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MODULES WITH MANY DIRECT SUMMANDS

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Let R be a ring and $\mathcal{X}$ a class of right R-modules. Let M be a right R-module such that for every submodule N of M there exists a direct summand K of M such that $N\subseteq K$ and $K/N\in\mathcal{X}$. The structure of M is investigated in the cases that $\mathcal{X}$ consists of Noetherian right R-modules, right R-modules with Krull dimension and right R-modules with finite uniform dimension, respectively.

1. Classes of modules

Throughout this note, all rings considered have an identity and all modules are unital right modules. Let R be a ring. By a *class of R-modules* we mean a collection of R-modules containing a zero module such that if $M\in\mathcal{X}$ and $M'\cong M$ then $M'\in\mathcal{X}$. Any member of $\mathcal{X}$ will be called an *$\mathcal{X}$-module*. Let

$$0\to M'\to M\to M''\to 0$$

be an exact sequence of R-modules. A class $\mathcal{X}$ of R-modules will be called

S-closed provided $M'\in\mathcal{X}$ whenever $M\in\mathcal{X}$,

Q-closed provided $M''\in\mathcal{X}$ whenever $M\in\mathcal{X}$, and

P-closed provided $M\in\mathcal{X}$ whenever both $M'\in\mathcal{X}$ and $M''\in\mathcal{X}$.

Moreover, $\mathcal{X}$ is called *$\{P, S\}$-closed* provided it is both P-closed and S-closed, and so on (this terminology is taken from [15]).

Let n be a positive integer and $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}_1, \cdots, \mathcal{X}_n$ classes of R-modules. Then $\mathcal{X}\mathcal{Y}$ is the class of R-modules M which contain a submodule N such that $N\in\mathcal{X}$ and $M/N\in\mathcal{Y}$. In particular $\mathcal{X}^2$ will denote $\mathcal{X}\mathcal{X}$. Thus $\mathcal{X}$ is P-closed if and only if $\mathcal{X}^2=\mathcal{X}$. Moreover $\mathcal{X}_1\oplus\cdots\oplus\mathcal{X}_n$ is the class of R-modules consisting of all R-modules $M_1\oplus\cdots\oplus M_n$, where $M_i\in\mathcal{X}_i$ $(1\leqslant i\leqslant n)$. In case $\mathcal{X}=\mathcal{X}_i$ $(1\leqslant i\leqslant n)$ we shall denote $\mathcal{X}_1\oplus\cdots\oplus\mathcal{X}_n$ by $\mathcal{X}^{(n)}$. It is clear that

$$\mathcal{X}\cup\mathcal{Y}\subseteq\mathcal{X}\oplus\mathcal{Y}\subseteq\mathcal{X}\mathcal{Y}, \tag{1}$$

for any classes $\mathcal{X}$ and $\mathcal{Y}$ of R-modules.

Let $\mathfrak{X}$ be a class of R-modules. Then $H\mathfrak{X}$ is the class of R-modules M such that $M/N\in\mathfrak{X}$ for every submodule N of M. On the other hand, $E\mathfrak{X}$ is the class of R-modules M such that $M/N\in\mathfrak{X}$ for every essential submodule N of M. Moreover, $D\mathfrak{X}$ is the class of R-modules M such that for each submodule N of M there exists a direct summand K of M containing N such that $K/N\in\mathfrak{X}$. It is clear that

$$H\mathfrak{X}\subseteq D\mathfrak{X}\subseteq E\mathfrak{X}, \tag{2}$$

for any class $\mathfrak{X}$. Moreover,

$$\mathfrak{X}\cap E\mathfrak{X}=H\mathfrak{X}, \tag{3}$$

for any $\{P, S\}$-closed class $\mathfrak{X}$. In order to establish (3) we first recall:

Lemma 1.1. *Let R be a ring and N any submodule of an R-module M. Then there exists a submodule K of M such that $N\cap K=0$ and $N\oplus K$ is an essential submodule of M.*

Proof. See [1, Proposition 5.21].

Consider (3). Let $\mathfrak{X}$ be any $\{P, S\}$-closed class of R-modules. Note first that, by (2), $H\mathfrak{X}\subseteq\mathfrak{X}\cap E\mathfrak{X}$. Now let $M\in\mathfrak{X}\cap E\mathfrak{X}$. Let N be any submodule of M. By Lemma 1.1 there exists a submodule N' such that $N\cap N'=0$ and $N\oplus N'$ is an essential submodule of M. Now $N'\in\mathfrak{X}$ (because $\mathfrak{X}$ is S-closed) and $M/(N\oplus N')\in\mathfrak{X}$ (because $M\in E\mathfrak{X}$). Thus $M/N\in\mathfrak{X}$, because $\mathfrak{X}$ is P-closed. It follows that $M\in H\mathfrak{X}$. This proves (3).

In this section we shall investigate further relationships between such classes. First of all we shall give examples to show that (3) fails if $\mathfrak{X}$ is not $\{P, S\}$-closed.

EXAMPLE 1. Let R be a right nonsingular ring which is not semiprime Artinian, and let $\mathfrak{I}$, $\mathfrak{I}'$ denote the classes of singular R-modules and nonsingular R-modules, respectively. Let $\mathfrak{X}=\mathfrak{I}\cup\mathfrak{I}'$. Then $\mathfrak{X}$ is S-closed but not P-closed because if M_1 is a non-zero $\mathfrak{I}$-module and M_2 a non-zero $\mathfrak{I}'$-module then $M=M_1\oplus M_2$ does not belong to $\mathfrak{X}$. Let M' denote the R-module $R\oplus R$. Then $M'\in\mathfrak{X}\cap E\mathfrak{X}$. Let E be a proper essential right ideal of R and N the submodule $E\oplus 0$ of M'. Then M'/N does not belong to $\mathfrak{X}$. Thus M' does not belong to $H\mathfrak{X}$.

EXAMPLE 2. Let R be any ring and $\mathfrak{X}$ the class of all R-modules of finite (composition) length n, where n is even. Then $\mathfrak{X}$ is P-closed but not S-closed. Let U be any simple R-module. Then $M=U\oplus U\in\mathfrak{X}\cap E\mathfrak{X}$, but M does not belong to $H\mathfrak{X}$.

For any ring R, it will be convenient to denote the classes of zero R-modules, semisimple R-modules, singular R-modules, nonsingular R-modules, Noetherian R-modules, R-modules with Krull dimension, and R-modules of finite uniform dimension by $\mathcal{Z}$, $\mathcal{C}$, $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{N}$, $\mathcal{K}$, and $\mathcal{U}$, respectively. In addition $\mathcal{G}$ will denote the class of all R-modules M such that every submodule is an essential submodule of a direct summand of M. The class $\mathcal{G}$ has been studied by a number of authors ([3], [4], [6]–[13]). Note that, for any ring R,

$$\mathcal{G} \subseteq D\mathcal{S} \quad \text{and} \quad \mathcal{S}' \cap D\mathcal{S} \subseteq \mathcal{G}\,. \tag{4}$$

The first statement is clear. For the second, let $M \in \mathcal{S}' \cap D\mathcal{S}$. Let N be a submodule of M. Then there exists a direct summand K of M containing N such that $K/N \in \mathcal{S}$. If L is a submodule of K and $N \cap L = 0$ then L embeds in K/N, so that L is singular and hence $L=0$. Thus N is essential in K. It follows that M belongs to $\mathcal{G}$.

Lemma 1.2. *Let R be a ring and $\mathcal{X}$ any class of R-modules. Then*
(i) $\mathcal{G} \cap E\mathcal{X} \subseteq D\mathcal{X}$, *and*
(ii) *if $M \in D\mathcal{X}$ and M contains no non-zero submodule in $\mathcal{X}$ then $M \in \mathcal{G}$.*

Proof. (i) Let $M \in \mathcal{G} \cap E\mathcal{X}$. Let N be any submodule of M. Then there exist submodules K, K' of M such that $M = K \oplus K'$ and N is an essential submodule of K. Then $N \oplus K'$ is an essential submodule of M and hence $K/N \cong M/(N \oplus K') \in \mathcal{X}$. Thus $M \in D\mathcal{X}$. (ii) follows by the proof of (4).

For any R-module M, the socle of M will be denoted soc M. Next we note the following well known result.

Lemma 1.3. *Let R be a ring and M an R-module. Then*
(a) $\operatorname{soc} M = \cap \{N$: *$N$ is an essential submodule of M*$\}$.
(b) *The following statements are equivalent.*
(i) $M \in \mathcal{C}$ (*i.e. M is semisimple*).
(ii) *Every submodule of M is a direct summand of M.*
(iii) *M is the only essential submodule of M.*

Proof. By [1, Theorem 9.6 and Proposition 9.7].

Lemma 1.3 has the following immediate consequence.

Corollary 1.4. *For any ring R and class $\mathcal{X}$ of R-modules, $D\mathcal{Z} = E\mathcal{Z} = \mathcal{C} \subseteq D\mathcal{X}$.*

The next result generalises [8, Proposition 4.3] where it is proved that if R is a ring such that $R_R \in D\mathcal{C}$ (in particular, this implies that R is right Noetherian by [2, Theorem 3.1]) then any cyclic right R-module belongs to $\mathcal{G}$. (Note that $D\mathcal{C}$ is Q-closed.)

Proposition 1.5. *For any ring R, $D\mathcal{C} \subseteq \mathcal{G}$.*

Proof. Let $M \in DC$. Let N be a submodule of M and let K be a maximal essential extension of N in M. We shall show that K is a direct summand of M. Since $M \in DC$ it follow that there exists a direct summand L of M such that $K \subseteq L$ and $L/K \in C$. There exist an index set Λ and submodules $U_\lambda (\lambda \in \Lambda)$ of M, each containing K, such that U_λ / K is simple for each λ in Λ and $L = \sum_{\lambda \in \Lambda} U_\lambda$. Note that, for each $\lambda \in \Lambda$, K is not essential in U_λ and hence there exists a simple submodule V_λ of M such that $U_\lambda = K \oplus V_\lambda$. Let $V = \sum_{\lambda \in \Lambda} V_\lambda$. Then $L = K + V$ and V is semisimple. By Lemma 1.3 there exists a submodule W of V such that $V = (K \cap V) \oplus W$, and hence $L = K \oplus W$. Thus K is a direct summand of M. It follows that $M \in \mathcal{J}$.

Combining Lemma 1.2, Proposition 1.5 and (2) we conclude

$$DC = \mathcal{J} \cap EC \,,$$

for any ring R. We have already noted that DC is Q-closed. Now we prove:

Proposition 1.6. *Let R be a ring and $\mathcal{X}$ a class of R-modules. Then*
(i) *$H\mathcal{X}$, $E\mathcal{X}$ and $D\mathcal{X}$ are all Q-closed, and*
(ii) *$H\mathcal{X}$ and $E\mathcal{X}$ are S-closed provided $\mathcal{X}$ is S-closed.*

Proof. (i) Let $M \in E\mathcal{X}$. Let N be any submodule of M. Let K be any essential submodule of M/N. Then $K = L/N$ for some essential submodule L of M containing N. By hypothesis, $M/L \in \mathcal{X}$, and hence $(M/N)/K \in \mathcal{X}$. It follows that $M/N \in E\mathcal{X}$. Thus $E\mathcal{X}$ is Q-closed. Similarly $H\mathcal{X}$ and $D\mathcal{X}$ are Q-closed.

(ii) Suppose that $\mathcal{X}$ is S-closed. Let $M \in H\mathcal{X}$. Let N be a submodule of M. Let K be any submodule of N. Then N/K is a submodule of M/K and $M/K \in \mathcal{X}$. Thus $N/K \in \mathcal{X}$. Thus $N \in H\mathcal{X}$.

Now suppose $M \in E\mathcal{X}$. Let N be a submodule of M. Let K be any essential submodule of N. By Lemma 1.1 there exists a submodule L of M such that $K \cap L = 0$ and $K \oplus L$ is an essential submodule of M. Note that K essential in N implies $N \cap L = 0$ and hence $N/K \cong (N \oplus L)/(K \oplus L)$. But $M/(K \oplus L) \in \mathcal{X}$ and hence so too does $(N \oplus L)/(K \oplus L)$. Thus $N/K \in \mathcal{X}$. It follows that $N \in E\mathcal{X}$.

Next we give an example to show that $D\mathcal{X}$ is not S-closed in general.

EXAMPLE 3. Let $R = \mathbf{Z}[x]$. Then $\mathcal{T}$ consists of all torsion R-modules and $\mathcal{T}$ is $\{P, Q, S\}$-closed. Let $M = R_R$. Then $M \in \mathcal{J} \subseteq D\mathcal{T}$, by (4), but $M \oplus M \notin \mathcal{J}$ (see [4, Example 2.4]). Let $E = E(M)$, the injective hull of M. Then $E \oplus E$ is injective and hence $E \oplus E \in \mathcal{J} \subseteq D\mathcal{T}$. Thus $D\mathcal{T}$ is not S-closed and $D\mathcal{T} \oplus D\mathcal{T} \neq D\mathcal{T}$.

Proposition 1.7. *Let R be a ring and $\mathcal{X}$ any class of R-modules. Then*
(i) *$C \oplus E\mathcal{X} = E\mathcal{X}$, and*

(ii) $\mathcal{C}\oplus D\mathcal{X}=D\mathcal{X}$.

Proof. (i) Let $M\in\mathcal{C}\oplus E\mathcal{X}$. Then there exist submodules M_1, M_2 of M such that $M=M_1\oplus M_2$, $M_1\in\mathcal{C}$ and $M_2\in E\mathcal{X}$. Let N be an essential submodule of M. Since M_1 is semisimple, it follows that $M_1\subseteq N$ (Lemma 1.3). Thus $N=M_1\oplus(N\cap M_2)$, and

$$M/N=(M_1\oplus M_2)/[M_1\oplus(N\cap M_2)]\cong M_2/(N\cap M_2)\,.$$

But $N\cap M_2$ is an essential submodule of M_2 and $M_2\in E\mathcal{X}$. Thus $M/N\in\mathcal{X}$. It follows that $M\in E\mathcal{X}$.

(ii) Let $M\in\mathcal{C}\oplus D\mathcal{X}$. Then there exist submodules M_1, M_2 such that $M=M_1\oplus M_2$, $M_1\in\mathcal{C}$ and $M_2\in D\mathcal{X}$. Let N be any submodule of M. Note that $N+M_2=[(N+M_2)\cap M_1]\oplus M_2$. Because M_1 is semisimple, it follows that

$$M_1=[(N+M_2)\cap M_1]\oplus L\,,$$

for some submodule L of M_1 (Lemma 1.3). Thus $N+M_2$ is a direct summand of M.

Since $M_2\in D\mathcal{X}$ it follows that there exist submodules K, K' of M_2 such that $M_2=K\oplus K'$, $N\cap M_2\subseteq K$ and $K/(N\cap M_2)\in\mathcal{X}$. Now $(K+N)/N\cong K/(K\cap N)$, and $K\cap N=K\cap M_2\cap N=N\cap M_2$. Thus

$$(K+N)/N\in\mathcal{X}\,. \tag{5}$$

Moreover,

$$\begin{aligned}K'\cap(K+N)&=K'\cap M_2\cap(K+N)\\&=K'\cap[K+(N\cap M_2)]=K'\cap K=0\,.\end{aligned}$$

Thus $M_2+N=K'\oplus(K+N)$, and hence $K+N$ is a direct summand of M. By (5) it follows that $M\in D\mathcal{X}$.

Note that $\mathcal{C}\oplus H\mathcal{X}=H\mathcal{X}$ implies $\mathcal{C}\subseteq H\mathcal{X}$ and hence $\mathcal{C}\subseteq\mathcal{X}$. Thus $\mathcal{C}\oplus H\mathcal{X}\neq H\mathcal{X}$ in general. On the other hand, by (2) and Proposition 1.7,

$$\mathcal{C}\oplus H\mathcal{X}\subseteq D\mathcal{X}\,, \tag{6}$$

for any class $\mathcal{X}$. We have already seen in Example 3 that $D\mathcal{X}\oplus D\mathcal{X}\neq D\mathcal{X}$, even when $\mathcal{X}$ is $\{P,Q,S\}$-closed.

Proposition 1.8. *Let R be a ring and $\mathcal{X}$ a P-closed class of R-modules. Then*

(i) $(H\mathcal{X})\oplus(H\mathcal{X})=(H\mathcal{X})^2=H\mathcal{X}$,

(ii) $(E\mathcal{X})\oplus(E\mathcal{X})=(E\mathcal{X})(H\mathcal{X})=E\mathcal{X}$, *and*

(iii) $(H\mathcal{X})\oplus(D\mathcal{X})=(D\mathcal{X})$.

Proof. (i) By (1), $(H\mathcal{X})\oplus(H\mathcal{X})\subseteq(H\mathcal{X})^2$, and $H\mathcal{X}\subseteq(H\mathcal{X})\oplus(H\mathcal{X})$ is clear. Let $M\in(H\mathcal{X})^2$. Then there exists a submodule N of M such that N

and M/N both belong to $H\mathscr{X}$. Let K be a submodue of M. Then $(N+K)/K \cong N/(N\cap K)\in\mathscr{X}$, and $M/(N+K)\in\mathscr{X}$. Thus M/K belongs to $\mathscr{X}$. Thus $M\in H\mathscr{X}$.

(ii) The proof of $(E\mathscr{X})(H\mathscr{X})=E\mathscr{X}\subseteq(E\mathscr{X})\oplus(E\mathscr{X})$ is similar to (i). Let $M\in(E\mathscr{X})\oplus(E\mathscr{X})$. Then there exist submodules M_1, M_2 of M such that $M=M_1\oplus M_2$ and $M_i\in E\mathscr{X}$ (i=1, 2). Let N be an essential submodule of M. Then $N\cap M_1$ is an essential submodule of M_1 so that $M_1/(N\cap M_1)\in\mathscr{X}$. Thus $(M_1+N)/N\in\mathscr{X}$. But $M_1+N=M_1\oplus[M_1+N)\cap M_2]$, so that

$$M/(M_1+N)\cong M_2[(M_1+N)\cap M_2]\,,$$

which belongs to $\mathscr{X}$ since $(M_1+N)\cap M_2$ is an essential submodule of M_2. Since $\mathscr{X}$ is P-closed it follows that $M/N\in\mathscr{X}$. Thus $M\in E\mathscr{X}$.

(iii) Let $M\in(H\mathscr{X})\oplus(D\mathscr{X})$. Then there exist submodules M_1, M_2 of M such that $M=M_1\oplus M_2$, $M_1\in H\mathscr{X}$ and $M_2\in D\mathscr{X}$. Let N be any submodule of M. Then $(M_1+N)/N\cong M_1/(M_1\cap N)\in\mathscr{X}$. Moreover $M_1+N=M_1\oplus[(M_1+N)\cap M_2]$. By hypothesis there exists a direct summand K of M_2 such that $(M_1+N)\cap M_2\subseteq K$ and $K/[(M_1+N)\cap M_2]\in\mathscr{X}$. It follows that $M_1\oplus K$ is a direct summand of M and

$$(M_1\oplus K)/(M_1+N)\cong K/[(M_1+N)\cap M_2]\in\mathscr{X}\,.$$

Thus $(M_1\oplus K)/N\in\mathscr{X}$. It follows that $M\in D\mathscr{X}$.

Corollary 1.9. *Let R be a ring and $\mathscr{X}$ a P-closed class of R-modules. Then $E\mathscr{X}=[\mathcal{C}\oplus(E\mathscr{X})^{(n)}](H\mathscr{X})$, for any positive integer n.*

Proof. By Propositions 1.7 and 1.8.

Note that

$$\mathcal{C}(H\mathscr{X})\subseteq E\mathscr{X} \tag{7}$$

for any class $\mathscr{X}$ of R-modules. For, let $M\in\mathcal{C}(H\mathscr{X})$. Then there exists a submodule N of M such that $N\in\mathcal{C}$ and $M/N\in H\mathscr{X}$. If K is any essential submodule of M then $N\subseteq K$ by Lemma 1.3 and hence $M/K\in\mathscr{X}$. It follows that $M\in E\mathscr{X}$. In general, $(E\mathscr{X})^2\neq E\mathscr{X}$ and $(D\mathscr{X})^2\neq D\mathscr{X}$. For example, $\mathcal{C}=E\mathscr{Z}=D\mathscr{Z}$ (Corollary 1.4), but $\mathcal{C}^2\neq\mathcal{C}$ in generall. (Example 3 also shows $(D\mathscr{X})^2\neq D\mathscr{X}$.)

The next two examples illustrate Proposition 1.8.

EXAMPLE 4. Let R be a ring and n any positive integer. Let $\mathscr{X}$ denote the class of R-modules of finite length at most n. Then $\mathscr{X}$ is $\{S, Q\}$-closed but not P-closed. Thus $H\mathscr{X}=\mathscr{X}$ and

$$\mathscr{X}\subset\mathscr{X}\oplus\mathscr{X}\subseteq\mathscr{X}^2\,.$$

If $R=\boldsymbol{Z}$ then $\mathscr{X}\oplus\mathscr{X}\neq\mathscr{X}^2$. Staying with $R=\boldsymbol{Z}$, note that for any prime p, $A=\boldsymbol{Z}/\boldsymbol{Z}_{p^{n+1}}\in E\mathscr{X}$ so that $A\oplus A\in E\mathscr{X}\oplus E\mathscr{X}$ but $A\oplus A\notin E\mathscr{X}$. Also $B=\boldsymbol{Z}/\boldsymbol{Z}_{p^{n+2}}\in(E\mathscr{X})\mathscr{X}$, but $B\notin E\mathscr{X}$.

EXAMPLE 5. Consider the ring $\boldsymbol{Z}$ of rational integers and let $\mathscr{T}$ denote the class of torsion $\boldsymbol{Z}$-modules. Then $H\mathscr{T}=\mathscr{T}$, and

(i) $(D\mathscr{T})(H\mathscr{T})=(D\mathscr{T})\mathscr{T}\subseteq D\mathscr{T}$, and

(ii) $E\mathscr{T}\subseteq(D\mathscr{T})(H\mathscr{T})=(D\mathscr{T})\mathscr{T}$.

First consider (i). Let M be any $\boldsymbol{Z}$-module with finite rank. Then there exists a free submodule F of M of finite rank such that $M/F\in\mathscr{T}$. If N is a submodule of F and K/N is the torsion submodule of F/N then F/K is finitely generated torsion free, so free, and hence K is a direct summand of F. Thus $F\in D\mathscr{T}$ and $M\in(D\mathscr{T})\mathscr{T}$. However, in general, $M\notin D\mathscr{T}$; consider M in $\mathscr{T}'$ and use (4) and [9, Theorem 14].

For (ii), let M be any free $\boldsymbol{Z}$-module of infinite rank. Then $M\in E\mathscr{T}$, because any $\boldsymbol{Z}$-module belongs to $E\mathscr{T}$, but $M\notin(D\mathscr{T})\mathscr{T}$, by Lemma 1.2 (ii) and [9, Theorem 5].

We complete this section by giving an example to show that $C\mathscr{N}\nsubseteq D\mathscr{N}$, in contrast to (7).

EXAMPLE 6. Let $\boldsymbol{Q}$, $\boldsymbol{R}$ denote the fields of rational and real numbers, respectively, and let R denote the subring of the ring of all 2×2 real matrices consisting of all matrices of the form

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

where $a\in\boldsymbol{Q}$, $b\in\boldsymbol{R}$, Then $R_R\in C\mathscr{N}$. However, it can easily be checked that the only idempotents of R are 0, 1, and hence $R_R\notin D\mathscr{N}$.

2. Modules with finite uniform dimension

Let R be a ring. An R-module M *has finite uniform* (*Goldie*) *dimension* provided M does not contain an infinite direct sum of non-zero submodules. The class of all such modules will be denoted $\mathscr{U}$. It is well known that a module M is a $\mathscr{U}$-module if and only if there exist a positive integer n and uniform submodules U_i $(1\leqslant i\leqslant n)$ of M such that $U_1\oplus\cdots\oplus U_n$ is an essential submodule of M, and in this case n is an invariant of the module called the *uniform dimension* of M (see, for example, [1, p. 294 ex. 2]). Therefore $\mathscr{U}\oplus\mathscr{U}=\mathscr{U}$, for any ring R. Clearly $\mathscr{U}$ is S-closed. Moreover, $\mathscr{U}$ is P-closed. For, let $M\in\mathscr{U}^2$. Then there exists a submodule N of M such that both N and M/N belong to $\mathscr{U}$. By Lemma 1.1. there exists a submodule K of M such that $K\cap N=0$ and $N\oplus K$ in an essential submodule of M. Since K is isomorphic to a submodule of M/N it follows that $K\in\mathcal{U}$. Thus $N\oplus K\in\mathcal{U}\oplus\mathcal{U}=\mathcal{U}$. It follows that $M\in\mathcal{U}$. Hence $\mathcal{U}$ is P-closed.

Theorem 2.1. *For any ring R, $E\mathcal{U}=\mathcal{C}(H\mathcal{U})$.*

Proof. By (7), $\mathcal{C}(H\mathcal{U})\subseteq E\mathcal{U}$. Conversely, suppose that $M\in E\mathcal{U}$. Let N denote the socle of M. Let K be any submodule of M containing N. By Lemma 1.1 there exists a submodule K' of M such that $K\cap K'=0$ and $K\oplus K'$ is an essential submodule of M. Thus

$$M/(K\oplus K')\in\mathcal{U}\,, \tag{8}$$

by hypothesis. Let $L=L_1\oplus L_2\oplus L_3\oplus\cdots$ be a direst sum of non-zero submodules of K'. Since $N\cap K'=0$ it follows that, for each $i\geqslant 1$, L_i is not semisimple and hence contains a proper essential submodule H_i (Lemma 1.3). Let $H=H_1\oplus H_2\oplus H_3\oplus\cdots$. Then H is an essential submodule of L and

$$L/H\cong(L_1/H_1)\oplus(L_2/H_2)\oplus(L_3/H_3)\oplus\cdots$$

is an infinite direct sum of non-zero submodules. But the submodule L of M belongs to $E\mathcal{U}$, by Proposition 1.6, a contradiction. Thus $K'\in\mathcal{U}$. Since $\mathcal{U}$ is P-closed it follows, by (8), that $M/K\in\mathcal{U}$. Thus M/N belongs to $H\mathcal{U}$. Hence $M\in\mathcal{C}(H\mathcal{U})$.

Let $\mathcal{X}$ be a class of R-modules such that $\mathcal{X}\subseteq\mathcal{U}$. Then $F\mathcal{X}$ will denote the class consisting of all $\mathcal{Z}$-modules together with all R-modules M such that there exist a positive integer n and uniform submodules U_i $(1\leqslant i\leqslant n)$ of M with $M=U_1\oplus\cdots\oplus U_n$ and $U_i\in E\mathcal{X}$ $(1\leqslant i\leqslant n)$. Note that a uniform module $U\in E\mathcal{X}$ if and only if $U/V\in\mathcal{X}$ for all non-zero submodules V of U. Note that

$$F\mathcal{N}\subseteq\mathcal{N} \quad\text{and}\quad F\mathcal{K}\subseteq\mathcal{K}\,, \tag{9}$$

for any ring R. For any ordinal $\alpha\geqslant 0$, let $\mathcal{K}_\alpha$ denote the class of all R-modules with Krull dimension at most α. Then $F\mathcal{K}_\alpha\subseteq\mathcal{K}_{\alpha+1}$, and a module $M\in F\mathcal{K}_\alpha$ if and only if M is a direct sum of $\mathcal{K}_\alpha$-submodules and $(\alpha+1)$-critical submodules (see [5]). Note that if $\mathcal{X}$ is a P-closed class of R-modules then

$$(\mathcal{C}\oplus F\mathcal{X})(H\mathcal{X})\subseteq E\mathcal{X}\,, \tag{10}$$

by Corollary 1.9.

Corollary 2.2. *Let R be a ring and $\mathcal{X}$ an S-closed class of R-modules such that $\mathcal{X}\subseteq\mathcal{U}$. Then $E\mathcal{X}\subseteq[\mathcal{C}\oplus F\mathcal{X}](H\mathcal{X})$.*

Proof. Let $M\in E\mathcal{X}$. Then $M\in E\mathcal{U}$. By the theorem there exists a submodule N of M such that $N\in\mathcal{C}$ and $M/N\in\mathcal{U}$. By Lemma 1.1 there exists a submodule K of M such that $N\cap K=0$ and $N\oplus K$ is an essential submodule of M. By [1, p. 294 ex. 2], there exist a positive integer n and uniform submodules U_i $(1 \leqslant i \leqslant n)$ of K such that $U = U_1 \oplus \cdots \oplus U_n$ is an essential submodule of K. By Proposition 1.6, $U_i \in E\mathcal{X}$ $(1 \leqslant i \leqslant n)$ and hence $U \in F\mathcal{X}$. Finally $N \oplus U$ is an essential submodule of M and hence $M/(N \oplus U) \in H\mathcal{X}$.

Note that if $\mathcal{X}$ is a $\{P, S\}$-closed class of R-modules, such that $\mathcal{X} \subseteq \mathcal{U}$, then

$$E\mathcal{X} = (\mathcal{C} \oplus F\mathcal{X})(H\mathcal{X}) \tag{11}$$

by (10) and Corollary 2.2. Now suppose further that $F\mathcal{X} \subseteq H\mathcal{X} = \mathcal{X}$ (for example this happens when $\mathcal{X} = \mathcal{N}$ or $\mathcal{K}$). Then

$$\mathcal{C}\mathcal{X} \subseteq (\mathcal{C} \oplus F\mathcal{X})(H\mathcal{X}) \subseteq (\mathcal{C} \oplus \mathcal{X})\mathcal{X} \subseteq \mathcal{C}\mathcal{X}^2 = \mathcal{C}\mathcal{X},$$

and hence $E\mathcal{X} = \mathcal{C}\mathcal{X}$.

Corollary 2.3. *For any ring R and ordinal $\alpha \geqslant 0$,*

$$E\mathcal{N} = \mathcal{C}\mathcal{N}, \quad E\mathcal{K} = \mathcal{C}\mathcal{K} \quad \text{and} \quad E\mathcal{K}_\alpha \subseteq \mathcal{C}\mathcal{K}_{\alpha+1}.$$

Proof. $E\mathcal{N} = \mathcal{C}\mathcal{N}$ and $E\mathcal{K} = \mathcal{C}\mathcal{K}$ by the above argument. Moreover, by (11),

$$\begin{aligned} E\mathcal{K}_\alpha &= (\mathcal{C} \oplus F\mathcal{K}_\alpha)(H\mathcal{K}_\alpha) = (\mathcal{C} \oplus F\mathcal{K}_\alpha)\mathcal{K}_\alpha \\ &\subseteq (\mathcal{C} \oplus \mathcal{K}_{\alpha+1})\mathcal{K}_\alpha \subseteq \mathcal{C}(\mathcal{K}_{\alpha+1})^2 = \mathcal{C}\mathcal{K}_{\alpha+1}. \end{aligned}$$

3. $D\mathcal{U}$-modules

The main result of this section is the following theorem.

Theorem 3.1. *For any ring R, $D\mathcal{U} = \mathcal{C} \oplus H\mathcal{U}$.*

In order to prove this result we first establish:

Lemma 3.2. *Let $M \in D\mathcal{U}$. Then $M \in \mathcal{U}$ if and only if the socle of M is contained in a finitely generated submodule of M.*

Proof. Let $S = \operatorname{soc} M$, the socle of M. If $M \in \mathcal{U}$ then S is itself finitely generated. Conversely, suppose S is contained in a finitely generated submodule N of M. By (2) and the proof of Theorem 2.1, $M/S \in \mathcal{U}$. We shall prove that $M \in \mathcal{U}$ by induction on the uniform dimension n of M/S. If $n=0$ then $M=S$ and M is finitely generated, so that $M \in \mathcal{U}$. Suppose $n>0$. Suppose M is not a $\mathcal{U}$-module. Then S is not finitely generated. There exist non-finitely generated submodules S_1, S_2 of S such that $S = S_1 \oplus S_2$. Since M is a $D\mathcal{U}$-module it follows that there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, $S_1 \subseteq M_1$ and M_1/S_1 belongs to $\mathcal{U}$. Note that $\operatorname{soc} M_1 = S_1 \oplus S'$ for some submodule S' of M_1. Since S' can be embedded in M_1/S_1 it follows that

$S'\in\mathcal{U}$ and hence S' is finitely generated. Now

$$S_1\oplus S_2=\operatorname{soc} M=\operatorname{soc} M_1\oplus\operatorname{soc} M_2=S_1\oplus S'\oplus\operatorname{soc} M_2\,,$$

and this implies $S_2\cong S'\oplus\operatorname{soc} M_2$. Thus $S'\oplus\operatorname{soc} M_2$, and hence $\operatorname{soc} M_2$, is not finitely generated.

Thus $M=M_1\oplus M_2$ and soc M_i is not finitely generated for $i=1$, 2. Note that

$$M/S\cong[M_1/(\operatorname{soc} M_1)]\oplus[M_2/(\operatorname{soc} M_2)]\,.$$

If $M_1=\operatorname{soc} M_1$ then $M_1\subseteq N$ and hence $N=M_1\oplus(N\cap M_2)$. It follows that M_1, and hence soc M_1, is finitely generated. Thus $M_1\neq\operatorname{soc} M_1$, and similarly $M_2\neq\operatorname{soc} M_2$. Therefore the modules $M_1/(\operatorname{soc} M_1)$ and $M_2/(\operatorname{soc} M_2)$ have smaller uniform dimensions than M/S. By induction on the uniform dimension of M/S it follows that $M_1\in\mathcal{U}$ and $M_2\in\mathcal{U}$. Thus $M\in\mathcal{U}$, a contradiction. Thus $M\in\mathcal{U}$, as required.

Proof of Theorem 3.1. By (6), $\mathcal{C}\oplus H\mathcal{U}\subseteq D\mathcal{U}$. Conversely, suppose that $M\in D\mathcal{U}$. By (2) and the proof of Theorem 2.1, $M/S\in\mathcal{U}$, where $S=\operatorname{soc} M$. We shall prove that M belongs to $\mathcal{C}\oplus H\mathcal{U}$ by induction on the uniform dimension n of M/S. If $n=0$ then $M=S\in\mathcal{C}\subseteq\mathcal{C}\oplus H\mathcal{U}$. Suppose $n>0$. Suppose M does not belong to $\mathcal{C}\oplus H\mathcal{U}$.

Suppose $M=M_1\oplus M_2$ for some submodules M_1, M_2 of M. Then $S=(\operatorname{soc} M_1)\oplus(\operatorname{soc} M_2)$, so that

$$M/S\cong[M_1/(\operatorname{soc} M_1)]\oplus[M_2/(\operatorname{soc} M_2)]\,.$$

If $M_1\neq\operatorname{soc} M_1$ and $M_2\neq\operatorname{soc} M_2$ then both $M_1/(\operatorname{soc} M_1)$ and $M_2/(\operatorname{soc} M_2)$ have smaller uniform dimensions than M/S, so that both M_1 and M_2 belong to $\mathcal{C}\oplus H\mathcal{U}$, and in this case $M\in\mathcal{C}\oplus H\mathcal{U}$. Thus $M_1=\operatorname{soc} M_1\in\mathcal{C}$ or $M_2=\operatorname{soc} M_2\in\mathcal{C}$.

Because $M\neq S$ there exists $m\in M$, $m\notin S$. By hypothesis, there exist submodules M_1, M_2 of M such that $M=M_1\oplus M_2$, $mR\subseteq M_1$ and $M_1/mR\in\mathcal{U}$. By the argument in the previous paragraph it follows that $M_2\in\mathcal{C}$. Let $S_1=\operatorname{soc} M_1$. Then $S_1=(S_1\cap mR)\oplus S'$ for some submodule S' of M_1. Now $S'\cong(S_1+mR)/mR$, a submodule of M_1/mR, so that $S'\in\mathcal{U}$ and hence S' is finitely generated. Thus $S_1\subseteq mR+S'$, a finitely generated submodule of M_1. By Proposition 1.6 and Lemma 3.2 it follows that $M_1\in\mathcal{U}$. Now $M_1\in\mathcal{U}\cap E\mathcal{U}=H\mathcal{U}$ by (3). Hence $M=M_1\oplus M_2\in\mathcal{C}\oplus H\mathcal{U}$, a contradiction. Thus $M\in\mathcal{C}\oplus H\mathcal{U}$.

Corollary 3.3. *Let R be a ring and $\mathcal{X}$ a $\{P, S\}$-closed class of R-modules contained in $\mathcal{U}$. Then $D\mathcal{X}=\mathcal{C}\oplus(H\mathcal{X})\oplus(\mathcal{J}\cap E\mathcal{X})$.*

Proof. Let $M\in D\mathcal{X}$. In particular, this means that $M\in D\mathcal{U}$, so that $M\in\mathcal{C}\oplus\mathcal{U}$, by Theorem 3.1. Thus we can suppose, without loss of generality, that $M\in\mathcal{U}$. We claim that

$$M\in(H\mathcal{X})\oplus(\mathcal{G}\cap E\mathcal{X})\,. \tag{12}$$

We shall prove (12) by induction on the uniform dimension of M. Suppose first that there exists a non-zero submodule N of M such that $N\in\mathcal{X}$. By hypothesis, there exist submodules K, K' of M such that $M=K\oplus K'$, $N\subseteq K$ and $K/N\in\mathcal{X}$. Since $\mathcal{X}$ is P-closed it follows that $K\in\mathcal{X}$. By Proposition 1.6, K and K' both belong to $D\mathcal{X}$. By (2) and (3), $K\in H\mathcal{X}$. Moreover, K' has smaller uniform dimension than M so that, by induction, $K'\in(H\mathcal{X})\oplus(\mathcal{G}\cap E\mathcal{X})$. It follows that $M\in(H\mathcal{X})\oplus(H\mathcal{X})\oplus(\mathcal{G}\cap E\mathcal{X})=(H\mathcal{X})\oplus(\mathcal{G}\cap E\mathcal{X})$, by Proposition 1.8. Now suppose that M does not contain any non-zero submodule in $\mathcal{X}$. By (2) and Lemma 1.2, $M\in\mathcal{G}\cap E\mathcal{X}$. This proves (12).

Conversely, note that $\mathcal{G}\cap E\mathcal{X}\subseteq D\mathcal{X}$, by Lemma 1.2, and hence

$$C\oplus(H\mathcal{X})\oplus(\mathcal{G}\cap E\mathcal{X})\subseteq C\oplus(H\mathcal{X})\oplus(D\mathcal{X})\subseteq C\oplus(D\mathcal{X})\subseteq D\mathcal{X}\,,$$

by Propositions 1.7 and 1.8.

Note that, in fact, the proof of Corollary 3.3, gives:

$$D\mathcal{X}=C\oplus(\mathcal{U}\cap H\mathcal{X})\oplus(\mathcal{U}\cap\mathcal{G}\cap E\mathcal{X})\,, \tag{13}$$

for any $\{P, S\}$-closed class $\mathcal{X}$ of R-modules such that $\mathcal{X}\subseteq\mathcal{U}$. Let $M\in\mathcal{U}\cap\mathcal{G}$. Let V be any uniform submodule of M. Because $M\in\mathcal{G}$, there exists a direct summand K of M such that V is an essential submodule of K. It follows that K is uniform. Thus, by induction on the uniform dimension of M, M is a finite direct sum of uniform submodules. Thus, (13) gives

$$D\mathcal{X}\subseteq C\oplus(\mathcal{U}\cap H\mathcal{X})\oplus(F\mathcal{X})\,, \tag{14}$$

for any $\{P, S\}$-closed class $\mathcal{X}$ of R-modules such that $\mathcal{X}\subseteq\mathcal{U}$, by Proposition 1.6.

Combining (9), (13), and (14), the above discussion gives, at once, the following theorem which extends [2, Theorems 3.1 and 4.1] and [15, Corollary 2.8].

Theorem 3.4. *For any ring R and ordinal $\alpha\geqslant 0$,*

$$D\mathcal{N}=C\oplus\mathcal{N}\,,\quad D\mathcal{K}=C\oplus\mathcal{K}\,,\quad \textit{and}\quad D\mathcal{K}_\alpha\subseteq C\oplus\mathcal{K}_{\alpha+1}\,.$$

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