 2.7 lists the groups in the remaining case (III): $\dim U_0=3$. We register the findings in a proposition.

Proposition 2.15. *Every maximal abelian subgroup G of $A(2)$ falls into one and only one conjugate class of the groups described below, where the matrix $\begin{pmatrix} a & b & u \\ c & d & v \end{pmatrix}$ denotes the affine transformation: $(x,y) \mapsto (ax+by+u, cx+dy+v)$, expressed with an affine coordinate system:*

$$(\text{I, 1}) \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \end{pmatrix}; \quad (\text{I, 2}) \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \end{pmatrix}; \quad (\text{I, 3}) \begin{pmatrix} a & b & 0 \\ -b & a & 0 \end{pmatrix}$$

$$(\text{II}) \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & v \end{pmatrix};$$

(III, 1) $\begin{pmatrix} 1 & b & u \\ 0 & 1 & b \end{pmatrix}$; (III, 2) $\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \end{pmatrix}$; (III, 3) $\begin{pmatrix} 1 & b & u \\ 0 & 1 & 0 \end{pmatrix}$,

the entries $a, \cdots, v$ being arbitrary except for $ad - bc \neq 0$. In particular, dim $G=2$ in cases.

The notations (I, 1) through (III, 3) will be freely used in the sequel. We state without proof the corollaries (2.16)-(2.17a) below to be used later, in which G_e denotes as before the identity component of an arbitrary maximal abelian subgroup G of $A(2)$.

(2.16)   *G has an open, dense (and hence unique) orbit in A^2, if G is not in the class* (III, 3).

(2.16a)   *G/G_e permutes the open G_e-orbits freely and transitively.*

(2.16b)   *The exponential map of G is a diffeomorphism of its Lie algebra onto G_e except for class* (I, 3).

(2.16c)   *G_e is simply transitive on each open orbit if G is not in the class* (I, 3).

(2.16d)   *If G is in the class* (I, 3), *then G is connected $G = G_e$, and its universal covering group acts on the universal covering space of the open G-orbit exactly as R^2 does on itself as the translation group.*

(2.17)   *There are only finitely many 1-dimensional G-orbits on A^2, none for* (III, 1) *and* (III, 2), *one of* (I, 1) *and* (II), *two for* (I, 2), *provided G is not in the class* (III, 3).

(2.17a)   *The 1-dimensional G_e-orbits are lines or half-lines.*

We introduce a few notations to state the last lemma in this section. Let $GL(2, R)^+$ be the indentity component of $GL(2, R)$, and π the projection of the universal covering group of $GL(2, R)^+$ onto $GL(2, R)^+$. And let G^+ denote the intersection $G \cap GL(2, R)^+$. When G lines in the class (I) (so that it has a fixed point), we agree that $GL(2, R)$ is the subgroup of $A(2)$ having the same fixed point as G.

Lemma 2.18.   *With these notations, the set of the connected components of $\pi^{-1}G^+$ is in a one-to-one correspondence with $\pi^{-1}\{-1, 1\}$ by the inclusion map, if G is in the class* (I, 1) *or* (I, 2).

Proof. G^+ has two connected components. One contains $+1 (=$ the identity) and the other (-1). The identity component G_e of G^+ is 1-connected.

<div align="right">Q.E.D.</div>

Notation 2.19.   We denote by ν the locally constant continuous homomorphism of the group $\pi^{-1}(G^+)$ onto $\pi^{-1}\{+1, -1\} \simeq Z$ which is defined by (2.18). Note that the kernel of ν is the identity component $(\simeq G_e)$ of $\pi^{-1}(G^+)$.

3. The homogeneous case

Let T^n denote the real n-dimensional torus. Suppose we are given an affine structure on T^n. This is a special kind of A-structure discussed in the first section, $A=A(n)=$the affine group. Thus we are in this situation; T^n has the holonomy covering T^n with the induced affine structure so that the projection $p\colon T^n \to T^n$ is an affine map as well as the developing map $d\colon T^n \to A^n$. Furthermore the holonomy group H acts on both T^n and A^n effectively and on T^n trivially as an affine automorphism group in any case, with respect to which d and p are equivariant. Since H is the image of the fundamental group $\pi_1(T^n) \cong Z^n$ under an eqimorphism, H is abelian and hence contained in a maximal abelian subgroup G of $A(n)$. H is an infinite group by (1.5).

Proposition 3.0. *Following the notations of* (2.11), (2.14b) *and the above, one has codim* $(NU_0)=1$ *in* U_0; *in particular, when $n=2$, H cannot be contained in the maximal abelian group G of the class* (*III*, 3) *in the sence of* (2.15).

Proof. We have dim $U_0 \leq 1+\dim \mathcal{R}U_0$ by (2.14b) and (1.5b). Thus codim $(NU_0)=\mathrm{codim}(N_0U_0)=1$ in U_0 by (2.1) and (2.4).   Q.E.D.

Hereafter we confine ourselves to the case $n=2$. In the rest of this section we will study the simpler case, assuming.

(3.1) (The homogeneity condition) *There exists some maximal abelian group G of the affine group $A(2)$ whcih contains H such that the development $d(T^2)=\mathcal{D}T^2$ does not meet any 1-dimensional G-orbit.*

The hypothesis (3.1) together with (1.25) implies that $\mathcal{D}T^2$ is contained in an open G-orbit, since $\mathcal{D}T^2$ is connected by (1.3d). Hence by (2.14), $\mathcal{D}T^2$ is actually an open G_e-orbit. Under the convention (1.9), the identity component G_e acts transitively on both T^2 and T^2. T^2 is thus a homogenous affine manifold. It is for this reason that the present case of (3.1) is easy to handle. Indeed d is covering by (1.16). Thus it follows from (1.11) that we have

(3.2) Our affine torus T^2 is affine isomorphic with A^*/H^*, where A^* is, as we remember, the universal covering space of $\mathcal{D}T^2$ with the induced affine structure and H^* is the holonomy group of the (A^*, A^*)-manifold T^2, A^* denoting the affine automorphism group of A^*.

We will make a free use of the classification (2.15) of G in order to reach the conclusion quickly; the class (*III*, 3) has been ruled out by (3.0). So we already know enough about G. Our next task is to locate the position of H^* in A^*. The main point is to see that H^* is contained in the identity component of A^*. We regard R^2 as the universal covering group of G_e in common to all the classes. We fix a Lie group epimorphism $\pi\colon R^2 \to G_e$. The action of G_e on $\mathcal{D}T^2$ lifts to that of R^2 on A^*. The action of R^2 depends on π but always it is simply transitive by (2.16c) and (2.16d). Since H centralizes G_e, H^* centralizes R^2 with

the action. H^* is a lattice subgroup (viz. R^2/H^* is a compact manifold of the same dimension.) Indeed the quotient space $R^2/H^* \approx A^*/H^*$ is diffeomorphic with T^2. And we have proved a half of the main theorem of this section.

Theorem 3.3. *An affine 2-torus satisfying* (3.1) *is characterized as a quotient affine space A^*/H^*, where A^* is the universal covering group of a maximal connected abelian subgroup G_e of $A(2)$, not in the class* (III, 3), *H^* is a lattice subgroup of A^* and the affine structure A^* is induced from any open G_e-orbit in the affine plane. (See the proof for how to induce it).*

Proof. We have only to prove the second half. Let A^* and H^* be the groups in the statement. Let π be the projection: $A^* \cong R^2 \rightarrow G_e$. Take a point x in any open G_e-orbit, which exists by (2.16) and (2.14). Then we have an immersion $d^*: A^* \rightarrow G_e(x)$ carrying $g \in A^*$ into $\pi(g)(x)$. The map induces the affine structure on A^* from the open subset of the plane. The structure is independent of the choice of x by (1.12a) and (2.16a). Obviosuly the group $A^* \cong R^2$ acts on itself as an affine automorphism group. Therefore A^*/H^* is naturally given an affine structure. This is an an affine torus, since H^* is a lattice group. The torus A^*/H^* has the holonomy group $H = \pi(H^*)$. And $G_e(x)$ is its development $\mathscr{D}(A^*/H^*)$, since A^*/H^* is a compact homogenous affine space (as in the proof of (3.2)). Therefore the torus A^*/H^* satisfies the condition (3.1) by (2.16a).

<div align="right">Q.E.D.</div>

Given an affine homogenus torus, the group G_e in Theorem 3.3 is uniquely determined by H^*, which is not always the case for H as Proposition 3.4 below shows; in fact the inverse image $(\exp)^{-1}(H^*)$ in the Lie algebra of the universal covering group of $A(2)$ necessarily spans a 2-dimensional subalgebra by Theorem 3.3, whose image under the exponential map can thus be called *the* group G_e. We thus conclude the next two corollaires, which virtually complete the classification of the affine homogeneous 2-tori.

Corollary 3.3a. *The set $\{\Gamma_h\}$ of all the affine 2-tori satisfying* (3.1) *module Diff* (T^2), *the diffeomorphism group, is in a one-to-one correspondence with the set of the lattice groups of the universal covering groups of the maximal connected abelian subgroups G_e of $A(2)$, not in the class* (III, 3), *module the inner automorphism $ad(g)$, $g \in A(2)$, carrying X ito gXg^{-1} with $ad(g)G = G$.*

REMARK. Because of these inner automorphisms, there corresponds a unique affine structure (the standard one) for G in the class (III, 2) whereas different lattice groups give different affine structures for G in the class (I, 2). In this paper, however, we will not go into the details.

We denote by $\mathrm{Diff}(T^2)_e$ the subgroup of $\mathrm{Diff}(T^2)$ whose member is a diffeomorphism inducing the identity on the fundamental group $\pi_1(T^2)$. Then we have

Corollary 3.3b *The set* $\{\Gamma_h\}$ *of all affine 2-tori satisfying* (3.1) *modulo Diff* $(T^2)_e$ *is in a one-to-one correspondence with the positively-oriented bases of the Lie algebras of the maximal abelian subgroup* G *of* $A(2)$, *not in the class* $(III, 3)$, *modulo the inner automorphisms in the preceding Corollary* 3.3a.

Proposition 3.4. *Under the assumption of* (3.1), *the maximal abelian subgroup* G *of* $A(2)$ *which contains* H *is not unique if and only if* H *fixes a point* $0 \in A^2$ *and is contained in the center of the subgroup* $GL(2, R)$ *of* $A(2)$ *which fixes* 0.

Proof. Although this can be directly proved with an algebraic method, we prefer to employ a geometric one. Let G be the group in (3.1). Suppose G is not in the class $(I, 3)$. Then we know from the above that the torus is G_e/H. Therefore it is easy to see that G is the centralizer of H in $A(2)$. If H is contained in more than one maximal abelian subgroup, then G is thus in the class $(I, 3)$ and H must leave fixed a point 0 and a line passing through it. It follows that H is obviously a subgroup of the center of $GL(2, R)$. The converse is self-evident.   Q.E.D.

The exceptional tori in the proposition above have so many remarkable peculiarities that they deserve a special name:

DEFINITION 3.5. An affine torus is called *a Hopf torus* if the holonomy group is contained in maximal abelian groups of all of the classes $(I, 1)$, $(I, 2)$ and $(I, 3)$.

A typical example of a Hopf torus is obtained from $R^2 - \{0\}$, regarded as an affine space in the usual fashion, by identifying every point (x, y) with the point (cx, cy) where c is a positive constant. The holonomy groud H is a cyclic group generated by the dilation. Note that there are infinitely many different Hopf tori with the same holonomy group H.

Another feature of Hopf tori is illustarted by

Corollary 3.6. *Let* Aut $(T^2, \Gamma_h)_e$ *be the identity component of the affine automorphism group of the 2-torus with homogenous affine structure* Γ_h. *If* (T^2, Γ_h) *is a Hopf torus, then* Aut $(T^2, \Gamma_h)_e$ *is locally isomorphic with* $GL(2, R)$ *and its maximal compact subgroup is the 2-dimensional toral group. Otherwise* Aut $(T^2, \Gamma_h)_e$ *is the 2-dimensional toral group.*

Later we will return to the Hopf tori, which lie in a certain sense, some where between the homogenous and the inhomogenous affine structures or in a penumbra overlapping both types.

Here are two applications of the classification, presented without proof.

Proposition 3.7. *Every complex structure on* T^2 *admits complex affine structures (which are necessarily homogenous; See Vitter* [17] *for instance) and they are parametralized by* C. *Moreover different complex structures admit different complex structures if regarded as real affine structures, expect for the standard one (viz.*

the one in $(III, 2)$).

REMARK. Similarly, the affine structures on the circle S^1 modulo $\mathrm{Diff}(S^1)_e$ [resp. $\mathrm{Diff}(S^1)$] are in a one-to-one correspondence with R[rsp. the interval $[0, \infty)$], in a certain natural way, where $\mathrm{Diff}(S^1)_e$ is the group of the orientation-preserving diffeomorphisms: $S^1 \to S^1$.

REMARK. The argument in the section, combined with the list of (2.15), should give the classification of the complex affine structures on the complex 2-dimensional tori. (See Vitter [17]).

Proposition 3.8. R^2 *has exactly four affine structures which are invariant under the usual action of* R^2 *on itself; i.e. the whole plane, the half plane, the sector, and the universal covering space of the punctured plane.*

REMARK. It should not be difficult to determine all the homogenerous affine structures on R^2. However the inhomogeneous ones are too abundant to remain under our control.

It is our plan in the next paper to build up the picture of the affine structures on T^2 as a whole. But we like to draw it for the homogeneous ones modulo Diff $(T^2)_e$. By (3.6), the toral group T^2 leaves in variant every homogebeous affine structure modulo $\mathrm{Diff}(T^2)_e$ on T^2. And so we fix the usual projection $p\colon R^2 \to T^2$ (which is a Lie group epimorphism). The map defines an atlas of the smooth manifold T^2. Then it is well known that the affine structure is uniquely expressed with a flat linear connection $(\Gamma^i_{jk})_{1\leq i, j, k\leq 2}$ having vanishing curvature and torsion, where the components Γ^i_{jk} can be thought of as functions defined globally on T^2. Since the connection has been assumed to be invariant under T^2, the functions Γ^i_{jk} are actually constant. Now the vanishing of the torsion and the curvature is quivalent to say $\Gamma^i_{jk}=\Gamma^i_k$ along with

(3.9) $\sum_{a=1}^{2}(\Gamma^i_{a1}\Gamma^a_{j2}-\Gamma^i_{a2}\Gamma^a_{j1})=0$, $1\leq i$, $j\leq 2$, since Γ^i_{jk} are constant. This is a quadratic cone in R^6, whose vertex is the orgin. In particular the set is connected. This fact would allude that the homogebeous affine structures are deformable to one another. We will be back to this point in the next paper. The cone (3.9), called C, is 4-diemsnsional analytic set and smooth everywhere except at 0, as we are about to prove. Consider two vectors $X=(\Gamma^1_{12}, \Gamma^2_{11}, \Gamma^1_{21}-\Gamma^1_{11})$ and $Y=(\Gamma^1_{22}, \Gamma^2_1, \Gamma^1_{22}-\Gamma^1_{12})$ in R^3. Then (3.9) is equivalent to say that the vectors X and Y are linearly dependent. The set $\{(X,Y)\,|\,X,Y\in R^3,\ X$ and Y are linearly dependent, but $(X,Y)\neq(0,0)\}$ is a homogeneous space, as one easily sees that one makes $(g, \theta)\in GL(3, R)\times R$ act on the set by assigning to (X,Y) the member $(\cos\theta\cdot gX+\sin\theta\cdot gY,\ -\sin\theta\cdot gX+\cos\theta\cdot gY)$. Finally we want to show that the cone C is in a one-to-one correspondence with the homogeneous affine structures on T^2 modulo $\mathrm{Diff}(T^2)_e$. Certainly C represents all the homogeneous affine

structures. It remains to show that there does not exist a diffeomorphism $f \in$ Diff$(T^2)_e$ which carries one point of C to a different point. Recall (3.6): every point admits T^2 as an automorphism group. Therefore f is an automorphism of the Lie group T^2 (carrying $t \in T^2$ into ftf$^{-1} \in T^2$) which induces the identity on $\pi_1(T^2)$. Then f must be the identity. And we have proved:

Theorem 3.10. *The totality of the homogeneous affine structures on T^2 modulo* Diff$(T^2)_e$ *is in a one-to-one correspondence with the real algebraic variety C defined by (3.9) in $\mathbf{R}^6$. C is connected, 4-dimensional and has no singularities except at 0.*

4. The inhomogeneous case

In this section we study the affine structures whose automorphism groups are not transitive on the torus. To be more precise, let H be the holonomy group $\subset A(2)$ of the given affine structure on the 2-torus T^2 and assume throughout this section, as opposed to (3.1), that

(4.1) (The inhomogeneity condition). *Every maximal abelian subgroup G of the affine group $A(2)$ whcih contains H has a one-dimensional orbit which meets the development $\mathscr{D}T^2$.*

Perhaps it is helpful to note at this stage that the group G then belongs to one of the classes $(I, 1)$, $(I, 2)$ and (II) defined in (2.14) by (2.17) since the class $(III, 3)$ has been ruled out by (3.0). Our utmost effort will be directed toward the demonstration of the fact that the developing map $\boldsymbol{d}$ is covering so that our torus will be A^*/H^* by (1.11). Then we will show that H^* determines the affine structure and will classify all H^*. We will employ Proposition 1.20 to establish that $\boldsymbol{d}$ is covering.

Now we begin with fixing G in (4.1) as well as $\boldsymbol{d}$. The development $\mathscr{D}T^2$ is the union of open G_e-orbits and one-dimensional ones by (1.25) and (2.14), where G_e is the identity component of G. We let G_e act on T^2 and its holonomy covering $\tilde{T}^2$ by (1.8) under the convention (1.9).

(4.2) *The one-dimensional G_e-orbits on $\tilde{T}^2$ are all closed geodesics $(=$ "envelopments" of line segments$)$ and they are finite in number.*

Proof. Immediate from (2.17), (2.17a) and (1.10).

Hereafter we call those geodesics in (4.2) *the invariant geodesics.* Note that every invriant geodesic is closed and has no self-intersection, or in other words, the geodesic is an imbedding of the circle into T^2. G acting on the affine plane leaves invariant those G_e-invariant lines (which are not necessarily G_e-orbits). And so does H consequently.

(4.3) *All the invariant geodesics belong to one and the same homotopy class $\alpha \in \pi_1$ (T^2), provided they are suitably oriented. And α generates a direct summand in*

$\pi_1(T^2)$, *viz. there exists some class β such that the natural map*: $Z\alpha \oplus Z\beta \rightarrow \pi_1(T^2)$ *is an isomorphism.*

Proof. Those closed geodesics L are not contractible, since otherwise their developments $\mathscr{D}L$ would be a closed curve by (1.4a$\mathscr{D}$) and hence $\mathscr{D}L$ could not be contained in lines. The rest is more or less well known (See Reinhart [13], for instance.) Briefly, an a proof will go like this. Passing to the homology groups, an easy intersection number argument will give the existence of some α such that the invariant geodesics are integral multiplies of α, since the invariant geodesics are disjoint from one another. The complement $T^2 - L$ is connected, since L is an imbedded circle and not contractible. Therefore there is a closed curve which meets L at exactly one point. Again an intersection member argument shows that L belongs to the class $\pm \alpha$.   Q.E.D.

Let h_α be the member of H which corresponds to α in (4.3) by (1.3c). Let L be an H-invariant line into which an invariant geodesic L is developed by (1.4$\mathscr{D}$), $\mathscr{D}L \subseteq L$. L has an induced affine structure from L. Its holonomy group is generated by h_α restricted to L. The eigenvalue $h_\alpha(L)$ of $h_\alpha \mid L$ (or of its rotation part) is positive by (1.7a), since L and L are orientable. The other eigenvalue of h_α is positive too by the same (1.7a), since T^2 and the plane are orientable. Thus we have proved

(4.4)   The eigenvalues of h_α are positive.

(4.4a) h_α belongs to the componet G_e of G.

Proof.   Immediate from (4.4) and (2.13).

(4.4b) h_α belongs to a unique 1-parameter subgroup G_1 of G.

Proof.   Immediate from (4.4a) and (2.14a).

(4.5)   The eigenvalue $h_\alpha(L) \neq 1$.

Proof. Restricted to L, h_α is not the identity by (1.5) since L is compact but L is not. Suppose $h_\alpha(L) = 1$. Then h_α acts on L as a nontrivial translation. Therefore G is in the class (II). Then G contains a 1-parameter subgroup $\{(x,y) \rightarrow (x,cy) \mid c > 0\}$ where L is given by $y = 0$. This 1-parameter group is thought of as the positive scaler mutliplication group of the line bundle with the fibres $x =$ const. and over L. Thus it follows from Proposition (1.2) and Remark 1.22 that L cannot meet the development $\mathscr{D}T^2$, contrary to (4.1) or the choice of L.   Q.E.D.

(4.5a)   The class (II) can not occur.

We orient all the invariant geodesics L so that we have

(4.6) $h_\alpha(L) > 1$ for every invariant geodeisc.

(4.7) G belongs to the class $(I, 1)$ or $(I, 2)$.

Proof. This is another way of saying (4.5a), which was verified in the proof of (4.5).

Next we like to show that the other eigenvalue of h_α is also greater than one. To express such a property, we say a linear transformation h to have *node type* if h is "expanding", viz. if the eigenvalues of h are all real and greater than one (so that h belongs to a 1-parameter group generated by a vector field $\sum_i a_i x^i \partial/\partial x^i$, $a_i > 0$, which has a node at 0.) That h_α has node type will mean that the domains bounded by the consecutive invariant geodesics in T^2 are composed of compact G_1-orbits or else noncompact G_1-orbits whose semiorbits have opposite signs (See proposition 3 in Reinhart [13]). (In other words, there are no Reeb components.)

(4.8) h_α belongs to G_1 as a transformation of the holonomy covering space T^2; see (1.3a) and (1.8) for the action and (4.4b) for the symbol G_1.

Prior to the proof, let us note that (4.8) implies:

(4.8a) The G_1-orbirs on $T^2 = T^2/H$ are all compact. And they are all in the homotopy class α.

Proof of (4.8). Take a point x on $L \subset T^2$. Regard the closed curve L as a map from a closed interval $[0, 1]$ into T^2 with $L(0) = x = L(1)$, which is injective on $(0, 1)$. Consider its lift starting at $\boldsymbol{x} \in p^{-1}(x)$, where p is the projection: $T^2 \to T^2$. The end point is $h_\alpha(\boldsymbol{x})$; this is due to a geometric interpretation of (1.3c). On the other hand, both L and $\boldsymbol{L}$ are (contained in) G_1-orbits, G_1 acting both on T^2 and $\boldsymbol{T}^2$. Therefore we have $h_\alpha(\boldsymbol{x}) = g(\boldsymbol{x})$ for some $g \in G_1$. Passing to the plane $\boldsymbol{A}^2$, we have $h_\alpha \, \boldsymbol{d}(\boldsymbol{x}) = \boldsymbol{d} \, h_\alpha(\boldsymbol{x}) = \boldsymbol{d} \, g(\boldsymbol{x}) = g \, \boldsymbol{d}(\boldsymbol{x})$. This point lies on $\boldsymbol{L}$ by (4.6). Recall that G_1 is 1-connected. Then we have $h_\alpha = g$ as a transformation of $\boldsymbol{A}^2$, since h_α belongs to G_1 as such. Back to $\boldsymbol{T}^2$, this gives $h_\alpha = g$ on some neighborhood of $\boldsymbol{x}$ in $\boldsymbol{T}^2$, since $\boldsymbol{d}$ is a local homeomorphism and equivaraint with respect to both H and G_1. Then we conclude $h_\alpha = g$ on the whole $\boldsymbol{T}^2$ from the above and the analyticity (of all transformations we are observing).
                                                                         Q.E.D.

(4.9) There is a curve c in the homotopy class β in (4.3) which is transversal to all the G_1-orbits and meets each of them exactly once.

A few remarks about the statement of (4.9). The class β in (4.3) is not unique but it does not matter. The proof below will reveal this point patently. Some parts of (4.9) are known (e.g. [16], [13]). At any rate, (4.9) is important, we can easily derive (4.10) from it.

Proof of (4.9). Our method is to use Riemannian geometry. There is a G_1-invariant Riemannian metric on T^2, since G_1 acts as a compact group G_1/H_α by (4.8), where H_α is the group generated by h_α. Since G_1 has no fixed points in T^2 by (1.8c) and (1.25), we may assume that the G_1-orbits are all geodesics

for the Riemannian metric; alter the metric, if necessary, in such a way that the parameter of G_1 becomes the arc-length of each G_1-orbit. There is a smooth curve c of the shortest length in the free homotopy class β since we have the intersection number $I(\beta, \alpha) = \pm 1 \neq 0$. c is transversal to the G_1-orbits, because both c and each G_1-orbit are geodesics and hence they cannot be tangent to each other at any point without being the same curve. Finally c meets every G_1-orbit exactly once. In fact, the well known geometric interpretation of the intersection number tells us that $|I(\beta, \alpha)| = 1$ equals the carinal number of the intersection of the set c and an arbitrary G_1-orbit, by (4.8a), transversality, and the obvious orientation between the tangent vectors to these curves.   Q.E.D.

(4.9a)   The G_1-orbits give rise to a trivial circle bundle structure on T^2.

Proof.   Immediate from the above. For this matter let us remark this theorem: Let G be a connected isometry group of a connected Riemannian manifold M. Assume that the action is free and there exists a closed G-orbit. Then the orbit space M/G is a smooth manifold in the natural way and M has a principal G-bundle structure over M/G. (W. Dydo's thesis in prepartion).

(4.10) h_α is of node type, viz. its eigenvalues are both greater than one.

Proof.   Suppose the contrary. Then (the eigenvalues are distinct and) the fixed point 0 of G_1 is the saddle point of the flow of the G_1-orbits, contrary to (4.9). To be more precise, G_1 consists of the linear transformations: $(x, y) \rightarrow (e^{ta}x, e^{tb}y)$, $t \in \mathbf{R}$, for some constants $a > 0 > b$. in some affine coordinates. Hence the orbits are given by $|x|^b |y|^{-a} = $ constant. These are convex toward 0 except for the x' and the y' axes, and they are asymptotic to both axes. Now consider the development $\mathscr{D}c$ of the curve c in (4.9). $\mathscr{D}c$ must contain an arc γ with one end on the x-axis and the other on the y-axis by (4.9) and $\mathscr{D}T^2 = A^2 - \{0\}$. Obviously γ must be tangenet to some G_1-orbit in the sector, contrary to the transversality (4.9).   Q.E.D.

Summarizing what we have seen so far, we state:

Proposition 4.11.   *Assume* (4.1) *for the given affine 2-torus. Then the torus has invariant geodesics in it. All of them belong to a single homotopy class* $\alpha \in \pi_1(T^2)$. *The corresponding member* h_α *of the holonomy group* H *is linear and of node type. In particular* H *is either in the class* $(I, 1)$ *or* $(I, 2)$.

We infer from (4.10) or (4.9) that

(4.12) $G_1 \in h_\alpha$ leaves in variant a Riemannian metric on $A^2 - \{0\}$, $0 = $ fixed point of G_1, and the orbit space $A^2 - \{0\}/G_1$ is the circle.

Therefore it follows from (1.20) that

(4.13)   The developing mapping d is covering under the hypothesis (4.1).

(4.14)   Our T^2 is affine-isomorphic with A^*/H^*.

Here A^* is the universal covering space of $A^2-\{0\}$ with the induced affine structure and H^* is the holobomy group for the (A^*, A^*)-structure on T^2, where A^* is the affine automorphism group of A^*.

Proof of (4.14).   Immediate from (1.11) and (4.13).   Q.E.D.

REMARK. H^* is thus closed and discrete in A^*. And H^* acts on A^* as a free, proprly discontinuous group.   But H itself does not always have these properties, acting on $A^2-\{0\}$ as a subgroup of $GL(2\cdot R)$.

Explotiting the notations in (2.18) and (2.19), we now state the main theoem in this section:

Theorem 4.15. *An affine 2-torus satisfying* (4.1) *is characterized as the affine manifold* A^*/H^* *such that* (1) H^* *is generated by two members* h_α^* *and* h_β^* *in* $\pi^{-1}(G^+)$, (2) $\pi(h_\alpha^*)$ *is of node type (viz. "expanding"),* (3_α) $\nu(h_\alpha^*)=0$, (3_β) $\nu(h_\beta^*)$ >0 *and* (4) *the projection* $H=\pi(H^*)$ *is not contained in the center of* $GL(2, R)$, *where* $G^+=G\cap GL(2, R)^+$ *and* G *is a maximal abelian subgroup of* $A(2)$ *in the class* $(I, 1)$ *or* $(I, 2)$.

REMARK.   The condition (4) is meant to exclude the Hopf tori.   And see (1.12a) for the uniqueness of the correspondence: $T^2\rightarrow A^*/H^*$ above.

Proof of (4.15).   Assume that the affine torus T^2 satisfies (4.1). By (4.14), T^2 is affine isomorphic with A^*/H^*, which clearly satisfies the condition (1). The second condition (2) follows from (4.10) and (4.3).   We have (3_α) by (4.8), or by its proof.   As to (3_β), suppose $\nu(h_\beta^*)=0$.   Then h_β^* lies in the identity component of $\pi^{-1}(G^+)$, which is the universal covering group of G_e and is isomorphic with G_e by π.   Therefore H^* can be thought of as a subgroup of G_e acting on A^*. Since the G_e-orbits in the plane A^2 are 1-connected by (2.17a), (2.16c) and (4.7) and the space $A^2-\{0\}$ has the fundamental group isomorphic with Z, G_e acting on A^* has infinitely many orbits and consequently A^*/H^* cannot be compact. This absurdity gives (3_β).   We have (4) also, since otherwise our torus would be a Hopf torus and hence G can be in the class $(I, 3)$, contrary to (4.7).   Conversely, let us assume that H^* is a subgroup of $\pi^{-1}(G^+)$, G in $(I, 1)$ or $(I, 2)$, which satisfies the conditions (1) though (4).   By (2), $h_\alpha=\pi(h_\alpha^*)$ is contained in a (unique) 1-parameter subgroup G_1 of G and G_1 is 1-connected.   Thus, by (3_α), h_α^* belongs to the 1-parameter group $\subset\pi^{-1}(G_1)\subset\pi^{-1}(G^+)$, denoted by G_1^*, since the restriction to the identity component $\nu^{-1}(0)$ of $\pi^{-1}(G^+)$ is injective.   By (1), $h_\beta=\pi(h_\beta^*)$ centralizes G_1.   Therefore the commutator $[h_\beta^*, g^*(t)]$, $g^*(t)\in G_1^*$, $t\in R$, belongs to the kernel of π, which is discrete.   Therefore h_β^* centralizes G_1^*.   In particular H^* is abelian.   By (3_α) and (3_β), H^* is thus isomorphic with $Z^2\simeq\pi_1(T^2)$.   Also we see by (3_β) that H^* has no fixed points as a transformation group of A^*/G_1^*.

The orbit space A^*/G_1^* is diffeomorphic with the real line, since A^*/G_1^* is the universal covering space of $(A^2-\{0\})/G_1$ and G_1 contains h_a of node type, which implies that $(A^2-\{0\})/G_1$ is a circle. Therefore H^* acts on $A^*/G_1^*\approx R$ as a (nontrivial cyclic) translation group by the lemma 4.16 below yet to be proved. (The lemma is concerned with the C^∞-category. But it does not matter, what we need is a topological fact.) Moreover the subgroup of H^* generated by h_a^* acts on each G_1^*-orbit $\subset A^*$ as a translation group, since G_1^* acts on it simply trnasitively, i.e. exactly as R does on itself in the usual manner. So we infer that H^* acts on A^* $\approx R^2$ as Z^2 does on R^2 in the usual way. Therefore A^*/H^* is diffeomorphic with the 2-torus. The affine torus A^*/H^* has the affine holonomy group $H=\pi(H^*)$, since the covering map d is the projection of A^* onto $A^2-\{0\}$. Thus it follows from (1) that A^*/H^* satisfies the condition in (4.1) for the given G. But the assumption (4) implies that the given G is the only maximal abelian subgroup of $A(2)$ that contains H.   Q.E.D.

The following lemma we have just used may be known; at least a local study was done by Sternberg [16].

Lemma 4.16. *Let f be a C^∞-diffeomorphism: $R\to R$ without fixed point. Then there is a C^∞-diffeomorphism φ: $R\to R$ such that the composite $\varphi^{-1}\circ f\circ\varphi$ is the translation: $x\mapsto x+1$.*

Proof. We want to construct a strictly monotone C^∞-function φ: $R\to R$ such that we have

(4.17) $f(\varphi(x))=\varphi(x+1)$ for all $x\in R$. We assume $f(0)>0$, since the other case is easy to handle with. We then have $f'>0$ everywhere, since otherwise we would have $f'<0$ everywhere and f would certainly hae a fixed point in view of $f(0)>0$. Now consider a strictly increasing C^∞-function $\varphi=\left[0,\frac{1}{2}\right)\to\left[0,\frac{1}{2}f(0)\right)$ with $\varphi(0)=0$. We intend to extend φ to $[0, 1]$. The condition (4.17) poses a compatibility condition; to wit, (4.17) completely determines the ∞-th order jet $j_1^\infty\varphi$(=the "Taylor seies" of φ at 1) as a function of the jets j_0^∞ and $j_0^\infty\varphi$ at 0. In particular we must have $\varphi(1)=f(\varphi(0))$ and $\varphi'(1)=f'(0)\varphi'(0)$ For these we observe $\varphi(1)>\frac{1}{2}f(0)>0$ and $\varphi'(1)>0$. Now it is well known that, given any formal power seies $\sum_{n=1}^\infty a_n(x-1)^n$, there exists a C^∞-function ψ with $j_1^\infty\psi=\sum_{n=1}^\infty a_n(x-1)^n$. Therefore φ extends to a C^∞-function defined on a small neighborhood of $\left[0,\frac{1}{2}\right)\cup\{1\}$ in $[0, 1]$ with the required value of $j_1^\infty\varphi$. Then clearly φ extends to a strictly increasing C^∞-function on $[0, 1]$ by $\varphi(1)>\frac{1}{2}f(0)$ and $\varphi'(1)>0$. Further we extend φ to a C^∞-function defined on the whole line R by means of (4.17). Then φ is the desired function. In fact φ is, in the first place, strictly monotone since φ has the property on $[0, 1]$ and (4.17) implies

$\varphi'(x)\varphi'(x+1)>0$ because of $f'>0$. And φ is surjective since $\varphi(\mathbf{R})$ is an f-invariant interval by (4.17) and so a point on its boundary is, if any, a fixed point of f. Therefore φ is a C^∞-diffeomorphism: $\mathbf{R}\to\mathbf{R}$ with (4.17) satisfied.

<div align="right">Q.E.D.</div>

A version of Theorem 4.15 may be formulated in terms of H rather than H^* as follows:

Proposition 4.18. *The inhomogeneous affine 2-torus is characterized by the triplet (h_α, h_β, ν) of a member $h_\alpha \in G$ of node type, a member h_α or h_β is not a member of the center of $GL(2, \mathbf{R})$, and a positive integer ν such that the eigenvalues of h_β have the sign $(-1)^\nu$, G is a maximal abelian subgroup of $GL(2, \mathbf{R})$ in the class $(I, 1)$ or $(I, 2)$.*

Proof. Let (h_α, h_β, ν) correspond to (h_α^*, h_β^*) in (4.5) such that $h_\alpha = \pi(h_\alpha^*)$, $h_\beta^* = \pi(h_\beta^*)$ and $\nu = \nu(h_\beta^*)$, where the ν in the right hand side is the map in (4.5) or (2.19). Then everything should be obvious.   Q.E.D.

(4.19) A geometric picture. The number ν in the above has a natural geometric meaning; the number of the invariant geodesics is equal to $k\nu_0$ if G is in the class (I, k), $k = 1$ or 2. Let $T(h_\alpha, h_\beta, \nu_0)$ denote the affine torus which corresponds to $(h_\alpha, h_\beta, \nu_0)$ as in (4.18); we call it "the" affine torus, though the uniqueness does not exactly obtain. By (2.16b), there is a unique $h_\gamma \in G^+$ such that we have $h_\beta = (h_\gamma)^\nu$ and that h_γ has negative eigenvalues. The affine torus $T(h_\alpha, h_\beta, \nu)$ is then a ν-fold covering space of $T(h_\alpha, h_\gamma, 1)$ with an affine map as the covering map, as is easily seen. The space $T(h_\alpha, h_\gamma, 1)$ can be constructuted from the subset $D^+ = \{(x, y) \in \mathbf{R}^2 \mid y \geqq 0, (x, y) \neq 0\}$ of the plane as follows. Consider the group $H_\alpha = \{(h_\alpha)^n \mid n \in \mathbf{Z}\} \subset GL(2, \mathbf{R})$. H_α acts on D^+ naturally. Assume that G leaves the x-axis invariant, as a metter of convention. To construct a quotient space, identify every point $(x, 0)$, $x > 0$, on the positive x-axis with the point $h_\gamma(x, 0)$ on the negative axis, and all the other points in D^+ with themselves alone. The space obtained is a topological cylinder C. H_α acts on C too, since h_α commutes with h_γ. The orbit space C/H_α is then a topological torus since h_α is nodal. On C, we introduce a manifold structure in the following way. First the projection: $D^+ \to C$ shall be a local diffeomorphism when restricted to the interior ($=$ the open upper half-plane). For a point $(x, 0)$, $x > 0$, on the boundary of D^+, we take its neighborhood U in $\mathbf{R}^2$ (but not in D^+) so that we can regard U as a neighborhood U^C of the point $\in C$ corresponding to $(x, 0)$ by identifying every point $p \in U$ with $h_\gamma(p)$. Then the inclusion map: $U \to \mathbf{R}^2$ shall give rise to a local chart: $U^C \to U \to \mathbf{R}^2$. Thus we have a manifold C. $T(h_\alpha, h_\gamma, 1)$ is then the manifold C/H_α with the affine structure induced from $\mathbf{R}^2 = A^2$; in fact H_α is an affine automorphism roup of C. From this picture we immediately obtain.

Corollary 4.20. *In an inhomogeneous affine torus, the domains bounded by consecutive invariant geodesics are affine isomorphic with one another. These*

isomorphisms extend to automorphisms of the torus itself.

REMARK. Those automorphisms are given by $(h_\gamma^*)^m$, $m=1, 2, \cdots, \nu$, and, in the case of $(I, 2)$, the one induced from the transformation: $(x, y) \rightarrow (-x, y)$ of D^+.

5. Concluding remarks

Now we see by Theorems 3.3 and 4.15 that every affine torus is completely determined by its holonomy group H^* in the modified sense. In particular we learn everything about the torus through the knowledge of the transformation group H^*, at least in principle. Another conspicuous common feature is that the automorphism group has an open dense orbit.

As to other A-structures, we like to point out without proof that different affine 2-tori are different as projective tori.

The Hopf tori, (3.5), differ from all other affine 2-tori in that (1) the automorphism group is 4-dimensional, rather than 2-diemsional, (2) its identity component is not abelian, and (3) the holonomy group H leaves infinitely many lines invariant. They differ together with the inhomogeneous tori, from all the other homogeneous tori in that (1) the automorphism group is not compact, (2) its component is not compact, and (3) its maximal compact subgroup is the circle group.

The homogeneous affine tori are characterized as the affine tori which can be regular covering affine tori of some affine tori with $Z_m \times Z_n$ as the covering groups for any positive integers m, n: Here Z_n is the cyclic group of order n.

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