Title	Killing tensor fields on the standard sphere and spectra of SO(n+1)/(SO(n-1)×SO(2) and O(n+1)/O(n-1)×O(2)
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KILLING TENSOR FIELDS ON THE STANDARD SPHERE AND SPECTRA OF $SO(n+1)/(SO(n-1)\times SO(2)$ AND $O(n+1)/O(n-1)\times O(2)$

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0. Introduction. The principal purpose of the present paper is to exhibit the eigen-space decomposition of the Laplacian of the Grassmann manifolds $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})=SO(n+1)/SO(n-1)\times SO(2)$ and $\boldsymbol{G}_{2,n-1}(\boldsymbol{R})=O(n+1)/O(n-1)\times O(2)$ with their canonical Riemannian metrics respectively and to clarify the relation between the eigen-space decompositions above and of the Lichnerowicz operator Δ on the standard sphere (S^n, g_0), restricted to the graded algebras $\boldsymbol{K}^*(S^n, g_0)$ of symmetric tensor fields on S^n, generated by Killing vector fields.

In **1**, we obtain fundamental properties of differential operators δ^*, δ, T^*, T and the Lichnerowicz operator Δ acting on the graded algebra $\boldsymbol{S}^*(M)=\sum_{p\geqq 0}\boldsymbol{S}^p(M)$ (direct sum) of symmetric tensor fields on a Riemannian manifold (M,g). In **2**, a pseudo-connection of infinite order on M is defined as a collection of linear differential operators Γ^p: $\boldsymbol{S}^p(M)\to\mathfrak{D}^p(M)$ $(p\geqq 1)$ splitting

$$(1.3)_p \qquad 0\longrightarrow \mathfrak{D}^{p-1}(M)\xrightarrow{\iota^p}\mathfrak{D}^p(M)\xrightarrow{\sigma^p}\boldsymbol{S}^p(M)\longrightarrow 0\,,$$

viewed as the short exact sequences of $\boldsymbol{R}$-modules, where $\mathfrak{D}^p(M)$ is the module of C^∞-differential operators of order at most p on M. In virtue of the existence of a certain pseudo-connection on (S^n, g_0), $\boldsymbol{K}^*(S^n, g_0)$ is characterized as the kernel of δ^* in $\boldsymbol{S}^*(S^n)$. In **3**, the Radon-Michel transform $^\wedge$: $\boldsymbol{S}^*(S^n)\to C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$, is defined by

$$(3.1) \qquad \xi^\wedge(\Gamma)=\frac{1}{2\pi}\int_{\gamma=\iota(\Gamma)}\langle\xi, (\dot\gamma)^p/p!\rangle ds\,,$$

where ι maps an oriented 2-plane $\Gamma\in\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ to the geodesic $\gamma=\Gamma\cap S^n$ on (S^n, g_0) with the orientation induced from Γ. The key theorem is Theorem 3.1; $\Delta^\wedge\xi^\wedge=(\Delta\xi)^\wedge$, where $\Delta^\wedge$ is the Laplacian on $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ with the canonical Riemannian metric, which was first proved by R. Michel [9] for $\boldsymbol{S}^2(S^n)$. In **4**, a linear differential operator S of degree -2 is defined, with the aid of which we obtain the eigen-space decomposition of the Lichnerowicz operator Δ restricted to $\boldsymbol{K}^*(S^n, g_0)$. This eigen-space decomposition of the Lichnerowicz operator yields the one of the Laplacian $\Delta^\wedge$ on $C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ via the Radon-Michel transform.

In the appendix a linear differential operator D_k^* of order k is introduced so that an eigen-function f of the Laplacian of (S^n, g_0) for the eigen-value $k(n+k-1)$ satisfies the differential equation

$$D_k^* f = 0 .$$

1. Fundamental operators. Let (M, g) be an oriented C^∞-Riemannian manifold of dimension n. Let $\mathrm{x} \in M$. The space of p-jets of C^∞-mapping $f\colon U_f \to \boldsymbol{R}$ at $\mathrm{x} \in U_f \subset M$ with U_f open is denoted by $T_{\mathrm{x}}^{*(p)}(M)$ and the union $T^{*(p)}(M) = \bigcup_{\mathrm{x} \in M} T_{\mathrm{x}}^{*(p)}$ forms a vector bundle over M. The dual vector bundle of $T^{*(p)}(M)$ is denoted by $T^{(p)}(M)$. As is well known the sequence of vector bundles

$$(1.1)_p \qquad 0 \longrightarrow T^{(p-1)}(M) \xrightarrow{\iota^p} T^{(p)}(M) \xrightarrow{\sigma^p} S^p T(M) \longrightarrow 0$$

is exact, where ι^p, σ^p and $S^p T(M)$ are the canonical inclusion, the symbol map and the symmetric tensor product of order p of the tangent bundle $T(M)$ of M, respectively. We mean by an (ascending) *filtered Lie algebra* ([4], p. iii) a Lie algebra L with an ascending chain of subspaces

$$\cdots \subset L^{p-1} \subset L^p \subset L^{p+1} \subset \cdots$$

satisfying $[L^p, L^q] \subset L^{p+q}$ and $L = \bigcup_{p \in \boldsymbol{Z}} L^p$. *The associated graded Lie algebra* $G(L)$ [4] with an filtered Lie algebra L is the Lie algebra

$$G(L) = \sum_{p \in \boldsymbol{Z}} L^p / L^{p-1} \qquad \text{(direct sum)}$$

with the *bracket* $[X \bmod L^{p-1}, Y \bmod L^{q-1}] = [X, Y] \bmod L^{p+q-1}$ for $X \in L^p$ and $Y \in L^q$. A C^∞-section D of $T^{(p)}(M)$ is called *a linear differential operator of order p*. It is written as

$$(1.2) \qquad D = \sum_{k=0}^{p} (1/k!) \xi^{i_1 \cdots i_k} \partial^k / \partial x^{i_1} \cdots \partial x^{i_k}$$

with respect to a local coordinate system $x^1, \cdots, x^n$. The $C^\infty(M)$-module of linear differential operators of order p on M is denoted by $\mathfrak{D}^p(M)$. We put $\mathfrak{D}(M) = \bigcup_{p \geq 0} \mathfrak{D}^p(M)$. As $[\mathfrak{D}^p(M), \mathfrak{D}^q(M)] \subset \mathfrak{D}^{p+q-1}(M)$ for the *bracket product* $[D_1, D_2] = D_1 D_2 - D_2 D_1$, $\mathfrak{D}(M)$ is a filtered Lie algebra if we put $L^p = \mathfrak{D}^{p+1}(M)$ $(p \geqq -1)$, $L^p = \{0\}$ $(p \leqq -2)$. From $(1.1)_p$ follows the exact sequence

$$(1.3)_p \qquad 0 \longrightarrow \mathfrak{D}^{p-1}(M) \xrightarrow{\iota^p} \mathfrak{D}^p(M) \xrightarrow{\sigma^p} S^p(M) \longrightarrow 0$$

of $C^\infty(M)$-modules, where the $C^\infty(M)$-module of symmetric tensor fields of degree p on M is denoted by $S^p(M)$. Put $S^*(M)=\sum_{p\geqq 0} S^p(M)$ (direct sum). The *symmetric tensor product* of $\xi\in S^p(M)$ and $\eta\in S^q(M)$ is defined as the symmetrization of the tensor product $\xi\otimes\eta$ which we denote by $\xi\circ\eta$. For latter use we give an interpretation in terms of the symbol maps;

$$(1.4) \qquad \xi\circ\eta=\sigma^{p+q}(D_\xi\cdot D_\eta),$$

where $D_\xi\in\mathfrak{D}^p(M)$ and $D_\eta\in\mathfrak{D}^q(M)$ are such that $\sigma^p(D_\xi)=\xi$ and $\sigma^q(D_\eta)=\eta$. With respect to a local coordinate system $\{x^1, \cdots, x^n\}$, $\xi\in S^p(M)$ is expressed as

$$(1.5) \qquad \xi=(1/p!)\xi^{i_1\cdots i_p}\partial/\partial x^{i_1}\circ\cdots\circ\partial/\partial x^{i_p},$$

where $\xi^{i_1\cdots i_p}$ will be called the *components* of ξ. The symmetric tensor product of $\xi\in S^p(M)$ and $\eta\in S^q(M)$ expeessed as in (1.5) is written as

$$\xi\circ\eta=\frac{1}{(p+q)!}(\xi\circ\eta)^{i_1\cdots i_{p+q}}\partial/\partial x^{i_1}\cdots\partial/\partial x^{i_{p+q}},$$

where

$$(1.6) \qquad (\xi\circ\eta)^{i_1\cdots i_{p+q}}=\sum_{\pi\in\mathfrak{S}_{p+q}}\frac{\xi^{i_{\pi(1)}\cdots i_{\pi(p)}}\eta^{i_{\pi(p+1)}\cdots i_{\pi(p+q)}}}{p!\,q!}$$

and $\mathfrak{S}_{p+q}$ is the symmetric group of degree $p+q$. We define the *bracket product* on $S^*(M)$ by

$$(1.7) \qquad [\xi, \eta]=\sigma^{p+q}[D_\xi, D_\eta],$$

where D_ξ and D_η are as in (1.4). Notice that $[\xi, \eta]\in S^{p+q-1}(M)$. $S^*(M)$ is identified with the associated graded algebra of the filtered Lie algebra $\mathfrak{D}(M)$. The componentwise expression of (1.7) is given by

$$(1.8) \qquad [\xi, \eta]^{i_1\cdots i_{p+q-1}}=\frac{1}{(p-1)!\,q!}\sum_{\pi\in\mathfrak{S}_{p+q-1}}\xi^{i_{\pi(1)}\cdots i_{\pi(p-1)}h}\cdot\partial\eta^{i_{\pi(p)}\cdots i_{\pi(p+q-1)}}/\partial x^h-\frac{1}{p!(q-1)!}\sum_{\pi\in\mathfrak{S}_{p+q-1}}\eta^{i_{\pi(1)}\cdots i_{\pi(q-1)}h}\cdot\partial\xi^{i_{\pi(q)}\cdots i_{\pi(p+q-1)}}/\partial x^h$$

for $\xi\in S^p(M)$ and $\eta\in S^q(M)$.

Assume from now on that M be compact. $S^p(M)$ is equipped with the positive definite inner product

$$(1.9) \qquad (\xi, \eta)=p!\int_M\langle\xi, \eta\rangle d\sigma \qquad \xi, \eta\in S^p(M),$$

where

$$\langle \xi, \eta \rangle = g_{i_1 j_1} \cdots g_{i_p j_p} \xi^{i_1 \cdots i_p} \eta^{j_1 \cdots j_p}$$

and $d\sigma$ is the volume element of (M, g). Let $g = g^{ij} \partial/\partial x^i \circ \partial/\partial x^j$ be the *contravariant Riemannian metric* of (M, g). We define a linear operator T^*: $S^*(M) \to S^*(M)$ of degree 2 by

$$T^* \xi = (1/2)\, g \circ \xi \in S^{p+2}(M) \tag{1.10}$$

for $\xi \in S^p(M)$. Let T be the adjoint operator of T^* with respect to the inner product (1.9). T is of degree -2. The components of $T^*\xi$ and $T\xi$ are given by

$$(T^*\xi)^{i_1 \cdots i_{p+2}} = \sum_{1 \leq h < k \leq p+2} g^{i_h i_k} \xi^{i_1 \cdots \hat{i}_h \cdots \hat{i}_k \cdots i_{p+2}} \tag{1.11}$$

$$(T\xi)^{i_1 \cdots i_{p-2}} = (1/2) g_{ab} \xi^{ab i_1 \cdots i_{p-2}}. \tag{1.12}$$

Lemma 1.1. (i) $[T, T^*] = ((n/2) + p) \mathbf{1}_p$
(ii) $[T^m, T^*] = m((n/2) + p - m + 1) T^{m-1}$

on $S^p(M)$, where $\mathbf{1}_p$ is the identity operator of $S^p(M)$.

Proof. From (1.11) and (1.12)

$$\begin{aligned} (TT^*\xi)^{i_1 \cdots i_p} &= (1/2) g_{ab} \sum_{1 \leq h < k \leq p} g^{i_k j_h} \xi^{i_1 \cdots \hat{i}_h \cdots \hat{i}_k \cdots i_p ab} \\ &\quad + (1/2) g_{ab} \sum_{1 \leq h \leq p} 2 g^{i_h a} \xi^{i_1 \cdots \hat{i}_h \cdots i_p b} + (1/2) \xi^{i_1 \cdots i_p} g^{ab} g_{ab} \\ &= (T^*T\xi)^{i_1 \cdots i_p} + ((n/2) + p) \xi^{i_1 \cdots i_p}. \end{aligned}$$

Thus (i) is obtained. (ii) follows from (i) by induction on m. Q.E.D.

Define δ^*: $S^*(M) \to S^*(M)$ by

$$\delta^* \xi = (1/2)\, [g, \xi]. \tag{1.13}$$

δ^* is a linear differential operator of degree 1. Define δ: $S^*(M) \to S^*(M)$ as the adjoint operator of δ^* with respect to the inner product (1.9). The componentwise expression of δ^* and δ are given by

$$\text{(1.14)} \quad \text{(i)} \quad (\delta^*\xi)^{i_1 \cdots i_{p+1}} = \sum_{h=1}^{p+1} g^{i_h a} \nabla_a \xi^{i_1 \cdots \hat{i}_h \cdots i_{p+1}}$$

$$\text{(ii)} \quad (\delta\xi)^{i_1 \cdots i_{p-1}} = -\nabla_a \xi^{i_1 \cdots i_{p-1} a},$$

where ∇ is the Riemannian connection on (M, g).

Lemma 1.2. (i) $[T, \delta] = 0$ (i)* $[T^*, \delta^*] = 0$
(ii) $[\delta^*, T] = \delta$ (ii)* $[T^*, \delta] = \delta^*$.

Proof. From (1.12) and (1.14) (i) is immediately obtained. (ii) is also obtained from (1.12) and (1.14) by direct calculations. (i)* and (ii)* follow from (i) and (ii), respectively. Q.E.D.

Lemma 1.3. *δ^* is a derivation on the associative algebra $S^*(M)$.*

Proof. From (1.4) we have

$$[g, \xi\circ\eta] = -\sigma^{p+q+1}[\Delta, D_\xi \cdot D_\eta] = -\sigma^{p+q+1}([\Delta, D_\xi]D_\eta + D_\xi[\Delta, D_\eta])$$
$$= [g, \xi]\circ\eta + \xi\circ[g, \eta],$$

where $\Delta = -g^{ij}\nabla_i\nabla_j$ is the Laplacian of (M, g) and D_ξ's are as in (1.7). Q.E.D.

We define self-adjoint linear differential operators $\square$, $\bar{\Delta}$, and Δ on $S^*(M)$ by

(1.15) (i) $\square = [\delta, \delta^*]$ (ii) $\bar{\Delta} = -\nabla^a\nabla_a$
(iii) $\Delta = 2\bar{\Delta} - \square$.

$\bar{\Delta}$ and Δ are called the *rough Laplacian* and the *Lichnerowicz operator* on (M, g), respectively [7]. The componentwise expression of $\square$ and Δ for ξ as in (1.5) are

(1.16) (i) $(\square\xi)^{i_1\cdots i_p} = (\bar{\Delta}\xi)^{i_1\cdots i_p} - (\kappa\xi)^{i_1\cdots i_p}$
(ii) $(\Delta\xi)^{i_1\cdots i_p} = (\bar{\Delta}\xi)^{i_1\cdots i_p} + (\kappa\xi)^{i_1\cdots i_p}$,

where $(\bar{\Delta}\xi)^{i_1\cdots i_p} = -\nabla^a\nabla_a\xi^{i_1\cdots j_p}$ and $(\kappa\xi)^{i_1\cdots i_p} = \sum_{h=1}^{p} R^{i_h}{}_a\xi^{ai_1\cdots \hat{i}_h\cdots i_p} - 2\sum_{1\leq h<k\leq p} R^{i_h}{}_{ab}{}^{i_k}\xi^{abi_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_p}$ and R^i_{jkh} (resp. R_{ij}) are the components of the curvature tensor (resp. the Ricci tensor) of (M, g).

Lemma 1.4 (i) $[\square, T] = 0$ (i)* $[\square, T^*] = 0$
(ii) $[\Delta, T] = 0$ (ii)* $[\Delta, T^*] = 0$.

Proof. Since $\square$ and Δ are self-adjoint, it suffices to prove (i) and (ii). By Lemma 1.2

$$[\square, T] = [\delta\delta^*, T] - [\delta^*\delta, T] = (\delta[\delta^*, T] + \delta T\delta^* - T\delta\delta^*)$$
$$-\delta^*\delta T + [T, \delta^*]\delta + \delta^* T\delta = 0$$

which proves (i). (ii) follows from (i) by virtue of $[\bar{\Delta}, T] = 0$. Q.E.D.

Lemma 1.5. *Let (M, g) be locally symmetric.*

(i) $[\Delta, \delta] = 0$ (i)* $[\Delta, \delta^*] = 0$.

Proof. For $\xi \in S^p(M)$ as in (1.5)

$$([\delta, \bar{\Delta}]\xi)^{i_1\cdots i_{p-1}} = [\nabla_b, \nabla_a]\nabla\xi^{abi_1\cdots i_{p-1}} - \nabla^a[\nabla_a, \nabla_b]\xi^{bi_1\cdots i_{p-1}}$$
$$= \nabla^a(R_{ab}\xi^{bi_1\cdots i_{p-1}} - 2\sum_{k=1}^{p-1} R_{abc}{}^{i_k}\xi^{bci_1\cdots \hat{i}_k\cdots i_{p-1}}).$$

On the other hand, we have

$$([\delta, \kappa]\xi)^{i_1\cdots i_{p-1}} = -\nabla^a(R_{ab}\xi^{bi_1\cdots i_{p-1}} - 2\sum_{k=1}^{p-1} R_{abc}{}^{i_k}\xi^{bci_1\cdots \hat{i}_k\cdots i_{p-1}}).$$

Adding these equalities, we obtain (i). (i)* follows from (i) directly. Q.E.D.

The Lichnerowicz operator Δ in a locally symmetric Riemannian manifold is regarded as a generalization of the Laplacian Δ. We denote them by the same notation, because the former acts on $S^*(M)$ while the latter is the restriction of the former on $S^0(M) = C^\infty(M)$.

An element of $(\boldsymbol{Ker}\ \boldsymbol{\delta}^*)(M, g)$ is called a *Killing tensor field.* $\boldsymbol{Ker}\ \boldsymbol{\delta}^*$ (M, g) is the graded subalgebra of $S^*(M)$ by virtue of Lemma 1.3. The graded subalgebra of $S^*(M)$ generated by $(\boldsymbol{Ker}\ \boldsymbol{\delta}^*)(M, g) \cap S^1(M)$ is denoted as $\boldsymbol{K}^*(M, g) = \sum_{p\geqq 0} \boldsymbol{K}^p(M, g)$ (direct sum). Obviously $(\boldsymbol{Ker}\ \boldsymbol{\delta}^*)(M, g) \supset \boldsymbol{K}^*(M, g)$. We are interested in (M, g) for which the equality holds (cf. Theorem 2.3).

2. Pseudo-connections. A splitting γ^p of the sequence $(1.3)_p$ of $C^\infty(M)$-modules is called an *affine connection of order p*. An infinite set $\{\gamma^p\}_{p\geqq 1}$ is called an *affine connection of infinite order*.

EXAMPLE 1. Let ∇' be an affine connection on M. Then

$$(2.1) \qquad \gamma_0^p((1/p!)\xi^{i_1\cdots i_p}\partial/\partial x^{i_1}\circ\cdots\circ\partial/\partial x^{i_p}) = (1/p!)\xi^{i_1\cdots i_p}\nabla'_{i_1}\cdots\nabla'_{i_p}$$

defines an affine connection γ_0^p of order p on M.

We define a *pseudo-connection of order p* by a linear differential operator Γ^p: $S^p(M) \to \mathfrak{D}^p(M)$ which is a splitting of $(1.3)_p$ as an exact sequences of $\boldsymbol{R}$-modules. A set $\Gamma = \{\Gamma^p\}_{p\geqq 1}$ of pseudo-connections Γ^p of order p is called a *pseudo-connection of infinite order* on M or simply a *pseudo-connection* for the sake of brevity. Γ^p is called *self-adjoint* if

$$(2.2) \qquad \Gamma^p(\xi)^* = (-1)^p\Gamma^p(\xi)$$

for any $\xi \in S^p(M)$. A pseudo-connection Γ is called *self-adjoint* if Γ^p is self-adjoint for each $p \geqq 1$.

EXAMPLE 2. Let ∇' be as in Example 1. Put

$$\Gamma_{\nabla'}^p(\xi) = \sum_{k=0}^{p} \frac{1}{2^p p!}\binom{p}{k}\nabla'_{i_1}\cdots\nabla'_{i_k}(\xi^{i_1\cdots i_p}\nabla'_{i_{k+1}}\cdots\nabla'_{i_p}),$$

where $\binom{p}{k}$'s are the binomial coefficients. $\Gamma_{\nabla'}=\{\Gamma^p_{\nabla'}\}$ is a self-adjoint pseudo-connection on M. Notice that $\Gamma_{\nabla'}(\xi)$ can also be expressed as

$$\Gamma^p_{\nabla'}(\xi)=\sum_{k=0}^{p}\frac{(-1)^{p-k}}{k!}\frac{(\delta^{p-k}\xi)^{i_1\cdots i_k}}{2^{p-k}(p-k)!}\nabla'_{i_1}\cdots\nabla'_{i_k}\,. \tag{2.3}$$

We write formally as $\Gamma_{\nabla'}=\gamma_0\cdot\exp(-\delta/2)$. A pseudo-connection $\Gamma=\{\Gamma^p\}$ is called an *extension* of an affine connection ∇' if $\Gamma^p=\Gamma^p_{\nabla'}$ for $p\geqq 1$. Given a pseudo-connection Γ, we have an isomorphism $S^*(M)\to\mathfrak{D}(M)$ as $\boldsymbol{R}$-modules, which we can not expect to be an isomorphism of Lie algebras. However, we might expect the formula:

$$[\Gamma(\xi),\Gamma(\eta)]=\Gamma([\xi,\eta])$$

for a certain fixed $\xi\in S^p(M)$. This situation leads us to the following

PROBLEM, Does there exist a self-adjoint pseudo-connection of infinite order extending the Riemannian connection ∇ on (M,g), letting both of the diagrams

$$(2.4)_p\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{D}^{p-1}(M) & \xrightarrow{\iota^p} & \mathfrak{D}^p(M) & \underset{\Gamma^p}{\overset{\sigma^p}{\rightleftarrows}} & S^p(M) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle [\rho,\ ]} & & \downarrow{\scriptstyle [\rho,\ ]} & & \downarrow{\scriptstyle L_\rho} & & \\ 0 & \longrightarrow & \mathfrak{D}^{p-1}(M) & \xrightarrow{\iota^p} & \mathfrak{D}^p(M) & \underset{\Gamma^p}{\overset{\sigma^p}{\rightleftarrows}} & S^p(M) & \longrightarrow & 0 \end{array}$$

(L_ρ: Lie derivative by $\rho\in\boldsymbol{K}^1(M,g)$) and

$$(2.5)_p\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{D}^{p-1}(M) & \xrightarrow{\iota^p} & \mathfrak{D}^p(M) & \underset{\Gamma^p}{\overset{\sigma^p}{\rightleftarrows}} & S^p(M) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle -[\Delta/2,\ ]} & & \downarrow{\scriptstyle -[\Delta/2,\ ]} & & \downarrow{\scriptstyle \delta^*} & & \\ 0 & \longrightarrow & \mathfrak{D}^{p}(M) & \xrightarrow{\iota^{p+1}} & \mathfrak{D}^{p+1}(M) & \underset{\Gamma^{p+1}}{\overset{\sigma^{p+1}}{\rightleftarrows}} & S^{p+1}(M) & \longrightarrow & 0 \end{array}$$

commutative?

Theorem 2.1. *Let (M, g) be locally flat. For the pseudo-connection Γ^p_∇ in Example 2 with respect to the Riemannian connection ∇ of (M,g) (2.4) and (2.5) are commutative diagrams for $p\geqq 1$.*

Proof. It suffices to show for $\xi\in S^p(M)$

$$(2.6)\quad (\mathrm{i})\ \ [\rho,\Gamma^p(\xi)]=\Gamma^p(L_\rho\xi)\quad (\mathrm{ii})\ \ -(1/2)[\Delta,\Gamma^p_\nabla(\xi)]=\Gamma^{p+1}_\nabla(\delta^*\xi)\,.$$

(i) is a matter of straightforward calculations. For (ii) we have $[\delta^*,\delta^{l+1}]=-(l+1)\delta^l\bar{\Delta}$ $(l\geqq 1)$ ($\bar{\Delta}=\Delta$ is the Lichnerowicz operator). Hence for $\xi\in S^*(M)$,

$$(2.7)\quad \frac{1}{2}\left(\gamma_0 \exp\left(-\frac{\delta}{2}\right)\right)(\Delta\xi) = \sum_{k=0}^{p} \frac{1}{(k+1)!} \frac{(-1)^{p-k}(\delta^*\delta^{p-k}\xi)^{i_1\cdots i_{k+1}}}{(p-k)!\,2^{p-k}} \nabla_{i_1}\cdots\nabla_{i_{k+1}} - (\gamma_0 \exp(-\delta/2))(\delta^*\xi)\,.$$

On the other hand, in a locally flat space we can easily verify $[\Delta, \gamma_0^p(\xi)] = -2\gamma_0^{p+1}(\delta^*\xi)+\gamma_0^p(\Delta\xi)$ for $\xi\in \boldsymbol{S}^p(M)$. From these we obtain

$$\begin{aligned}[\Delta, \Gamma_\nabla^p(\xi)] &= -2A+\sum_{k=0}^{p}\frac{1}{k!}\frac{(-1)^{p-k}(\Delta\delta^{p-k}\xi)^{i_1\cdots i_k}}{(p-k)!\,2^{p-k}}\nabla_{i_1}\cdots\nabla_{i_k} \\ &= -2A+2A-2(\gamma_0\exp(-\delta/2))(\delta^*\xi) = -2(\Gamma_\nabla^{p+1}(\delta^*\xi))\,,\end{aligned}$$

where A is the differential operator given as the first term of the right-hand side of (2.7). Q.E.D.

Lemma 2.1 (K. Tandai, T. Sumitomo [10]). *Let M_i $(i=1,2)$ be differentiable manifolds. Then there are subalgebras $\tilde{\mathfrak{D}}(M_i)$ of $\mathfrak{D}(M_1\times M_2)$ $(i=1,2)$, canonically isomorphic to $\mathfrak{D}(M_i)$ respectively, and one of them is the centralizer of the other in $\mathfrak{D}(M_1\times M_2)$.*

Let $\iota\colon S^n\to \boldsymbol{R}^{n+1}$ be the canonical imbedding of S^n onto the unit sphere in a Euclidean space $\boldsymbol{R}^{p+1}$. Then $\tilde{\iota}\colon S^n\times\boldsymbol{R}\to\boldsymbol{R}^{n+1}-\{0\}$ defined by $(x,t)\mapsto e^t\iota(x)$ is a trivialization of the real line bundle $\boldsymbol{R}^{n+1}-\{0\}$ over S^n with the projection $\pi\colon \pi(y)=y/\langle y,y\rangle^{1/2}$. We identify $f\in C^\infty(S^n)$ with $\pi^*f\in C^\infty(\boldsymbol{R}^{n+1}-\{0\})$. By Lemma 2.1 a vector field ξ on S^n is uniquely identified with the vector field $\tilde{\xi}$ on $S^n\times\boldsymbol{R}$ such that

$$(2.8)\qquad [\tilde{\xi}, t]=0 \quad\text{and}\quad [\tilde{\xi}, \partial/\partial t]=0\,.$$

$\tilde{\xi}$ is obtained as the vector field $\tilde{\iota}_*\xi$ via the diffeomorphism $\tilde{\iota}$. The mapping defined by $\xi\mapsto\tilde{\xi}$ is a monomorphism of Lie algebras. The condition (2.8) for $\tilde{\xi}=\sum_{A=0}^{n}\tilde{\xi}^A\partial/\partial y^A\in\mathfrak{D}(\boldsymbol{R}^{n+1}-\{0\})$ is equivalent to

$$(2.9)\qquad \sum_{A=0}^{n}\tilde{\xi}^Ay^A=0 \quad\text{and}\quad \sum_{B=0}^{n}(\partial\tilde{\xi}^A/\partial y^B)y^B=\tilde{\xi}^A\,.$$

since $r\cdot\tilde{\iota}(x,t)=e^t$ and $\tilde{\iota}_*(\partial/\partial t)=\sum_{A=0}^{n}y^A\partial/\partial y^A$ $(r^2=\sum_{A=0}^{n}(y^A)^2)$. From the latter condition of (2.8) $\tilde{\xi}^A$ is a homogeneous function of degree 1 with respect to y's. Owing to Lemma 2.1 we can identify $\mathfrak{D}(S^n)$ with the subalgebra $\tilde{\mathfrak{D}}(S^n)=\{D\in\mathfrak{D}(\boldsymbol{R}^{n+1}-\{0\})\colon [D,r^2]=0,\ [D,\sum_{A=0}^{n}y^A\partial/\partial y^A]=0\}$ of $\mathfrak{D}(\boldsymbol{R}^{n+1}-\{0\})$. Every coefficient $\tilde{\xi}^{A_1\cdots A_k}$ of $D\in\tilde{\mathfrak{D}}^p(S^n)$ is a homogeneous function of degree k $(p\geqq k\geqq 0)$ with respect to the variables $y^0,\cdots,y^n$. This identification is transferred to the identification of the two algebras $S^*(S^n)$ and $\tilde{S}^*(S^n)=\tilde{\sigma}^*(\tilde{\mathfrak{D}}(S^n))$, where $\tilde{\sigma}$ is the symbol map of $\mathfrak{D}(\boldsymbol{R}^{n+1}-\{0\})$. Namely,

$$(2.10)\qquad (1/p!)\Xi^{A_1\cdots A_p}\partial/\partial y^{A_1}\circ\cdots\circ\partial/\partial y^{A_p}\in S^p(\boldsymbol{R}^{n+1}-\{0\})$$

is in $\tilde{S}^p(S^n)$ if and only if

$$(2.11)\qquad \sum_{A=0}^{n}\Xi^{A_1\cdots A_{p-1}A}y^A=0\quad\text{and}\quad\sum_{A=0}^{v}\frac{\partial\Xi^{A_1\cdots A_p}}{\partial y^A}y^A=p\Xi^{A_1\cdots A_p}.$$

The canonical identification between $\mathfrak{D}(S^n)$ and $\tilde{\mathfrak{D}}(S^n)$ (resp. $S^*(S^n)$ and $\tilde{S}^*(S^n)$) preserves their algebraic structures of associative algebras and of filtered Lie algebras (resp. of graded Lie algebras). Notice that the identification between $\mathfrak{D}(S^n)$ and $\tilde{\mathfrak{D}}(S^n)$ preserves the adjointness of differential operators. In the following, for an operator D in $\mathfrak{D}(S^n)$ the corresponding operator in $\tilde{\mathfrak{D}}(S^n)$ will be denoted by $\tilde{D}$.

Lemma 2.2 *Let* $\xi\in\tilde{S}^p(S^n)$ *expressed as in* (2.10).

(i) $\tilde{T}\xi=(1/(2r^2\cdot(p-2)!))\sum_{A,A_1\cdots A_{p-2}=0}^{n}\xi^{AAA_1\cdots A_{p-2}}\partial/\partial y^{A_1}\circ\cdots\circ\partial/\partial y^{A_{p-2}}$

(ii) $\tilde{\delta}^*\xi=(r^2/(p+1)!)\sum_{A_1\cdots A_{p+1}=0}^{n}\sum_{k=1}^{p+1}\dfrac{\partial\xi^{A_1\cdots\hat{A}_k\cdots A_{p+1}}}{\partial y^{A_k}}\partial/\partial y^{A_1}\circ\cdots\circ\partial/\partial y^{A_{p+1}}$

(iii) $$\tilde{\delta}\xi=(-1/(p+1)!)\sum_{A,A_1\cdots A_{p-1}=0}^{n}\left(\frac{\partial\xi^{AA_1\cdots A_{p-1}}}{\partial y^A}+\sum_{k=1}^{p-1}(1/r^2)y^{A_k}\xi^{A_1\cdots\hat{A}_k\cdots A_{p-1}AA}\right)\partial/\partial y^{A_1}\circ\cdots\circ\partial/\partial y^{A_{p-1}}$$

(iv) $\Delta\xi=(1/p!)\sum_{A_1,\cdots,A_p=0}^{n}\{(\tilde{\Delta}\xi)^{A_1\cdots A_p}+p(n+p-2)\xi^{A_1\cdots A_p}-4(\tilde{T}^*\tilde{T}\xi)^{A_1\cdots A_p}\}\partial/\partial y^{A_1}$ $\circ\cdots\circ\partial/\partial y^{A_p}$, *where*

$$(\tilde{\Delta}\xi)^{A_1\cdots A_p}=-\sum_{A,B=0}^{n}(r^2\delta^{AB}-y^Ay^B)\frac{\partial^2\xi^{A_1\cdots A_p}}{\partial y^A\partial y^B}+(n-1)p\xi^{A_1\cdots A_p}$$
$$-2\sum_{k=1}^{p}y^{A_k}\frac{\partial\xi^{A_1\cdots\hat{A}_k\cdots A_{p+1}}}{\partial y^{A_{p+1}}}-2\sum_{\substack{h\neq k\\ h,k=1}}^{p}(1/r^2)y^{A_h}y^{A_k}\sum_{A=0}^{n}\xi^{A_1\cdots\hat{A}_k\cdots\hat{A}_h\cdots A_pAA}.$$

Proof. (i) As the canonical contravariant Riemannian metric g_0 on S^n is given by $(r^2\delta^{AB}-y^Ay^B)\partial/\partial y^A\circ\partial/\partial y^B$ (δ^{AB}; Kroneker's, symbol) we have from (1.12) and the canonical identification between $S^*(S^n)$ and $\tilde{S}^*(S^n)$

$$(\tilde{T}\xi)^{A_1\cdots A_{p-2}}=\sum_{A,B=0}^{n}\frac{(r^2\delta^{AB}-y^Ay^B)}{2r^4}\xi^{ABA_1\cdots A_{p-2}}=\frac{1}{2r^2}\sum_{A=0}^{n}\xi^{AAA_1\cdots A_{p-2}}.$$

(ii) From the definition of δ^*,

$$\tilde{\delta}^*\tilde{\xi} = \sigma^{p+1}[-\tilde{\Delta}/2, \gamma_0^p(\tilde{\xi})],$$

where $\tilde{\Delta} = -(r^2\delta^{AB} - y^A y^B)\partial^2/\partial y^A \partial y^B + n y^A \partial/\partial y^A$ and γ_0^* is the Riemannian connection of (S^n, g_0). From (1.8) and (2, 11) we obtain

$$\begin{aligned}(\tilde{\delta}^*\tilde{\xi})^{A_1\cdots A_{p+1}} &= \sum_{\pi\in\mathfrak{S}_{p+1}} \frac{(r^2\delta^{A_{\pi(1)}B} - y^{A_{\pi(1)}}y^B)}{p!\cdot 1!} \frac{\partial\tilde{\xi}^{A_{\pi(2)}\cdots A_{\pi(p+1)}}}{\partial y^B} \\ &- \sum_{\pi\in\mathfrak{S}_{p+1}} \frac{\tilde{\xi}^{A_{\pi(1)}\cdots A_{\pi(p-1)}B}}{2!\cdot(p-1)!} \frac{\partial(r^2\delta^{A_{\pi(p)}A_{\pi(p+1)}} - y^{A_{\pi(p)}}y^{A_{\pi(p+1)}})}{\partial y^B} \\ &= r^2\sum_{k=1}^{p+1} \frac{\partial\tilde{\xi}^{A_1\cdots\hat{A}_k\cdots A_{p+1}}}{\partial y^{A_k}}.\end{aligned}$$

(iii) From (ii) and Lemma 1.2

$$\begin{aligned}(\tilde{\delta}\tilde{\xi})^{A_1\cdots A_{p-1}} &= (\tilde{\delta}^*\tilde{T}\tilde{\xi})^{A_1\cdots A_{p-1}} - (\tilde{T}\tilde{\delta}^*\tilde{\xi})^{A_1\cdots A_{p-1}} \\ &= r^2\sum_{k=1}^{p-1} \frac{\partial}{\partial y^{A_k}}\left(\frac{1}{2r^2}\tilde{\xi}^{AAA_1\cdots\hat{A}_k\cdots A_{p-1}}\right) - \frac{\delta^{A_pA_{p+1}}}{2r^2}\sum_{k=1}^{p+1} r^2 \frac{\partial\tilde{\xi}^{A_1\cdots\hat{A}_k\cdots A_{p+1}}}{\partial y^{A_k}}.\end{aligned}$$

From this we obtain the desired expression of $\tilde{\delta}\tilde{\xi}$ with the aid of (2.11) (ii).

(iv) From (ii) and (iii)

$$\begin{aligned}(\tilde{\delta}\tilde{\delta}^*\tilde{\xi})^{A_1\cdots A_p} &= -\frac{\partial}{\partial y^A}(\tilde{\delta}^*\tilde{\xi})^{AA_1\cdots A_p} - r^{-2}\sum_{A=0}^{n}\sum_{k=1}^{p} y^{A_k}(\tilde{\delta}^*\tilde{\xi})^{A_1\cdots\hat{A}_k\cdots A_pAA} \\ &= -2\sum_{A=0}^{n} y^A\sum_{k=1}^{p}\frac{\partial\tilde{\xi}^{AA_1\cdots\hat{A}_k\cdots A_p}}{\partial y^{A_k}} - r^2\sum_{A=0}^{n}\frac{\partial^2\tilde{\xi}^{A_1\cdots A_p}}{(\partial y^A)^2} - r^2\sum_{A=0}^{n}\sum_{k=1}^{p}\frac{\partial^2\tilde{\xi}^{AA_1\cdots\hat{A}_k\cdots A_p}}{\partial y^A\partial y^{A_k}} \\ &- \sum_{A=0}^{n} 2y^A\frac{\partial\tilde{\xi}^{A_1\cdots A_p}}{\partial y^A} - \sum_{A=0}^{n}\sum_{\substack{h\neq k\\ h,k=1}}^{p} y^{A_k}\frac{\partial\tilde{\xi}^{A_1\cdots\hat{A}_k\cdots\hat{A}_h\cdots A_pAA}}{\partial y^{A_h}} \\ &- 2\sum_{A=0}^{n}\sum_{k=1}^{p} y^{A_k}\frac{\partial\tilde{\xi}^{A_1\cdots\hat{A}_k\cdots A_pA}}{\partial y^A}.\end{aligned}$$

In the right-hand side of the above equality the first and the fourth terms are cancelled out. On the other hand, we have

$$\begin{aligned}(\tilde{\delta}^*\tilde{\delta}\tilde{\xi})^{A_1\cdots A_p} &= -r^2\sum_{A=0}^{n}\sum_{k=1}^{p}\frac{\partial^2\tilde{\xi}^{A_1\cdots\hat{A}_k\cdots A_pA}}{\partial y^{A_k}\partial y^A} - r^2\sum_{\substack{h,k=1\\ h\neq k}}^{p}\sum_{A=0}^{n} r^{-4}(r^2\delta^{A_hA_k} - 2y^{A_h}A_k)\tilde{\xi}^{A_1\cdots\hat{A}_h\cdots\hat{A}_k\cdots A_pAA} \\ &- \sum_{\substack{h,k=1\\ h\neq k}}^{p}\sum_{A=0}^{n} y^{A_k}\frac{\partial\tilde{\xi}^{A_1\cdots\hat{A}_h\cdots\hat{A}_k\cdots A_pAA}}{\partial y^{A_h}}\end{aligned}$$

From (1.16)

$$(\tilde{\Box}\tilde{\xi})^{A_1\cdots A_p} = ((\tilde{\delta}\tilde{\delta}^* - {}^*\tilde{\delta}\tilde{\delta})\tilde{\xi})^{A_1\cdots A_p} = -\sum_{A,B=0}^{n}(r^2\delta^{AB} - y^Ay^B)\cdot$$

$$\cdot\frac{\partial^2\xi^{A_1\cdots A_p}}{\partial y^A\partial y^B}+p(n-1)\xi^{A_1\cdots A_p}-2\sum_{k=1}^{p}\frac{\partial\xi^{AA_1\cdots\hat{A}_k\cdots A_p}}{\partial y^A}y^{A_k}$$

$$-\sum_{\substack{h,k=1\\h\neq k}}^{n}\sum_{A=0}^{n}\frac{y^{A_h}y^{A_k}}{r^2}\xi^{A_1\cdots\hat{A}_k\cdots\hat{A}_h\cdots A_pAA}-p(n+p-2)\xi^{A_1\cdots A_p}$$

$$+\sum_{\substack{h,k=1\\h\neq k}}^{p}\sum_{A=0}^{n}r^{-2}(r^2\delta^{A_hA_k}-y^{A_h}y^{A_k})\xi^{A_1\cdots\hat{A}_h\cdots\hat{A}_k\cdots A_pAA}.$$

Notice that the last term above equals $4(\tilde{T}^*\tilde{T}\xi)^{A_1\cdots A_p}$. The desired expression of $\tilde{\delta}\xi$ is obtained from (1.15) (iii). Q.E.D.

We define

$$\tilde{\delta}'\xi=\frac{-1}{(p-1)!}\sum_{A=0}^{n}\frac{\partial\xi^{AA_1\cdots A_{p-1}}}{\partial y^A}\,\partial/\partial y^{A_1}\circ\cdots\circ\partial/\partial y^{A_{p-1}}$$

which is nothing but the first term of the right-hand side of (iii) in Lemma 2.2. Let Γ'_0 be the pseudo-connection defined by

$$\Gamma_0'^p(\xi)=\sum_{k=0}^{p}(-1)^{p-k}\frac{((\tilde{\delta}')^{p-k}\xi)^{A_1\cdots A_k}}{2^{p-k}(p-k)!\,k!}\,\partial^k/\partial y^{A_1}\cdots\partial/\partial y^{A_k}, \tag{2.12}$$

where $\xi\in\tilde{S}^p(S^n)$.

Theorem 2.2. *Γ'_0 is a pseudo-connection of (S^n, g_0) making the diagrams (2.4) and (2.5) commutative.*

Proof. Let $\xi\in\tilde{S}^p(S^n)$. Then

$$[\Gamma_0'^p(\xi)\cdot r^2]=0\,. \tag{2.13}$$

which is obtained by straightforward calculations. We have also

$$[\Gamma_0'^p(\xi),\ \sum_{A=0}^{n}y^A\partial/\partial y^A]=0 \tag{2.14}$$

as an immediate consequence of the homogeneity of $\Gamma_0'^p(\xi)$. Since the Laplacian $\tilde{\Delta}_{(S^n,g_0)}$ is represented as

$$\tilde{\Delta}=\sum_{A,B=0}^{n}(y^Ay^B-r^2\delta^{AB})\partial^2/\partial y^A\partial y^B+n\sum_{A=0}^{n}y^A\partial/\partial y^A\,,$$

it follows from (2.13) and (2.14) that

$$[\tilde{\Delta}_{(S^n,g_0)},\ \Gamma_0'^p(\xi)]=-r^2[\sum_{A=0}^{n}\frac{\partial^2}{(\partial y^A)^2},\ \Gamma_0'^p(\xi)]\,.$$

Hence again from (2.13) and (2.14) we see easily that Theorem 2.2 is reduced to Theorem 2.1 for $\boldsymbol{R}^{n+1}-\{0\}$ with the flat metric $\sum_{A=0}^{n}(dy^A)^2$. Q.E.D.

Theorem 2.3. *Let $\xi \in S^p(S^n)$. The following three conditions are mutually equivalent*

(i) $\xi \in \boldsymbol{K}^p(S^n, g_0)$

(ii) $\delta^* \xi = 0$

(iii) $\xi = \sigma^p(D)$ *for some* $D \in \mathfrak{D}^p(S^n)$ *such that* $[\Delta, D] = 0$.

Proof. (i)⇒(ii) follows from Lemma 1.3. (ii)⇒(iii) is a direct consequence of Theorem 2.2. (iii)⇒(i) is proved in Theorem 1 in the previous paper [10]. Q.E.D.

3. In this section we assume $n \geqq 2$. By a *2-frame* in $\boldsymbol{R}^{n+1}$ we mean an ordered pair of two linearly independent vectors in $\boldsymbol{R}^{n+1}$. Denote by $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$, the manifold of all 2-frames in $\boldsymbol{R}^{n+4}$. Let L_n be the linear group of regular $n \times n$ matrices with positive determinants. L_{n+1} acts transitively on $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ from the left. $\boldsymbol{W}_2(\boldsymbol{R}^{n+1}) \cong L_{n+1}/H_q$, where H_q is the isotropy subgroup of L_{n+1} at $q \in \boldsymbol{W}_2(\boldsymbol{R}^{n+1})$. L_2 acts on $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ from the right in the obvious manner. The submanifold of $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ consisting of all orthonormal 2-frames with respect to the canonical inner product is designated as $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$. $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$ is identified with the homogeneous space $SO(n+1)/SO(n-1)$. Let $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ be the Grassmann manifold of all oriented 2-planes through the origin of $\boldsymbol{R}^{n+1}$. $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ is identified with $SO(n+1)/SO(n-1) \times SO(2)$. $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$ is the principal bundle over $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ with the structure group $SO(2)$, where the projection π_V is identified with the canonical one:

$$SO(n+1)/SO(n-1) \to SO(n+1)/SO(n-1) \times SO(2) .$$

For $q = \{\boldsymbol{q}_0, \boldsymbol{q}_1\} \in \boldsymbol{W}_2(\boldsymbol{R}^{n+1})$, the (2,2)-matrix $\rho^2 = (\rho^2_{\alpha\beta}) = (\langle \boldsymbol{q}_\alpha, \boldsymbol{q}_\beta \rangle)$ is positive definite. Let $\rho = (\rho_{\alpha\beta})$ be the positive definite square root of (2,2)-matrix ρ^2.

Lemma 3.1. *There is a diffeomorphism* $\phi = (\psi, \pi_W)$:

$$\boldsymbol{W}_2(\boldsymbol{R}^{n+1}) \cong P_2 \times \boldsymbol{V}_2(\boldsymbol{R}^{n+1})$$

with $\psi(q) = \rho$ *and* $\pi_W(q) = q\rho^{-1}$, *where* P_2 *is the space of real positive definite* (2,2)-*matrices.*

Proof. $\pi_W q$ is easily proved to be an element of $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$. The rest of the proof is obvious. Q.E.D.

Let $\mathbf{Geod}(S^n)$ be the space of oriented geodesics on (S^n, g_0). $\mathbf{Geod}(S^n)$ can be identified with $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ by the canonical map ι, attaching an oriented 2-plane Γ through the origin to the geodesic $\iota(\Gamma) = S^n \cap \Gamma$ with the induced orientation. For $\xi \in \tilde{S}^p(S^n)$ we define a function $\xi^\wedge \in C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ by

$$\xi^\wedge(\Gamma) = (1/2\pi) \int_{\gamma = \iota(\Gamma)} \langle \xi, \dot{\gamma}^p / p! \rangle ds , \tag{3.1}$$

where $\dot{\gamma}^p$ is the p-th symmetric power in $S^*(\gamma)$ of the unit tangent vector field $\dot{\gamma}$ along $\gamma=\iota(\Gamma)$. For $\xi\in S^p(S^n)$ we define $\xi^\wedge\in C^\infty(SG_{2,n-1}(R))$ by $\xi^\wedge=\tilde{\xi}^\wedge$, where $\tilde{\xi}$ is the element of $\tilde{S}^p(S^n)$ corresponding to ξ. We call $\xi^\wedge$ the *Radon-Michel transform* of ξ.

Lemma 3.2. *Let $\tilde{\xi}\in\tilde{S}^p(S^n)$ correspond to $\xi\in K^p(S^n, g_0)$. Then the integrand of* (3.1) *is constant along γ. Consequently, $\xi^\wedge(\Gamma)=\langle\tilde{\xi}, \dot{\gamma}^p/p!\rangle$ for $\dot{\gamma}$ as above.*

Proof. As $\nabla^2\gamma/ds^2=0$, we have

$$(d/ds)\langle\tilde{\xi}, \dot{\gamma}^p\rangle = (p+1)\langle\tilde{\delta}^*\tilde{\xi}, \dot{\gamma}^{p+1}\rangle = 0 . \qquad \text{Q.E.D.}$$

Lemma 3.3. *Let $\underset{AB}{\tilde{\xi}}=y^A\partial/\partial y^B-y^B\partial/\partial y^A\,(0\leqq A<B\leqq n)$. Then $P^{AB}=\underset{AB}{\xi}^\wedge$ are the Plücker coordinates of $SG_{2,n-1}(R)$ satisfying*

$$\sum_{A<B}(P^{AB})^2 = 1 . \tag{3.2}$$

Proof. Let p be a point on the geodesic $\gamma=\iota(\Gamma)$. Put

$$\dot{\gamma}\,|_{\mathrm{p}} = \sum_{A=0}^{n} Z^A\partial/\partial y^A\,|_{\mathrm{p}} .$$

Then by Lemma 3.2

$$\underset{AB}{\xi}^\wedge(\Gamma) = \langle\underset{AB}{\tilde{\xi}}, \sum_{C=0}^{n} Z^C\partial/\partial y^C\rangle = Z^B y^A - Z^A y^B .$$

The rest of the proof is obvious. Q.E.D.

The Plücker coordinates $\{P^{AB}\}$ satisfying (3.2) are called *normalized Plücker coordinates* in the following.

Lemma 3.4. *Let $P(M, G)$ be a principal bundle with the Lie group G as its fibre. Let $\mathfrak{D}^G(P)$ be the subalgebra of G-invariant differential operators of $\mathfrak{D}(P)$. Then $\mathfrak{D}^G(P)/\mathcal{J}\simeq\mathfrak{D}(M)$, where $\mathcal{J}$ is the two-sided ideal of $\mathfrak{D}^G(P)$ generated by G-invariant vertical vector fields on P.*

Proof. The proof is essentially given in ([5] Chapter VI, Prop. II), where only the module of vector fields is treated. Our assertion follows from this special case as an application of the theory of the universal envelopping algebra ([8] I-2-4). Q.E.D.

Applying Lemma 3.4 to the principal bundle $V_2(R^{n-1})\overset{\pi_V}{\to}SG_{2,n-1}(R)$ with $SO(2)$ as its fibre, we obtain

$$\mathfrak{D}(SG_{2,n-1}(R))\simeq\mathfrak{D}^{SO(2)}(V_2(R^{n-1}))/\mathcal{J}' ,$$

where $\mathcal{J}'$ is the principal ideal in $\mathfrak{D}^{SO(2)}(V_2(R^{n+1}))$ generated by an $SO(2)$-invariant vertical vector field.

Lemma 3.5. (i) *$C^\infty(V_2(\boldsymbol{R}^{n+1}))$ is identified with the subalgebra $C^\infty(V_2(\boldsymbol{R}^{n+1}))^\dagger=\{f\in C^\infty(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))\,|\,f$ is constant along each fibre of $\pi_W\}$ of the algebra $C^\infty(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$.*

(ii) *$\mathfrak{D}(V_2(\boldsymbol{R}^{n+1}))$ is identified with the subalgebra $\mathfrak{D}(V_2(\boldsymbol{R}^{n+1}))^\dagger=\{D\in \mathfrak{D}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))\,|\,[\rho_{\alpha\beta}, D]=0$ and $[\partial/\partial\rho_{\alpha\beta}, D]=0$ $(\alpha, \beta=0, 1)\}$ of $\mathfrak{D}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$.*

(iii) *$\boldsymbol{S}^*(V_2(\boldsymbol{R}^{n+1}))$ is identified with the subalgebra $\boldsymbol{S}^*(V_2(\boldsymbol{R}^{n+1}))^\dagger$ of $\boldsymbol{S}^*(\boldsymbol{W}_2(\boldsymbol{R}^{n-1}))$ generated by $\sigma_W^*(\mathfrak{D}(V_2(\boldsymbol{R}^{n+1})))^\dagger$, where σ_W^* denotes the symbol map of $\mathfrak{D}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$.*

Proof (i) is evident. Applying Lemma 2.1 to the decomposition in Lemma 3.1, we obtain (ii) and (iii). Notice that P_2 in Lemma 3.1 is of dimension 3 and is parameterized by ρ_{00}, $\rho_{01}(=\rho_{10})$ and ρ_{11}. Q.E.D.

Lemma 3.6. *Let $\bar{g}=\sum_{A<B}(dP^{AB})^2$ be the canonical metric on $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$. Then*

$$\text{(i)}\quad \pi_V^*(\bar{g})=\sum_{A,B=0}^{n}(\delta^{AB}-p_\alpha^A p_\beta^B\delta^{\alpha\beta})\delta^{\gamma\delta}dp_\gamma^A dp_\delta^B,$$

where $p_\alpha^A(\alpha=0, 1)$ are components of $p=\{p_0, p_1\}\in V_2(\boldsymbol{R}^{n+1})$.

$$\text{(ii)}\quad (\pi_V\cdot\pi_W)^*(\bar{g})=\sum_{A,B=0}^{n}(\delta^{AB}-q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta})(\rho^2)^{\gamma\delta}dq_\gamma^A dq_\delta^B\,,$$

where $(\rho^2)^{\alpha\beta}=(\rho^{-2})_{\alpha\beta}$.

Proof. We can easily obtain (i). Making use of Lemma 3.1, we obtain (ii) by straightforward calculations. Notice that

$$\delta^{\alpha\beta}-q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta}=\delta^{AB}-p_\alpha^A p_\beta^B\delta^{\alpha\beta}\,,\tag{3.5}$$

since $\boldsymbol{q}_\beta=\sum_{\alpha=0}^{1}\boldsymbol{p}_\alpha\rho_{\alpha\beta}$ Q.E.D.

The Laplacian with respect to $\bar{g}$ is denoted by $\Delta^\wedge$.

Lemma 3.7. *A representative (mod $\mathcal{J}$) in $\mathfrak{D}(V_2(\boldsymbol{R}^{n+1}))^\dagger$ of $\Delta^\wedge$ is $(\Delta^\wedge)^\dagger$*

$$\overset{\text{def}}{=}-(\delta^{AB}-q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\frac{\partial^2}{\partial q_\gamma^A\,\partial q_\delta^B}+(n-1)q_\alpha^A\frac{\partial}{\partial q_\alpha^A}\,.$$

Proof. $(\Delta^\wedge)^\dagger\in\mathfrak{D}(V_2(\boldsymbol{R}^{n+1}))^\dagger$, since $[\rho_{\alpha\beta}, (\Delta^\wedge)^\dagger]=0$ and $\left[\frac{\partial}{\partial\rho_{\alpha\beta}}, (\Delta^\wedge)^\dagger\right]=0$ are easily verified. Moreover, $(\Delta^\wedge)^\dagger$ is found to be L_2-invariant, since $\left[q_\alpha^A\frac{\partial}{\partial q_\alpha^A}, (\Delta^\wedge)^\dagger\right]=0$ $(\alpha, \beta=0, 1)$. Consequently, $(\Delta^\wedge)^\dagger$ represents a differential operator in $\mathfrak{D}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$. Notice that

$$-\sigma_W^2(\Delta^\wedge)^\dagger=(\delta^{AB}-q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\,\partial/\partial q_\gamma^A\,\partial/\partial q_\delta^B\,.$$

Comparing with Lemma 3.6, we can easily verify that $-\sigma_W^2(\Delta^\wedge)^\dagger$ represents g^*, where g^* is the contravariant metric tensor corresponding to g. As $(\Delta^\wedge)^\dagger$ is self-adjoint in $\mathfrak{D}(V_2(\boldsymbol{R}^{n+1}))^\dagger$ and annihilates constants, we conclude that $(\Delta^\wedge)^\dagger$ represents the Laplacian $\Delta^\wedge$ of $(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}), g)$. Q.E.D.

Lemma 3.8. (i) $(\Delta^\wedge)^\dagger(\rho^{\alpha\beta}f) = \rho^{\alpha\beta}(\Delta^\wedge)^\dagger f$, *for* $f\in C^\infty(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$

(ii) $(\Delta^\wedge)^\dagger(q_\alpha^A q_\beta^B \rho^{\alpha\gamma}\rho^{\beta\delta}) = -2\delta^{\gamma\delta}\delta^{AB} + 2\delta^{\gamma\delta}q_\alpha^A q_\beta^B\delta^{\alpha\beta} + 2(n-1)q_\alpha^A q_\beta^B\rho^{\alpha\gamma}\rho^{\beta\delta}$

(iii) $\dfrac{\partial(q_\delta^C\rho^{\delta\gamma})}{\partial q_\alpha^A}\dfrac{\partial(q_\eta^D\rho^{\eta\varepsilon})}{\partial q_\beta^B}(\rho^2)_{\alpha\beta}(-\delta^{AB}+q_\tau^A q_\mu^B(\rho^2)^{\tau\mu}) = -\delta^{\gamma\varepsilon}(\delta^{CD}-q_\alpha^C q_\beta^D(\rho^2)^{\alpha\beta})$.

Proof. (i) and (ii) follow from $[\rho_{\alpha\beta}, (\Delta^\wedge)^\dagger]=0$. (iii) follows immediately from (ii). Q.E.D.

By Lemma 3.5 we identify $f\in C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ with $(\pi_V\cdot\pi_W)^*f\in C^\infty(\boldsymbol{W}_2(\boldsymbol{R}^{n-1}))$.

Theorem 3.1. *Let* $\xi\in \boldsymbol{S}^p(S^n)$. *Then*

$$\Delta^\wedge\xi^\wedge = (\Delta\xi)^\wedge .$$

Proof. It is enough to show

$$(\tilde{\Delta}^\wedge)^\dagger(\tilde{\xi}^\wedge)^\dagger = ((\tilde{\Delta}\tilde{\xi})^\wedge)^\dagger \tag{3.6}$$

for $\tilde{\xi}$ as in (2.10).

Recall $(\tilde{\xi}^\wedge)^\dagger(q)=(1/2\pi)\int_\gamma\langle\tilde{\xi},\dot{y}^p/p!\rangle ds$ for $q\in \boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ such that $(\pi_V\cdot\pi_W)(q)=\Gamma$ $(\gamma=\iota(\Gamma))$, where $y^A=\sum_{\alpha,\beta=0}^{1} q_\beta^A\rho^{\beta\alpha}u_\alpha$ $((u_0)^2+(u_1)^2=1)$. Interchanging the order of the integration and the differential operator $(\Delta^\wedge)^\dagger$, we obtain

$$\begin{aligned}(\tilde{\Delta}^\wedge)^\dagger(\tilde{\xi}^\wedge)^\dagger = {}& (1/2\pi)\int_\gamma \sum_{A_1\cdots A_p=0}^{n} -(\delta^{AB}-q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\\ &\cdot[(\partial^2 y^C/\partial q_\gamma^A\partial q_\delta^B)(\partial\tilde{\xi}^{A_1\cdots A_p}/\partial y^C)\dot{y}^{A_1}\cdots\dot{y}^{A_p}+(\partial y^C/\partial q_\gamma^A)(\partial y^D/\partial q_\delta^B)\\ &\cdot(\partial^2\tilde{\xi}^{A_1\cdots A_p}/\partial y^C\partial y^D)\dot{y}^{A_1}\cdots\dot{y}^{A_p}+2p(\partial y^C/\partial q_\gamma^A)(\partial\tilde{\xi}^{A_1\cdots A_p}/\partial y^C)\\ &\cdot\dot{y}^{A_1}\cdots\dot{y}^{A_{p-1}}\partial\dot{y}^{A_p}/\partial q_\delta^B+p(p-1)\tilde{\xi}^{A_1\cdots A_p}\dot{y}^{A_1}\cdots\dot{y}^{A_{p-2}}\\ &\cdot(\dot{y}^{A_{p-1}}/\partial q_\gamma^A)(\partial\dot{y}^{A_p}/\partial q_\delta^B)+p\tilde{\xi}^{A_1\cdots A_p}\dot{y}^{A_1}\cdots\dot{y}^{A_p}\partial^2\dot{y}^{A_p}/\partial q_\gamma^A\partial q_\delta^B]ds\\ &+((n-1)/2\pi)\int_\gamma\sum_{A_1\cdots A_p=0}^{0} q_\alpha^A[(\partial y^C/\partial q_\alpha^A)(\partial\tilde{\xi}^{A_1\cdots A_p}/\partial y^C)\\ &\cdot\dot{y}^1\cdots\dot{y}^{A_p}+p\tilde{\xi}^{A_1\cdots A_p}\dot{y}^{A_1}\cdots\dot{y}^{A_{p-1}}\partial\dot{y}^{A_p}/\partial q_\alpha^A]ds ,\end{aligned}\tag{3.7}$$

where $\dot{y}^A=\sum_{\alpha,\beta=0}^{1} q_\alpha^A\rho^{\alpha\beta}\dot{u}_\beta$ and $\partial\dot{y}^C/\partial q_\alpha^A=(\partial\sum_{\gamma=0}^{1} q_\gamma^C\rho^{\gamma\delta}/\partial q_\alpha^A)\dot{u}_\delta$. On the other hand, we see easily

$$\text{(3.8)}\quad \text{(i)}\ \sum_{\gamma=0}^{1}(\dot{u}_\gamma)^2=0 \quad \text{(ii)}\ \sum_{\gamma=0}^{1}u_\gamma\dot{u}_\gamma=0$$

(iii) $\ddot{u}_\gamma = -u_\gamma$ (iv) $\delta_{\alpha\beta} = u_\alpha u_\beta + \dot{u}_\alpha \dot{u}_\beta$.

The first term of the first integral in (3.7) together with the first term of the second integral, becomes from Lemma 3.8

$$(1/2\pi)\int_\gamma \sum_{A_1\cdots A_p=0}^{n}((\tilde{\Delta}^\wedge)^\dagger(\sum q_\alpha^C \rho^{\alpha\beta}u_\beta))\frac{\partial\xi^{A_1\cdots A_p}}{\partial y^C}\dot{y}^{A_1}\cdots\dot{y}^{A_p}ds$$
$$= p(n-1)\,(\xi^\wedge)^\dagger\,((\pi_V\cdot\pi_W)^{-1}\cdot\iota^{-1}(\gamma))\,.$$

Similarly by Lemma 3.8 the last term of the first integral together with the second term of the second integral in (3.7) is reduced to

$$(p/2\pi)\int_\gamma \sum_{A_1\cdots A_p=0}^{n}((\tilde{\Delta}^\wedge)^\dagger y^{A_p})y^{A_1}\cdots y^{A_{p-1}}\xi^{A_1\cdots A_p}ds$$
$$= p(n-1)\,(\xi^\wedge)^\dagger((\pi_V\cdot\pi_W)^{-1}\iota^{-1}(\gamma))\,.$$

The fourth term of the first integral becomes

$$(p(p-1)/2\pi)\int_\gamma \sum_{A_1\cdots A_p=0}^{n}\xi^{A_1\cdots A_p}\dot{y}^{A_1}\cdots\dot{y}^{A_{p-2}}(-\delta^{AB}+q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta})$$
$$\cdot(\partial y^{A_{p-1}}/\partial q_\gamma^A)\,(\partial y^{A_p}/\partial q_\delta^B)\,(\rho^2)_{\gamma\delta}ds = (p(p-1)\,(\xi^\wedge)^\dagger - 4((\tilde{T}^*\tilde{T}\xi)^\wedge)^\dagger)$$
$$((\pi_V\cdot\pi_W)^{-1}\,\iota^{-1}(\gamma))\,.$$

Similarly, the second term of the first integral in (3.7) is calculated as

$$-(1/2\pi)\int_\gamma \sum_{A_1\cdots A_p=0}^{n}(\delta^{AB}-q_\alpha^A q_\beta^B(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}(\partial y^C/\partial q_\gamma^A)\,(\partial y^D/\partial q_\delta^B)$$
$$(\partial^2\xi^{A_1\cdots A_p}/\partial y^C\partial y^D)\dot{y}^{A_1}\cdots\dot{y}^{A_p}ds = -(1/2\pi)\int_\gamma(\delta^{AB}r^2 - y^A y^B)(\partial^2\xi^{A_1\cdots A_p}/\partial y^A\partial y^B)$$
$$\cdot\dot{y}^{A_1}\cdots\dot{y}^{A_p}ds$$

because of the identity;

$$\text{(3.9)}\qquad \int_\gamma \sum_{A_1\cdots A_p=0}^{n}\dot{y}^A\dot{y}^B(\partial^2\xi^{A_1\cdots A_p}/\partial y^A\partial y^B)\dot{y}^{A_1}\cdots\dot{y}^{A_p}ds = 0\,.$$

(3.9) is deduced from

$$2\pi((\tilde{\delta}^*)^2\xi)^\wedge = \int_\gamma [r^4\sum_{\substack{h,k=1\\h\neq k}}^{p+2}\partial^2\xi^{A_1\cdots\hat{A}_h\cdots\hat{A}_k\cdots A_{p+2}}/\partial y^{A_h}\partial y^{A_k}$$
$$+2r^2\sum_{\substack{h,k=1\\h\neq k}}^{p+2}y^{A_k}\partial\xi^{A_1\cdots\hat{A}_h\cdots\hat{A}_k\cdots A_{p+2}}/\partial y^{A_h}]\dot{y}^{A_1}\cdots\dot{y}^{A_{p+2}}ds$$

and $\sum_{A=0}^{n}\dot{y}^A y^A = 0$. As Im δ^* is annihilated by the Radon-Michel transform,

$$((\tilde{\delta}^*)^2\xi)^\wedge = 0\,.$$

This proves (3.9). Comparing these results with Lemma 2.2 (iv) we obtain the theorem. Q.E.D.

4. Eigen-space decomposition of Lichnerwicz operator Δ on $K^*(S^n, g_0)$

From now on, if no confusion arise, we omit the symbols $\tilde{}$ and $\dagger$. For example we write $\delta\xi$ instead of $\tilde{\delta}\tilde{\xi}$ and $\Delta^\wedge$ instesd of $(\Delta^\wedge)^\dagger$.

On (S^n, g_0) the curvature tensor and the Ricci tensor are given respectively by

$$R^i{}_{jkl} = \delta^i_l(g_0)_{kl} - \delta^i_k(g_0)_{jl} \quad \text{and} \quad R_{jk} = (n-1)(g_0)_{jk}\,,$$

So the Lichnerowicz operator on $S^n(S^n)$ is expressed as

$$\Delta = 2p(n+p-2)\mathbf{1}_p - 8T^*T + \Box\,, \tag{4.1}$$

where $\mathbf{1}_p$ is the identity operator on $S^p(S^n)$.

Put

$$\lambda_{p,k} = 2(p-k)n + 2p^2 - 4(k+1)p + 4k^2 + 6k\,,$$

where p and k are integers such that $p \geqq 2k \geqq 0$. As

$$\lambda_{p,i} - \lambda_{p,k} = 2(k-i)(n+2p-2k-2i-3), \tag{4.2}$$

we find that

$$\lambda_{p,k} \geqq \lambda_{p,i} \quad (k \geqq i)\,.$$

Let $S\colon S^*(S^n) \to S^*(S^n)$ be the differential operator of degree -2 defined by

$$S = \Delta T - \lambda_{p,1}T + (1/3)(16T^*T^2 + [\delta^*, T\delta]) \tag{4.3}$$

on $S^p(S^n)$.

Lemma 4.1. $[\delta^*, S] = (4/3)(n+2p)T\delta^*$ *on* $S^p(S^n)$. *In particular, S induces an endomorphism on* $K^*(S^n, g_0)$.

Proof. Owing to Lemma 1.4 and (4.1), we can express $\Delta\delta$ restricted to $S^p(S^n)$ in three ways as

(i) $\Delta\delta = 2(p-1)(n+p-3)\delta - 8T^*T\delta + \Box\delta$

(ii) $\Delta\delta = 2p(n+p-2)\delta - 8\delta T^*T + \delta\Box$

(iii) $\Delta\delta = ((2p-1)n + 2p^2 - 6p + 3)\delta - 4T^*T\delta - 4\delta T^*T + (1/2)(\Box\delta + \delta\Box)$.

By Lemma 1.4 and (4.3), we have

$$\begin{aligned}[\delta^*, S] = {} & ((2p-1)n + 2p^2 - 6p + 3)[\delta^*, T] - 2((p-1)n + p^2 - 4p + 5)\delta^*T \\ & + 2(pn + p^2 - 2p + 2)T\delta^* - 4T^*T\delta - 4\delta T^*T + \frac{1}{2}(\Box\delta + \delta\Box)\end{aligned}$$

$$+\frac{16}{3}T^*[\delta^*, T^2]+\frac{1}{3}[\delta^*, [\delta^*, \delta T]]$$
$$=(n+2p-7)\delta^*T+(n+2p+1)T\delta^*-4[T^*, \delta]T-8\delta T^*T$$
$$+\frac{1}{2}(\Box\delta+\delta\Box)+\frac{16}{3}T^*[\delta^*, T^2]+\frac{1}{3}[\delta^*, \delta^2-\Box T].$$

Since $[\delta^*, \delta^2]=-(\Box\delta+\delta\Box)$, we have

$$(4.4)\qquad [\delta^*, S]=(n+2p-11)\delta^*T+(n+2p+1)T\delta^*+\frac{1}{6}(\Box\delta+\delta\Box)$$
$$-8\delta T^*T+\frac{32}{3}T^*\delta T-\frac{1}{3}[\delta^*, T\Box].$$

On the other hand, we can obtain the fourth expression of $\Delta\delta$:

$$\delta\Delta=[\delta^*, \Delta T]=2p(n+p-2)\delta^*T-8\delta^*TT^*T+[\delta^*, T\Box]$$
$$-2(p-1)(n+p-3)T\delta^*+8T^*T^2\delta^*.$$

From this and the third equality of (4.3) we have

$$((2p-1)n+2p^2-6p+3)\delta-4T^*T\delta+\frac{1}{2}(\Box\delta+\delta\Box)=$$
$$2p(n+p-2)\delta^*T-2(p-1)(n+p-3)T\delta^*-8\delta^*TT^*T+8T^*T^2\delta^*+[\delta^*, \Box T].$$

Eliminating the term $[\delta^*, T\Box]$ from the equality above and (4.4), we obtain the desired formula Q.E.D.

Lemma 4.2. *Let* $p\geqq 2k\geqq 0$.

$$\frac{3k}{2k+1}T^{k-1}S=\Delta T^k-\lambda_{p,1}T^k+\frac{1}{2k+1}\{8(k+1)T^*T^{k+1}+[\delta^*, T^k\delta]\}\quad on\ S^p(S^n).$$

Proof. From the definition of S, Lemma 1.1 and Lemma 1.4 we have

$$(4.5)\quad T^{k-1}S=\Delta T^k-\lambda_{p,1}T^k+(1/3)(16T^{k-1}T^*T^2-\delta T^k\delta^*+T^{k-1}\delta^*T\delta)$$
$$=\Delta T^k-\lambda_{p,1}T^k+(1/3)(16T^*T^{k+1}+8(k-1)(n+2p-2k-4)T^k$$
$$+[\delta^*, \delta T^k]-(k-1)T^{k-1}\delta^2).$$

On the other hand, we have

$$(4.6)\quad \Delta T^k=\lambda_{p-2k,0}T^k-8T^*T^{k+1}+[\delta, \delta^*]T^k$$
$$=\lambda_{p-2k,0}T^k-8T^*T^{k+1}+[\delta T^k, \delta^*]+kT^{k-1}\delta^2.$$

Eliminating $T^{k-1}\delta^2$ from (4.5) and (4.6), we obtain the desired formula. Q.E.D.

Put

$$(4.7)\qquad B_k^*=2k^2T^*+(\delta^*)^2\quad\text{and}\quad A_k^*=(\prod_{i=1}^{k}B_{2i}^*)T^k$$

for a non-negative integer k.

Lemma 4.3. $(3k/2k+1)T^{k-1}S=\Delta T^k-\lambda_{p,k}T^k+(1/(2k+1)(k+1))B^*_{2(k+1)}T^{k+1}$ *on* $\boldsymbol{K}^p(S^n,g_0)$ *for* $p\geqq 2k\geqq 0$. *In particular,* A^*_k *leaves* $\boldsymbol{K}^*(S^n,g_0)$ *invariant.*

Proof. The first assertion follows immediately from Lemma 4.2. We prove the second one by induction on k. For $k=0$ the assertion coincides with (4.1). Suppose that A^*_i $(0\leqq i\leqq k)$ leave $\boldsymbol{K}^p(S^n,g_0)$ invariant for each p. Applying $\prod_{i=1}^{k} B^*_{2i}$ to the equality of the first assertion, we obtain

$$(4.8)\qquad (\prod_{i=1}^{k} B^*_{2i})(3i/2i+1)T^{k-1}S=\Delta A^*_k-\lambda_{p,k}A^*_k+A^*_{k+1}/((2k+1)(k+1))\,.$$

As the left-hand side of (4.8) can be expressed as $(3k/2k+1)B^*_{2k}A^*_{k-1}S$, with the aid of Lemma 4.1 we conclude from the induction hypothesis and (4.8) that A^*_{k-1} leaves $\boldsymbol{K}^*(S^n g_0)$ invariant. Q.E.D.

Let Π_0: $\boldsymbol{K}^*(S^n,g_0)\rightarrow \boldsymbol{K}^*(S^n,g_0)\cap(\mathrm{Im}\ T^*)^{\perp}$ be the orthogonal projection with respect to the inner product (1.9). Π_0 commutes with Δ. Put

$$(4.9)\qquad H_k=\Pi_0 A^*_k\,.$$

As the image of $B^*_{2k}A^*_{k-1}S$ restricted to $\boldsymbol{K}^p(S^n,g_0)$ is contained in $T^*(\boldsymbol{K}^{p-2}(S^n,g_0))$

$$(4.10)\qquad \Delta H_k-\lambda_{p,k}H_k+(1/(2k+1)(k+1))H_{l+1}=0$$

on $\boldsymbol{K}^p(S^n,g_0)$. Put

$$(4.11)\quad P_{p,k}=\frac{n+2p-4k-3}{k!\cdot(n+2p-2k-3)!!}\sum_{i=k}^{[p/2]}\frac{(-1)^{i-k}\,(n+2p-2k-2i-5)!!}{(2i)!\cdot(i-k)!}H_i\,,$$

where $p\geqq 2k\geqq 0$ and $k!!=2^{[k/2]}\dfrac{\Gamma(1+(k/2))}{\Gamma(1+(k/2)-[k/2])}$.

Notice that

$$(2k)!!=2^k\cdot k!\,,\quad (2k+1)!!=(2k+1)!/(2^k\cdot k!)$$

for a non-negative integer k, $(-1)!!=1$, $(-3)!!=-1$.

Lemma 4.4. $\Delta P_{p,k}=\lambda_{p,k}P_{p,k}$ *on* $\boldsymbol{K}^p(S^n,g_0)$.

Proof.

$$\sum_{i=k}^{[p/2]}\frac{(-1)^{i-k}(n+2p-2k-2i-5)!!}{(2i)!\cdot(i-k)!}\Delta H_i=\sum_{i=k}^{[p/2]}\frac{(-1)^{i-k}}{(2i)!\cdot(i-k)!}$$
$$\cdot(n+2p-2k-2i-5)!!(\lambda_{p,i}H_i-H_{i+1}/((2i+1)\,(i+1)))$$
$$=(n+2p-4k-5)!!\lambda_{p,k}H_k/(2k)!$$

$$+\sum_{i=k+1}^{[p/2]}\Big(\frac{(-1)^{i-k}(n+2p-2k-2i-5)!!}{(2i)!\cdot(i-k)!}\lambda_{p,i}$$

$$-\frac{(-1)^{i-k-1}(n+2p-2k-2i-3)!!}{i\cdot(2i-2)!\cdot(i-k-1)!\cdot(2i-1)}\Big)H_i$$

$$=\frac{(n+2p-4k-5)!!}{(2k)!}\lambda_{p,k}H_k+\sum_{i=k+1}^{[p/2]}\Big((-1)^{i-k}(n+2p-2k-2i-5)!!$$

$$\cdot\frac{\lambda_{p,i}+2(i-k)(n+2p-2k-2i-3)}{(2i)!\cdot(i-k)!}H_i\Big).$$

By (4.2) the right-hand side of the equality above coincides with

$$\lambda_{p,k}\sum_{i=k}^{[p/2]}\frac{(-1)^{i-k}(n+2p-2k-2i-5)!!}{(2i)!\cdot(i-k)!}H_i. \qquad \text{Q.E.D.}$$

Lemma 4.5. *Let* $\mathbf{1}_p$ *be the identity operator on* $\boldsymbol{K}^p(S^n,g_0)\cap(\mathrm{Im}\,T^*)^\perp$. *Then we have* $\sum_{k=0}^{[p/2]}P_{p,k}=\mathbf{1}_p$ *on* $\boldsymbol{K}^p(S^n,g_0)\cap(\mathrm{Im}\,T^*)^\perp$.

Proof. The proof can be reduced to the following identity.

$$(4.12)\qquad \sum_{k=0}^{m}\frac{(-1)^k\binom{j}{k}(x-2k)}{\prod_{i=0}^{j}(x-k-i)}=\frac{(-1)^m\binom{j-1}{m}}{\prod_{i=1}^{j}(x-m-i)}\quad (j\geqq 1).$$

In particular, putting $m=j$ in (4.12), we obtain

$$(4.13)\qquad \sum_{k=0}^{j}\frac{(-1)^k\binom{j}{k}(x-2k)}{\prod_{i=0}^{j}(x-k-i)}=0\quad (j\geqq 1).$$

(4.12) is proved by induction on j. Substituting $x=(n+2p-3)/2$ into (4.13) we obtain

$$(4.13)'\quad \sum_{k=0}^{j}\frac{2^j(-1)^k\binom{j}{k}(n+2p-4k-3)(n+2p-2k-2j-5)!!}{(n+2p-2k-3)!!}=0\quad (j\geqq 1).$$

On the other hand,

$$\sum_{k=0}^{[p/2]}P_{p,k}=\sum_{k=0}^{[p/2]}\frac{(-1)^k\binom{j}{k}(n+2p-4k-3)}{(n+2p-2k-3)!}\sum_{j=k}^{[p/2]}\frac{(-1)^j\binom{j}{k}(n+2p-2k-2j-5)!!}{j!\cdot(2j)!}$$

$$=\sum_{k=0}^{[p/2]}\frac{(-1)^j}{j!\cdot(2j)!}\sum_{k=0}^{j}\frac{(-1)^k\binom{j}{k}(n+2p-4k-3)\cdot(n+2p-2k-2j-5)!!}{(n+2p-2k-3)!!}H_j.$$

From this and (4.13)′ we obtain

$$\sum_{k=0}^{[p/2]} P_{p,k} = H_0 = \mathbf{1}_p$$

on $\boldsymbol{K}^p(S^n, g_0) \cap (\operatorname{Im} T^*)^\perp$. Q.E.D.

Theorem 4.1. (i) *The operator* $\sum_{k=0}^{[p/2]} \lambda_{p,k} P_{p,k}$ *on* $\boldsymbol{K}^p(S^n, g_0) \cap (\operatorname{Im} T^*)^\perp$ *gives the eigen-space decomposition of* Δ *restricted to* $\boldsymbol{K}^p(S^n, g_0) \cap (\operatorname{Im} T^*)^\perp$.

(ii) $P_{p,k} \neq 0$ *for* $n \geqq 3$, *i.e., the* $\lambda_{p,k}$ *eigen-subspace is non-trivial on* $\boldsymbol{K}^p(S^n, g_0) \cap (\operatorname{Im} T^*)^\perp$. $P_{p,k} = \delta_{k,[p/2]} \mathbf{1}_p$ *and* $\lambda_{p,[p/2]} = p(p+1)$ *for* $n=2$, *where* δ_{ij} *is the Kronecker's symbol.*

(iii) $\boldsymbol{K}^p(S^n, g_0) = \sum_{k=0}^{[p/2]} (T^*)^k (\boldsymbol{K}^{p-2k}(S^n, g_0) \cap (\operatorname{Im} T^*)^\perp)$ *(direct sum) together with* (i) *and* (ii) *gives the eigen-space decomposition of* Δ *on* $\boldsymbol{K}^p(S^n, g_0)$.

Proof of (i) follows from Lemmas 4.4 and 4.5. In fact

$$\Delta = \Delta \mathbf{1}_p = \sum_k \Delta P_{p,k} = \sum_k \lambda_{p,k} P_{p,k}\,.$$

Since Δ is self-adjoint, (i) follows. (iii) follows from $[T^*, \Delta]=0$. In order to prove the rest of Theorem 4.1 we need the following five lemmas.

Lemma 4.6. *Let* ϕ *and* ϕ_i $(i=1, 2)$ *be eigen-functions of the Laplacian of* (S^n, g_0) *for the first eigen-value n. Then*

(i) $B_1^* \phi = 0$ (ii) $B_2^* \phi^2 = 4T^* \phi^2 + 2(\delta^* \phi)^2 \in \boldsymbol{K}^2(S^n, g_0)$.

(iii) $B_2^*(\phi_1 \phi_2) = 4T^* \phi_1 \phi_2 + 2(\delta^* \phi_1)(\delta^* \phi_2)$.

Proof. From (4.7) and a known theorem (cf. [2]) we have

$$B_1^* \phi = 2T^* \phi + (\delta^*)^2 \phi = 0\,.$$

By this equality

$$\begin{aligned} B_2^* \phi^2 &= (8T^* + (\delta^*)^2) \phi^2 = 8T^* \phi^2 + 2(\delta^* \phi)^2 + 2\phi (\delta^*)^2 \phi \\ &= 4T^* \phi^2 + 2(\delta^* \phi)^2\,. \end{aligned}$$

Moreover, $\delta^*(B_2^* \phi^2) = 8T^*(\phi \delta^* \phi) + 4((\delta^*)^2 \phi) \circ (\delta^* \phi) = 0$. (iii) follows from (ii) by polarization. Q.E.D.

Lemma 4.7. *Let* ϕ *be as in Lemma* 4.6. *For* $k \geqq 2j \geqq 0$

$$\prod_{i=0}^{j-1} B_{k-2i}^* \phi^k = (k!/(k-2j)!)(1/2^j)(B_2^* \phi^2)^j \phi^{k-2j}\,.$$

Proof. We prove the lemma by induction on j. For $j=1$

$$\begin{aligned} B_k^*\phi^k &= 2k^2T^*\phi^2\cdot\phi^{k-2}+k\delta^*(\delta^*\phi\cdot\phi^{k-1}) \\ &= 2k^2T^*\phi^2\cdot\phi^{k-2}+k(\delta^*)^2\phi\cdot\phi^{k-1}+k(k-1)\delta^*\phi\circ(\delta^*\phi)\phi^{k-2} \\ &= (2k^2-2k)T^*\phi^2\cdot\phi^{k-2}+k(k-1)(\delta_*\phi)^2\phi^{k-2} \\ &= (k!/(k-2)!)(1/2)(4T^*\phi^2+2(\delta^*\phi)^2)\phi^{k-2}\,. \end{aligned}$$

Suppose that the assertion be true for $j\geqq 1$. Then

$$\begin{aligned} (\prod_{i=0}^{j} B_{k-2i}^*)\phi^k &= B_{k-2j}^*((k!/(k-2j)!)(1/2^j)(B_2^*\phi^2)^j\phi^{k-2j}) \\ &= 2(k-2j)^2(k!/(k-2j)!)(1/2^j)T^*(B_2^*\phi^2)^j\cdot\phi^{k-2j} \\ &\quad +(k!/(k-2j-2)!)(\delta^*\phi)^2(B^*\phi^2)^j\phi^{k-2j-2}/2^j \\ &\quad +(k!/(k-2j-1)!)(\delta^*)^2\phi\circ\phi^{k-2j-1}\circ(B_2^*\phi^2)^j/2^j \\ &= (k!/(k-2j-2)!)\phi^{k-2j-2}(B_2^*\phi^2)^{j+1}/2^{j+1}\,. \end{aligned}$$

Q.E.D.

Lemma 4.8. *Let* ϕ_i $(i=1, 2)$ *be as in Lemma* 4.6.

$$\prod_{i=1}^{k} B_{2i}^*((\phi_1)^2+(\phi_2)^2)^k \equiv ((2k)!/2^k)(B_2^*((\phi_1)^2+(\phi_2)^2)^k \quad (\mathit{mod\ Im}\ T^*\cap \boldsymbol{K}^{2k}(S^n, g_0))\,.$$

Proof. At first we remark that when either $k=1$ or none of ϕ_i's annihilates our assertion coincides with Lemma 4.7. We write $\xi_1\sim\xi_2(\xi_i\in \boldsymbol{S}^*(S^n),\ i=1, 2)$ if and only if $\xi_1-\xi_2\in \mathrm{Im}\ T^*$. Obviously this is an equivalence relation. From the definition of B^*'s we have

$$\prod_{i=1}^{k} B_{2i}^*((\phi_1)^2+(\phi_2)^2)^k\sim(\delta^*)^{2k}((\phi_1)^2+(\phi_2)^2)^k=$$
$$\sum_{i=1}^{k}\binom{2k}{s}(\delta^*)^s((\phi_1)^2+(\phi_2)^2)\circ(\delta^*)^{2k-s}((\phi_1)^2+(\phi_2)^2)^{k-1}\,.$$

On the other hand, we have $(\delta^*)^k((\phi_1)^2+(\phi_2)^2)\sim 0$ $(k\geqq 3)$. Hence

$$\prod_{i=1}^{k} B_{2i}^*((\phi_1)^2+(\phi^2))^k\sim\binom{2k}{2}(\delta^*)^2((\phi_1)^2+(\phi_2)^2)\circ(\delta^*)^{2k-2}((\phi_1)^2+(\phi_2)^2)^{k-1}\,.$$

As $\prod_{i=0}^{k-1}\binom{2k-2i}{2}=(2k)!/2^k$, from the formula above we obtain the assertion by induction on k. Q.E.D.

REMARK. $(\phi_1)^2+(\phi_2)^2$ in Lemma 4.8 can be replaced by any quadratic form of ϕ_j's.

Lemma 4.9. $(\delta^*(y^A/r))^2+(y^A/r)^2g_0=\sum_{\substack{B=0\\B\neq A}}^{n}(y^A\partial/\partial y^B-y^B\partial/\partial y^A)^2,\ (A=0, \cdots, n)$.

The proof is a matter of straight-forward calculations.

Lemma 4.10 *Let* $\underset{AB}{\xi} = y^A \frac{\partial}{\partial y^B} - y^B \frac{\partial}{\partial y^A}$. For $\xi = \underset{01}{\xi} \in \boldsymbol{K}^1(S^n)$

(i) $H_i \xi^p = \frac{(2i)! \cdot p!}{2^i \cdot (p-2i)!} \Pi_0 [(\sum_{A=1}^{n} (\underset{0A}{\xi})^2 + \sum_{\substack{A=0 \\ A \neq 1}}^{n} (\underset{1A}{\xi})^2)^i \circ \xi^{p-2i}]$

(ii) $P_{p,k} \xi^p = \frac{p! \cdot (n+2p-4k-3)}{k! \cdot (n+2p-2k-3)!!} \Pi_0 \Big[\sum_{i=k}^{[p/2]} \frac{(-1)^{i-k} \cdot (n+2p-2k-2i-5)!!}{(i-k)! \cdot (p-2i)! \, 2^i} \cdot (\sum_{A=1}^{u} (\underset{0A}{\xi})^2 + \sum_{\substack{A \neq 1 \\ A=0}}^{n} (\underset{1A}{\xi})^2)^j \circ \xi^{-2j} \Big]$,

where ξ^p *is the p-th symmetric power of* ξ.

Proof. As $T^i \xi^p = \frac{p!}{2^i (p-2i)!} ((y^0/r)^2 + (y^1/r)^2)^i \xi^{p-2i}$,

$$A_i^* \xi^p \equiv \frac{(2i)! \cdot p!}{4^i \cdot (p-2i)!} (B_2^*(y^0/r)^2 + B_2^*(y^1/r)^2)^i \circ \xi^{p-2i} \quad (\mathrm{mod\ Im}\ T^* \cap \boldsymbol{K}^p(S^n, g_0)).$$

By Lemma 4.6 (ii) and Lemma 4.9 we have

$$A_i^* \xi^p \equiv \frac{(2i)! \cdot p!}{2^i \cdot (p-2i)!} (\sum_{j=1}^{n} \underset{0j}{\xi}^2 + \sum_{\substack{j=0 \\ j \neq 1}}^{n} \underset{1j}{\xi}^2)^i \circ \xi^{p-2i}.$$

From this (i) and (ii), respectively, follow immediately. Q.E.D.

Now we prove (ii) in Theorem 4.1 for $n=2$. We recall the following expansion formula for the Legendre polynomials $P_n(z) = (1/(2^n \cdot n!)) \frac{d^n}{dz^n} (z^2-1)^n$:

$$P_{2m}(z) = \sum_{j=0}^{m} (-1)^{m-j} \frac{(2m+2j-1)!!}{(2j)! \, (2m-2j)!!} z^{2j}$$

(4.14)

$$P_{2m+1}(z) = \sum_{j=0}^{m} (-1)^{m-j} \frac{(2m+2j+1)!!}{(2j+1)! \, (2m-2j)!!} z^{2j+1},$$

with $P_n(1) = 1$.

Lemma 4.11. *On* $\boldsymbol{K}^p(S^2, g_0) \cap (\mathrm{Im}\ T^*)^\perp$ *we have*

(i) $H_i = \frac{p! \cdot (2i)!}{(p-2i)! \cdot 2^i} \mathbf{1}_p$

(ii) $P_{p,k} = \frac{p! \cdot (2p-4k-1)}{k! \cdot (2p-2k-1)!!} \sum_{i=k}^{[p/2]} \frac{(-1)^{i-k} \cdot (2p-2k-2i-3)!!}{(i-k)! \cdot (p-2i)! \, 2^i} \mathbf{1}_p$.

Proof. It suffices to prove (i), because (ii) follows immediately from (i). Let $\xi = A\underset{01}{\xi} + B\underset{12}{\xi} + C\underset{02}{\xi}$ be a Killing vector field on (S^2, g_0), where $\underset{ij}{\xi}$'s $(i<j)$ are as in Lemma 4.11 and A, B and C are constants. Then

$$T^i\xi^p=\frac{p!\,(2i)!}{2^i(p-2i)!}[(A^2+C^2)(y^0/r)^2+(A^2+B^2)(y^1/r)^2$$
$$+(B^2+C^2)(y^2/r)^2-2ABy^0y^2/r^2-2BCy^0y^1/r^2-2CAy^2y^1/r^2]^i\xi^{p-2i}\,.$$

From Remark to Lemma 4.8 we obtain

$$A_i^*\xi^p\sim\frac{p!\cdot(2i)!}{2^i\cdot(p-2i)!}[(A^2+C^2)(\nabla(y^0/r))^2+(A^2+B^2)(\nabla(y^1/r))^2$$
$$+(B^2+C^2)(\nabla(y^2/r))^2-2AB(\nabla(y^0/r))\nabla(y^2/r)-2BC(\nabla(y^0/r))\nabla(y^1/r)$$
$$-2AC(\nabla(y^2/r))\nabla(y^1/r)]^i\xi^{p-2i}\sim\frac{p!\cdot(2i)!}{2^i\cdot(p-2i)!}\xi^p\,.$$

Applying Π_0 to the last relation, we obtain (i). Q.E.D.

Proof of (ii) *in Theorem* 4.1. *for* $n=2$. Let $p=2p'$.

$$P_{2p',k}=\sum_{i=k}^{p'}(-1)^{i-k}\frac{(2p')!\cdot[(4p'-2k-2i-1)!!+2(i-k)\cdot(4p'-2k-2i-3)!!]}{2^i\cdot k!\,(4p'-2k-1)!!\,(2p'-2i)!\,(i-k)!}$$
$$=\frac{(2p')!}{k!\cdot(4p'-2k-1)!!}\sum_{i=k}^{p'}(-1)^{i-k}\frac{2^{i-k}\cdot(4p'-2k-2i-1)!!}{2^i\cdot(2p'-2i)\cdot(2i-2k)!!}\cdot\mathbf{1}_p$$
$$+\frac{(2p')!}{2k!\cdot(4p'-2k-1)!!}\sum_{i=k}^{p'}(-1)^{i-k}\frac{2(i-k)\cdot(4p'-2k-2i-3)!!\,2^{i-k}}{2^i\cdot(2p'-2i)!\cdot(2i-2k)!!}\cdot\mathbf{1}_p$$
$$=\frac{(2p')!}{2^k\cdot k!\cdot(4p'-2k-1)!!}\sum_{p'-i=0}^{p'-i=p'-k}\frac{[(2p'-k)+2(p'-i)]!!\,(-1)^{p'-k-(p'-i)}}{(2p'-2i)!\cdot(2i-2k)!!}\cdot\mathbf{1}_p$$
$$-\frac{(2p')!}{2^k\cdot k!\cdot(4p'-2k-1)!!}\sum_{p'-i=0}^{p'-i=p'-k-1}\frac{(4p'-2k-2i-3)!!}{(2p'-2i)!\cdot(2i-2k-2)!!}$$
$$\cdot(-1)^{p'-k-1-(p'-i)}\cdot\mathbf{1}_p\,.$$

Substituting the first equality of (4.14) into the fromula above, we obtain for $k<p'$

$$P_{2p',k}=\frac{(2p')!}{2^k\cdot k!\cdot(4p'-2k-1)!!}P_{2(p'-k)}(1)\cdot\mathbf{1}_p$$
$$-\frac{(2p')!}{2^k\cdot k!\cdot(4p'-2k-1)!!}P_{2(p'-k-1)}(1)\cdot\mathbf{1}_p=0\,.$$

For $k=p'$, we obtain from Lemma 4.11 (ii)

$$P_{2p',p'}=\frac{(2p')!}{(2p')!!\cdot(2p'-1)!!}\cdot\mathbf{1}_p=\mathbf{1}_p\,.$$

For p odd the proof is analogous as above except the employment of the second equality of (4.14) in place of the first one, and is omitted. Q.E.D.

In order to prove the first half of Theorem 4.1 (ii), we need the following two lemmas.

Lemma 4.12. (i) *The image of the Radon-Michel transform restricted to* $\boldsymbol{K}^*(S^h, g_0)$ *is the subalgebra of* $C^\infty(\boldsymbol{S}G_{2,n-1}(\boldsymbol{R}))$ *consisting of the polynomials of the normalized Plücker coordinates.* (ii) *The kernel of the Radon-Michel transform is the principal ideal generated by* $g_0 - 1$ *in* $\boldsymbol{K}^*(S^n, g_0)$.

Proof. The algebra of polynomials of the normalized Plücker coordinates is isomorphic to $\boldsymbol{R}[X]/I$, where $\boldsymbol{R}[X]$ is the polynomial algebra generated by indeterminates X_{ij}'s $(1 \leqq i < j \leqq n)$ and I is the ideal generated by

$$(4.15)\quad \begin{array}{ll} \text{(i)} & \Pi_{ijkl} \overset{\text{def}}{=} X_{ij}X_{kl} - X_{ik}X_{jl} + X_{il}X_{jk} \quad (0 \leqq i < j < k < l \leqq n) \\ \text{(ii)} & \sum_{i<j} (X_{ij})^2 - 1 . \end{array}$$

Π_{ijkl}'s are Plücker polynomials. (4.15) (ii) arises from the normalization of Plücker coordinates.

Hence (i) is obvious. From (i) and Lemma 3.3, (ii) follows immediately.

Q.E.D.

Let J be the Plücker ideal generated by Π_{ijkl}'s in $\boldsymbol{R}[X]$.

Lemma 4.13. $\boldsymbol{K}^*(S^n, g_0) \cong \boldsymbol{R}[X]/J$

Proof. Let $\Phi: \boldsymbol{R}[X] \to \boldsymbol{K}^*(S^n, g_0)$ be given by $\Phi(X_{ij}) = \underset{ij}{\xi}$. Φ can be extended to a surjective homomorphism of graded algebrs. Obviously $\Pi_{ijkl} \in \operatorname{Ker} \Phi$. If we consider the homomorphism Φ followed by the Radon-Michel transform, Lemma 4.12 tells us that the Ker Φ is exactly generated by Π_{ijkl}'s.

Q.E.D.

Proof of (ii) *in Theorem* 4.1. *for* $n \geqq 3$, Lemma 4.10 is restated as in the following form:

$$P_{p,k}(\underset{01}{\xi})^p = \Pi_0 \sum_{i=k}^{[p/2]} c_{p,k,i} [\sum_{j=1}^{n} (\underset{0j}{\xi})^2 + \sum_{\substack{j \neq 1 \\ j=0}}^{n} (\underset{1j}{\xi})^2]^i (\xi)^{p-2i} ,$$

where $c_{p,k,i} = \dfrac{p! \cdot (n+2p-4k-3)(-1)^{i-k}(n+2p-2k-2i-5)!!}{k! \cdot (n+2p-2k-3)!! \cdot (i-k)! \cdot (p-2i)!\, 2^i}$

If this were identically zero, then

$$\sum_{i=k}^{[p/2]} c_{p,k,i} [2(X_{01})^2 + \sum_{j \geqq 2} ((X_{0j})^2 + (X_{1j})^2)]^i (X_{01})^{p-2i}$$

should be annihilated by X_{ij}'s which annihilate Π_{ijkl} and $\sum_{j>i} (X_{ij})^2$. This is not the case. For if we put

$$X_{01}=\sqrt{2}, \quad X_{23}=\sqrt{-2}, \quad X_{02}=X_{03}=1, \quad -X_{12}=X_{13}=\sqrt{-1},$$
$$X_{ij}=0 \quad (\text{Max}\ \{i, j\} \geqq 4),$$

we should have $c_{p,k,i}=0$. This is a contradiction. Q.E.D.

Theorem 4.2. *The spectra of* $(SG_{2,n-1}(\boldsymbol{R}), \bar{g})$ *are*

$$\lambda_{p,k}=2(p-k)n+2p^2-4(k+1)p+4k^2+6k \qquad n\geqq 3$$
$$\lambda_{p,[p/2]}=p(p+1) \qquad n=2,$$

where p and k are integers such that $p\geqq 2k\geqq 0$. The eigen-space for the eigen-value $\lambda_{p,k}$ is the image by the Radon-Michel transform of the eigen-subspace in $\boldsymbol{K}^*(S^n, g_0)$ of the Lichnerowicz operator for the eigen-value $\lambda_{p,k}$.

Proof. As is well known the polynomial algebra, generated by the normalized Plücker coordinates P^{AB}'s separates two points in $SG_{2,n-1}(\boldsymbol{R})$ By the Stone-Weierstrass theorem it is uniformly dense in $C^\infty(SG_{2,n-1}(\boldsymbol{R}))$. Thus, from Theorem 3.1, Theorem 4.1 and the non-triviality of the image by the Radon-Michel transform of the non-trivial eigen-subspace of $\boldsymbol{K}^*(S^n, g_0)$ which is essentially contained in the proof of (ii) in Theorem 4.1, we conclude the proof of Theorem 4.2. Q.E.D.

The Grassmann manifold $G_{2,n-1}(\boldsymbol{R})=O(n+1)/O(n-1)\times O(2)$ which is the space of 2-planes in $\boldsymbol{R}^{n+1}$ has $SG_{2,n-1}(\boldsymbol{R})$ as its 2-fold covering:

$$SG_{2,n-1}(\boldsymbol{R}) \xrightarrow{\pi_s} G_{2,n-1}(\boldsymbol{R}).$$

$C^\infty(SG_{2,n-1}(\boldsymbol{R}))$ is identified with the subalgebra $\{g\in C^\infty(SG_{2,n-1}(\boldsymbol{R})\,|\,g=(\pi_s)^*f, f\in C^\infty(G_{2,n-1}(\boldsymbol{R})\}$ of $C^\infty(SG_{2,n-1}(\boldsymbol{R}))$. On the other hand, π_s being local isometry, the Laplacian of $G_{2,n-1}(\boldsymbol{R})$ can be viewed as the canonical one of $SG_{2,n-1}(\boldsymbol{R})$ restricted to the subalgebra above. Hence we obtain

Theorem 4.3. *The spectra of* $(G_{2,n-1}(\boldsymbol{R}), \bar{g})$ *are*

$$\lambda_{p,k}=2(p-k)n+2p^2-4(k+1)p+4k^2+6k \qquad n\geqq 3$$
$$\lambda_p=p(p+1) \qquad n=2$$

for even integer p and integer k ($p\geqq k\geqq 0$).

Appendix. Differential equations for spherical ploynomials

Let D_k^* be the linear differential operator of order $k+1$ defined by

$$D_k^* = \begin{cases} \delta^* \prod_{i=0}^{(k/2)-1} B_{k-2i}^* & (k;\ \text{non-negative even integer}) \\ \prod_{i=0}^{[k/2]-1} B_{k-2i}^* & (k;\ \text{non-negative odd integer}). \end{cases}$$

Lemma A. $D_k^*(\xi/r^k)=r^{k+2}(\partial^*)^{k+1}\xi$,
where $\xi/r^k\in S^p(S^n)$ and $(\partial^*\xi)^{A_1\cdots A_{p+1}}=\sum_{j=1}^{p+1}\frac{\partial\xi^{A_1\cdots\hat{A_j}\cdots A_{p+1}}}{\partial y^{A_j}}$.

Proof. For $k=0$, $D_0^*(\xi/r^0)=\delta^*\xi=r^2\partial^*\xi$. For $k=1$, $D_1^*(\xi/r)=B_1^*(\xi/r)=(\delta^*)^2(\xi/r)+2T^*(\xi/r)=r^3(\partial^*)^2\xi$. Suppose that the assertion be affirmative for $k>0$. Then

$$D_{k+2}^*(\xi/r^{k+2}) = D_k^* B_{k+2}^*(\xi/r^{k+2}) = r^{k+4}(\partial^*)^{k+3}\xi$$

by virtue of the Leibniz's formula. Q.E.D.

Theorem A. *Let* $f\in C^\infty(S^n)$. $D_k^*f=0$ *if and only if*

$$f\in\sum_{i=0}^{[k/2]} E_{k-2i} \quad (\textit{direct sum}),$$

where E_k *is the eigen-space of the Laplacian for the eigen-value* $k(n+k-1)$ *on* (S^n, g_0).

Proof. Put $\Psi=fr^k$. Then from Lemma A, $D_k^*f=0$ if and only if $r^{k+2}(\partial^*)^{k+1}\Psi=0$. Thus $D_k^*=0$ if and only if Ψ is a homogeneous polynomial of degree k. Q.E.D.

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