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RIGIDITY AND INFINITESIMAL DEFORMABILITY
OF EINSTEIN METRICS

Dedicated to Professor Yozo Matsushima on his 60th birthday

NORIHITO KOISO*

(Received August 15, 1980)

1. Introduction and results

Let \((M,g)\) be a compact Einstein manifold. If all Einstein metrics on \(M\) near \(g\) are homothetic to \(g\), then the Einstein metric \(g\) is said to be rigid. The first result concerning the rigidity of Einstein metrics is given by Berger [1; Proposition 6.4]. He proved that all Einstein metrics on the sphere \(S^n\) whose sectional curvature is \((\dim M-2)/((\dim M-1))\)-pinched are homothetic to \(g\). Berger and Ebin [2; §7] considered generalizations of this result and introduced “infinitesimal deformations”. The result they gave is, roughly speaking, that the space of all Einstein metrics on \(M\) is locally finite dimensional. By their method, Koiso [7; Proposition 3.3] gave the following Proposition (for the definition, see 2) and applied it to locally symmetric spaces of non-compact type without 2-dimensional factor ([7; Theorem 1.1]) and to some irreducible locally symmetric spaces of compact type ([7; Theorem 1.2]).

**Proposition 2.5.** If there is no essential Einstein \(i\)-deformation of an Einstein metric \(g\), then \(g\) is rigid.

One of the purposes of this paper is to generalize Koiso [7; Theorem 1.2]. For that, we shall classify essential Einstein \(i\)-deformations on simply connected symmetric spaces of compact type (Theorem 5.7). The result is as follows.

**Corollary 5.8.** Let \((M,g)\) be a locally symmetric Einstein manifold of compact type. Let \((\tilde{M},\tilde{g})\) be its universal riemannian covering and \((\tilde{M},\tilde{g})=\coprod_{a=1}^{N}(M_a,g_a)\) the irreducible decomposition as symmetric space. If \(N=1\) and \((\tilde{M},\tilde{g})\) is neither \(SU(p+q)/S(U(p)\times U(q))\) \((p\geq q\geq 2)\), \(SU(l)/SO(l)\) \((l\geq 3)\), \(SU(2l)/Sp(l)\) \((l\geq 3)\), \(E_6/F_4\) nor \(SU(l)\) \((l\geq 3)\), then \(g\) is rigid. If \(N=2\) and \(M_0\) are neither one of the above, \(G_2\) nor any hermitian space except \(S^2\), then \(g\) is rigid. If \(N\geq 3\) and \(M_0\) are neither one of the above nor \(S^2\), then \(g\) is rigid. Another purpose is to decide whether the converse of Proposition 2.5 holds or not. We expect that the converse holds, because if so, we would get

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many examples of Einstein metrics by Theorem 5.7. In the case of Kähler metrics, i.e., if we consider only Kähler metrics on a compact complex manifold, then it is not difficult to show that the converse holds (cf. Yau [13]). But, unfortunately, we shall give counter-examples to the converse in the real case. To analyze this problem, we shall introduce "infinitesimal deformations of second order" (Definition 4.4) and check whether each essential Einstein \( i \)-deformation has an Einstein \( i \)-deformation of second order or not (Theorem 6.2). As a result, we shall give the following

**Theorem 6.12.** There exist Einstein metrics which is infinitesimally deformable but rigid.

This paper is organized as follows: after some preliminaries in 2, we consider infinitesimal Einstein deformations in 3 and infinitesimal Einstein deformations of second order in 4, in general case. We apply the results in 3 and 4 to symmetric spaces of compact type in 5 and 6. Theorem 5.7 and Corollary 5.8 are proved in 5 and Theorem 6.12 in 6.

2. Preliminaries

In this section, we recall some fundamental definitions and some known facts concerning the space of riemannian metrics and deformations of Einstein metrics. Let \( M \) be a compact connected \( C^\infty \)-manifold with \( n=\dim M \geq 3 \). Riemannian metrics on \( M \), etc. are all to be in \( C^\infty \)-category, unless otherwise stated. When we fix a riemannian metric on \( M \), we identify covariant tensors and contravariant tensors with each other by the fixed metric as usual, and denote by \( T^pM, S^2M \) the \( p \)-tensor bundle over \( M \), the symmetric 2-tensor bundle over \( M \), respectively. Moreover, we denote by \( <,> \) the inner product on tensors on \( M \) and by \( <,>,> \) the global inner product for tensor fields.

For a fibre bundle \( F \) over \( M \), we denote by \( H'(F) \) the set of all \( H^s \)-cross sections of \( F \). We denote by \( \mathcal{M}^s \), \( \mathcal{D}^s \) the Hilbert manifold of all \( H^s \)-riemannian metrics on \( M \), the group of all \( H^s \)-diffeomorphisms of \( M \), respectively. (Here, we assume that \( s \) is sufficiently large.) The group \( \mathcal{D}^{s+1} \) acts on \( \mathcal{M}^s \) by pull-back and this action admits a slice (Ebin [6;8.20 Théorème]). For a riemannian metric \( g \) on \( M \), we denote by \( S_g^2 \) this slice. Recall that \( S_g^2 \) is a submanifold of \( \mathcal{M}^s \) containing \( g \) such that \( T_gS_g^2=\text{Ker} \; \delta \), where \( \delta \) is the differential operator: \( H^s(S^2M)\rightarrow H^{s-1}(TM) \) defined by

\[
(\delta h)_i = -\nabla^i h_{ii}.
\]

Denote by \( \mathcal{M}_v^s \) the Hilbert manifold of all \( H^s \)-riemannian metrics on \( M \) with volume \( c \). The tangent space of \( \mathcal{M}_v^s \) at \( g \in \mathcal{M}_v^s \) is given by \( \text{Ker} \; f \), where the function \( f \) on \( H^s(S^2M) \) is defined by \( fh=\langle h, g \rangle \).
DEFINITION 2.1. Let \( g \in \mathcal{M}_c \) be an Einstein metric. If there exists a \( \mathcal{D}^{\ast+1} \)-invariant open set \( U \) of \( \mathcal{M}_c \) containing \( g \) such that every \( H^s \)-Einstein metric in \( U \) is an element of \( (\mathcal{D}^{\ast+1})^*g \), then \( g \) is said to be rigid.

If we use Ebin's slice, we get the following

**Lemma 2.2** (Koiso [8; Lemma 3.1]). Let \( g \in \mathcal{M}_c \) be an Einstein metric. If there exists an open neighbourhood \( V \) of \( g \) in \( S^*_e \cap \mathcal{M}_c \) such that \( g \) is the unique \( H^s \)-Einstein metric in \( V \), then \( g \) is rigid.

For \( g \in \mathcal{M}_c \), we define
\[
T(g) = \int_M u_g^* u_g,
\]
\[
E(g) = S(g) - (T(g)/ng) \cdot g,
\]
where \( u_g \) is the \( H^{s-2} \)-function on \( M \) defined by the scalar curvature of \( g \) and \( S(g) \) the Ricci tensor of \( g \). Remark that \( g \) is an Einstein metric if and only if \( E(g) = 0 \). Following Lichnerowicz [10; (19.4)], the differential \( S'_e \) of the map \( S: \mathcal{M} \rightarrow H^{s-2}(S^2M) \) at \( g \in \mathcal{M}_c \) is given by

\[
2S'_e(h) = (\Delta + 2L + 2Q - 2\delta \delta - \text{Hess tr}) h,
\]
where
\[
(\Delta h)_{ij} = -\nabla^i \nabla_j h_{ij} \quad \text{for } h \in H'(S^2M),
\]
\[
(Lh)_{ij} = R^k_{ij} h_{ki} \quad \text{for } h \in H'(S^2M),
\]
\[
2(Qh)_{ij} = S_k^i h_{kj} + S_k^j h_{ki} \quad \text{for } h \in H'(S^2M),
\]
\[
2(\delta \delta h)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i \quad \text{for } \xi \in H'(TM),
\]
and the sign convention of the curvature tensor \( R \) is taken in such a way that \( R_{ij} \leq 0 \) for the standard sphere. Since an Einstein metric is a critical point of the function \( T \) on \( \mathcal{M}_c \), the differential \( E' \) of \( E \) at an Einstein metric \( g \in \mathcal{M}_c \) is given by

\[
2E'_e(h) = (\Delta + 2L - 2\delta \delta - \text{Hess tr}) h.
\]

Since \( T_e(S^*_e \cap \mathcal{M}_c) = \text{Ker } \delta \cap \text{Ker } \delta \), if \( h \in T_e(S^*_e \cap \mathcal{M}_c) \), then

\[
2E'_e(h) = (\Delta + 2L - \text{Hess tr}) h.
\]

**Definition 2.3.** Let \( g \in \mathcal{M}_c \) be an Einstein metric. We denote by \( \text{EID}(M) \) or simply \( \text{EID} \) the kernel of the map \( E'_e| T_e(S^*_e \cap \mathcal{M}_c) \). A non-zero element \( h \in \text{EID} \) is called an **essential Einstein i-deformation**. If \( \text{EID} \) vanishes, then \( g \) is said to be **infinitesimally non-deformable**, otherwise **infinitesimally deformable**.

The Lichnerowicz operator \( \Delta \) is defined by
\[
\Delta \psi = \delta \psi + 2L \psi + pQ \psi \quad \text{for } \psi \in H'(T^pM),
\]
where

\[(\bar{\Delta}\varphi)_{i_1\cdots i_p} = -\nabla^i \nabla_i \varphi,\]

\[(L\varphi)_{i_1\cdots i_p} = \sum_{t=3}^p R_{i_1\cdots i_t} \psi_t^{(t-1)},\]

and

\[p(Q\varphi)_{i_1\cdots i_p} = \sum_i S_{i}^{k} \psi_{i_{k-1}i_{k-2}\cdots i_1}.\]

Remark that this definition does not contradict the previous definitions and
the ordinary Laplace-Bertrami operator (Lichnerowicz [10; §10]). Moreover,
we can check that \(\Delta\) commutes with \(\delta\), \(\delta^*\), Hess, tr and \(d\) on an Einstein
manifold.

**Lemma 2.4** (Berger and Ebin [2; Lemma 7.1]). Let \(g^<3H7\) be an Einstein
metric. The space \(\mathcal{E}(M)\) coincides with \(\text{Ker}(\Delta_s - 2\varepsilon) \cap \text{Ker} \text{tr} \cap \text{Ker} \delta\), where \(\Delta_s\)
is the restriction of the Lichnerowicz operator \(\Delta\) to \(H'(S^2 M)\) and \(\varepsilon\) the Einstein
constant, i.e., \(S(g) = \varepsilon \cdot g\).

**Proposition 2.5** (Koiso [8; Proposition 3.3]). Let \(g\) be an Einstein metric
on \(M\). If \(g\) is infinitesimally non-deformable, then \(g\) is rigid.

3. Einstein \(i\)-deformation

Let \(g \in \mathcal{M}^\varepsilon\) be an Einstein metric with Einstein constant \(\varepsilon\), i.e., \(S(g) = \varepsilon \cdot g\). We define differential operators \(\gamma: H'(S^2 M) \to H'^1(TM)\) and \(\beta: H'(S^2 M) \to H'^2(S^2 M)\) by

\[\gamma = \delta + \frac{1}{2} d \text{tr},\]

\[\beta = \Delta_s - 2\varepsilon \cdot \text{Hess \ tr}.\]

Remark that \(\beta\) is an elliptic operator.

**Lemma 3.1.** \(\beta(\text{Ker } \delta \cap \text{Ker } f) = \text{Im } \beta \cap \text{Ker } \gamma \cap \text{Ker } f\).

Proof. Denote by \(\Delta_i\) the Lichnerowicz operator on \(H'(TM)\). By Koiso
[8; Lemma 3.2],

\[(3.1.1) \quad \gamma \beta = (\Delta_i - 2\varepsilon) \delta.\]

Since \(\text{tr } \beta = 2(\Delta - \varepsilon) \text{tr}\),

\[\beta(\text{Ker } \delta \cap \text{Ker } f) \subset \text{Im } \beta \cap \text{Ker } \gamma \cap \text{Ker } f.\]

Let \(\beta h \in \text{Ker } \gamma \cap \text{Ker } f\) and decompose \(h\) into \(\psi + \delta^* \xi; \delta \psi = 0\), by Ebin [6;8.8 Proposition]. Then

\[0 = \gamma \beta h = (\Delta_i - 2\varepsilon) \delta(\psi + \delta^* \xi) = \delta \delta^*(\Delta_i - 2\varepsilon) \xi,\]

and so, \(\delta^*(\Delta_i - 2\varepsilon) \xi = 0, \delta(\Delta_i - 2\varepsilon) \xi = 0\). Since we can easily check that
\( (3.1.2) \quad \delta \delta^* = \frac{1}{2} (\Delta - 2\varepsilon + d\delta), \)

\[ 0 = \delta \delta^*(\Delta - 2\varepsilon) \xi = \frac{1}{2} (\Delta - 2\varepsilon + d\delta) (\Delta - 2\varepsilon) \xi = \frac{1}{2} (\Delta - 2\varepsilon)^2 \xi, \]

which implies that \( (\Delta - 2\varepsilon) \xi = 0. \)

\( (3.1.3) \quad \beta \delta^* \xi = (\Delta - 2\varepsilon - \text{Hess tr}) \delta^* \xi \\
= \delta^*(\Delta - 2\varepsilon) \xi + \text{Hess} \delta \xi = \text{Hess} \delta \xi. \)

Set \( \phi = \text{Hess} \delta \xi + \varepsilon \delta \xi \cdot g. \) Then

\( (3.1.4) \quad \delta \phi = \delta \delta^* d \delta \xi - \varepsilon d \delta \xi \)

\[ = \frac{1}{2} (\Delta - 2\varepsilon + d\delta) d \delta \xi - \varepsilon d \delta \xi \]

\[ = \frac{1}{2} d \delta(\Delta - 2\varepsilon) \xi + \frac{1}{2} d \Delta \delta \xi - \varepsilon d \delta \xi \]

\[ = \frac{1}{2} d \delta(\Delta - 2\varepsilon) \xi = 0, \]

\( (3.1.5) \quad \beta \phi = (\Delta - 2\varepsilon - \text{Hess tr}) (\text{Hess} \delta \xi + \varepsilon \delta \xi \cdot g) \\
= \text{Hess} \Delta \delta \xi - n\varepsilon \text{Hess} \delta \xi \\
= (2-n)\varepsilon \text{Hess} \delta \xi. \)

Since \( \Delta \xi = 2\varepsilon \xi \) and so \( \Delta \delta \xi = 2\varepsilon \delta \xi, \) if \( \varepsilon = 0 \) then \( \Delta \xi = 0 \) and \( \delta \xi = 0. \) Therefore \( 2\delta^* \xi = (\Delta - 2\varepsilon + d\delta) \xi = 0, \) which implies that \( \delta^* \xi = 0. \) In this case the equalities \( \delta h = 0 \) and \( \beta h = \beta(h - (h/nc) \cdot g) \) hold, and so \( \beta(\text{Ker} \delta \cap \text{Ker} f) \supset \text{Im} \beta \cap \text{Ker} \gamma \cap \text{Ker} f. \) Thus we may assume that \( \varepsilon \neq 0. \) Then

\[ \beta \psi = \beta \psi + \beta \delta^* \xi = \beta \psi + \text{Hess} \delta \xi \quad (3.1.3) \]

\[ = \beta(\psi + \phi/(2-n)\varepsilon), \quad (3.1.5) \]

and

\[ \delta(\psi + \phi/(2-n)\varepsilon) = 0, \quad (3.1.4) \]

\[ \int \psi = \int h - \int \delta^* \xi = -\frac{1}{2\varepsilon} \int \beta h = 0, \]

\[ \int \phi = \int \text{Hess} \delta \xi + \varepsilon \int \delta \xi \cdot g = 0. \quad \text{Q.E.D.} \]

**Proposition 3.2.** Let \( g \) be an Einstein metric on \( M. \) Then

\( \text{Im}(E'_\xi | \text{Ker} f) \oplus \text{EID} = \text{Ker} \gamma \cap \text{Ker} f \)

(orthogonal direct sum), where \( \text{Im}(E'_\xi | \text{Ker} f) \) is a closed subspace.

**Proof.** First we see that \( \text{Im}(E'_\xi | \text{Ker} f) = E'_\xi(\text{Ker} \delta \cap \text{Ker} f) \oplus E'_\xi(\text{Im} \delta^*) \)

and \( E'_\xi(\text{Ker} \delta \cap \text{Ker} f) = \beta(\text{Ker} \delta \cap \text{Ker} f) \) by (2.2.4) and \( E'_\xi(\text{Im} \delta^*) = 0, \) and so
Im\( (E_1^*|\text{Ker} f) = \beta(\text{Ker} \delta \cap \text{Ker} f) \). Next we remark that the formal adjoint \( \beta^* \) of \( \beta \) is given by \( \Delta_\gamma - 2\varepsilon - g \cdot \delta \delta \) and see, by Lemma 2.4, that
\[
\langle \beta(\text{Ker} \delta \cap \text{Ker} f), \text{EID} \rangle = \langle \text{Ker} \delta \cap \text{Ker} f, \beta^*\text{EID} \rangle = 0.
\]
Moreover, by Lemma 2.4 and Lemma 3.1, it is easy to see that \( \beta(\text{Ker} \delta \cap \text{Ker} f) \oplus \text{EID} \subset \text{Ker} \gamma \cap \text{Ker} f \).

Now, let \( k \in \text{Ker} \gamma \cap \text{Ker} f \). Since \( \beta \) is elliptic, we can decompose \( h \) into \( \beta \phi + \psi; \beta^* \psi = 0 \). Then
\[
0 = \delta \delta \beta^* \psi = \delta (\Delta_\gamma - 2\varepsilon - g \cdot \delta \delta) \psi = (\Delta - 2\varepsilon) \delta \delta \psi + \delta \delta \delta \psi = 2(\Delta - \varepsilon) \delta \delta \psi.
\]
But here \( \varepsilon = 0 \) or \( \varepsilon \) is not an eigenvalue of \( \Delta \) on a compact Einstein manifold (Lichnerowicz [9; p. 135]). Then \( \delta \delta \psi = 0 \), and so \( (\Delta_\gamma - 2\varepsilon) \psi = 0 \).

\[
0 = \delta \gamma h = \delta \gamma \beta \phi + \delta \gamma \psi = (\Delta - 2\varepsilon) \delta \delta \phi + \delta (\delta + \frac{1}{2} \text{tr} \psi)
\]

Therefore \( (\Delta - 2\varepsilon)^2 \delta \delta \phi = -\frac{1}{2} \text{tr} (\Delta_\gamma - 2\varepsilon) \psi \), and so \( (\Delta - 2\varepsilon) \delta \delta \phi = 0, 0 = \Delta \text{tr} \psi = 2\varepsilon \text{tr} \psi \), and \( \phi = 0 \). If \( \varepsilon \neq 0 \), then \( \text{tr} \psi = 0 \). Even if \( \varepsilon = 0 \), \( \int \psi = \int h - \int \beta \phi = 0 \) implies that \( \text{tr} \psi = 0 \). Thus
\[
0 = \gamma h = \gamma \beta \phi + \gamma \psi = (\Delta_\gamma - 2\varepsilon) \delta \delta \phi + \delta \psi
\]

which implies that \( (\Delta_\gamma - 2\varepsilon)^2 \delta \phi = -\delta (\Delta_\gamma - 2\varepsilon) \psi = 0 \) and so \( (\Delta_\gamma - 2\varepsilon) \delta \phi = 0 \) and \( \delta \psi = 0 \). These formulae implies that \( \psi \in \text{EID} \) and \( \beta \phi \in \text{Ker} \gamma \cap \text{Ker} f \), and so \( \beta \phi \in \beta(\text{Ker} \delta \cap \text{Ker} f) \) by Lemma 3.1. Q.E.D.

**Proposition 3.3.** Let \( g \) be an Einstein metric with Einstein constant \( \varepsilon \). Then \( \text{dim EID} \)
\[
= \text{dim}(\text{Ker}(\Delta_\gamma - 2\varepsilon) \cap \text{Ker tr}) - \text{dim}(\text{Ker}(\Delta_\gamma - 2\varepsilon)) + \text{dim}(\text{Ker} \delta^*)
\]

**Proof.** Define a differential operator \( \theta : H'(TM) \to H^{*\cdot}(S^2 M) \) by
\[
\theta g = (\delta^* g) = \frac{1}{n} \delta \delta \cdot g.
\]
Remark that \( \text{tr} \theta = 0 \) and the formal adjoint \( \theta^* \) of \( \theta \) is given by
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\[ \theta^* h = \delta h + \frac{1}{n} \text{tr} h. \]

Let \( h \in \text{Ker}(\Delta_s - 2\varepsilon) \cap \text{Ker} \text{tr}. \) Since \( \theta \) has injective symbol, we can decompose \( h \) into \( \theta \xi + \psi; \theta^* \psi = 0 \) (Ebin [6; 8.5 Théorème]). Then \( 0 = \text{tr} h = \text{tr} \theta \xi + \text{tr} \psi = \text{tr} \psi, \) and

\[ \delta \psi = \theta^* \psi - \frac{1}{n} \text{tr} \psi = 0. \]

Moreover

\[ 0 = (\Delta_s - 2\varepsilon)h \]
\[ = (\Delta_s - 2\varepsilon)\theta \xi + (\Delta_s - 2\varepsilon)\psi \]
\[ = \theta(\Delta_1 - 2\varepsilon)\xi + (\Delta_s - 2\varepsilon)\psi, \]
\[ \theta^* (\Delta_s - 2\varepsilon)\psi = (\Delta_1 - 2\varepsilon) \theta^* \psi = 0, \]

and so \( \theta^* \theta(\Delta_1 - 2\varepsilon)\xi = 0, \theta(\Delta_1 - 2\varepsilon)\xi = 0 \) and \( (\Delta_s - 2\varepsilon)\psi = 0, \) which implies that \( \psi \in \text{EID}. \) In this correspondence: \( h \rightarrow \psi, \) if \( h \in \text{EID} \) then \( \psi = h. \) Thus we have a projection \( P: \text{Ker}(\Delta_s - 2\varepsilon) \cap \text{Ker} \text{tr} \rightarrow \text{EID}; P(h) = \psi. \) Then

\[ \dim \text{EID} = \dim(\text{Ker}(\Delta_s - 2\varepsilon) \cap \text{Ker} \text{tr}) - \dim(\text{Ker} P). \]

Here, if we remark that \( \text{tr} \theta = 0, \) then we see that

\[ \text{Ker} P = \text{Im} \theta \cap \text{Ker}(\Delta_s - 2\varepsilon). \]

We easily see that \( \theta(\text{Ker}(\Delta_1 - 2\varepsilon)) \subset \text{Ker} P. \) Conversely, let \( \theta \xi \in \text{Ker}(\Delta_s - 2\varepsilon) \) for \( \xi \in H'(TM) \) and decompose \( \xi \) into \( \xi + (\Delta_1 - 2\varepsilon)\eta; (\Delta_1 - 2\varepsilon)\eta = 0. \) Then

\[ 0 = (\Delta_s - 2\varepsilon)\theta \xi = (\Delta_s - 2\varepsilon)\theta \xi + (\Delta_s - 2\varepsilon)\theta \eta = (\Delta_1 - 2\varepsilon)\theta \eta, \]

and so \( (\Delta_1 - 2\varepsilon)\eta = 0. \) Therefore \( \xi \in \text{Ker}(\Delta_1 - 2\varepsilon) + \text{Ker} \theta, \) which implies that \( \theta \) gives a surjection from \( \text{Ker}(\Delta_1 - 2\varepsilon) \) to \( \text{Ker} P. \) Thus

\[ \dim \text{Ker} P = \dim(\text{Ker}(\Delta_1 - 2\varepsilon) \cap \text{Ker} \theta). \]

Here we easily see that \( \text{Ker} \delta^* \subset \text{Ker}(\Delta_1 - 2\varepsilon) \cap \text{Ker} \theta \) by (3.1.2). Conversely, if \( \xi \in \text{Ker}(\Delta_1 - 2\varepsilon) \cap \text{Ker} \theta, \) then

\[ 0 = \delta \theta \xi = \delta(\delta^* + \frac{1}{n} g \cdot \delta) \xi \]
\[ = \frac{1}{2} (\Delta_1 - 2\varepsilon + d\delta) \xi - \frac{1}{n} d\delta \xi \]
\[ = \left( \frac{1}{2} - \frac{1}{n} \right) d\delta \xi, \]

and so \( \delta \xi = 0, \delta^* \xi = 0, \) which implies that \( \text{Ker} \delta^* \supset \text{Ker}(\Delta_1 - 2\varepsilon) \cap \text{Ker} \theta. \) Q.E.D.

4. Infinitesimal Einstein deformation of second order

In this section, we discuss about the second derivative of the map \( E. \) Let
$g \in \mathcal{M}^r$ and $h \in H^r(S^2M)$. Regarding $h$ as an infinitesimal deformation of $g$, i.e., $h \in T_g \mathcal{M}^r$, we set

$$X(\xi, \eta) = (\nabla \xi \eta)' \quad \text{for} \; \xi, \eta \in TM.$$  

Then $X$ is a well-defined 3-tensor field (of type (1,2)) and given by

$$X_{ij}^k = \frac{1}{2} (\nabla_i h_j^k + \nabla_j h_i^k - \nabla^k h_{ij})$$

(see Lichnerowicz [9; (17.2)]).

**Lemma 4.1.** Let $g$ be an Einstein metric and $h$ an essential Einstein $i$-deformation of $g$. Then we have

\begin{align}
&g^{ik} X_{ki}^i = 0, \\
&\nabla^i X_{ki}^j = (Lh)_{ij}, \\
&(R_{ijk})' = \nabla_i X_{jk}^l - \nabla_j X_{ik}^l,
\end{align}

and the symmetric part of $X_{ikj}$ with respect to $i$ and $j$ is $(1/2)\nabla_i h_{ij}$.

**Proof.** That is easy to check by tensor calculus. For (4.1.3), see Lichnerowicz [9; (17.5)].

**Proposition 4.2.** Let $g \in \mathcal{M}^r$ be an Einstein metric and $h$ an essential Einstein $i$-deformation of $g$. Then the second derivative $E''(h, h)$ is given by

\begin{align}
2E''(h, h)_{ij} &= 2h^{kl} \nabla_k \nabla_i h_{lj} + 2\nabla_j h_i^l \cdot \nabla^k h_{lj} - 2\nabla^l h_{ik} \cdot \nabla^j h_{lj} - 4R_{ijkl}^m h_{mk} h_{kl} \\
&\quad - 2(h^{kl} \nabla_i h_{lj} + h^{ij} \nabla_j h_{li}) - \nabla^k h^m \cdot \nabla_j h^m \\
&\quad + 2((Lh)_{kl}^i h_{kj} + (Lh)_{ij}^k h_{kj}) + \nabla_i \nabla_j (h, h). \tag{4.1.2}
\end{align}

**Proof.** Since $g$ is a critical point of the function $T$ on $M^r$, $T'(h) = 0$. Moreover, (Hess $T$) $(h, h) = 0$ by Koiso [7; Theorem 2.4, Theorem 2.5]. Thus we see $E''(h, h) = S''(h, h)$. We calculate $S''(h, h)$ by Lemma 4.1.

\begin{align}
(\Delta h)'_{ij} &= -g^{kl} \nabla_k \nabla_l h_{ij} \\
&= h^{kl} \nabla_k \nabla_l h_{ij} + g^{kl}(X_{ki}^m \nabla_m h_{lj} + X_{kj}^m \nabla_m h_{ij} + X_{kj}^m \nabla_m h_{im}) \\
&\quad + g^{kl} \nabla_k (X_{ij}^m h_{mj} + X_{ij}^m h_{im}) \tag{4.1.1}, \tag{4.1.2} \\
&= h^{kl} \nabla_k \nabla_l h_{ij} + 2X_{ki}^m \nabla^m h_{lj} + 2X_{kj}^m \nabla^m h_{ij} + \nabla^k X_{ki}^m \cdot h_{mj} + \nabla^k X_{kj}^m \cdot h_{im} \\
&= h^{kl} \nabla_k \nabla_l h_{ij} + (\nabla_i \nabla_l h^m_k + \nabla_j \nabla^m h_{kj} - \nabla^m h_{kj}) \cdot \nabla^k h_{mj} \\
&\quad + (\nabla^k h^m_i + \nabla^k h^m_j - \nabla^m h_{kj}) \cdot \nabla^k h_{mj} + (Lh)_{ij}^m h_{mj} + (Lh)_{ij}^m h_{im} \\
&= h^{kl} \nabla_k \nabla_l h_{ij} + 2\nabla_i h^m_j \cdot \nabla^m h_{mj} + (\nabla_i h^m_j \cdot \nabla^m h_{mj} + \nabla_j h^m_i \cdot \nabla^m h_{im}) \\
&\quad - 2\nabla_i h^m_j \cdot \nabla^m h_{mj} + ((Lh)_{ij}^m h_{mj} + (Lh)_{ij}^m h_{im}),
\end{align}
\[(Lh)_{ij} = (g^{km}R_{im,j}h_{kl})' = -h^{km}R_{im,j}h_{kl} + g^{km}(\nabla_l X_{mj} - \nabla_m X_{lj}) \cdot h_{kl} \quad (4.1.3)\]
\[= -R_{ij}^m h^m_k h_{kl} + \nabla_l X_{mj} \cdot h_{ml} - \frac{1}{2} \nabla_m(\nabla_l h_{ij} + \nabla_j h_{li} - \nabla^l h_{ij}) \cdot h_{ml} \]
\[= -R_{ij}^m h^m_k h_{kl} + \frac{1}{2} \nabla_l \nabla_j h_{ml} \cdot h_{ml} + \frac{1}{2} \nabla_m(\nabla_l h_{ij} + \nabla_j h_{li} - \nabla^l h_{ij}) \cdot h_{mi} \]
\[= -\nabla_l \nabla_j h_{ml} + \frac{1}{2} \nabla_l \nabla_j h_{ml} \cdot h_{ml} + \frac{1}{2} \nabla_m(\nabla_l h_{ij} + \nabla_j h_{li} - \nabla^l h_{ij}) \cdot h_{mi} \]
\[= -\nabla_l \nabla_j h_{ml} + \frac{1}{2} \nabla_l \nabla_j h_{ml} \cdot h_{ml} + \frac{1}{2} \nabla_m(\nabla_l h_{ij} + \nabla_j h_{li} - \nabla^l h_{ij}) \cdot h_{mi} \]
\[= -2R_{ij}^m h^m_k h_{kl} + \frac{1}{2} \nabla_l \nabla_j h_{ml} \cdot h_{ml} + \frac{1}{2} \nabla_m(\nabla_l h_{ij} + \nabla_j h_{li} - \nabla^l h_{ij}) \cdot h_{mi} \]
\[+ \frac{1}{2} \left( (Lh)_{ik} h^k_j + (Lh)_{jk} h^k_i \right) - \frac{1}{2} (\nabla_l \nabla_j h_{ml} \cdot h_{ml} + \nabla_m(\nabla_l h_{ij} + \nabla_j h_{li} - \nabla^l h_{ij}) \cdot h_{mi} ) ,\]
\[(Qh)_{ij} = \frac{1}{2} (g^{kl}S_{ij,k}h_{kl} + g^{kl}S_{kl,j}h_{kl})' \]
\[= -\frac{1}{2} (h^{kl}S_{ij,k} + h^{kl}S_{kl,j}h_{kl}) + \frac{1}{2} (S_{ik}h^k_j + S_{jk}h^k_i) = 0 ,\]
\[(\text{Hess } \text{tr } h)' = (\text{Hess})' \text{ tr } h + \text{Hess } (\text{tr } h)' = \text{Hess } (\text{tr } h)' \]
\[(\text{tr } h)' = (g^{kl}h_{kl})' = -h^{kl}h_{kl} = -(h, h) ,\]
\[(\text{Hess } h)'_{ij} = -\nabla_i \nabla_j (h, h) ,\]
\[(\delta * h)' = (\delta * h) + \delta * (\delta h)' = \delta * (\delta h)' ,\]
\[(\delta h)'_{ij} = -(g^{kl} \nabla_k h_{ij})' \]
\[= h^{kl} \nabla_k h_{ij} + g^{kl}(X_{kl} h_{mi} + X_{mi} h_{kl}) \]
\[= h^{kl} \nabla_k h_{ij} + \frac{1}{2} \nabla_l h^m_i h^{kl} \cdot h_{ml} \]
\[
+ \frac{1}{2} h^{ik} \nabla_i \nabla_k h_{ij} + \frac{1}{2} \nabla_i \nabla_j h^m_{m} \cdot h^k_m + \frac{1}{4} (R_{ji}^j h^m_{m} + R_{ji}^k h^k_m) h^k_m
\]

\[
= \frac{1}{2} (\nabla_i h^{ik} \cdot \nabla_k h_{ij} + \nabla_j h^{ik} \cdot \nabla_i h_{ij}) + \frac{1}{2} \nabla_i h^k_m \cdot \nabla_j h^m_{m}
\]

\[
+ \frac{1}{2} (h^{ik} \nabla_i h_{ij} + h^{ik} \nabla_j h_{ij}) + \frac{1}{2} \nabla_i \nabla_j h^m_{m} \cdot h^k_m.
\]

Therefore, \( 2E''(h,h)_{ij} \)

\[
= h^{ik} \nabla_i \nabla_j h_{ij} + 2 \nabla_i h^m_{m} \cdot \nabla_j h_{ij} + (\nabla_i h^m_{m} \cdot \nabla_j h_{ij} + \nabla_j h^m_{m} \cdot \nabla_i h_{ij})
\]

\[
- 2 \nabla_i h^m_{m} \cdot \nabla_j h_{ij} + (Lh)_i^m h^m_{m} + (Lh)_j^m h^m_{m}
\]

\[
- 4 R_{ij}^k h^m_{m} h^k_m + \nabla_i \nabla_j h^m_{m} \cdot h^k_m + h^m_i \nabla_m \nabla_j h_{ij}
\]

\[
+ (Lh)_i^j h^k_m + (Lh)_j^i h^k_m - (\nabla_i h^m_{m} \cdot \nabla_j h_{ij} + \nabla_j h^m_{m} \cdot \nabla_i h_{ij})
\]

\[
- (\nabla_i h^m_{m} \cdot \nabla_j h_{ij} + \nabla_j h^m_{m} \cdot \nabla_i h_{ij} - \nabla_i h^m_{m} \cdot \nabla_j h_{ij} - \nabla_j h^m_{m} \cdot \nabla_i h_{ij} + \nabla_i h^m_{m} \cdot h^k_m)
\]

\[
+ \nabla_i \nabla_j (h,h).
\]

Q.E.D.

Now, we calculate \( \langle E''(h,h), h \rangle \) which we use in 6.

**Lemma 4.3.** Let \( g \) and \( h \) be as in Proposition 4.2. Then

\[
2 \langle E''(h,h), h \rangle = 2 \varepsilon(h_{ij}, h_i^i h_{ij}) + 3 \langle \nabla_i \nabla_j h_{kl}, h_{ij} h_{kl} \rangle - 6 \langle \nabla_i \nabla_j h_{kl}, h_{ij} h_{kl} \rangle.
\]

Proof.

\[
\langle h^{ik} \nabla_i \nabla_j h_{ij}, h_{ij} \rangle = \langle \nabla_i \nabla_j h_{kl}, h_{ij} h_{kl} \rangle,
\]

\[
\langle \nabla_i h^i_j \cdot \nabla^i_j h_{ij}, h_{ij} \rangle = -h^{i}_i \nabla_i h_{ij} \cdot \nabla^i_j h_{ij} + h^{i}_i \nabla^i_j h_{ij} \cdot \nabla_i h_{ij},
\]

\[
\langle \nabla_i h^i_j \cdot \nabla^i_j h_{ij}, h_{ij} \rangle = \frac{1}{2} \langle h_i^i (\Delta h)_{ij}, h_{ij} \rangle = -\langle (Lh)_i^i, h_i^i h_{ij} \rangle,
\]

\[
\langle \nabla^i_j h_{kl} \cdot \nabla^i_j h_{ij}, h_{ij} \rangle = -\langle h_i^i \nabla_i h_{kl} \cdot \nabla^i_j h_{ij}, h_{ij} \rangle - \langle h_i^i \nabla^i_j h_{ij}, \nabla_i h_{kl} \rangle
\]

\[
= -\langle h_i^i (R^k_m h^k_m + R^k_m h^k_m), h_{ij} \rangle + \langle h_i^i h_{ij}, \nabla_i h_{ij} \rangle
\]

\[
= \langle \nabla_i \nabla h_{kl}, h_{ij} h_{kl} \rangle - \langle (Lh)_i^i, h_i^i h_{ij} \rangle - \varepsilon(h_{ij}, h_i^i h_{ij}),
\]

\[
\langle R^i_{ij}^k h^m_{m} h^k_m, h_{ij} \rangle = \langle (Lh)_i^i, h_i^i h_{ij} \rangle,
\]

\[
\langle h^{ik} \nabla_i \nabla_j h_{ij} + h^{ik} \nabla_i \nabla_j h_{ij}, h_{ij} \rangle = 2 \langle h^{ik} \nabla_i \nabla_j h_{ij}, h_{ij} \rangle
\]

\[
= 2 \langle \nabla_i \nabla h_{ij}, h_{ij} \rangle + \langle \nabla^m_{i} h^m_{m} \cdot \nabla_j h_{ij}, h_{ij} \rangle = -\langle h^m_{m} \nabla_i \nabla_j h_{ij}, h_{ij} \rangle = -\langle \nabla_i \nabla_j h_{kl}, h_i^i h_{kl} \rangle
\]

\[
\langle (Lh)_i^i h_{ij} + (Lh)_j^i h_{kl}, h_{ij} \rangle = 2 \langle (Lh)_i^i, h_i^i h_{ij} \rangle,
\]

\[
\langle \text{Hess}(h,h), h \rangle = 0.
\]

Q.E.D.

Now, let \( g \in \mathcal{M}^c \) be an Einstein metric and \( g(t) \) a curve in \( S^k_g \cap \mathcal{M}^c \) such
that \( g(0) = g \) and each \( g(t) \) is an Einstein metric. Then,

\[
E(g(0)) = 0, \\
\frac{d}{dt} E(g(t)) = 0, \text{ i.e., } E'_t(g'(0)) = 0, \\
\frac{d^2}{dt^2} E(g(t)) = 0, \text{ i.e., } E''_t(g'(0), g''(0)) + E'_t(g''(0)) = 0.
\]

Therefore, for an Einstein metric \( g \), we call a pair \((h, h')\) \( C^\infty(S^2M) \times C^\infty(S^2M) \) an essential Einstein \( i \)-deformation of second order of \( g \) if \( h \) is an essential Einstein \( i \)-deformation of \( g \) and \( h' \) satisfies that \( E''(h, h') + E'(h') = 0 \).

**Definition 4.4.** Let \( g \) be an Einstein metric and \( h \) an essential Einstein \( i \)-deformation of \( g \). If there exists \( h' \in C^\infty(S^2M) \) such that \( (h, h') \) is an essential Einstein \( i \)-deformation of second order, \( h \) is said to be **integrable up to second order**. If there is an Einstein deformation \( g(t) \) of \( g \) such that \( g'(0) = h, h \) is said to be **integrable**.

We easily see the following

**Proposition 4.5.** Let \( g \) be an Einstein metric and \( h \) an essential Einstein \( i \)-deformation of \( g \). If \( h \) is not integrable up to second order, then \( h \) is not integrable.

Moreover the following proposition holds.

**Proposition 4.6.** Let \( g \in \mathcal{M}_c^\infty \) be an Einstein metric. If all essential Einstein \( i \)-deformations are not integrable up to second order, then \( g \) is rigid.

**Proof.** By Lemma 2.2, it is sufficient to prove that \( g \) is isolated in \( S^*_t \cap \mathcal{M}_c^\infty \). Consider the map \( E|S^*_t \cap \mathcal{M}_c^\infty : S^*_t \cap \mathcal{M}_c^\infty \to H^{-2}(S^2M) \). By formula (2.2.3) and Lemma 3.1, \( \text{Im}(E|S^*_t \cap \mathcal{M}_c^\infty) \) is closed in \( H^{-2}(S^2M) \). Denote by \( P \) the orthogonal projection: \( H^{-2}(S^2M) \to \text{Im}(E|S^*_t \cap \mathcal{M}_c^\infty) \). Then \( \text{Im}(P \circ E|S^*_t \cap \mathcal{M}_c^\infty) = \text{Im}(E|S^*_t \cap \mathcal{M}_c^\infty) \) and, by the implicit function theorem, there is an open neighbourhood \( U \) of \( g \) in \( S^*_t \cap \mathcal{M}_c^\infty \) such that all \( H^t \)-Einstein metrics in \( U \) are elements of \( (P \circ E|S^*_t \cap \mathcal{M}_c^\infty)^{-1}(0) \cap U \). Here, since the operator \( \beta \) is elliptic, \( (P \circ E|S^*_t \cap \mathcal{M}_c^\infty)^{-1}(0) \cap U \) becomes a finite dimensional submanifold of \( S^*_t \cap \mathcal{M}_c^\infty \). If we apply the condition to the map \( E|(P \circ E|S^*_t \cap \mathcal{M}_c^\infty)^{-1}(0) \cap U \), then the result is obvious.

Q.E.D.

**Lemma 4.7.** Let \( g \in \mathcal{M}_c^\infty \) be an Einstein metric and \( h \) an essential Einstein \( i \)-deformation. Then \( h \) is integrable up to second order if and only if \( E''(h, h) \) is orthogonal to EID.

**Proof.** By the definition, \( h \) is integrable up to second order if and only if
Remark that the formulae $\gamma E = 0$ (by the Bianchi identity) and $\int E = 0$ on $\mathcal{M}$ hold. By differentiating the formulae, we get that

$$
\gamma'_{(h,h)}(E(g)) + 2\gamma'_{(h)}(E'_{(h)}) + \gamma E''_{(h,h)} = 0,
$$

$$
\int_{(h,h)}'(E(g)) + 2\int_{(h)}'(E'_{(h)}) + \int E''_{(h,h)} = 0
$$

for all $g \in \mathcal{M}$ and $h \in H(S^2 M)$. Therefore the assumption of $g$ and $h$ implies that $\gamma E''_{(h,h)} = 0$ and $\int E''_{(h,h)} = 0$, i.e., $E''_{(h,h)} \in \text{Ker} \gamma \cap \text{Ker} \int$. Thus by Proposition 3.2, the result is obvious. Q.E.D.

5. Classification of essential Einstein $\mathfrak{i}$-deformations

In this section and the following, we use the representation theory of Lie groups. For fundamental data concerning root systems of simple Lie algebras (resp. of irreducible symmetric pairs), see Bourbaki [4; Planche I-IX] (resp. Murakami [11]).

First we show some facts concerning a compact semi-simple Lie group $G$ and $G$-modules. Modules are all taken to be complex modules, unless otherwise stated. Let $\mathfrak{g}$ be the Lie algebra of $G$ with a $G$-invariant inner product $B$ and $\mathfrak{a}$ a Cartan subalgebra of $\mathfrak{g}$ with a linear order. We denote by $2\delta_{\mathfrak{g}}$ the sum of all positive roots of $\mathfrak{g}^c$ and by $V(\lambda)$ the irreducible $G$-module with highest weight $\lambda$. Then the Casimir operator on $V(\lambda)$ coincides with the scalar operator $e(V(\lambda)) = B(\lambda + 2\delta_{\mathfrak{g}}, \lambda)$. If $G$ is decomposed into $\Pi_i G_i$ where $G_i$ are simple groups, we denote by $\mathfrak{g}_i$ the Lie algebra of $G_i$ and $\mathfrak{b}_i$ the restriction of $B$ on $\mathfrak{g}_i$. An irreducible $G$-module $V$ has the form $\otimes_i V_i$ where each $V_i$ is an irreducible $G_i$-module or a trivial $G_i$-module $C$. Then we see $e(V) = \sum e(V_i)$. Assume that all $\mathfrak{b}_i$ satisfy $e(V_i) = 2\epsilon$. By an easy computation, we can check

**Lemma 5.1.** Let $G$ be a compact simple Lie group. Then for any irreducible $G$-module, $e(V) > (2/3)\epsilon$ holds.

By this lemma, we can classify irreducible $G$-modules $V$ such that $e(V) = 2\epsilon$, for a semi-simple Lie group $G$. Assume that $V$ has the form $\otimes V_i$ and that each $V_i$ is not trivial. Then the equality $e(V) = 2\epsilon$ implies that $G$ has at most two simple factors. For the case that $G$ is simple, we can check

**Lemma 5.2.** Let $G$ be a compact simple Lie group and $V$ an irreducible $G$-module. If $e(V) = 2\epsilon$, then $V$ is isomorphic to $\mathfrak{g}^c$.

For the case where $G$ has two simple factors, we list all pairs of irreducible $G_i$-modules $V_i (i = 1, 2)$ such that $e(V_1) + e(V_2) = 2\epsilon$ and $e(V_1) \leq e(V_2)$. In the following table, $\omega_i$ means the highest weight of $V_1$ and $V_2$. 

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
</tr>
</tbody>
</table>

The following table, $\omega_i$ means the highest weight of $V_1$ and $V_2$. 

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
</tr>
</tbody>
</table>
Table 5.3.

<table>
<thead>
<tr>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1/\Lambda_2$</td>
<td>$\omega_i/B_{2l}^{l+1}$ (l$\geq$1)</td>
</tr>
<tr>
<td>$\omega_1/\Lambda_{2l-1}$</td>
<td>$\omega_i/D_{2l}^{l}$ (l$\geq$2)</td>
</tr>
<tr>
<td>$\omega_i/C_i$</td>
<td>$\omega_i/D_{l+2}$ (l$\geq$1)</td>
</tr>
<tr>
<td>$\omega_i/B_2$</td>
<td>$\omega_i/B_3$</td>
</tr>
<tr>
<td>$\omega_i/C_3$</td>
<td>$\omega_i/D_2$</td>
</tr>
<tr>
<td>$\omega_i/G_2$</td>
<td>$\omega_i/G_2$</td>
</tr>
</tbody>
</table>

Next, we show some facts concerning a simply connected irreducible symmetric space $G/K$ of compact type. Let $\mathfrak{f}$ be the Lie algebra of $K$ and $\mathfrak{g}=\mathfrak{f}+\mathfrak{m}$ the canonical decomposition. We compute the dimension of $\text{Hom}_K(\mathfrak{g}^C, S_0^2(\mathfrak{m}^C))$, where $S_0^2$ means the traceless part of the symmetric tensor product. If $G/K$ is of group type, then $\mathfrak{g}^C=\mathfrak{t}^C+\mathfrak{m}^C$, $\mathfrak{t}^C=\mathfrak{t}^C$ as $K$-modules. So we have to compute $\text{dim}_C \text{Hom}_K(\mathfrak{t}^C, S_0^2(\mathfrak{t}^C))$, where $K$ is a compact simple Lie group.

**Lemma 5.4.** If $K$ is not of type $\Lambda_l$ (l$\geq$2), then $\text{dim}_C \text{Hom}_K(\mathfrak{t}^C, S_0^2(\mathfrak{t}^C))=0$. If $K$ is of type $\Lambda_l$ (l$\geq$2), then $\text{dim}_C \text{Hom}_K(\mathfrak{t}^C, S_0^2(\mathfrak{t}^C))=1$.

If $G/K$ is not of group type, we can check

**Lemma 5.5.** The dimension of $\text{Hom}_K(\mathfrak{g}^C, S_0^2(\mathfrak{m}^C))$ is (H1) two if $(G,K)=(SU(p+q), S(U(p)\times U(q)))$ [AIII] (p$\geq$q$\geq$2), (H2) zero if $(G,K)=(SU(2), S(U(1)\times U(1)))$ [S$^2$], (H3) one if $(G,K)$ is of another hermitian type, (N1) one if $(G,K)=(SU(l), SO(l))$ [AI] (l$\geq$3), $(SU(2l), Sp(l))$ [AII] (l$\geq$3) or $(E_6, F_4)$ [EIV] and (N2) zero if $(G,K)$ is of another non-hermitian type.

Now, we come back to our Einstein manifold $(M,g)$ and assume that $(M,g)$ is a simply connected symmetric space $G/K$. The tangent space $T_0M$ of $M$ at the origin is identified with $\mathfrak{m}$ and the metric $g$ is induced by a $G$-invariant inner product $B$ on $\mathfrak{g}$.

Generally, for a finite dimensional $K$-module $U$, a cross section $s$ of the homogeneous vector bundle $G\times_k U$ over $M$ may be identified with a $U$-valued function $s$ on $G$ such that $s(xy)=y^{-1}s(x)$ for all $x\in G$ and $y\in K$. Let $C^\omega(G,U)_K$ be the space of all such $s$ and enlarge this space to $H^0(G,U)_K$. Then $C^\omega(G,U)_K$ and $H^0(G,U)_K$ canonically become $G$-modules and $H^0(G,U)_K$ is decomposed into $\bigoplus_i V_i$ as Hilbert space, where $V_i$ are irreducible $G$-modules contained in $C^\omega(G,U)_K$. Let $V$ be an irreducible $G$-module and denote by $W$ the direct sum of all irreducible $G$-modules $V_i$ which are isomorphic to $V$. Then we see, by the Frobenius reciprocity theorem (cf. Wallach [12; Theorem 8.2]), that

\[
\dim W = \dim \text{Hom}_C(V,C^\omega(G,U)_K) = \dim V \cdot \dim \text{Hom}_K(V,U).
\]

Lemma 5.6 (Koiso [8; Proposition 5.3]). The Lichnerowicz operator $\Delta$ regarded as an endomorphism of $C^\infty(G, \otimes^3 m^c)_K$ coincides with the Casimir operator.

Let $M = \prod_{a} M_a$ be the irreducible decomposition of the symmetric space $M$. Remark that all $(M_a, g_a)$ are Einstein manifolds with the same Einstein constant $\varepsilon$. Let $(G_a, K_a)$ be the symmetric pair of each $M_a$, $g_a$ (resp. $\mathfrak{f}_a$) the Lie algebra of $G_a$ (resp. $K_a$) and $g_a = \mathfrak{f}_a + m_a$ the canonical decomposition. Since $\ker\delta^* \subset \ker(\Delta - 2\varepsilon)$, Lemma 5.6 implies that $e(g_a^\circ) = 2\varepsilon$. Therefore we see, combining Proposition 3.3, that

$$(5.6.1) \quad \dim_{R EID} = \sum_{a} \dim_{\mathfrak{c}} V_a \cdot \dim_{\mathfrak{c}} \text{Hom}_K(V_a, S_0^2(m^c))$$

$- \sum_{a} \dim_{\mathfrak{c}} V_a \cdot \dim_{\mathfrak{c}} \text{Hom}_K(V_a, m^c) + \dim_{\mathfrak{c}} g^c,$

where $V_a$ runs through the set of all equivalence classes of irreducible $G$-modules whose Casimir operators are $2\varepsilon$. Let

$V^a = \mathcal{C}^a \oplus \bigoplus_{i=1}^n V_i^a$

be the irreducible decomposition of $V^a$ as $K$-module. Each $V_i^a$ has the form

$\bigotimes_{s \in \mathfrak{f}_a} V_i^a_s$,

where $\mathfrak{f}_a$ is a subset of $\{b \in \mathbb{Z}; 1 \leq b \leq N\}$ and $V_i^a_s$ are irreducible $K_a$-modules. Then we see that

$$\text{Hom}_K(V^a, S_0^2(m^c)) = \text{Hom}_K(V^a, \bigoplus_{s \in \mathfrak{f}_a} S_0^2(m^c_s) \oplus m_0 \otimes m_0 + C^{N-1})$$

$$= \bigoplus_{s \in \mathfrak{f}_a} \text{Hom}_K(V^a, S_0^2(m^c_s)) \oplus \text{Hom}_K(V^a, m_0 \otimes m_0) \oplus \text{Hom}_K(V^a, C).$$

Here, by Frobenius reciprocity, if $\text{Hom}_K(V^a, m^c_0 \otimes m^c_0)$ does not vanish, then there is a non-zero 2-tensor field $h$ on $M$ such that $\Delta h = 2\varepsilon h$ and $h \in T(M_a)^c \otimes T(M_a)^c$ at each point of $M$. Then $\Delta h = -2Lh = 0$ and so $h$ is parallel. But a parallel symmetric 2-tensor field is a linear combination of the metrics $g_a$ on $M_a$. Therefore

$$\text{Hom}_K(V^a, m^c_0 \otimes m^c_0) = 0 \quad \text{for} \quad a \neq b.$$ 

Thus

$$\text{Hom}_K(V^a, S_0^2(m^c)) = \bigoplus_{i=1}^n \text{Hom}_K(V_i^a, S_0^2(m^c_s)) \bigoplus \text{Hom}_K(C^a, S_0^2(m^c_s))$$

$$\bigoplus \bigoplus \text{Hom}_K(V_i^a, C) \bigoplus \text{Hom}_K(C^a, C).$$

If $\text{Hom}_K(C, S_0^2(m^c)) \neq 0$, then there is a $G$-invariant symmetric 2-tensor field $h$ such that $h \in S_0^2(M_a)^c$ at each point. Since there is no such $h$, ...
\[ \text{Hom}_K(C^a, S_0^2(m^c)) = 0. \]

Thus

\[ \text{Hom}_K(V^a, S_0^2(m^c)) = \bigoplus_{i=1}^{n_a} \text{Hom}_K(\bigotimes_{j=1}^N V_{t_i}^j, S_0^2(m^c)) \bigoplus_{\sigma=1}^{\nu_c(N-1)} \text{Hom}_K(C, C) \]

\[ = \bigoplus_{a, i : [T_i] = [\sigma]} \text{Hom}_K(V^a, S_0^2(m^c)) \bigoplus C^a(N-1). \]

Moreover,

\[ \text{Hom}_K(V^a, m^c) = \text{Hom}_K(\bigoplus_{i=1}^{n_a} V_{t_i}^j, m^c) \bigoplus \text{Hom}_K(C^a, \bigoplus_{a=1}^{N} m^c). \]

Here, since there is no parallel 1-tensor field on \( M \), \( \text{Hom}_K(C, \bigoplus_{a=1}^{N} m^c) = 0 \). Therefore,

\[ \text{Hom}_K(V^a, m^c) = \bigoplus_{i=1}^{n_a} \text{Hom}_K(V_{t_i}^j, m^c) \bigoplus \text{Hom}_K(C^a, m^c). \]

Thus we see

\[ \text{dim EID} = \sum_a N(V^a) \cdot \text{dim } V^a, \]

where

\[ N(V^a) = \sum_{a, i : [T_i] = [\sigma]} [\text{dim}_C \text{Hom}_K_{C^a}(V_{t_i}^j, S_0^2(m^c))] - \text{dim}_C \text{Hom}_K(V_{t_i}^j, m^c) + \nu^a(N-1) + \kappa^a, \]

and \( \kappa^a = 1 \) if \( V^a \) or \( V^a \oplus V^a \) is isomorphic to some \( g^f \), \( \kappa^a = 0 \) if not. (The case \( V^a \oplus V^a \) occurs if \( M \) is of group type.)

Now, we compute \( N(V^a) \). By Lemma 5.1 and remarks following it, the number of elements of \( I^a = \bigcup_{t_i} I^a_1 \) is one or two.

Case 1: the number of elements of \( I^a \) is one. We may assume that \( I^a = \{1\} \). First we assume that \( M \) is not of group type. Then Lemma 5.2 implies that \( V^a \) is isomorphic to \( g^f \).

Case 1-H (\( M \) is hermitian). The module \( V^a \) is decomposed into \( \mathfrak{t}^c \oplus m^f \otimes m^f \oplus C \) as \( K_1 \)-module, where \( \mathfrak{t}^c \) is the semisimple part of \( \mathfrak{t}^c \), \( m^f \) is the \( \pm \sqrt{-1} \)-eigenspace of \( m^f \) with respect to the almost complex structure of \( M \). Then \( \dim \text{Hom}_K(V^a, m^f) = 2, \nu^a = 1, \kappa^a = 1 \). Therefore,

\[ N(V^a) = \dim \text{Hom}_K(g^f, S_0^2(m^f)) + N - 2. \]

Combining with Lemma 5.5 (H), we see that

\[ N(V^a) = N \text{ if } M \text{ is of type AIII } (p \geq q \geq 2), \]
\[ N(V^a) = N - 2 \text{ if } M \text{ is } S^2, \]
\[ N(V^a) = N - 1 \text{ if } M \text{ is of another hermitian type.} \]
Case 1-N ($M_1$ is not hermitian). The module $V^*$ is irreducibly decomposed into $\mathfrak{t}_1^c \oplus m_1^c$ as $K_1$-module. Then $\dim c \text{Hom}_{K_1}(V^c, m_1^c) = 1$, $\nu^c = 0$, $\kappa^c = 1$. Therefore, 

$$N(V^*) = \dim \text{Hom}_{K_1}(g_1^c, S_0^2(m_1^c)).$$

By Lemma 5.5(N), we see that

$N(V^*) = 1$ if $M_1$ is of type AI ($l \geq 3$), AII ($l \geq 3$) or EIV,

$N(V^*) = 0$ if $M_1$ is of another non-hermitian type.

Next we assume that $M_1$ is of group type. Then Lemma 5.2 implies that $V^*$ is isomorphic to $\mathfrak{t}_1^c$ or to $W_1 \otimes W_2$ as $G_1$-module, where $W_1$ and $W_2$ are irreducible modules of simple factors of $G_1$.

Case 1-G ($V^*$ is isomorphic to $\mathfrak{t}_1^c$). The modules $V^c$, $m_1^c$ and $\mathfrak{t}_1^c$ are isomorphic to each other as $K_1$-modules. Then $\dim \text{Hom}^c_{K_1}(V^c, m_1^c) = 1$, $\nu^c = 0$ and $\kappa^c = 1$. Therefore,

$$N(V^*) = \dim \text{Hom}^c_{K_1}(\mathfrak{t}_1^c, S_0^2(\mathfrak{t}_1^c)).$$

By Lemma 5.4, we see that

$N(V^*) = 1$ if $M_1$ is $SU(l)$ ($l \geq 3$),

$N(V^*) = 0$ if $M_1$ is another group.

Case 1'-G ($V^*$ is isomorphic to $W_1 \otimes W_2$). Table 5.3 implies that this case occurs only if $M_1$ is the group of type $G_2$. By computing, we see that $\dim \text{Hom}_{K_1}(V^c, S_0^2(m_1^c)) = 1$, $\dim \text{Hom}_{K_1}(V^c, m_1^c) = 1$, $\nu^c = 1$ and $\kappa^c = 0$. Therefore, 

$N(V^*) = N - 1$ if $M_1$ is of type $G_2$,

$N(V^*) = 0$ if $M_1$ is another group.

Case 2: the number of elements of $I^a$ is two. We may assume that $I^a = \{1, 2\}$ and $V^a = W_1 \otimes W_2$, where $W_a$ is an irreducible $G_a$-module such that $e(W_1) \leq e(W_2)$. Then, since the first non-zero eigenvalue of $\Delta$ on $C^\infty(M_1)$ is greater than $\varepsilon$ (Lichnerowicz [9; p. 135]), $\text{Hom}_{K_1}(W_1, C^\infty(G_1, C)) = 0$ and so $\text{Hom}_{K_1}(W_1, C) = 0$. Let $W_1 = \bigoplus W_{1,i}$ and $W_2 = C^\otimes \bigoplus \bigoplus W_{2,i}$ be the irreducible decompositions as $K_1$ and $K_2$-modules. Then $V^a$ is irreducibly decomposed into 

$$\bigoplus \mu W_{1,i} \bigoplus \bigoplus W_{1,i} \otimes W_{2,j}$$

as $K_1 \times K_2$-module. Therefore, since $\nu^a = 0$ and $\kappa^a = 0$, we see that 

$$N(V^a) = \mu \cdot \left[ \dim \text{Hom}_{K_1}(W_1, S_0^2(m_1^c)) - \dim \text{Hom}_{K_1}(W_1, m_1^c) \right].$$

If $M_2$ is of group type, then $W_2$ is irreducible as $K_2$-module, and so $\mu = 0$, which implies that $N(V^a) = 0$. Let $G_2$ and $W_2$ be in the list of $V_2$ in Table 5.3 and assume that $(G_2, K_2)$ is a symmetric pair. We can check that if $\text{Hom}_{K_2}(W_2, C) = 0$, then $G_2/K_2$ is the standard sphere, i.e., of type $B$ or $D$, and $W_2 = V(\omega_1)$. On the other hand, if $G_1$ is of type $A_1$ and $W_1 = V(\omega_1)$ or $V(\omega_2)$, or $G_1$ is of type $C_1$ and $W_1 = V(\omega_1)$, then we can check that there is no symmetric pair $(G_1, K_1)$ such that $\text{Hom}_{K_1}(W_1, S_0^2(m_1^c)) = 0$ or $\text{Hom}_{K_1}(W_1, m_1^c) = 0$. Moreover if
$M_1$ is of group type, we easily see that the $K_1$-module $W_1$ does not admit zero as weight and $S^3(m_1)$ and $m_1^*$ admits zero as weight, and so Hom$_{K_1}(W_1, S^3(m_1^*))=0$ and Hom$_{K_1}(W_1, m_1^*)=0$. Thus in this case we see that $N(V^*_1)=0$.

Let $M, M_a$ and $G_a$ be as above. Assume that $M_1$ is a hermitian space or the group of type $G_2$. Then there is a unique irreducible $G_1$-module $V_1$ such that $e(V_1)=2\varepsilon$ and Hom$_{K_1}(V_1, C)=0$. Moreover dim Hom$_{K_1}(V_1, C)=1$. Therefore $2\varepsilon$ is an eigenvalue of $\Delta$ on $C^\infty(M_1)$ and the corresponding eigenspace $F$ becomes an irreducible real $G_1$-module. Let $g_a$ be the metric on each $M_a$ and $f_a \in F$ and set

$$h = \text{Hess} f_1 + \varepsilon \sum_{a=1}^N f_a \cdot g_a.$$ 

Then,

$$\Delta h = \text{Hess} \Delta f_1 + \varepsilon \sum_{a=1}^N 2\varepsilon f_a \cdot g_a = 2\varepsilon h,$$

$$\delta h = \delta(\text{Hess} f_1 + \varepsilon f_1 \cdot g_1) + \varepsilon \sum_{a=1}^N \delta(f_a \cdot g_a) = 0,$$

$$\text{tr } h = -\Delta f_1 + \varepsilon \sum_{a=1}^N n_a f_a = -2\varepsilon f_1 + \varepsilon \sum_{a=2}^N n_a f_a,$$

where $n_a = \dim M_a$. If $\sum_{a=1}^N n_a f_a - 2f_1 = 0$, then $h \in \text{EID}(M)$. Remark that if $M_1 = S^2$, then Hess $f_1 + \varepsilon f_1 \cdot g_1 = 0$. Since EID($M_1) \subset \text{EID}(M)$, we get the following

**Theorem 5.7.** Let $(M, g)$ be a compact simply connected symmetric Einstein

<table>
<thead>
<tr>
<th>type</th>
<th>$V_1$</th>
<th>$N_1$</th>
<th>form of $h \in W_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(p+q)/S(U(p) \times U(q)) \ (p \geq q \geq 2)$</td>
<td>$\mathfrak{su}(p+q)C$</td>
<td>$N$</td>
<td>$h_0 + \text{Hess} f_1 + \varepsilon \sum_{a=1}^N f_a g_a$ (partitioned)</td>
</tr>
<tr>
<td>$S^2$</td>
<td>$\mathfrak{su}(2)C$</td>
<td>$N-2$</td>
<td>$\sum_{a=2}^N f_a g_a$ (partitioned)</td>
</tr>
<tr>
<td>other hermitian</td>
<td>$\mathfrak{g}_l^C$</td>
<td>$N-1$</td>
<td>$\text{Hess} f_1 + \varepsilon \sum_{a=1}^N f_a g_a$ (partitioned)</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$V(\omega_3) \otimes V(\omega_3)$</td>
<td>$N-1$</td>
<td>$\text{Hess} f_1 + \varepsilon \sum_{a=1}^N f_a g_a$ (partitioned)</td>
</tr>
<tr>
<td>$SU(l) \ (l \geq 3)$</td>
<td>$\mathfrak{su}(l)C$</td>
<td>$2$</td>
<td>$h_0^{(*)}$</td>
</tr>
<tr>
<td>$SU(l)/SO(l) \ (l \geq 3)$</td>
<td>$\mathfrak{su}(l)C$</td>
<td>$1$</td>
<td>$h_0$</td>
</tr>
<tr>
<td>$SU(2l)/Sp(l) \ (l \geq 3)$</td>
<td>$\mathfrak{su}(2l)C$</td>
<td>$1$</td>
<td>$h_0$</td>
</tr>
<tr>
<td>$E_6/F_4$</td>
<td>$\mathfrak{e}_6$</td>
<td>$1$</td>
<td>$h_0$</td>
</tr>
<tr>
<td>other type</td>
<td></td>
<td></td>
<td>$0$</td>
</tr>
</tbody>
</table>

(*) decomposes into right invariant form and left invariant form
manifold, \((M,g) = \Pi_{a=1}^{N_a} (M_a, g_a)\) its irreducible decomposition as symmetric space \((\dim M_a = n_a)\) and \((G_a, K_a)\) the symmetric pair attached to \(M_a\). Then \(EID(M)\) becomes a real \(G\)-module and is decomposed into \(\bigoplus W_a\) where each \(W_a^\epsilon\) is a \(G_a\)-module (which may be 0). Each \(W_a^\epsilon\) is the direct sum of \(N_a\) copies of an irreducible \(G_a\)-module \(V_a\) \((N_a\) may be 0). The \(G_a\)-module \(V_a, N_a\) and the form of elements of \(W_a\) are listed above \(\langle\text{we may assume that } a=1\rangle\). There \(h_0\) means an element of \(EID(M_1)\) \((\subset \text{EID}(M))\), \(f_\epsilon\) eigenfunctions of \(\Delta\) on \(C^\infty(M_1)\) with eigenvalues \(2\epsilon\).

**Corollary 5.8.** Let \((M, g)\) be a locally symmetric Einstein manifold of compact type and \(\Pi_{a=1}^{N_a} M_a\) be the irreducible decomposition of the universal riemannian covering manifold \(\bar{M}\) of \(M\). If \(N=1\) and \(\bar{M}\) is neither \(SU(p+q)/SU(p) \times SU(q)\) \((p \geq q \geq 2)\), \(SU(l)/SO(l)\) \((l \geq 3)\), \(SU(2l)/Sp(l)\) \((l \geq 3)\), \(E_6/F_4\) nor \(SU(l)\) \((l \geq 2)\), then \(g\) is rigid. If \(N=2\) and \(M_a\) are neither one of the above, the group of type \(G_2\) nor any hermitian space except \(S^2\), then \(g\) is rigid. If \(N \geq 3\) and \(M_a\) are neither one of the above nor \(S^2\), then \(g\) is rigid.

**Proof.** It is obvious that infinitesimal non-deformability of an Einstein metric reduces to that of its riemannian covering. So Proposition 2.5 implies this result.

**Q.E.D.**

### 6. Second order Einstein \(i\)-deformation on symmetric spaces

Let \((M, g)\) be a compact simply connected symmetric space \(G/K\) where \(g\) is an Einstein metric with Einstein constant \(\epsilon\). Let \(M = \Pi_{a=1}^{N_a} M_a\) be its irreducible decomposition and \((G_a, K_a)\) the symmetric pair of \(M_a\). By Theorem 5.7, \(EID(M) = \bigoplus_{a=1}^{N_a} W_a\) where each \(W_a\) is a real \(G_a\)-module (which may be 0). By Lemma 4.7, if we denote by \(\psi(h_1, h_2)\) the \(EID\)-component of \(E''(h_1, h_2)\) for \(h_1, h_2 \in \text{EID}\), then \(h\) is integrable up to second order if and only if \(\psi(h, h) = 0\). We easily see that \(\psi\) is a \(G\)-homomorphism. Therefore we get

**Lemma 6.1.** In the above situation, if \(\text{Hom}_G(S^2(\bigoplus W_a), \oplus W_a) = 0\), then all essential Einstein \(i\)-deformations are integrable up to second order.

\[
\text{Hom}_G(S^2(\bigoplus W_a), \oplus W_a) = \text{Hom}_G(\bigoplus S^2(W_a) \oplus \bigoplus W_a \otimes W_b, \oplus W_c)
\]

\[
= \bigoplus_{a,b} \text{Hom}_G(S^2(W_a), W_b) \oplus \bigoplus_{a,b} \text{Hom}_G(W_a \otimes W_b, W_c).
\]

Since each \(W_a\) has no trivial component as \(G_a\)-module, the last form equals to \(\bigoplus_{a=1}^{N_a} \text{Hom}_G(S^2(W_a), W_a)\). Thus the integrability of \(h \in \text{EID}(M)\) up to second order reduces to the integrability of its components in each \(W_a\).

If \(M_1\) is \(E_6/F_4\), then by Theorem 5.7, \(W_1\) is isomorphic to \(g_1\) and Lemma 5.4 implies that \(\text{Hom}_G(S^2(W_1), W_1) = 0\).

Let \(M_1\) be the group of type \(G_2\) or a hermitian space except \(A_{III} (p \geq q \geq 2)\).
and denote by $F$ the $2\varepsilon$-eigenspace of $\Delta$ on $C^\infty(M)$. Then by Theorem 5.7, an element $h$ of $W_1$ has the form

$$h_1 + \sum_{a=2}^N f_a \cdot g_a,$$

where $h_1 \in C^\infty(S^2M)$ and $f_a \in F$. We calculate $E''(h,h)$.

$$h^k_1 \nabla_1 \nabla j h_{kl} = (h_1)^k_1 \nabla_1 \nabla j (h_1)_{kl} + \sum_{a=2}^N (h_1, \text{Hess } f_a) \cdot g_a,$$

$$\nabla^k h_{ij} \cdot \nabla j h_i^j = \nabla^k (h_1)_{ij} \cdot \nabla j (h_1)^j_i + \sum_{a=2}^N (df_a, df_a) \cdot g_a,$$

$$\nabla^k h_{ij} \cdot \nabla^l h_{jk} = \nabla^k (h_1)_{ij} \cdot \nabla^l (h_1)_{jk},$$

$$R_i^\lambda j h_{km} h_i^m = R_i^\lambda j (h_1)_{km} (h_1)^m_i - \varepsilon \sum_{a=2}^N (f_a)^2 \cdot g_a,$$

$$h^k_1 \nabla_1 \nabla j h_{kl} = (h_1)^k_1 \nabla_1 \nabla j (h_1)_{kl},$$

$$\nabla_i h^k_1 \cdot \nabla_j h_{kl} = \nabla_i (h_1)^k_1 \cdot \nabla_j (h_1)_{kl} + \sum_{a=2}^N n_a \nabla_i f_a \cdot \nabla_j f_a,$$

$$R_i^\lambda h_{km} h_j^m = R_i^\lambda h_{km} (h_1)^m_j - \varepsilon \sum_{a=2}^N (f_a)^2 \cdot g_a,$$

$$\text{Hess}(h,h) = \text{Hess}(h_1,h_1) + 2 \sum_{a=2}^N n_a df_a \otimes df_a + 2 \sum_{a=2}^N n_a f_a \cdot \text{Hess } f_a,$$

and so $2E''(h,h)$

$$= 2E''(h_1,h_1) + 2 \sum_{a=2}^N (h_1, \text{Hess } f_a) \cdot g_a$$

$$+ 2 \sum_{a=2}^N (df_a, df_a) \cdot g_a + 2 \sum_{a=2}^N n_a df_a \otimes df_a + 2 \sum_{a=2}^N n_a f_a \cdot \text{Hess } f_a.$$

Let $h' = h_1 + \sum_{a=2}^N 2^a f_a \cdot g_a \in W_1$. Then

$$\langle E''(h,h), h' \rangle = \langle E''(h_1,h_1), h_1' \rangle$$

$$+ \sum_{a=2}^N n_a \langle df_a \otimes df_a + 2 f_a \cdot \text{Hess } f_a, h_1' \rangle$$

$$+ 2 \sum_{a=2}^N n_a \langle (h_1, \text{Hess } f_a) + (df_a, df_a), f_a \rangle.$$

Assume that $M_1$ is not of type AIII $(p+q \geq 3)$. Then we can set $h_1 = \text{Hess } f + \varepsilon f \cdot g_1$ and $h_1' = \text{Hess } f' + \varepsilon f' \cdot g_1$, where $f, f' \in F$. Moreover, by Lemma 5.4, $\text{Hom}_{G_1}(S^2(g_1), g_1) = 0$ holds. Therefore

$$\langle E''(h_1,h_1), h_1' \rangle = 0,$$

$$\langle df_a \otimes df_a + 2 f_a \cdot \text{Hess } f_a, h_1' \rangle = 0,$$

$$\langle (df_a, df_a), f_a \rangle = 0,$$

$$\langle (h_1, \text{Hess } f_a), f_a \rangle = \langle (\text{Hess } f, \text{Hess } f_a), f_a \rangle - \varepsilon \langle f \cdot f_a, f_a \rangle = 0,$$

which implies that $\psi(h,h) = 0$ for $h \in W_1$.

**Theorem 6.2.** Let $(M,g)$ be a compact simply connected symmetric Einstein
manifold. If all irreducible factors of $M$ are neither $SU(p+q)/S(U(p) \times U(q))$ ($p+q \geq 3$), $SU(l) 
mid l \geq 3$, $SU(l)/SO(l)$ ($l \geq 3$) nor $SU(2l)/Sp(l)$ ($l \geq 3$), then all essential Einstein $\iota$-deformations are integrable up to second order.

Now, we treat the case where $M_i = P^l(C)$ ($l \geq 2$). For $f, f' \in F$, we decompose $f \cdot f'$ into eigenfunctions of $\Delta$ and denote by $\varphi(f, f')$ the $F$-component. The map $\varphi$ becomes a real $SU(l+1)$-homomorphism: $S^2(F) \to F$.

**Lemma 6.3.** Let $\varphi$ and $F$ be as above. Then $\varphi = 0$. Moreover, if $l$ is even, $\varphi(f, f) \neq 0$ for all non-zero $f \in F$.

**Proof.** Let $S_{2l+1} \subset C^{l+1}$ be the unit sphere. Then $U(1) = \{w \in C; |w| = 1\}$ acts on $S_{2l+1}$ and $C^{l+1}$ by $w(z) = wz$ and $S_{2l+1}/U(1)$ becomes the projective space $P^l(C)$. The spectrum of $\Delta$ on $C^\infty(P^l(C))$ is given by $\{2m(l+m)e((l+1); m \in \mathbb{Z}, m \geq 0\}$. Denote by $F^m$ the eigenspace with eigenvalue $2m(l+m)e((l+1)$ and $H^m(C^{l+1})$ the space of all homogeneous harmonic polynomials of degree $2m$ on $C^{l+1}$ which are invariant under the action of $U(1)$. If $f \in F^m$, then $f$ is extended canonically to an element $\bar{f} \in H^m(C^{l+1})$. This correspondence $\sim$ is an $SU(l+1)$-isomorphism (cf. Berger, Gauduchon and Mazet [3; pp. 172–173]). Let $f \in F$. Since $F$ is isomorphic to $\mathfrak{su}(l+1)$ as a real $SU(l+1)$-module, we may assume that $f$ is an element of the subspace of $F$ which corresponds to a Cartan subalgebra of $\mathfrak{su}(l+1)$. That is,

$$ f(z) = \sum_{i=1}^{l+1} a_i |z^i|^2; a_i \in \mathbb{R}, \sum_{i=1}^{l+1} a_i = 0. $$

Set $\Delta' = \Delta/4$ on $C^{l+1}$. Then $\Delta' = \sum_{i=1}^{l+1} (\partial^i/\partial z^i \partial \bar{z}^i)$. 

$$ \Delta' \bar{f}^2 = \Delta' \sum_{i=1}^{l+1} a_i^2 |z^i|^2 + \Delta' \sum_{i \neq j} a_i a_j |z^i|^2 |z^j|^2 $$

$$ = 4 \sum_{i=1}^{l+1} a_i^2 |z^i|^2 + 2 \sum_{i \neq j} a_i a_j |z^i|^2 $$

$$ = 2 \sum_{i=1}^{l+1} a_i^2 |z^i|^2, $$

and,

$$ \Delta' \left( \sum_{i} b_i |z^i|^2 \sum_{i} |z^i|^2 \right) = \Delta' \sum_{i} b_i |z^i|^2 + \Delta' \sum_{i \neq j} b_i |z^i|^2 |z^j|^2 $$

$$ = 4 \sum_{i} b_i |z^i|^2 + \sum_{i} b_i (|z^i|^2 + |z^i|^2) $$

$$ = \sum_{i} ((l+3)b_i + \sum_{j} b_j) |z^i|^2. $$

Therefore,

$$ f^2 = \frac{1}{l+3} \sum_{i} (2a_i^2 - \frac{1}{l+2} \sum_{j} a_j^2) |z^i|^2 \sum_{i} |z^i|^2 \in H^2(C^{l+1}), $$

and

$$ \varphi(f, f) = \frac{2}{l+3} \sum_{i} (a_i^2 - \frac{1}{l+1} \sum_{j} a_j^2) |z^i|^2. $$
Thus $\psi(f,f) = 0$ if and only if $|a_i|$ is independent of $i$. Q.E.D.

**Lemma 6.4.** Let $\psi'$ be any real $SU(l+1)$-homomorphism: $S^p(F) \to F$. If $\langle \psi'(f,f), f \rangle = c\langle \psi(f,f), f \rangle$ for all $f \in F$, then $\psi' = c\psi$.

Proof. That is easy to see by Lemma 5.4 and the fact that $F$ is isomorphic to $\mathfrak{su}(l+1)$ as real $SU(l+1)$-module. Q.E.D.

**Lemma 6.5.** The Lichnerowicz operator $\Delta$ commutes with the covariant derivative $\nabla$ on a locally symmetric space.

Proof. The operators $\Delta$ and $\nabla$ may be regarded as the Casimir operator (Lemma 5.6) and a $G$-homomorphism, respectively. Q.E.D.

Denote by $D^p f$ the $p$-tensor field defined by

$$(D^p f)_{i_1 \cdots i_p} = \nabla_{i_1} \cdots \nabla_{i_p} f .$$

**Lemma 6.6.** Let $N$ be a locally symmetric Einstein manifold with Einstein constant $\varepsilon$. If $f \in C^\infty(N)$ satisfies $\Delta f = 2\varepsilon f$, then

\begin{align*}
(6.6.1) & \quad \langle D^{p+1} f, df \otimes D^p f \rangle = \varepsilon \langle (D^p f, D^p f), f \rangle , \\
(6.6.2) & \quad \langle (D^{p+1} f, D^{p+1} f), f \rangle \\
& \quad = (1-p)\varepsilon \langle (D^p f, D^p f), f \rangle - 2\langle (LD^p f, D^p f), f \rangle .
\end{align*}

Proof.\[6.6.1\] \begin{align*}
\langle \nabla_i \nabla_{i_1} \cdots \nabla_{i_p} f, \nabla_i f \cdot \nabla_{i_1} \cdots \nabla_{i_p} f \rangle \\
& \quad = -\langle \nabla_{i_1} \cdots \nabla_{i_p} f, \nabla_i \nabla_{i_1} \cdots \nabla_{i_p} f + \nabla_i f \cdot \nabla_{i_1} \cdots \nabla_{i_p} f \rangle \\
& \quad = \langle D^p f, \Delta f \cdot D^p f \rangle - \langle df \otimes D^p f, D^{p+1} f \rangle .
\end{align*}

\begin{align*}
[6.6.2] & \quad \langle \nabla_i \nabla_{i_1} \cdots \nabla_{i_p} f, \nabla_i \nabla_{i_1} \cdots \nabla_{i_p} f \rangle \\
& \quad = -\langle \nabla_{i_1} \cdots \nabla_{i_p} f, \nabla^i f \cdot \nabla_i \nabla_{i_1} \cdots \nabla_{i_p} f + \nabla^i f \cdot \nabla_i \nabla_{i_1} \cdots \nabla_{i_p} f \rangle \\
& \quad = -\langle df \otimes D^p f, D^{p+1} f \rangle + \langle D^p f, f \Delta D^p f \rangle ,
\end{align*}

and so

\begin{align*}
\langle (D^{p+1} f, D^{p+1} f), f \rangle \\
& \quad = -\varepsilon \langle (D^p f, D^p f), f \rangle + \langle f D^p f, (\Delta - 2L - pQ) D^p f \rangle \quad (6.6.1) \\
& \quad = -\varepsilon \langle (D^p f, D^p f), f \rangle + \langle f D^p f, D^p \Delta f \rangle \\
& \quad - 2\varepsilon \langle (D^p f, f L D^p f), f \rangle - p\varepsilon \langle f D^p f, D^p f \rangle . \quad (6.5)
\end{align*}

Q.E.D.

**Lemma 6.7.** If $f \in C^\infty(P'(C))$ satisfies $\Delta f = 2\varepsilon f$, then

\begin{align*}
(6.7.1) & \quad L \text{Hess } f = -c(\text{Hess } f - \varepsilon f \cdot g) , \\
(6.7.2) & \quad R_{i i}^j R_{j i}^{i t} \nabla_i f = 2\varepsilon (\text{Hess } f - \varepsilon f \cdot g)_{i i} ,
\end{align*}

where $2c$ is the holomorphic sectional curvature.
Proof. Denote by $z^α$, $z^β$ etc. holomorphic coordinate functions. Since $\nabla f$ is a holomorphic vector field, $\nabla_α\nabla_β f = 0$. We know that the curvature tensor has the form
\[
R^γ_αβ_\delta = c(δ^γ_αδ^β_\delta + δ^β_αδ^γ_\delta)
\]
(cf. Calabi and Vesentini [5; (3.5)]). Therefore,

\[
(L \text{ Hess } f)_{αβ} = R^α_βγδ_\delta \nabla_γ \nabla_δ f = 0,
\]

\[
(L \text{ Hess } f)^*_{αβ} = R^α_βγδ_\delta \nabla^*_γ \nabla^*_δ f = -R^α_βγδ_\delta \nabla_γ \nabla_δ f
\]

\[
= -c(δ^α_βδ^γ_δ + δ^γ_αδ^β_δ)\nabla_γ \nabla_δ f
\]

\[
= -c(\nabla_α \nabla_β f + \nabla^*_α \nabla^*_β f \cdot δ^α_β)
\]

\[
= -c(\text{Hess } f + ε f \cdot g)^*_{αβ}.
\]

And if we set $φ_{ij} = R_{ii}^i R_{jj}^j \nabla_k \nabla_i f$, then

\[
(6.7.2)
\]

\[
φ_{αβ} = R^α_βγδ_δ \nabla_γ \nabla_δ f = 0,
\]

\[
φ^*_{αβ} = R^α_βγδ_δ \nabla^*_γ \nabla^*_δ f = R^α_βγδ_δ \nabla_γ \nabla_δ f
\]

\[
= c(δ^α_β δ^γ_δ + δ^γ_α δ^β_δ)(δ^δ_α δ^γ_α + δ^α_β δ^δ_β)\nabla_γ \nabla_δ f
\]

\[
= 2c(δ^α_β δ^γ_δ + δ^γ_α δ^β_δ)\nabla_γ \nabla_δ f
\]

\[
= 2cR^α_βγδ_δ \nabla_γ \nabla_δ f
\]

\[
= -2c(L \text{ Hess } f)^*_{αβ}.
\]

Q.E.D.

Lemma 6.8. Let $f$ and $c$ be as above. Then

\[
(6.8.1) \quad \langle df, df \rangle = ε\langle f^2, f \rangle,
\]

\[
(6.8.2) \quad \langle \text{Hess } f, \text{Hess } f \rangle = 0,
\]

\[
(6.8.3) \quad \langle \text{Hess } f, df \otimes df \rangle = ε\langle f^2, f \rangle,
\]

\[
(6.8.4) \quad \langle D^2 f, df \otimes \text{Hess } f \rangle = 0,
\]

\[
(6.8.5) \quad \langle (D^2 f, D^2 f), f \rangle = 4cε^2\langle f^2, f \rangle,
\]

\[
(6.8.6) \quad \langle (L \text{ Hess } f, \text{Hess } f), f \rangle = -2cε^2\langle f^2, f \rangle,
\]

\[
(6.8.7) \quad \langle D^4 f, df \otimes D^2 f \rangle = 4cε^3\langle f^2, f \rangle,
\]

\[
(6.8.8) \quad \langle L \text{ Hess } f, df \otimes df \rangle = 0,
\]

\[
(6.8.9) \quad \langle D^3 f, df \otimes L \text{ Hess } f \rangle = -2cε^3\langle f^2, f \rangle,
\]

\[
(6.8.10) \quad \langle \nabla_i \nabla_j f \cdot \nabla^i \nabla_k f, \nabla_i \nabla_j f \rangle = ε\langle f^2, f \rangle,
\]

\[
(6.8.11) \quad Δdf = εdf,
\]

\[
(6.8.12) \quad Δ\text{Hess } f = 2c(\text{Hess } f - ε f \cdot g)
\]
Proof. Except (6.8.10), that is easy to show by Lemma 6.6 and Lemma 6.7.

\[ 6.8.10 \]
\[ \langle \nabla \nabla_i f \cdot \nabla_i \nabla_j f, \nabla \nabla_k f \rangle \]
\[ = -\langle \nabla \nabla_i f \cdot \nabla_i \nabla_k f + \nabla_i \nabla_j f \cdot \nabla_i \nabla_k f, \nabla i f \rangle \]
\[ = \langle \Delta f \otimes df, \text{Hess } f \rangle - \langle \nabla \nabla_i \nabla_k f, \nabla_i f \cdot \nabla_i f \rangle \]
\[ = \varepsilon \langle df \otimes df, \text{Hess } f \rangle - \langle R_{ikj} \nabla_i f + \nabla_k \nabla_i f, \nabla_i f \cdot \nabla_i f \rangle \]
\[ = \varepsilon \langle f^2, f \rangle + \langle L \text{ Hess } f, df \otimes df \rangle - \langle D^3_f, df \otimes \text{Hess } f \rangle \]
\[ = \varepsilon \langle f^2, f \rangle . \]

Q.E.D.

**Lemma 6.9.** Let \( f \) and \( c \) be as above. Then

\[ 6.9.1 \]
\[ \langle \nabla \nabla_i \nabla_j \nabla_k f, \nabla \nabla_i f \cdot \nabla \nabla_j f \rangle = -2c \varepsilon \langle f^2, f \rangle , \]

\[ 6.9.2 \]
\[ \langle \nabla \nabla_i \nabla_j \nabla_k f, \nabla \nabla_i f \cdot \nabla \nabla_j f \rangle = -c \varepsilon \langle f^2, f \rangle . \]

Proof.

\[ 6.9.1 \]
\[ \langle \nabla \nabla_i \nabla_j \nabla_k f, \nabla \nabla_i f \cdot \nabla \nabla_j f \rangle \]
\[ = -\langle \nabla \nabla_i \nabla_j f \cdot \nabla \nabla_i \nabla_j f + \nabla \nabla_i f \cdot \nabla \nabla_j f, \nabla \nabla_i f \rangle \]
\[ = \langle D^2 f, \Delta f \rangle \]
\[ + \langle R_{ij} \nabla \nabla_i f + \nabla_j \nabla \nabla_i f, \nabla \nabla_i f \rangle \]
\[ = \varepsilon \langle D^3 f, df \otimes \text{Hess } f \rangle \]
\[ = 2 \langle R_{ij} \nabla \nabla_i f, \nabla \nabla_i f \rangle + \langle D^3 f, df \otimes D^3 f \rangle \]
\[ - 2c \varepsilon \langle f^2, f \rangle \]
\[ = 2 \varepsilon \langle f^2, f \rangle + 4c \varepsilon \langle f^2, f \rangle \]
\[ = 2 \varepsilon \langle f^2, f \rangle . \]

\[ 6.8.11 \]
= -2\epsilon \langle df \otimes df, L \operatorname{Hess} f \rangle + 2\epsilon \langle D^2 f, df \otimes \operatorname{Hess} f \rangle - 2\epsilon \delta \langle f^2, f \rangle \tag{6.8.4}

= 2\epsilon \langle df \otimes df, \operatorname{Hess} f - \epsilon f \cdot g \rangle - 2\epsilon \delta \langle f^2, f \rangle \tag{6.8.3}

= 2\epsilon \delta \langle f^2, f \rangle - 2\epsilon \delta \langle (df, df), f \rangle - 2\epsilon \delta \langle f^2, f \rangle \tag{6.8.1}

= -2\epsilon \delta \langle f^2, f \rangle .

\[ [6.9.2] \]
\[ \langle \nabla_i \nabla_k \nabla_j f, \nabla_i \nabla_j f \cdot \nabla_k \nabla_i f \rangle \]
\[ = \langle \nabla_i (R_{ij} - 2c \delta f \delta f), \nabla_k \nabla_j f \cdot \nabla_i \nabla_k \nabla_i f \rangle \]
\[ = \langle R_{ij}, \nabla_k \nabla_j f \cdot \nabla_i \nabla_k \nabla_i f \rangle + \langle \nabla_i \nabla_k \nabla_j f, \nabla_i \nabla_j f \cdot \nabla_k \nabla_i f \rangle \]
\[ = -2\epsilon \langle f^2, f \rangle \]
\[ = -2\epsilon \delta \langle f^2, f \rangle . \]

Q.E.D.

Now, we come back to our symmetric space \((M, g)\) where \(M = \text{P}^1(C)\) \((r \geq 2)\) (below Theorem 6.2). We assume that \(N \geq 2\). Set \(h = \psi + \phi; \psi = \operatorname{Hess} f + \epsilon f \cdot g, \phi = \epsilon f^* \cdot g^*\), where \(f, f^* \in F\). Remark that \(\delta \psi = 0\). In the following calculation, we use Lemma 6.4, Lemma 6.8 and Lemma 6.9. If \(h = 0\), then \(h \in \text{EID}(M)\) and

\[ 2\epsilon \langle E''(h, h), h \rangle \]
\[ = 2\epsilon \langle h, h^* h \rangle + 3\epsilon \langle \nabla_i \nabla_j h, h_i h_{kl} \rangle - 6\epsilon \langle \nabla_i \nabla_j h, h_i h_{kl} \rangle \]
\[ = 2\epsilon \langle \psi, \nabla_i \nabla_j \psi \rangle + 2\epsilon \langle \phi, \nabla_i \nabla_j \phi \rangle + 3\epsilon \langle \nabla_i \nabla_j \psi, \nabla_i \nabla_j \psi \rangle \]
\[ + 3\epsilon \langle \nabla_i \nabla_j \phi, \nabla_i \nabla_j \phi \rangle - 6\epsilon \langle \nabla_i \nabla_j \psi, \nabla_i \nabla_j \psi \rangle . \]

Here,
\[ \langle \psi, \nabla_i \nabla_j \psi \rangle \]
\[ = \langle \nabla_i \nabla_j f, \nabla_i \nabla_k f \cdot \nabla_k \nabla_j f \rangle + 3\epsilon \langle \nabla_i \nabla_j f, f \cdot \nabla_i \nabla_j f \rangle \]
\[ + 3\epsilon \delta \langle \nabla_i \nabla_j f, f^2(g_{ij}) \rangle + \epsilon \delta \langle f \cdot (g_{ij})_j, f^2(g_{ij}) \rangle \]
\[ = \epsilon \langle f^2, f \rangle - 6\epsilon \langle f^2, f \rangle + n\epsilon \delta \langle f^2, f \rangle \]
\[ = (n - 5)\epsilon \delta \langle f^2, f \rangle , \]
\[ \langle \phi, \nabla_i \nabla_j \psi \rangle \]
\[ = \langle \phi, \nabla_i \nabla_j (g_2), (g_2)^2 \rangle \]
\[ = n_\epsilon \delta \langle (f^2)^2, f^2 \rangle , \]
\[ \langle \nabla_i \nabla_j \psi, \nabla_i \nabla_j \psi \rangle \]
\[ = \langle \nabla_i \nabla_j \psi, \nabla_i \nabla_j f \cdot \nabla_k \nabla_j f \rangle + \epsilon \langle \nabla_i \nabla_j \psi, \nabla_i \nabla_j f \cdot (g_{ij})_j \rangle \]
\[ + \epsilon \langle \nabla_i \nabla_j \psi, (g_{ij})_j \rangle + \epsilon \langle \nabla_i \nabla_j f, \nabla_i \nabla_j f \cdot \nabla_k \nabla_j f \rangle \]
\[ + \epsilon \langle \nabla_i \nabla_j \psi, \nabla_i \nabla_j f \rangle - \epsilon \langle A \psi, f \cdot \operatorname{Hess} f \rangle - \epsilon \delta \langle \Delta \psi, f^2 \rangle \]
\[ = -2\epsilon \delta \langle f^2, f \rangle - 2\epsilon \langle \operatorname{Hess} f, f \cdot \operatorname{Hess} f \rangle \]
\[ +(n - 2)\epsilon \langle \operatorname{Hess} f, f \cdot \operatorname{Hess} f \rangle \]
Thus, $<E''(h,h),h> = -2(n_1-2)\varepsilon <f^3,f> + n_2\varepsilon <(f')^3,f'$.}

Since $f' = -(n_1-2)/n_2) f$, we have

$$<E''(h,h),h> = -\frac{(n_1-2) (n_1+n_2-2) (n_1+2n_2-2)}{n_2^2} \varepsilon <f^3,f>.$$}

Therefore, by Lemma 6.4, we get

**Lemma 6.10.** Let $h$ be as above and $h''$ have the same form defined by $f''$. Then $<E''(h,h),h'> = r \cdot <f^3,f''>'$, where $r$ is a non-zero constant.

**Theorem 6.11.** Let $P^i(C) \times M' (l \geq 2)$ be a symmetric Einstein manifold. Then there exists an essential Einstein i-deformation which is not integrable.

**Proof.** That is easy to see by Proposition 4.5, Lemma 4.7, Lemma 6.3 and Lemma 6.10. Q.E.D.

Moreover, we have the following

**Theorem 6.12.** There exist rigid Einstein metrics which are infinitesimally deformable.
Proof. For example, let $M$ be $P^{2l}(C) \times S^2$. Then, by Theorem 5.7, all elements $h \in EID(M)$ have the form introduced above Lemma 6.10. Thus Proposition 4.6, Lemma 6.3 and Lemma 6.10 complete the proof. Q.E.D.

References