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A NEW LOOK AT CONDITION A

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Abstract

Ozeki and Takeuchi [14] introduced the notion of Condition A and Condition B to construct two classes of inhomogeneous isoparametric hypersurfaces with four principal curvatures in spheres, which were later generalized by Ferus, Karcher and Münzner to many more examples via the Clifford representations; we will refer to these examples of Ozeki and Takeuchi and of Ferus, Karcher and Münzner collectively as OT-FKM type throughout the paper. Dorfmeister and Neher [5] then employed isoparametric triple systems [3, 4], which are algebraic in nature, to prove that Condition A alone implies the isoparametric hypersurface is of OT-FKM type. Their proof for the case of multiplicity pairs {3, 4} and {7, 8} rests on a fairly involved algebraic classification result [9] about composition triples.

In light of the classification [2] that leaves only the four exceptional multiplicity pairs {4, 5}, {3, 4}, {7, 8} and {6, 9} unsettled, it appears that Condition A may hold the key to the classification when the multiplicity pairs are {3, 4} and {7, 8}. Thus Condition A deserves to be scrutinized and understood more thoroughly from different angles.

In this paper, we give a fairly short and rather straightforward proof of the result of Dorfmeister and Neher, with emphasis on the multiplicity pairs {3, 4} and {7, 8}, based on more geometric considerations. We make it explicit and apparent that the octonion algebra governs the underlying isoparametric structure.

1. Introduction

An isoparametric hypersurface M in the sphere S^n is one whose principal curvatures and their multiplicities are fixed. We shall not dwell on the history and development of the beautiful isoparametric story, and shall leave it to, e.g., [2], and the references therein. Through Münzner's work [12, 13] one knows that such a hypersurface can be characterized by a homogeneous polynomial $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ of degree $g = 1, 2, 3, 4$ or 6, satisfying

$$|\nabla F|^2(x) = g^2|x|^{2g-2}, \quad (\Delta F)(x) = (m_2 - m_1)g^2\frac{|x|^{g-2}}{2}$$

for two natural numbers m_1 and m_2. The interpretation of m_1 and m_2 is that if we arrange the principal curvatures $\lambda_1 > \cdots > \lambda_g$ with multiplicities $m_1, \ldots, m_g$, respectively,

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then $m_i = m_{i+2}$ with index mod (g); therefore, which one is m_1 or m_2 is only a matter of convention, by changing F to $-F$ if necessary. F is called the Cartan–Münzner polynomial, whose restriction f to S^n has values in the interval $[-1, 1]$. $f^{-1}(c)$, $-1 < c < 1$, is a one-parameter family of isoparemetric hypersurfaces to which M belongs. The family degenerates to two connected submanifolds $M_+ := f^{-1}(1)$ and $M_- := f^{-1}(-1)$, called the focal submanifolds of M, of codimension $m_1 + 1$ and $m_2 + 1$, respectively.

In the case when $g = 4$, Ozeki and Takeuchi [14] introduced what they called Conditions A and B to construct two classes of inhomogeneous isoparametric hypersurfaces. Later on, using representations of the symmetric Clifford algebras C'_{m_1+1} (following the notation of [8]), Ferus, Karcher and Münzner [7] generalized their work to construct many more isoparametric hypersurfaces in $S^{2(m_1+m_2)+1}$; we will refer to these examples of Ozeki and Takeuchi and of Ferus, Karcher and Münzner collectively as OT-FKM type throughout the paper. The OT-FKM hypersurfaces are of multiplicities $\{m_1, m_2\}$, where

$$m_2 = k\delta(m_1) - m_1 - 1 \tag{1}$$

for some integer $k > 0$, and $\delta(m_1)$ is the dimension of an irreducible module of the skew-symmetric Clifford algebra C_{m_1-1} (following the notation of [8]). These multiplicities, with the exception of $\{m_1, m_2\} = \{2, 2\}$ or $\{4, 5\}$, turn out to be exactly the multiplicities of isoparametric hypersurfaces in spheres by the work of Stolz [16]. We will refer to (1) as the multiplicity formula. The author and his collaborators recently established in [2] that if $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface is of OT-FKM type with m_1 and m_2 given in (1). This leaves open only the cases in which the multiplicities $\{m_1, m_2\} = \{4, 5\}, \{3, 4\}, \{7, 8\}$ or $\{6, 9\}$ by the multiplicity formula; we refer to them as the exceptional multiplicity pairs.

One peculiar feature of the exceptional multiplicity pairs is that they are the only pairs for which incongruent examples of OT-FKM type admit $m_1 > m_2$ in (1). A deeper reason for this phenomenon manifests in [2], where it is shown that the condition $m_2 \geq 2m_1 - 1$ warrants that an ideal generated by certain (complexified) components of the 2nd fundamental form is reduced, i.e., has no nilpotent elements, at any point of M_+. The reducedness property no longer holds, as seen by the examples of OT-FKM type, when it comes to the exceptional multiplicity pairs.

The aforementioned examples of Ozeki and Takeuchi are of multiplicities $(m_1, m_2) = (3, 4k), (7, 8k)$ of OT-FKM type. For the construction, Ozeki and Takeuchi first imposed Condition A on the isoparametric hypersurface. That is, they stipulated that at some point x of M_+, the shape operators S_n of M_+ in all normal directions n have the same kernel. Then they imposed Condition B, which says that at the same point x the components of the (cubic) 3rd fundamental form are linearly spanned by the components of the (quadratic) 2nd fundamental form, with coefficients being linear functions of the coordinates of the tangent space to M_+ at x.

Through the work of Ferus, Karcher and Münzner [7], one knows that Condition B always holds for the OT-FKM type. Moreover, for the OT-FKM type, Condition A is true at some points on the focal submanifold of the smaller codimension in the case of the exceptional multiplicity pair {3, 4} or {7, 8}.

Dorfmeister and Neher then showed [5] that in fact Condition A alone implies that the isoparametric hypersurface is of OT-FKM type. It seems therefore that Condition A holds the key to the unsettled cases when the multiplicity pairs are {3, 4} and {7, 8}. Condition A thus deserves to be scrutinized and understood more thoroughly from different angles.

Dorfmeister and Neher's approach was via the isoparametric triple systems [3, 4], which are algebraic in nature. The proof also relies on the fairly involved algebraic classification result [9] about composition triples.

In this paper, we give a fairly short and rather straightforward proof of the result of Dorfmeister and Neher, with emphasis on the multiplicity pairs {3, 4} and {7, 8}, based on more geometric considerations. We make it explicit and apparent that the governing force of isoparametricity is the octonion algebra.

In Section 2, we review the octonion algebra whose left and right multiplications by the standard purely imaginary basis elements $e_1, \dots, e_7$, with e_0 understood to be the multiplicative identity, give rise to the two inequivalent Clifford representations J_a and J'_a, $1 \leq a \leq 7$, of C_7 on $\mathbb{R}^8$. We also review normalized orthogonal multiplications on $\mathbb{R}^{n+1}$, which are those bilinear binary operations $x \circ y$ such that $|x \circ y| = |x||y|$ and $e_0 \circ y = y$ for all $x, y \in \mathbb{R}^{n+1}$, where $(e_0, \dots, e_n)$ is the standard basis. In $\mathbb{O}$ we characterize all the normalized orthogonal multiplications as either $x \circ y = (x(y\bar{\alpha}))\alpha$ or $x \circ y = \alpha((\bar{\alpha}y)x)$, where α is a unit vector in $\mathbb{O}$ with the octonion multiplication employed on the right hand side. In particular, restricting to $\mathbb{H}$, the associativity of the quaternions implies $x \circ y = xy$, or $= yx$ for all $x, y \in \mathbb{H}$. At this point, we introduce the angle θ by setting $\alpha = \cos(\theta)e_0 + \sin(\theta)e$ for some purely imaginary unit e.

In Section 3 we recall the expansion formula and Condition A of Ozeki and Takeuchi, and show that at a point $x \in M_+$ of Condition A, the 2nd fundamental form components can be assumed to be $p_a(U, U) = 2\langle e_a A, B\rangle$, $1 \leq a \leq 7$, associated with the standard octonion multiplication, up to an appropriate choice of bases of the eigenspaces of the shape operator S of M_+ at x. Here, $U = A \oplus B \oplus C$ and A, B, C are, respectively, eigenvectors of S with eigenvalues $1, -1, 0$.

Section 4 introduces two points, $x^\# \in M_+$ and $x^* \in M_-$, related to $x \in M_+$ of Condition A, referred to as the mirror points of x. Here, $x^\#$ is also of Condition A, whose 2nd fundamental form components are given by $p^\#_a(V, V) = 2\langle e_a \circ A, B\rangle$, $1 \leq a \leq 7$, for a tangent vector V at $x^\#$ with the same eigenvector components A and B as above, where $\circ$ is some normalized orthogonal multiplication on the octonion algebra. Furthermore, the 2nd fundamental matrices at x^* are appropriate combination of those at x and $x^\#$, so that the 2nd fundamental form p^* at x^* can be succinctly expressed in terms of $\circ$ and the octonion multiplication to read $p^*(W, W) = -\sqrt{2}(XZ + Y \circ Z)$, where $W = X \oplus Y \oplus Z$ is the eigenvector decomposition of the shape operator of a tangent vector W at x^* with eigenvalues $1, -1, 0$, respectively.

In Section 5 we first present the octonion setup of the isoparametric hypersurfaces constructed by Ferus, Karcher and Münzner. Our expression is slightly more general than that given in [6] to account for all possible normalized orthogonal multiplications $\circ$ at $x^{\#}$ as indicated above. We show that, for the hypersurfaces constructed by Ferus, Karcher and Münzner, we can in fact perturb the original mirror point x^* with arbitrary θ to one at which $\theta = 0$ or π, i.e., at which either $a \circ b = ab$ or $a \circ b = ba$ for all $a, b \in \mathbb{O}$, so that up to isometry there are only two such hypersurfaces. We calculate the 3rd fundamental form at x^* to be $\mathbf{q}^*(W,W,W) = X(Y \circ Z) - Y \circ (XZ)$ with $W = X \oplus Y \oplus Z$ the same eigenvector decomposition at x^* as before. We then introduce the octonion setup of the isoparametric hypersurface constructed by Ozeki and Takeuchi. This is a hypersurface of both Conditions A and B at the point x of Condition A, where the 3rd fundamental form is not linear in all variables, whereas converting to x^* the 3rd fundamental form $\mathbf{q}^*$ turns out to be $\mathbf{q}^*(W, W, W) = (XY - YX)Z$ (the orthogonal multiplication $\circ$ at $x^{\#}$ coincides with the octonion multiplication in this case). The fact that q^* is linear in the eigenvector components X, Y, Z in both Ozeki–Takeuchi and Ferus–Karcher–Münzner examples points to that it will be simpler to look at the 3rd fundamental form at x^*.

Section 6 paves the way for the classification of the 3rd fundamental form at x^*, and hence of the isoparametric hypersurface of Condition A, by verifying first that at x^* the 3rd fundamental form $\mathbf{q}^*(W, W, W)$, for a tangent vector $W = X \oplus Y \oplus Z$ with eigenvector decomposition as before, is indeed only linear in X, Y and Z; therefore, we may denote $\mathbf{q}^*$ by $\mathbf{q}^*(X, Y, Z)$ instead to treat it as a multilinear form. We observe, by the eighth identity of the ten equations of Ozeki and Takeuchi [14, pp. 529–530] defining an isoparametric hypersurface, that at least $|\mathbf{q}^*(X, Y, Z)| = |X(Y \circ Z) - Y \circ (XZ)|$. We then prove several identities of $\mathbf{q}^*(X, Y, Z)$ about what happens when one interchanges the variables X, Y, Z, based on the fifth of the ten equations of Ozeki and Takeuchi. These properties together enable us to classify, up to an ambiguity of sign, of the important special case $\mathbf{q}^*(X, Y, e_0)$ that the remaining classification hinges on.

In Section 7, we prove that, if $\theta \neq 0$ and π, then the aforementioned ambiguity of sign can be removed and the isoparametric hypersurface must be of the type constructed by Ferus, Karcher and Münzner, so that the classification is reduced to the case when $\theta = 0$ or π, where the ambiguity of sign persists to an advantage. The classification is first done for the quaternionic case. The octonion case then follows naturally from that the octonion algebra is two (twisted) copies of the quaternion algebra. The sign choices then differentiate the example constructed by Ozeki and Takeuchi from the two by Ferus, Karcher and Münzner.

Lastly, we remark that in [10], [11], Miyaoka proves exactly that Condition A holds for either focal submanifold, when the number of principal curvatures is six, to show that such isoparametric hypersurfaces are homogeneous.

2. The octonion algebra and Clifford representations

Let $\mathbb{H}$ be the quaternion algebra with the standard basis $1, i, j, k$. The octonion algebra $\mathbb{O}$ is $\mathbb{H} \oplus \mathbb{H}$ with the multiplication

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where overline denotes quaternionic conjugation. For $x = (a, b) \in \mathbb{O}$, the conjugate of x is $\bar{x} := (\bar{a}, -b)$, and the real and imaginary parts of x are $(x \pm \bar{x})/2$, respectively. The inner product

$$\langle x, y \rangle := \frac{x\bar{y} + y\bar{x}}{2} \tag{2}$$

satisfies

$$\begin{gathered} \langle \bar{x}, \bar{y} \rangle = \langle x, y \rangle, \\ \langle xy, z \rangle = \langle y, \bar{x}z \rangle = \langle x, z\bar{y} \rangle, \\ x(\bar{y}z) + y(\bar{x}z) = (zx)\bar{y} + (zy)\bar{x} = 2\langle x, y \rangle z. \end{gathered} \tag{3}$$

In particular, first of all, the above formulae are the rules to follow when we interchange two objects in the octonion multiplication. Secondly, when x and y are perpendicular and purely imaginary in $\mathbb{O}$, they satisfy

$$xy = -yx, \quad x(yz) = -y(xz), \quad (zx)y = -(zy)x \tag{4}$$

for all $z \in \mathbb{O}$. As a consequence of (4), if we let $\epsilon := (0, 1) \in \mathbb{O}$, the standard orthonormal basis

$$(e_0, e_1, \ldots, e_7) := (1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon) \tag{5}$$

gives rise to orthogonal matrices $J_1, \ldots, J_7$ over $\mathbb{O}$, where $J_i(z) = e_i z$, $1 \le i \le 7$, such that

$$J_i J_k + J_k J_i = -2\delta_{ik}\, Id.$$

Similarly, the orthogonal matrices $J'_1, \ldots, J'_7$, where $J'_i(z) = ze_i$, satisfies

$$J'_i J'_k + J'_k J'_i = -2\delta_{ik}\, Id.$$

Recall [8] that the Clifford algebra C_n (respectively, C'_n) is the algebra over $\mathbb{R}$ generated by $E_1, \ldots, E_n$ subject to only the conditions that $(E_i)^2 = -1$ (respectively, $(E_i)^2 = 1$) and $E_i E_j = -E_j E_i$ for $i \neq j$. The structure of C_n (respectively, C'_n, to be displayed later) is well known [8],

n	1	2	3	4	5	6	7	8
C_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$

subject to the periodicity condition $C_{n+8} = C_n \otimes \mathbb{R}(16)$, of which the most important ones for our purposes are $C_2 = \mathbb{H}$, $C_3 = \mathbb{H} \oplus \mathbb{H}$, $C_6 = \mathbb{R}(8)$, the matrix ring of size 8-by-8 over $\mathbb{R}$, and $C_7 = \mathbb{R}(8) \oplus \mathbb{R}(8)$. The generators $E_1, \ldots, E_n$ projected to each irreducible summand of C_n, $n = 2, 3, 6, 7$, give rise to n matrices $T_1, \ldots, T_n$ in $\mathbb{R}(4)$ for C_2 and C_3, and in $\mathbb{R}(8)$ for C_6 and C_7, satisfying $(T_i)^2 = -Id$ and $T_iT_j = -T_jT_i$ for $i \neq j$. These T_i make $\mathbb{R}^4$ and $\mathbb{R}^8$ into irreducible C_n-modules. For $n = 2, 6$, there is only one such irreducible module as the number of irreducible summands of C_n is one, whereas for $n = 3, 7$, there are two inequivalent such irreducible modules as the number of irreducible summands of C_n is two. $T_1, \ldots, T_n$ are called representations of C_n on the appropriate Euclidean spaces.

The upshot is that the octonion (respectively, quaternionic) left and right multiplications generated above, i.e., $J_1, \ldots, J_7$ vs. $J_1', \ldots, J_7'$ (respectively, J_1, J_2, J_3 vs. J_1', J_2', J_3') are precisely the inequivalent representations of C_7 on $\mathbb{R}^8$ (respectively, C_3 on $\mathbb{R}^4$). These two representations are inequivalent as $J_1 \cdots J_7 = -Id$ whereas $J_1' \cdots J_7' = Id$ (respectively, $J_1J_2J_3 = -Id$ whereas $J_1'J_2'J_3' = Id$).

Now the subalgebra of C_7 linearly spanned by the even products of the Clifford generators is isomorphic to $C_6 \simeq \mathbb{R}(8)$ having a single irreducible summand. We see $J_1J_7, J_2J_7, \ldots, J_6J_7$ and $J_1'J_7', J_2'J_7', \ldots, J_6'J_7'$ are equivalent representations of C_6. That is, there is an orthogonal matrix U over $\mathbb{R}^8$ such that $U^{-1}J_iJ_7U = J_i'J_7'$ for $1 \le i \le 6$. A similar discussion also holds true for $\mathbb{H}$ by forgetting $e_4, \ldots, e_7$, since $C_2 = \mathbb{H}$. As an application, we prove the following to be employed later.

Lemma 1. *Let $m = 3, 7$. Let A_a, $1 \le a \le m$, be $(m+1)$-by-$(m+1)$ matrices satisfying*

$$A_aA_b^{tr} + A_bA_a^{tr} = 2\delta_{ab}Id. \tag{6}$$

Then there are two orthogonal matrices $P, Q \in O(m+1)$ for which $E_a := P^{-1}A_aQ$ satisfy $E_m = Id$, and for $1 \le a, b \le m-1$,

$$E_aE_b + E_bE_a = -2\delta_{ab}\, Id.$$

Proof. Clearly we can find two orthogonal matrices P and Q such that $P^{-1}A_mQ = Id$. (Take, e.g., $P = Id$ and $Q = (A_m)^{-1}$.) Set $a = m$. Then (6) reduces to

$$E_bE_b^{tr} = Id,$$
$$E_b + E_b^{tr} = 0,$$

for $1 \le b \le m-1$. This says exactly that E_b, $1 \le b \le m-1$, are orthogonal matrices satisfying $(E_b)^2 = -Id$ and $E_bE_c = -E_cE_b$ for $1 \le b \neq c \le m-1$. □

Corollary 1. *Conditions and notations as in* Lemma 1, *then we may pick orthogonal P and Q so that $A_a = PJ_aQ^{-1}$, $1 \le a \le m$.*

Proof. As mentioned earlier C_{m-1} is generated by $J_1J_m, \dots, J_{m-1}J_m$. Since $C_2 = \mathbb{H}$ and $C_6 = \mathbb{R}(8)$, we know all the Clifford representations are equivalent. Thus, there is an $O \in O(m+1)$ such that $E_a = OJ_aJ_mO^{-1}$ for $1 \le a \le m-1$. Changing the P and Q in the above lemma to PO and QO, we may assume now that $E_a = J_aJ_m$, $1 \le a \le m-1$. But then changing the (new) P to PJ_m^{-1}, we see that we may assume $E_b = J_b$ for $1 \le b \le m$. □

Recall [8] that a binary operation $\circ$ defined on $\mathbb{R}^{m+1}$ is called an *orthogonal multiplication* if $|x \circ y| = |x||y|$ for all $x, y \in \mathbb{R}^{m+1}$. Let $e_0, e_1, \dots, e_m$ be the standard basis of $\mathbb{R}^{m+1}$. We say $\circ$ is *normalized* if $e_0 \circ x = x$ for all $x \in \mathbb{R}^{m+1}$; we call $(\mathbb{R}^{m+1}, \circ)$ a normed algebra. It is well known that if $\circ$ is normalized, then the orthogonal maps $U_i(x) = e_i \circ x$, $1 \le i \le m$, satisfy $U_iU_j + U_jU_i = -2\delta_{ij}Id$ for all $1 \le i, j \le m$. In particular, $\mathbb{R}^{m+1}$ is a C_m-module, which is the case only when $m = 1, 3, 7$. Conversely, if we have such U_i, $1 \le i \le m$, we let $U_0 = Id$, then $e_i \circ e_j := U_i(e_j)$, $0 \le i, j \le m$, extended by linearity, gives a normalized orthogonal multiplication with $e_0 \circ x = x$ for all x. We identify $\mathbb{R}^{m+1}$ with $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, respectively, for $m = 1, 3, 7$.

Lemma 2. *Notation as above, for all z, then there is an orthogonal transformation T such that*

$$
\begin{aligned}
e_a \circ T(z) &= T(e_a z) \quad \textit{or} \\
&= T(z e_a)
\end{aligned}
\tag{7}
$$

for $1 \le a \le m$ and for all z in the normed algebra; moreover, there is a unit vector α such that $T(z) = z\alpha$ in the former case, or $T(z) = \alpha z$ in the latter. It follows that

$$x \circ y = (x(y\bar{\alpha}))\alpha$$

in the former case, or

$$x \circ y = \alpha((\bar{\alpha}y)x)$$

in the latter. In particular, (2) *and* (3) *remain true for $\circ$.*

Proof. Let $U_a(x) := e_a \circ x$. There is an orthogonal matrix T such that either $U_a = TJ_aT^{-1}$, or $U_a = TJ_a'T^{-1}$, $1 \le a \le m$. The first statement follows.

To prove the second statement, we may assume $e_a \circ T(z) = T(e_a z)$ without loss of generality. Then by the first statement just established, we obtain

$$\langle T(u) \circ \overline{T(v)}, w\rangle = \langle T(u), w \circ T(v)\rangle = \langle u, wv\rangle = \langle u\bar{v}, w\rangle,$$

so that

$$T(u) \circ \overline{T(v)} = u\bar{v}.$$

In particular, setting $\alpha := T(e_0)$ we derive

$$T(u) = u \circ \alpha.$$

But then the identity $\langle uv, w\rangle = \langle u \circ T(v), T(w)\rangle$ implies

$$\langle uv, w\rangle = \langle u \circ (v \circ \alpha), w \circ \alpha\rangle,$$

so that when we set $v = \bar{\alpha}$ we deduce

$$\langle u, w\alpha\rangle = \langle u, w \circ \alpha\rangle = \langle u, T(w)\rangle$$

for all u, w. That is, $T(w) = w\alpha$.

In particular, in the former case without loss of generality, we obtain

$$x \circ y = x \circ T(T^{-1}(y)) = T(xT^{-1}(y)) = (x(y\bar{\alpha}))\alpha. \qquad \square$$

REMARK 1. It follows by the associativity of $\mathbb{H}$ that $x \circ y = xy$ or $= yx$ for all $x, y \in \mathbb{H}$.

Now decompose α as

$$\alpha = \cos(\theta)e_0 + \sin(\theta)e$$

for some θ and some purely imaginary unit e.

Lemma 3. *We assume $x \circ y = (x(y\bar{\alpha}))\alpha$. When orthonormal $a, b \in \mathrm{Im}(\mathbb{O})$ are such that $(ab)e = \pm e_0$, then $a \circ b = ab$. On the other hand, when a, b and ab are all perpendicular to e, we have*

$$a \circ b = \cos(2\theta)ab + \sin(2\theta)(ab)e.$$

Proof. Let us first recall equation (4) above to be employed in the following calculations. We assume $ab = e$ without loss of generality. Then $b\bar{e} = \bar{a}$, so that

$$\begin{aligned} a \circ b &= (a(b\bar{\alpha}))\alpha \\ &= (a(\cos(\theta)b + \sin(\theta)\bar{a}))\alpha \\ &= (\cos(\theta)e + \sin(\theta)e_0)(\cos(\theta)e_0 + \sin(\theta)e) \\ &= e = ab. \end{aligned}$$

When a, b, and ab are all perpendicular to e, we observe that

$$\begin{aligned} a \circ b &= (a(b\bar{\alpha}))\alpha \\ &= (\cos(\theta)ab - \sin(\theta)a(be))\alpha \\ &= (\cos(\theta)ab - \sin(\theta)a(be))(\cos(\theta)e_0 + \sin(\theta)e) \\ &= (\cos^2(\theta) - \sin^2(\theta))ab + 2\sin(\theta)\cos(\theta)(ab)e, \end{aligned}$$

where we invoke (4) to write $a(be) = -(ab)e$ and $(a(be))e = ab$. □

In passing, let us briefly remark that the table for C'_n,

n	1	2	3	4	5	6	7	8
C'_n	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

subject to the periodicity condition $C'_{n+8} = C'_n \otimes \mathbb{R}(16)$, gives that the dimension of an irreducible module of the Clifford algebra C'_{m+1}, $m \geq 1$, is $2\delta(m)$, where $\delta(m)$ is the dimension of an irreducible module of C_{m-1}. We have $\delta(m + 8) = 16\delta(m)$ and $\delta(m) = 1, 2, 4, 4, 8, 8, 8, 8$ for $m = 1, \ldots, 8$, respectively.

3. The expansion formula of Ozeki and Takeuchi

Let M be an isoparametric hypersurface with four principal curvatures in the sphere. To fix our notation, we let V_+, V_- and V_0 be the eigenspaces of the shape operator of M_+ in the normal direction $\mathbf{n}_0$ associated with the eigenvalues $1, -1$ and 0, of dimension m_2, m_2, m_1, respectively. Let us agree that objects of these eigenspaces are indexed by α, μ and p, respectively, so that, typical vectors (coordinates) of V_+, V_- and V_0 are denoted by e_α, e_μ, e_p (x_α, y_μ, z_p), respectively, etc.

With this understood, the 2nd fundamental matrices S_a of M_+ in the normal direction $\mathbf{n}_a$, $0 \leq a \leq m_1$, upon fixing orthonormal bases e_α, e_μ, e_p, are

$$S_0 = \begin{pmatrix} Id & 0 & 0 \\ 0 & -Id & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_a = \begin{pmatrix} 0 & A_a & B_a \\ A_a^{tr} & 0 & C_a \\ B_a^{tr} & C_a^{tr} & 0 \end{pmatrix}, \quad 1 \leq a \leq m_1, \tag{8}$$

where $A_a\colon V_-\to V_+$, $B_a\colon V_0\to V_+$ and $C_a\colon V_0\to V_-$.

Ozeki and Takeuchi [14, pp. 523–530] obtained the expansion formula for the Cartan–Münzner polynomial F of M as follows.

$$
\begin{aligned}
F(tx+y+w) &= t^4+(2|y|^2-6|w|^2)t^2+8\left(\sum_{a=0}^{m_1}p_aw_a\right)t\\
&\quad+|y|^4-6|y|^2|w|^2+|w|^4-2\sum_{a=0}^{m_1}(p_a)^2+8\sum_{a=0}^{m_1}q_aw_a\\
&\quad+2\sum_{a,b=0}^{m_1}\langle\nabla p_a,\nabla p_b\rangle w_aw_b.
\end{aligned}
\tag{9}
$$

Here, x is a point on M_+, y is tangent to M_+ at x, and w is normal to M_+ with coordinates w_i with respect to the chosen orthonormal normal basis $\mathbf{n}_0,\mathbf{n}_1,\ldots,\mathbf{n}_{m_1}$ at x. Moreover, $p_a(y)$ (respectively, $q_a(y)$) is the a-th component of the 2nd (respectively, 3rd) fundamental form of M_+ at x. Furthermore, p_a and q_a are subject to ten equations [14, pp. 529–530], of which the first three assert that, since $S_{\mathbf{n}}$, the 2nd fundamental matrix of M_+ in any unit normal direction $\mathbf{n}$, has eigenvalues $1,-1,0$ with fixed multiplicities, it must be that $(S_{\mathbf{n}})^3=S_{\mathbf{n}}$. From this we can derive [15, p. 45]

$$
\begin{aligned}
&A_aA_b^{tr}+A_bA_a^{tr}+2(B_aB_b^{tr}+B_bB_a^{tr})=2\delta_{ab}Id,\\
&A_a^{tr}A_b+A_b^{tr}A_a+2(C_aC_b^{tr}+C_bC_a^{tr})=2\delta_{ab}Id,\\
&B_a^{tr}B_b+B_b^{tr}B_a=C_a^{tr}C_b+C_b^{tr}C_a,
\end{aligned}
\tag{10}
$$

for $a\neq b$.

A point $x\in M_+$ is said to be of *Condition A* [14] if the kernel of $S_{\mathbf{n}}$ is V_0 for all $\mathbf{n}$, which amounts to the same as saying the matrices $B_a=C_a=0$ for all $1\le a\le m_1$ in (8), so that (10) now reads

$$
A_aA_a^{tr}=Id,\quad A_aA_b^{tr}+A_bA_a^{tr}=0,\quad A_a^{tr}A_b+A_b^{tr}A_a=0,
\tag{11}
$$

for $1\le a\neq b\le m_1$. It follows that the symmetric 2nd fundamental matrices S_a, $0\le a\le m_1$, satisfy

$$
(S_a)^2=Id,\quad S_aS_b=-S_bS_a,\quad \forall a\neq b
\tag{12}
$$

when they are restricted to $V_+\oplus V_-$. In other words, (12) asserts that $V_+\oplus V_-\simeq\mathbb{R}^{2m_2}$ is a C'_{m_1+1}-module. Hence, by the passing remark at the end of the preceding section, we see $m_2=k\delta(m_1)$ for some k; thus among $(m_1,m_2)=(2,2),(4,5),(5,4)$, only the first is possible. (In fact, Ozeki and Takeuchi established, in their outline [15, p. 54] of the classification of the $(2,2)$ case that had been indicated by Cartan without proof [1], that Condition A holds on one of the focal submanifolds, from which there follows the classification.) But then the multiplicity formula $m_1 + m_2 + 1 = s\delta(m_1)$ for some s, with $(m_1, m_2) \neq (2,2), (4,5), (5,4)$, implies $m_1 + 1 = (s-k)\delta(m_1)$, so that $m_1 = 1, 3$ or 7. In particular, for $m_1 = 3$ or 7 we always have $m_2 \geq 2(m_1+1)$ when $m_2 \neq m_1 + 1$, whereas clearly $m_2 \geq 2m_1 - 1$ for $m_1 = 1$; therefore, by the result in [2] M is of the type of multiplicity (m_1, m_2) constructed by Ozeki and Takeuchi [14] when either $m_1 = 1$ or $m_2 \neq m_1 + 1$.

Thus from now on, we assume $m_2 = m_1 + 1$ with $m_1 = 3, 7$. Then (11) and Corollary 1 give the following.

Corollary 2. *At a point $x \in M_+$ of* Condition A *we may assume, by picking appropriate bases for V_+ and V_-, that $A_a = J_a$, $1 \leq a \leq m_1$.*

Proof. The matrices P and Q are for the basis changes in V_+ and V_-. □

4. Mirror points on M_+ and M_-

Assume Condition A at $x \in M_+$ when $(m_1, m_2) = (3,4)$ or $(7,8)$. As above, let $\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_{m_1}$ be an orthonormal normal basis at x. We decompose the tangent space to M_+ at x into the eigenspaces V_+, V_-, V_0, with coordinates x_α, y_μ, z_p as aforementioned, of the shape operator $S_{\mathbf{n}_0}$. Traversing along the great circle spanned by x and $\mathbf{n}_0$ by length $\pi/2$, we end up again on M_+ at $\mathbf{n}_0$ with x as a normal vector. Accordingly, set $x^\# := \mathbf{n}_0 \in M_+$ and $\mathbf{n}_0^\# := x$ normal to M_+ at $x^\#$. Then the eigenspaces $V_+^\#, V_-^\#, V_0^\#$ of $S_{\mathbf{n}_0^\#}$ with eigenvalues $1, -1, 0$ are [2, p. 15], respectively, $V_+, V_-, \mathbf{n}_0^\perp := \mathrm{span}(\mathbf{n}_1, \ldots, \mathbf{n}_{m_1})$. Moreover, $\mathbb{R}x \oplus V_0$ is the normal space to M_+ at $x^\#$.

Lemma 4. *$x^\# \in M_+$ is also of* Condition A.

Proof. Although a straightforward proof can be given by the formulae on p. 15 of [2], we choose to give one based on the expansion formula (9). Since x is of Condition A, we know p_a, $0 \leq a \leq m_1$, are quadratic forms in x_α and y_μ only. If we denote, at $x^\#$, all the involved quantities in (9) with an additional #, then $t^\# = w_0$, $w_0^\# = t$, $w_1^\# = z_1, \ldots, w_{m_1}^\# = z_{m_1}$. The 3rd term of (9) at $x^\#$, which is

$$8\left(\sum_{a=0}^{m_1} p_a^\# w_a^\#\right) t^\#,$$

is what determines the 2nd fundamental form at $x^\#$.

One obtains $p_0^\# = p_0$ by the fact that $p_0 w_0 t = p_0 w_0^\# t^\#$, which is part of the 3rd term of (9) at x, and no other terms contribute $w_0 t$ of the 1st degree. Furthermore, expanding $8q_0w_0$ in $z_1, \ldots, z_{m_1}$, we have

$$(13)\qquad \begin{aligned} 8q_0w_0 &= 8(H_1z_1 + \cdots + H_{m_1}z_{m_1})w_0 \\ &= 8(H_1w_1^\# + \cdots + H_{m_1}w_{m_1}^\#)t^\#, \end{aligned}$$

where $H_1,\ldots,H_{m_1}$ are quadratic forms only in x_α and y_μ, because q_0 is homogeneous of degree 1 in all x_α, y_μ, z_p [14, Lemma 15 (ii), p. 537]. No other terms of (9) contribute $z_1w_0, \ldots, z_{m_1}w_0$ of the 1st degree. It follows that $p_1^\# = H_1, \ldots, p_{m_1}^\# = H_{m_1}$. Hence, $x^\#$ is of Condition A as well. □

In (8), we use an additional # to indicate the corresponding quantities in the 2nd fundamental matrices at $x^\#$.

REMARK 2. Actually, Lemma 4 proves more. It shows that in fact q_0 determines $A_a^\#$, $1 \le a \le m_1$, whose entries are the coefficients of $H_a/2$, $1 \le a \le m_1$.

Next, let

$$x^* = \frac{x + \mathbf{n}_0}{\sqrt{2}}, \quad \mathbf{n}_0^* = \frac{x - \mathbf{n}_0}{\sqrt{2}}.$$

Then $x^* \in M_-$. We decompose the tangent space to M_- at x^* into the eigenspaces V_+^*, V_-^*, V_0^*, of the shape operator $S_{\mathbf{n}_0^*}$ with eigenvalues $1, -1, 0$, respectively. Again, we use an additional $*$ to denote all involved quantities at x^*.

Lemma 5. *We have*
(1) *At x^*, there holds $V_+^* = \mathbf{n}_0^\perp$, $V_-^* = V_0$, $V_0^* = V_-$, and the normal space to M_- at x^* is $\mathbb{R}\mathbf{n}_0^* \oplus V_+$.*
(2) *The second fundamental matrices at $x^* \in M_-$ are given by the $m_1 + 1$ $(= m_2)$ matrices*

$$S_a^* := \begin{pmatrix} 0 & 0 & B_a^* \\ 0 & 0 & C_a^* \\ (B_a^*)^{tr} & (C_a^*)^{tr} & 0 \end{pmatrix},$$

where $1 \le a \le m_1 + 1$, $m_1 = 3, 7$, and B_a^* (respectively, C_a^*) is the m_1-by-$(m_1 + 1)$ matrix formed by stacking together, in order, the a-th row of each of the m_1 matrices $-A_1/\sqrt{2},\ldots,-A_{m_1}/\sqrt{2}$ (respectively, $-A_1^\#/\sqrt{2},\ldots,-A_{m_1}^\#/\sqrt{2}$) at x (respectively, at $x^\#$).

Proof. Again we explore (9) with a slight modification. Namely, since (9) is with respect to M_+ while $x^* \in M_-$, we must consider the expansion of $-F$ at x^* in order to apply (9). From the definition of x^* and $\mathbf{n}_0^*$, we see $t = (t^* + w_0^*)/\sqrt{2}$ and $w_0 = (t^* - w_0^*)/\sqrt{2}$.

The collection of $(t^*)^2$ terms for $-F$ will reveal the tangent and normal space at x^*. But these terms come from the first two terms, $8p_0w_0t$, $-6|y|^2(w_0)^2$, $|w|^4$ and $2\langle\nabla p_0, \nabla p_0\rangle w_0^2$ in the expansion of F. As a result, the 2nd term in the expansion of $-F$ at x^* is

$$(t^*)^2\left(2\left(\sum_\mu y_\mu^2 + \sum_p z_p^2 + \sum_{a\geq 1} w_a^2\right) - 6\left((w_0^*)^2 + \sum_\alpha x_\alpha^2\right)\right),$$

where as before x_α, y_μ, z_p, w_a parametrize V_+, V_-, V_0 and the normal space to M_+ at x. On the other hand, the collection of $w_0^*t^*$, which comes from the same terms, gives p_0^* so that we end up with

$$p_0^* = \sum_{a\geq 1} w_a^2 - \sum_p z_p^2.$$

Hence, the first statement follows.

We denote the Euclidean coordinates of V_+^*, V_-^*, V_0^* and the normal space $\mathbb{R}n_0^* \oplus V_+$ at x^* by x_α^*, y_μ^*, z_p^* and w_a^*, respectively. Then the first statement says $x_\alpha^* = w_\alpha$, $y_\mu^* = z_\mu$, $1 \leq \alpha,\ \mu \leq m_1$, and $z_p^* = y_p$, $w_a^* = x_a$, $1 \leq a,\ p \leq m_1 + 1$.

The collection of the terms $w_1^*t^* = x_1t^*, \ldots, w_{m_2}^*t^* = x_{m_2}t^*$, with coefficients being quadratic forms in y_μ, z_p, w_a, $a \geq 1$, gives rise to the 2nd fundamental form of M_- at x^*. But these terms come only from $8\big(\sum_{a\geq 1} p_a w_a\big)t^*/\sqrt{2}$ obtained by the third term of (9), and from $8q_0t^*/\sqrt{2}$ obtained by the eighth term in (9). Combining them yields, by (13),

$$8\sum_\alpha\left(\sum_{a,\mu} 2A_{\alpha\mu a}y_\mu w_a\right)\frac{x_\alpha}{\sqrt{2}} + 8\sum_\alpha\left(\sum_{a,\mu} 2A^\#_{\alpha\mu a}y_\mu z_a\right)\frac{x_\alpha}{\sqrt{2}},$$

where $A_a = (A_{\alpha\mu a})$, $A_a^\# = (A^\#_{\alpha\mu a})$. This is the 2nd statement, where the negative sign accounts for considering $-F$ at x^*. □

Recall by Corollary 2 we may assume $A_a = J_a$, $1 \leq a \leq m_1$, at a point x of Condition A. We now understand the structure of $A_a^\#$, $1 \leq a \leq m_1$.

Lemma 6. *Let $e_0, e_1, \ldots, e_{m_1}$ be the standard basis of $\mathbb{R}^{m_2} \simeq \mathbb{H}$ or $\mathbb{O}$. Then $\langle A_a^\#(e_0), e_0\rangle = 0$ for all $1 \leq a \leq m_1$. In particular, we may assume $A_a^\#(e_0) = e_a$ for all $1 \leq a \leq m_1$; as a result, $(A_a^\#)^{tr}(e_0) = -e_a$. It follows that we may further assume that $A_a^\#$ are skew-symmetric, i.e., that $A_a^\#$, $1 \leq a \leq m_1$, form a Clifford system.*

Proof. Since $A_a = J_a$, $1 \leq a \leq m_1$, the second item in Lemma 5 says that the a-th column of B_a^* is zero, $1 \leq a \leq m_1$. Now, the third equation of (10) applied to the point $x^* \in M_-$ says

$$(B_a^*)^{tr}B_b^* + (B_b^*)^{tr}B_a^* = (C_a^*)^{tr}C_b^* + (C_b^*)^{tr}C_a^*, \tag{14}$$

which implies that the a-th column of C_a^* is also zero, $1 \le a \le m_1$, when we set $a = b$ in the equation. Equivalently, this means the diagonal of $A_a^\#$, $1 \le a \le m_1$, is zero. So,

$$\langle A_a^\#(e_b), e_b\rangle = 0, \quad 1 \le a \le m_1,\ 0 \le b \le m_1. \tag{15}$$

Since $v_a := A_a^\#(e_0)$, $1 \le a \le m_1$, are perpendicular to each other by the third equation of (11) and Lemma 4, we deduce therefore that $v_a, 1 \le a \le m_1$, span$e_0^\perp$. Thus, there is an orthogonal matrix (θ_{ab}) of size m_1-by-m_1 such that $\sum_b \theta_{ab} v_b = e_a$. The matrices $\sum_b \theta_{ab} A_b^\#$, $1 \le a \le m_1$, which are the A-blocks of the 2nd fundamental matrices corresponding to the new normal basis $\mathbf{n}_0' := \mathbf{n}_0^\#$, $\mathbf{n}_a' := \sum_b \theta_{ab}\mathbf{n}_b^\#$, $1 \le a \le m_1$, at $x^\# \in M_+$, will serve as the new $A_a^\#$ mapping e_0 to e_a. Thus without loss of generality we may now assume $A_a^\#(e_0) = e_a$, $1 \le a \le m_1$.

In coordinates, (14) assumes the form

$$\sum_{a=1}^{m_1}(A_{\alpha\mu a}A_{\beta\nu a} + A_{\beta\mu a}A_{\alpha\nu a}) = \sum_{b=1}^{m_1}(A^\#_{\alpha\mu b}A^\#_{\beta\nu b} + A^\#_{\alpha\mu b}A^\#_{\beta\nu b}). \tag{16}$$

Hence, if we pick $\alpha = \mu = 0$ and $\beta = \nu = a$, $1 \le a \le m_1$, we see by the fact that $A_a = J_a$, $1 \le a \le m_1$, that the product of the $(a, 0)$-entry and the $(0, a)$-entry of $A_a^\#$ is -1, so that the latter is -1 since the former is 1. This forces all other entries of the first row of $A_a^\#$ to be zero as $A_a^\#$ is orthogonal. In conclusion, $(A_a^\#)^{tr}(e_0) = -e_a$. That is, $A_a^\#$ is skew-symmetric in the first row and column, $1 \le a \le m_1$.

Since $A_a^\#$, $1 \le a \le m_1$, leave $\langle e_0, e_a\rangle^\perp$ invariant and since the group of automorphism of $\mathbb{H}$ and $\mathbb{O}$, which are $SO(3)$ and G_2, respectively, are transitive on the unit sphere of $e_0^\perp$, we see that any purely imaginary unit vector e can serve as e_1. Therefore, $\langle A_a^\#(e), e\rangle = 0$ by (15). It follows that $A_a^\#$ restricted on $\langle e_0, e_a\rangle^\perp$ is also skew-symmetric. In particular, (11) says that $A_a^\#$, $1 \le a \le m_1$, form a Clifford system. □

DEFINITION 1. For notational ease, we let $A_0^\# = Id$. We define a normalized orthogonal multiplication $\circ$ on $\mathbb{R}^{m_2}$ by $e_a \circ e_b = A_a^\#(e_b)$ for $0 \le a, b \le m_1$, and extend it by linearity.

We can now determine the 2nd fundamental form at $x^* \in M_-$.

Proposition 1. *For $(m_1, m_2) = (7, 8)$, the 2nd fundamental form $\mathbf{p}^*$ at $x^* \in M_-$ is given by*

$$\mathbf{p}^*(W, W) = -\sqrt{2}(XZ + Y \circ Z) \tag{17}$$

for a tangent vector $W = X \oplus Y \oplus Z$ at x^*, where $X \in V_+^* \simeq \mathrm{Im}(\mathbb{O})$, the purely imaginary part of $\mathbb{O}$, $Y \in V_-^* \simeq \mathrm{Im}(\mathbb{O})$, $Z \in V_0^* \simeq \mathbb{O}$, and $\mathbf{p}^*$ lives in the normal space to M_-, which is $\mathbb{R}\mathbf{n}_0^* \oplus V_+ \simeq \mathbb{R} \oplus \mathbb{O}$.

For $(m_1, m_2) = (3, 4)$, one has the same formula by forgetting the orthogonal complement of $\mathbb{H}$ in $\mathbb{O}$.

Proof. It is an immediate consequence of the second item in the statement of Lemma 5, which can be rephrased as $\langle B_a^*(e_p), e_\alpha\rangle = \langle e_\alpha e_p, e_a\rangle$ and $\langle C_a^*(e_p), e_\mu\rangle = \langle e_\mu \circ e_p, e_a\rangle$. □

Henceforth, we will mainly study the structure of isoparametric hypersurfaces in the case when $(m_1, m_2) = (7, 8)$.

5. Octonion realization of the isoparametric hypersurfaces of OT-FKM type

5.1. Isoparametric hypersurfaces constructed by Ferus Karcher and Münzner. Let $\mathbb{R}^{32}$ be the direct sum of four copies of $\mathbb{O}$. We identify $(0, 0, -e_0, 0)$ with $x \in M_+$; $\{(0, 0, Y, 0)\colon Y \in \mathrm{Im}(\mathbb{O})\}$ with $V_0 = V_-^*$; $(0, e_0, 0, 0)$ with $\mathbf{n}_0 \in M_+$; and $\{(0, X, 0, 0)\colon X \in \mathrm{Im}(\mathbb{O})\}$ with V_+^*. We identify $V_- = V_0^*$ with $(Z, 0, 0, Z)$, $Z \in \mathbb{O}$, and identify V_+, which is the normal subspace perpendicular to $\mathbf{n}_0^*$ at x^*, with $(W, 0, 0, -W)$. The notation here is in accordance with Lemma 5 and Proposition 1.

Consider the orthogonal transformations

$$
\begin{aligned}
&P_{-1}\colon (A, X, Y, B) \mapsto (A, -X, Y, -B),\\
&P_a\colon (A, X, Y, B) \mapsto (-Xe_a, -A\bar{e}_a, -B \circ \bar{e}_a, -Y \circ e_a)
\end{aligned}
\tag{18}
$$

for $0 \le a \le 7$. It is immediate that $P_iP_j + P_jP_i = 2\delta_{ij}Id$, $-1 \le i, j \le 7$. Therefore, the symmetric Clifford system $P_{-1}, P_0, \ldots, P_7$ over M_- generates an isoparametric hypersurface M constructed by Ferus, Karcher and Münzner [6], [7].

It is readily checked that

$$
\begin{aligned}
&\langle P_a((Z, X, Y, Z)), (Z, X, Y, Z)\rangle\\
&= 2\langle XZ + Y \circ Z, e_a\rangle,
\end{aligned}
\tag{19}
$$

and $\langle P_{-1}((Z, X, Y, Z)), (Z, X, Y, Z)\rangle = -|X|^2 + |Y|^2$. That is, rescaling Z, $-P_i$, $-1 \le i \le 7$, restricted to the tangent space to M_- at x^* give exactly the 2nd fundamental form by Proposition 1.

Recall M_- is said to be of *Condition B* [14] at x^* if

$$
q_b^* = \sum_{a=-1}^{m_1} r_{ab}\, p_a^*,
\tag{20}
$$

where $r_{ab} = -r_{ba}$, $-1 \le a, b \le m_1$; here, we set $q_{-1}^* = 0$ and $p_{-1}^* = |X|^2 - |Y|^2$. An isoparametric hypersurface of OT-FKM type satisfies Condition B; it is well known [7] that

$$
r_{ab}(v) = \langle P_a(v), n_b\rangle,
\tag{21}
$$

where v is tangent to the focal submanifold, which is M_- in our case, defined by the symmetric Clifford matrices P_a as the zero locus of $\langle P_a(x), x\rangle = 0$, $-1 \le a \le 7$, and n_a are the normal basis elements. With $n_a = (e_a, 0, 0, -e_a)/\sqrt{2}$ and $v = X + Y + Z$, it is straightforward to find $r_{ab} = \langle e_a, Xe_b - Y \circ e_b\rangle$ and so

$$\mathbf{q}^*(W, W, W) = X(Y \circ Z) - Y \circ (XZ), \tag{22}$$

for a tangent vector $W = X \oplus Y \oplus Z$ at x^*, in the case of isoparametric hypersurfaces constructed by Ferus, Karcher and Münzner.

5.2. Perturbing the mirror point x^*.

Proposition 2. *There is a point x^* on M_- of the isoparametric hypersurfaces constructed by Ferus, Karcher and Münzner at which either $a \circ b = ab$ or $a \circ b = ba$ for all $a, b \in \mathbb{O}$, up to an isometry of the ambient Euclidean space.*

Proof. Similar to Lemma 2 we can apply an orthogonal transformation U such that

$$U(z) \circ e_a = U(ze_a) \quad \text{or} \quad U(e_a z)$$

for all a, z. With $x^\# = (0, e_0, 0, 0)$ and $\mathbf{n}^\# = (0, 0, n, 0)$ for $n = -U(e_0)$, the normal space to M_- at $x^*_{\mathbf{n}} := (x^\# + \mathbf{n}^\#)/\sqrt{2}$ is spanned by

$$P_{-1}(x^*_{\mathbf{n}}) = (0, -e_0, -U(e_0), 0)/\sqrt{2},$$

and

$$P_a(x^*_{\mathbf{n}}) = (-e_a, 0, 0, U(e_a))/\sqrt{2}, \quad 0 \le a \le 7,$$

whereas the tangent vectors, being perpendicular to $x^*_{\mathbf{n}}$ and the normal vectors, are thus of the form $(Z, X, U(Y), U(Z))$; therefore,

$$\begin{aligned} &-\langle P_a((Z, X, U(Y), U(Z)), (Z, X, U(Y), U(Z))\rangle \\ &= -2\langle XZ + YZ, e_a\rangle \quad \text{or} \quad -2\langle XZ + ZY, e_a\rangle, \end{aligned}$$

for $0 \le a \le 7$, give that the 2nd fundamental form at $x^*_{\mathbf{n}}$ is $-\sqrt{2}(XZ + YZ)$, or $-\sqrt{2}(XZ + ZY)$ after rescaling Z. □

5.3. Isoparametric hypersurfaces of the type constructed by Ozeki and Takeuchi. Let $\mathbb{R}^{32}$ be identified as the direct sum of four copies of $\mathbb{O}$. Let $x = (0, 0, e_0, 0)$ and at x identify V_+ as the first copy, V_- as the second copy and the normal space as the fourth copy of $\mathbb{O}$ in $\mathbb{R}^{32}$. Lastly, identify the imaginary part of the third copy of $\mathbb{O}$ as V_0 at x. Define

$$P_0 \colon (u, v, z, w) \mapsto (u, -v, w, z),$$
$$P_a \colon (u, v, z, w) \mapsto (e_a v, -e_a u, e_a w, -e_a z)$$

for $1 \le a \le 7$. A calculation similar to the above one gives that the symmetric Clifford system $P_0, P_1, \dots, P_7$ over M_+ defines an isoparametric hypersurface M, where $x \in M_+$ is of Condition A whose 2nd fundamental form is

$$p_0 = |u|^2 - |v|^2, \quad p_a = 2\langle e_a, u\bar{v}\rangle, \quad 1 \le a \le 7.$$

In particular, the orthogonal multiplication $\circ$ at $x^\#$ coincides with the octonian multiplication. By [14, 15], [7], we know x is also of Condition B. Indeed, with the normal basis $n_b = (0, 0, 0, e_b)$ and a tangent vector $x = (u, v, z, 0)$, where $u, v \in \mathbb{O}$ and $z \in \mathrm{Im}(\mathbb{O})$, we calculate by (20) to deduce $r_{0b} = \langle z, e_b\rangle$, $1 \le b \le 7$ and $r_{ab} = -\langle e_a z, e_b\rangle$, $0 \le a \ne b \le 7$. From this we obtain by (21)

$$q_0 = 2\langle z, u\bar{v}\rangle,$$
$$q_a = \langle z, e_a\rangle(|u|^2 - |v|^2 - 2\langle u, \bar{v}\rangle) - 2\langle z e_a, u\bar{v}\rangle,$$

for $1 \le a \le 7$ [14, p. 556].

Since q_0 gives $A_a^\#$, $1 \le a \le 7$, by Remark 2, we see $A_a = A_a^\# = J_a$, $1 \le a \le 7$. On the other hand, Remark 4, to be given later, gives that

$$\mathbf{q}^* = \sum_{a=0}^{m_1} w_a q_a^* = \langle 2z(u\bar{v}) - 2\langle u, v\rangle z, w\rangle$$

with $w = \sum_{a=0}^{m_1} w_a e_a$. The identification $X = w \in V_+^* \simeq \mathrm{Im}(\mathbb{O})$, $Y = -z \in V_-^* \simeq \mathrm{Im}(\mathbb{O})$, $Z = -v \in V_0^*$, and $W = u$ in the normal space to $x^* \in M_-$ derives that, for a tangent vector $U = X \oplus Y \oplus Z$ and a normal vector W at x^*,

$$\begin{aligned}
&\langle \mathbf{q}^*(U, U, U), W\rangle \\
&= \langle 2Y(W\bar{Z}) - 2\langle W, Z\rangle Y, X\rangle = \langle -2(YX)Z - 2\langle X, Y\rangle Z, W\rangle \\
&= \langle 2(XY)Z + 2\langle X, Y\rangle Z, W\rangle = \langle (XY)Z - (YX)Z, W\rangle.
\end{aligned}$$

We thus arrive at

$$\mathbf{q}^*(U, U, U) = (XY - YX)Z$$

for a tangent vector $U = X \oplus Y \oplus Z$ at x^*. The fact that the 3rd fundamental form at x of Condition A in the example of Ozeki and Takeuchi is not linear in all variables whereas the 3rd fundamental form is linear at x^*, in the cases of both Ozeki–Takeuchi and Ferus–Karcher–Münzner, in all variables points to that it will be simpler to look at the mirror point x^* instead.

6. The 3rd fundamental form at a mirror point on M_-

Henceforth, we concentrate on $x^* \in M_-$. It is understood $(m_1, m_2) = (7, 8)$. In coordinate calculations we use $x^*_\alpha, y^*_\mu, z^*_p$ to denote coordinates of V^*_+, V^*_-, V^*_0, respectively, so that $X = \sum_{\alpha=1}^{m_1} x^*_\alpha e_\alpha$, $Y = \sum_{\mu=1}^{m_1} y^*_\mu e_\mu$, and $Z = \sum_{p=0}^{m_1} z^*_p e_p$.

Lemma 7. *At $x^* \in M_-$, we have $q^*_0 = 0$.*

Proof. This follows from Remark 2. There, we see that q_0 at $x \in M_+$ determines $A^\#_a$, $1 \le a \le m_1$, and vice versa. Hence, if $A_a = 0$, $1 \le a \le m_1$, then (16) derives that $A^\#_a = 0$, $1 \le a \le m_1$, so that $q_0 = 0$. Now replace F by $-F$ and $x^\#$ by x^* and observe that $A^*_a = 0$, $1 \le a \le m_1$ by the second item of Lemma 5. □

Now that $q^*_0 = 0$, there will be no confusion for us to change our notation from now on to rename $q^*_1, \dots, q^*_{m_2}$, where $m_2 = m_1 + 1$, at x^* to be $q^*_0, \dots, q^*_{m_1}$, so that the 3rd fundamental form can be written as $\mathbf{q}^* = \sum_{a=0}^{m_1} q^*_a e_a$ in accordance with the standard octonion basis $e_0, e_1, \dots, e_{m_1}$.

Lemma 8. *At $x^* \in M_-$, the 3rd fundamental form $\mathbf{q}^*$ satisfies*

$$|\mathbf{q}^*(U, U, U)| = |X(Y \circ Z) - Y \circ (XZ)| \tag{23}$$

for a tangent vector $U = X \oplus Y \oplus Z$ at x^*.

Proof. Recall the identity for an isoparametric hypersurface [14, p. 530]

$$16|\mathbf{q}^*|^2 = 16G(|X|^2 + |Y|^2 + |Z|^2) - |\nabla G|^2, \tag{24}$$

where $G = \sum_{a=-1}^{m_1} (p^*_a)^2$, that an isoparametric hypersurface must satisfy. It is understood that $p^*_{-1} = |X|^2 - |Y|^2$.

For the isoparametric hypersurfaces of the type constructed by Ferus, Karcher and Münzner, we know the left hand side of (24) is $|X(Y \circ Z) - Y \circ (XZ)|$ by (22). On the other hand, the right hand side of (24) depends only on the 2nd fundamental form, which is exactly $-\sqrt{2}(XZ + Y \circ Z)$ for the type constructed by Ferus, Karcher and Münzner by (19) and in general by Proposition 1. □

REMARK 3. When $m_1 = 1$, the underlying normed algebra is $\mathbb{C}$. Therefore, Lemma 8 implies $\mathbf{q}^* = 0$.

When $m_1 = 2$, Ozeki and Takeuchi established [15, p. 54, Case (B_1)] that one can choose appropriate coordinates so that $\mathbf{p}^*$ is identical with that of the homogeneous example. The same argument as in Lemma 8 then implies that $\mathbf{q}^* = 0$ as it is so for the homogeneous example [15, p. 41], so that the isoparamentric hypersurface is exactly the homogeneous one.

Proposition 3. *For $0 \le a \le m_1$ at x^*, we have $q_a^* = \sum_{\alpha\mu p} q_a^{\alpha\mu p} x_\alpha^* y_\mu^* z_p^*$ for some coefficients $q_a^{\alpha\mu p}$. That is, $\mathbf{q}^*$ is homogeneous of degree 1 in X, Y, Z.*

Proof. We record the equation from Ozeki and Takeuchi [15, p. 529], with respect to $-F$, that

$$\langle \nabla p_i^*, \nabla q_j^* \rangle + \langle \nabla p_j^*, \nabla q_i^* \rangle = 0 \tag{25}$$

for all $-1 \le i \ne j \le m_1$. Picking $i = -1$ and $j = a$, we get

$$\langle \nabla p_{-1}^*, \nabla q_a^* \rangle = 0 \tag{26}$$

since $q_{-1}^* = 0$ by Lemma 7. Note that $p_{-1}^* = \sum_\alpha (x_\alpha^*)^2 - \sum_\mu (y_\mu^*)^2$.

For the component $\sum_{\mu\nu p} q_a^{\alpha\beta p} x_\alpha^* x_\beta^* z_p^*$ of q_a^*, where α, β are in the same index range over V_+^*, the left hand side of (26) gives $4\sum_{\alpha\beta p} q_a^{\alpha\beta p} x_\alpha^* x_\beta^* z_p^*$ (Euler's identity for homogeneous polynomials). Similarly for the component $\sum_{\alpha pq} q_a^{\alpha pq} x_\alpha^* z_p^* z_q^*$, where p, q are in the same index range over V_0^*, the left hand side of (26) derives $2\sum_{\alpha pq} q_a^{\alpha pq} x_\alpha^* z_p^* z_q^*$, etc. The vanishing of the right hand side of (26) therefore shows that all those components, exactly two of whose coordinates are in the same index range, are zero. The same reasoning gives zero to the components whose coordinates are either all in the α-range, or all in the μ-range (over V_-^*). The only component of repeated ranges not accounted for by this procedure is thus of the form $\sum_{pqr} q_a^{pqr} z_p^* z_q^* z_r^*$ with p, q, r in the same index range. However, Lemma 15 (i) of [14, p. 537] asserts that such components cannot exist. □

REMARK 4. $\mathbf{q}^*$ at $x^* \in M_-$ is determined by collecting the part of $\mathbf{q}$ at $x \in M_+$ linear in all variables. Explicitly, since $\mathbf{q}^*$ is of degree 1 in X, Y, Z, the term $8\sum_{a=0}^{m_1} q_a^* w_a^*$ is of the form $8\sum_{\alpha\mu pa} q_a^{\alpha\mu p} x_\alpha^* y_\mu^* z_p^* w_a^*$, which is also linear in $x_\alpha, y_\mu, z_p, w_a$. This is because by our convention, $x_\alpha, y_\mu, z_p, w_a$ parametrize, respectively, V_+, V_-, V_0 and the normal space to $x \in M_+$; we know by the first item of Lemma 5 that $x_\alpha^* = w_\alpha, y_\mu^* = z_\mu$, $1 \le \alpha, \mu \le m_1$, and $z_p^* = y_p$, $w_a^* = x_a$, $1 \le a, p \le m_2$. However, a glance at (9) shows that the only term of F that contributes to items linear in $x_\alpha, y_\mu, z_p, w_a$ comes from $8\sum_{a=1}^{m_1} q_a w_a$.

We denote $\mathbf{q}^*$ by $\mathbf{q}^*(X, Y, Z)$, where $X \in V_+^*$, $Y \in V_-^*$ and $Z \in V_0^*$; thanks to Proposition 3 we see that $\mathbf{q}^*$ is a multilinear form in X, Y, Z. We extend $\mathbf{q}^*(X, Y, Z)$ by requiring that $\mathbf{q}^*(e_0, Y, Z) = 0$ and $\mathbf{q}^*(X, e_0, Z) = 0$ for all $X, Y \in \mathbb{O}$. This is well-defined as the right hand side of (23) is 0 if either $X = e_0$ or $Y = e_0$. With this extension (23) continues to hold.

Lemma 9. *For $0 \le a,\ p \le m_1$ and $X, Y \in \mathbb{O}$, we have*

$$\begin{aligned}
&\langle \mathbf{q}^*(X, Y, e_a), e_a\rangle = 0,\\
&\langle \mathbf{q}^*(X, Y, e_0), X\rangle = \langle \mathbf{q}^*(X, Y, e_0), Y\rangle = 0,\\
&\langle \mathbf{q}^*(e_a, Y, e_p), e_a\rangle = -\langle \mathbf{q}^*(e_a\bar{e}_p, Y, e_0), e_a\rangle,\\
&\langle \mathbf{q}^*(X, e_a, e_p), e_a\rangle = -\langle \mathbf{q}^*(X, e_a \circ \bar{e}_p, e_0), e_a\rangle,\\
&\langle \mathbf{q}^*(e_a, Y, e_a), e_p\rangle = -\langle \mathbf{q}^*(e_p\bar{e}_a, Y, e_0), e_a\rangle,\\
&\langle \mathbf{q}^*(X, e_a, e_a), e_p\rangle = -\langle \mathbf{q}^*(X, e_p \circ \bar{e}_a, e_0), e_a\rangle.
\end{aligned} \tag{27}$$

Proof. Setting $i = a$, $j = b$ in (25) and considering the homogeneous part in Y and Z only, we obtain

$$\begin{aligned}
\sum_{\alpha=0}^{m_1} &\langle \mathbf{q}^*(e_\alpha, Y, Z), e_a\rangle\langle e_\alpha Z, e_b\rangle\\
&+ \langle \mathbf{q}^*(e_\alpha, Y, Z), e_b\rangle\langle e_\alpha Z, e_a\rangle = 0.
\end{aligned}$$

Equivalently, it is

$$\begin{aligned}
&\sum_{\alpha=0}^{m_1} \langle \mathbf{q}^*(e_\alpha, Y, e_p), e_a\rangle\langle e_\alpha e_q, e_b\rangle\\
&+ \sum_{\alpha=0}^{m_1} \langle \mathbf{q}^*(e_\alpha, Y, e_q), e_a\rangle\langle e_\alpha e_p, e_b\rangle\\
&+ \sum_{\alpha=0}^{m_1} \langle \mathbf{q}^*(e_\alpha, Y, e_p), e_b\rangle\langle e_\alpha e_q, e_a\rangle\\
&+ \sum_{\alpha=0}^{m_1} \langle \mathbf{q}^*(e_\alpha, Y, e_q), e_b\rangle\langle e_\alpha e_p, e_a\rangle = 0.
\end{aligned} \tag{28}$$

Setting $q = a = b$ in (28), we see the first and the third sums on the left are 0, since they are simplified to $\langle q^*(e_0, Y, e_p), e_a\rangle$. Hence we obtain $\langle \mathbf{q}^*(e_\alpha, Y, e_a), e_a\rangle = 0$, where e_α is parallel to $e_a\bar{e}_p$ for any p. Since $e_a\bar{e}_p$ runs through $e_0, \ldots, e_{m_1}$ when we vary p, we see $\langle \mathbf{q}^*(e_\alpha, Y, e_a), e_a\rangle = 0$ for all α. That is,

$$\langle \mathbf{q}^*(X, Y, e_a), e_a\rangle = 0 \tag{29}$$

for all X, Y, e_a. In particular, the first identity of (27) is true.

On the other hand, setting $a = b$ and $p = q = 0$ we deduce the identity $\langle \mathbf{q}^*(e_a, Y, e_0), e_a\rangle = 0$ for all a, which implies that

$$\langle \mathbf{q}^*(X, Y, e_0), X\rangle = 0 \tag{30}$$

for all $X \in \mathrm{Im}(\mathbb{O})$, because any unit imaginary X can serve as e_a, for some $a \neq 0$, since the group of automorphism of the normed algebra is transitive on the unit imaginary sphere. It follows from (30), (29) for $a = 0$, and $\mathbf{q}^*(e_0, Y, Z) = 0$ that $\langle \mathbf{q}^*(X, Y, e_0), X\rangle = 0$ for all $X, Y \in \mathbb{O}$. Hence, the second identity of (27) is true.

The third identity of (27) follows from setting $a = b$ and $q = 0$.

The fifth identity comes from setting $p = b$ and $q = 0$ and employing (29).

The fourth and sixth identities are derived from an equation similar to (28) when, in (25), we look at the homogeneous part in X and Z only. □

Corollary 3. *For* $X, Y \in \mathrm{Im}(\mathbb{O})$,

$$\begin{aligned}
&\langle \mathbf{q}^*(X, Y, Z), Z\rangle = 0, \quad Z \in \mathrm{Im}(\mathbb{O}) \quad \textit{or} \quad Z = e_0,\\
&\langle \mathbf{q}^*(X, Y, e_0), X\rangle = \langle \mathbf{q}^*(X, Y, e_0), Y\rangle = 0,\\
&\langle \mathbf{q}^*(X, Y, Z), X\rangle = -\langle \mathbf{q}^*(X\bar{Z}, Y, e_0), X\rangle, \quad Z \in \mathbb{O},\\
&\langle \mathbf{q}^*(X, Y, Z), Y\rangle = -\langle \mathbf{q}^*(X, Y \circ \bar{Z}, e_0), Y\rangle, \quad Z \in \mathbb{O},\\
&\langle \mathbf{q}^*(X, Y, X), Z\rangle = \langle \mathbf{q}^*(ZX, Y, e_0), X\rangle, \quad Z \in \mathbb{O},\\
&\langle \mathbf{q}^*(X, Y, Y), Z\rangle = \langle \mathbf{q}^*(X, Z \circ Y, e_0), Y\rangle, \quad Z \in \mathbb{O}.
\end{aligned}$$

Proof. It follows from the identities, in order, of Lemma 9 and the transitivity of the automorphism group of $\mathbb{O}$ on its imaginary unit sphere. □

In fact, we can strengthen the first identity of Corollary 3 as follows.

Lemma 10.

$$\langle \mathbf{q}^*(U\bar{V}, Y, V), W\rangle = -\langle \mathbf{q}^*(W\bar{V}, Y, V), U\rangle, \tag{31}$$

where $U, Y, W \in \mathbb{O}$ *and* V *is either* e_0 *or purely imaginary. In particular,* $\langle \mathbf{q}^*(X, Y, Z), W\rangle$ *is skew-symmetric for* Z *and* W *in* $\mathbb{O}$. *Moreover,* $\langle \mathbf{q}^*(X, Y, e_0), Z\rangle$ *is skew-symmetric in all* $X, Y, Z \in \mathbb{O}$.

Proof. Setting $p = q$ in (28), we obtain

$$\langle \mathbf{q}^*(e_b\bar{e}_p, Y, e_p), e_a\rangle = -\langle \mathbf{q}^*(e_a\bar{e}_p, Y, e_p), e_b\rangle.$$

The first statement follows.

Setting $U = e_0$ and $X := W\bar{V}$ for a purely imaginary V, we obtain

$$\begin{aligned}
\langle \mathbf{q}^*(X, Y, V), e_0\rangle &= \langle \mathbf{q}^*(V, Y, V), XV\rangle\\
&= -\langle \mathbf{q}^*(X, Y, e_0), V\rangle,
\end{aligned} \tag{32}$$

where the last equality follows from the fifth identity of Corollary 3.

The second statement is a consequence of (32) and the first identity of Corollary 3, which says that $\langle \mathbf{q}^*(X, Y, Z), W\rangle$ is skew-symmetric in Z and W when Z and W are purely imaginary.

The third statement follows from anti-symmetrizing the X and Y slots of the two equations, respectively, of the second identity of Corollary 3. □

Corollary 4. *For $W \in \mathbb{O}$, we have*

$$\langle \mathbf{q}^*(X, Y, W), XW\rangle = 0 \quad \textit{and} \quad \langle \mathbf{q}^*(X, Y, W), Y \circ W\rangle = 0,$$

so that anti-symmetrizing we get

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, U), XV\rangle &= -\langle \mathbf{q}^*(X, Y, V), XU\rangle,\\ \langle \mathbf{q}^*(X, Y, U), Y \circ V\rangle &= -\langle \mathbf{q}^*(X, Y, V), Y \circ U\rangle\end{aligned}$$

for $U, V \in \mathbb{O}$.

Proof. Setting $U = XW$ for $W \in \mathrm{Im}(\mathbb{O})$, we derive from (31)

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, W), XW\rangle &= \langle \mathbf{q}^*(U\bar{W}, Y, W), U\rangle\\ &= -\langle \mathbf{q}^*(U\bar{W}, Y, W), U\rangle = 0.\end{aligned}$$

We next calculate $\langle \mathbf{q}^*(X, Y, e_0), XW\rangle$ for a purely imaginary W. By the skew symmetry of $\langle \mathbf{q}^*(X, Y, e_0), Z\rangle$ for all $X, Y, Z \in \mathbb{O}$,

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, e_0), XW\rangle &= \langle \mathbf{q}^*(XW, Y, X), e_0\rangle\\ &= -\langle \mathbf{q}^*(\bar{W}\bar{X}, Y, X), e_0\rangle = \langle \mathbf{q}^*(e_0\bar{X}, Y, X), \bar{W}\rangle\\ &= -\langle \mathbf{q}^*(X, Y, X), \bar{W}\rangle = \langle \mathbf{q}^*(X, Y, X), W\rangle,\end{aligned}$$

which cancels $\langle \mathbf{q}^*(X, Y, W), X\rangle$ for an imaginary W. Putting all these together, it follows that

$$\langle \mathbf{q}^*(X, Y, W), XW\rangle = 0 \tag{33}$$

for all $W \in \mathbb{O}$.

Likewise, $\langle \mathbf{q}^*(X, Y, W), Y \circ W\rangle = 0$ for all $W \in \mathbb{O}$ by a similar argument. □

REMARK 5. In fact, the first two identities of Corollary 4 establish that $\langle \mathbf{p}^*, \mathbf{q}^*\rangle = 0$ by (17). This is the seventh of the ten equations of Ozeki and Takeuchi [14, p. 530] defining an isoparametric hypersurface.

We now come to a crucial observation. Recall the angle θ given before Lemma 3.

Proposition 4. *Assume $\theta \neq 0$ and π. Let $R(X, Y) := \mathbf{q}^*(X, Y, e_0)$. Then*

$$R(X, Y) = XY - Y \circ X,$$

if e is perpendicular to X, Y and XY, while

$$R(X, Y) = \pm(XY - Y \circ X)$$

if XY is parallel to e.

Proof. By Lemma 8 we see $|R(Z, Z)| = |ZZ - Z \circ Z| = 0$, so that $R(Z, W)$ is skew-symmetric in Z and W.

We may assume $X,Y \in \mathrm{Im}(\mathbb{O})$ are orthonormal vectors such that X,Y and XY are all perpendicular to e, where e is given before Lemma 3. Then $e_0, X, Y, XY, e, Xe, Ye, (XY)e$ form an octonion basis of $\mathbb{O}$. It follows that $R(X,Y)$ is a linear combination of the above basis elements. We know

$$\langle R(X, Y), e_0\rangle = \langle R(X, Y), X\rangle = \langle R(X, Y), Y\rangle = 0$$

by the first two identities of Corollary 3. Therefore, we conclude

$$R(X, Y) = a(XY) + fe + c(Xe) + d(Ye) + b((XY)e) \tag{34}$$

for some functions a, b, c, d, f defined in the Stiefel manifold $\mathcal{M}$ of orthonormal 2-frames over $\mathrm{Im}(\mathcal{O})$.

Let $X = g^{-1}(X'), Y = g^{-1}(Y')$ and $e = g^{-1}(e')$ for any automorphism g of $\mathbb{O}$. Then

$$\begin{aligned}(g \cdot R)(X', Y') &:= g(R(g^{-1}(X'), g^{-1}(Y'))) = g(R(X, Y))\\ &= a(X'Y') + fe' + c(X'e') + d(Y'e') + b((X'Y')e').\end{aligned}$$

The interpretation is that $(g \cdot R)(X', Y')$ is $R(X, Y)$ relative to the new octonion basis $e_0, g^{-1}(e_1), \ldots, g^{-1}(e_7)$ with coordinates X', Y' and e'. Since any such (X, Y, e) can be $(g^{-1}(X'), g^{-1}(Y'), g^{-1}(e'))$ for a fixed (X', Y', e') (think of it as (e_1, e_2, e_4)) as we vary g, we see that a, b, c, d, f are all constant. But then homogenizing X and Y in (34) shows that $c = d = 0$ for (polynomial) degree reason, and, moreover, that $f = 0$ since R(X,Y) is skew-symmetric. So now

$$R(X, Y) = a(XY) + b((XY)e). \tag{35}$$

To determine a and b, we note that by Lemma 10

$$\langle R(U, V), W\rangle = \langle \mathbf{q}^*(U, V, e_0), W\rangle$$

is skew-symmetric in all variables. Hence the 3rd identity of Corollary 3 gives

$$\langle R(X,Y), XY\rangle = \langle \mathbf{q}^*(X,Y,X), Y\rangle,$$

while the 4th identity of Corollary 3 gives

$$\langle R(X,Y), Y\circ X\rangle = -\langle \mathbf{q}^*(X,Y,X), Y\rangle.$$

Adding these two equations, incorporating Lemma 3 and bearing in mind that $a = \langle R(X,Y), XY\rangle$ and $b = \langle R(X,Y), (XY)e\rangle$, we obtain

$$a(1-\cos(2\theta)) - b\sin(2\theta) = 0.$$

But then

$$a^2 + b^2 = |R(X,Y)|^2 = |XY - Y\circ X|^2 = 2 + 2\cos(2\theta)$$

results in

$$a = \pm(1+\cos(2\theta)), \quad b = \pm\sin(2\theta).$$

(The signs for a and b agree.) By changing e to $-e$, we may assume the sign is positive. It follows that

$$R(X,Y) = (1+\cos(2\theta))XY + \sin(2\theta)(XY)e = XY - Y\circ X.$$

In the case when the orthonormal imaginary X and Y are such that $XY = e$, we form an octonian basis $e_0, X, Y, e, W, WX, WY, We$. We have, since $X\circ Y = XY = e$ by Lemma 3 and since $R(X,Y)$ is skew-symmetric, that

$$\langle R(X,Y), W\rangle = \langle R(W,X), Y\rangle = \langle WX - X\circ W, Y\rangle = 0$$

by the previous case. In other words, $R(X,Y)$ is in the span of e_0 and e since $\langle R(X,Y), X\rangle = \langle R(X,Y), Y\rangle = 0$. Write

$$R(X,Y) = ae + be_0.$$

Now, $b = \langle R(X,Y), e_0\rangle = 0$ by skew symmetry. Moreover, since $|R(X,Y)| = |XY - Y\circ X| = 2$, we see $a = \pm 2$ and

$$R(X,Y) = \pm 2e = \pm 2XY = \pm(XY - Y\circ X). \qquad \square$$

Corollary 5. *$R(X,Y) = XY - YX$ if $\theta = 0$ and $R(X,Y) = 0$ if $\theta = \pi$.*

Proof. e is arbitrary in (35) when $\theta = 0$ or π. Hence the real number $b = 0$, so that $R(X, Y) = aXY$. In the case when $\theta = \pi$ we have $a \circ b = ba$ for all a, b and $|R(X, Y)| = |XY - Y \circ X| = 0$. So $a = 0$. For $\theta = 0$, i.e., when $a \circ b = ab$ for all a, b, $|R(X, Y)| = 2|X||Y|$. So, $a = \pm 2$. Since changing X, Y, Z to $-X, -Y, -Z$ leaves the 2nd fundamental form fixed and changes the 3rd fundamental form by a sign, we may choose the positive sign. □

7. Classification of $\mathbf{q}^*$

We have seen in Lemma 8 that the 3rd fundamental form $\mathbf{q}^*$ satisfies

$$|\mathbf{q}^*(X, Y, Z)| = |X(Y \circ Z) - Y \circ (XZ)|. \tag{36}$$

We now prove that there are only three possibilities for $\mathbf{q}^*$.

Theorem 1. *Up to isometry, the possible* $\mathbf{q}^*$ *are either*

$$\mathbf{q}^*(X, Y, Z) = (XY - YX)Z$$

constructed by Ozeki and Takeuchi, where $\circ$ *coincides with the octonion multiplication, or*

$$\mathbf{q}^*(X, Y, Z) = X(Y \circ Z) - Y \circ (XZ)$$

constructed by Ferus, Karcher and Münzner, where either $a \circ b = ab$ *or* $a \circ b = ba$ *for all* $a, b \in \mathbb{O}$.

The proof of Theorem 1 consists of a series of lemmas and corollaries in the following subsections.

7.1. The case when $\theta \neq 0$ and π.

Lemma 11. *Suppose* $\theta \neq 0$ *and* π. *Let* X *and* Y *be purely imaginary and perpendicular vectors in* $\mathbb{O}$ *and let* W *be in the orthogonal complement of the quaternion algebra* $\mathcal{A}$ *generated by* X *and* Y. *Then*

$$\mathbf{q}^*(X, Y, W) = X(Y \circ W) - Y \circ (XW)$$

if e *is perpendicular to* $\mathcal{A}$, *while*

$$\mathbf{q}^*(X, Y, W) = \pm(X(Y \circ W) - Y \circ (XW))$$

if XY *is parallel to* e; *here, the sign agrees with that of* $R(X, Y)$.

Proof. We may assume X, Y are unit vectors. Suppose X, Y and XY are all perpendicular to e. Complete it to an octonion basis $e_0, X, Y, XY, e, Xe, Ye, (XY)e$ of $\mathbb{O}$. The third identity in Corollary 3 and Proposition 4 imply that

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, e), X\rangle &= \langle R(X, Y), Xe\rangle = \langle XY - Y \circ X, Xe\rangle \\ &= 2\sin(2\theta)\langle (XY)e, Xe\rangle = 0.\end{aligned}$$

Likewise, the fourth identity in Corollary 3 and Proposition 4 imply

$$\langle \mathbf{q}^*(X, Y, e), Y\rangle = \langle R(X, Y), Y \circ e\rangle = \langle XY - Y \circ X, Y \circ e\rangle = 0.$$

Meanwhile,

$$\langle \mathbf{q}^*(X, Y, e), e_0\rangle = -\langle \mathbf{q}^*(X, Y, e_0), e\rangle = -\langle XY - Y \circ X, e\rangle = 0.$$

On the other hand,

$$\langle \mathbf{q}^*(X, Y, e), Xe\rangle = \langle \mathbf{q}^*(X, Y, e), Ye\rangle = 0$$

by the first two identities of Corollary 4. Lastly, $\langle \mathbf{q}^*(X,Y,e),e\rangle = 0$ by the first identity of Corollary 3. In conclusion,

$$\mathbf{q}^*(X, Y, e) = a(XY) + b((XY)e). \tag{37}$$

To determine a and b, setting $U = e$ and $V = Y$ in the 3rd equation in Corollary 4, we deduce

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, e), XY\rangle &= -\langle \mathbf{q}^*(X, Y, Y), Xe\rangle \\ &= \langle \mathbf{q}^*(X, Y, e_0), (Xe) \circ Y\rangle \\ &= \langle XY - Y \circ X, (Xe) \circ Y\rangle = \sin(2\theta).\end{aligned} \tag{38}$$

In the same vein,

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, e), Y \circ X\rangle &= -\langle \mathbf{q}^*(X, Y, X), Y \circ e\rangle \\ &= \langle \mathbf{q}^*(X, Y, e_0), (Y \circ e)X\rangle = \langle XY - Y \circ X, (Y \circ e)X\rangle \\ &= \langle XY - Y \circ X, (Ye)X\rangle = \sin(2\theta),\end{aligned}$$

while its left hand side simplifies to

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, e), Y \circ X\rangle &= \langle \mathbf{q}^*(X, Y, e), \cos(2\theta)YX + \sin(2\theta)(YX)e\rangle \\ &= -\cos(2\theta)\sin(2\theta) - \sin(2\theta)\langle \mathbf{q}^*(X, Y, e), (XY)e\rangle\end{aligned}$$

by (38). So, when $\theta \neq \pi/2$, we end up with

$$\langle \mathbf{q}^*(X, Y, e), (XY)e \rangle = -(1 + \cos(2\theta)),$$

which is exactly

$$\mathbf{q}^*(X, Y, e) = X(Y \circ e) - Y \circ (Xe).$$

We then use the third identity of Corollary 4 to see that

$$\mathbf{q}^*(X, Y, W) = X(Y \circ W) - Y \circ (XW).$$

for $W = Xe, Ye, (XY)e$, and hence for all W perpendicular to $\mathcal{A}$.

When $\theta = \pi/2$, a straightforward calculation gives

$$|\mathbf{q}^*(X, Y, e)| = |X(Y \circ e) - Y \circ (Xe)| = 1 + \cos(2\theta) = 0, \tag{39}$$

so that once more

$$\mathbf{q}^*(X, Y, e) = X(Y \circ e) - Y \circ (Xe) \quad (= 0).$$

In the case when $XY = e$, we know $R(X, Y) = \pm(XY - YX) = \pm 2e$. We form an octonian basis $e_0, X, Y, e, W, WX, WY, We$. Then

$$\begin{aligned}
\langle \mathbf{q}^*(X, Y, W), e_0 \rangle &= -\langle R(X, Y), W \rangle = \langle \pm 2e, W \rangle = 0,\\
\langle \mathbf{q}^*(X, Y, W), X \rangle &= \langle R(X, Y), WX \rangle = 0,\\
\langle \mathbf{q}^*(X, Y, W), Y \rangle &= \langle R(X, Y), W \circ Y \rangle = 0,\\
\langle \mathbf{q}^*(X, Y, W), W \rangle &= 0,\\
\langle \mathbf{q}^*(X, Y, W), XW \rangle &= \langle \mathbf{q}^*(X, Y, W), YW \rangle = 0,
\end{aligned}$$

where the last identity follows from Corollary 4. It follows that

$$\mathbf{q}^*(X, Y, W) = a(XY) + b(W(XY))$$

for some $a, b \in \mathbb{R}$. But then for (polynomial) degree reason $a = 0$. Since

$$X(Y \circ W) - Y \circ (XW) = 2\cos(2\theta) W(XY),$$

we see by (23) that

$$\mathbf{q}^*(X, Y, W) = \pm(X(Y \circ W) - Y \circ (XW)). \qquad \square$$

Corollary 6. *Suppose $\theta \neq 0$ and π. Let X and Y be purely imaginary and perpendicular vectors in $\mathbb{O}$ and let W be in the quaternion algebra $\mathcal{A}$ generated by X and Y. Then*

$$\mathbf{q}^*(X, Y, W) = X(Y \circ W) - Y \circ (XW) \tag{40}$$

if e is perpendicular to $\mathcal{A}$, while

$$\mathbf{q}^*(X, Y, W) = \pm(X(Y \circ W) - Y \circ (XW)) \tag{41}$$

if XY is parallel to e; here, the sign agrees with that of $R(X, Y)$.

Proof. The proof follows the same line of thoughts as in the preceding lemma. Thus we shall only indicate the essential point.

We first assume that e is perpendicular to $\mathcal{A}$ so that by the preceding lemma

$$\mathbf{q}^*(X, Y, Z) = X(Y \circ Z) - Y \circ (XZ) \tag{42}$$

for Z perpendicular to $\mathcal{A}$. Then as before we construct an octonion basis e_0, X, Y, XY, e, Xe, Ye, $(XY)e$. We know $\langle \mathbf{q}^*(X,Y,X), e_0\rangle = -\langle R(X,Y), X\rangle = 0$ and $\langle \mathbf{q}^*(X,Y,X), X\rangle = 0$. By the 5th identity of Corollary 3,

$$\begin{aligned}\langle \mathbf{q}^*(X, Y, X), Y\rangle &= \langle R(X, Y), XY\rangle \\ &= \langle XY - Y \circ X, XY\rangle = 1 + \cos(2\theta).\end{aligned}$$

For Z perpendicular to $\mathcal{A}$, we use (42) to see

$$\langle \mathbf{q}^*(X, Y, X), Z\rangle = -\langle \mathbf{q}^*(X, Y, Z), X\rangle = \langle (Ze)(XY), X\rangle,$$

so that we derive

$$\langle \mathbf{q}^*(X, Y, X), e\rangle = \langle \mathbf{q}^*(X, Y, X), Xe\rangle = \langle \mathbf{q}^*(X, Y, X), (XY)e\rangle = 0, \tag{43}$$

while

$$\langle \mathbf{q}^*(X, Y, X), Ye\rangle = -\sin(2\theta). \tag{44}$$

Therefore, we conclude

$$\mathbf{q}^*(X, Y, X) = (1 + \cos(2\theta))Y - \sin(2\theta)Ye = X(Y \circ X) - Y \circ (XX). \tag{45}$$

(Note that $\mathbf{q}^* = 0$ if $\theta = \pi/2$.) When $XY = e$, we form the octonion basis e_0, X, Y, e, W, XW, YW, $(XY)W$ and we have $R(X, Y) = \pm 2XY$ and $\mathbf{q}^*(X, Y, Z) = \pm(X(Y \circ Z) -$

$Y\circ(XZ))$ for Z perpendicular to $\mathcal{A}$. We see $\langle \mathbf{q}^*(X,Y,X),Y\rangle = \pm 2$ and $\langle \mathbf{q}^*(X,Y,X),Z\rangle = 0$ for all Z perpendicular to $\mathcal{A}$. Hence

$$\mathbf{q}^*(X, Y, X) = \pm 2Y = \pm(X(Y \circ X) - Y \circ (XX)). \qquad \square$$

Theorem 2. *Suppose $\theta \neq 0$ and π. For all $X, Y \in \mathbb{O}$ and all $Z \in \mathbb{O}$ we have*

$$\mathbf{q}^*(X, Y, Z) = X(Y \circ Z) - Y \circ (XZ). \tag{46}$$

Thus the hypersurfaces are of the type constructed by Ferus, Karcher and Münzner.

Proof. Lemma 11 and Corollary 6 only deal with the case when the imaginary X and Y are perpendicular in $\mathbf{q}^*(X, Y, Z)$, which leaves an undetermined sign. We now remove the sign by considering the case when $X = Y$.

Let $X, Y \in \mathrm{Im}(\mathbb{O})$ be orthonormal such that e is perpendicular to X, Y and XY. Then the circles $X(t) := \cos(t)X + \sin(t)Y$ and $Y(t) := -\sin(t)X + \cos(t)Y$ satisfy that $X(t)$, $Y(t)$, $X(t)Y(t)$ are perpendicular to e. Differentiating (40) at $t = 0$, we obtain

$$\begin{aligned}&\mathbf{q}^*(Y, Y, W) - \mathbf{q}^*(X, X, W)\\ &= -(X(X \circ W) - X \circ (XW)) + (Y(Y \circ W) - Y \circ (YW)).\end{aligned}$$

Note that

$$|\mathbf{q}^*(X, X, Z)| = |\sin(2\theta)(X((XZ)e) - (X(XZ))e)| \neq 0 \tag{47}$$

unless $\theta = \pi/2$. Homogenizing and comparing polynomial types, we get

$$\mathbf{q}^*(X, X, W) = X(X \circ W) - X \circ (XW)$$

when $\theta \neq \pi/2$. On the other hand, when $\theta \neq \pi/2$, we fix the same X and choose a Y such that $XY = e$, differentiating (41) gives

$$\mathbf{q}^*(X, X, W) = \pm((X(X \circ W) - X \circ (XW)).$$

Therefore, the sign must be positive when $\theta \neq \pi/2$.

When $\theta = \pi/2$, the formula (47) implies $\mathbf{q}^*(X, X, Z) = 0$ for all $X, Z \in \mathbb{O}$, and so $\mathbf{q}^*$ is skew-symmetric in X and Y. So, a priori the sign is undetermined. However, by (39) and (45) we have seen $\mathbf{q}^*(X,Y,Z) = 0$ for all Z when e is perpendicular to X,Y and XY. The sign is ambiguous only in the case when $XY = e$. Now, set $e = e_4$. Then since any two different imaginary basis elements $e_a, e_b \neq e_4$ satisfy either $e_a e_b = e_4$, or e_a, e_b and $e_a e_b$ are all perpendicular to e_4, the analysis in Lemma 11 and Corollary 6 provides a recipe for writing down $\mathbf{q}^*(X, Y, Z)$ explicitly as follows.

$$\mathbf{q}^*(X, Y, Z) = \pm \sum (x_i y_j e_i(\circ(e_j Z)) - y_j x_i e_j \circ (e_i Z)),$$

where $i, j \geq 1$ run over the indexes where $e_i e_j e_4 = \pm e_0$.

Since changing X, Y, Z to $-X, -Y, -Z$ retains the 2nd fundamental form and changes the 3rd fundamental form by a sign, we might as well choose the positive sign.

Therefore, in any event, the 3rd fundamental form is the desired form given by (46). □

Proposition 2 implies that we can always perturb to find a mirror point $x^* \in M_-$ at which $\theta = 0$ or π, even when initailly the choice of x^* produces an angle θ different from 0 and π. Therefore, the classification is reduced to the case when $\theta = 0$ or π.

7.2. The case when $\theta = 0$ or π. By Corollary 5, we know $R(X,Y) = XY - YX$ for $\theta = 0$ and $R(X, Y) \equiv 0$ for $\theta = \pi$.

Corollary 7. *Suppose $a \circ b = ab, \forall a, b$. For $X, Y \in \mathrm{Im}(\mathbb{O})$, we have*

$$\begin{aligned}
&\langle \mathbf{q}^*(X, Y, Z), Z\rangle = 0,\\
&\langle \mathbf{q}^*(X, Y, e_0), X\rangle = \langle \mathbf{q}^*(X, Y, e_0), Y\rangle = 0,\\
&\langle \mathbf{q}^*(X, Y, Z), X\rangle = 2\langle X, Y\rangle\langle X, Z\rangle - 2|X|^2\langle Y, Z\rangle,\\
&\langle \mathbf{q}^*(X, Y, Z), Y\rangle = -2\langle X, Y\rangle\langle Y, Z\rangle + 2|Y|^2\langle X, Z\rangle.
\end{aligned}$$

Proof. This follows from $R(X, Y) = XY - YX$ and Corollary 3. □

Corollary 8. *If the normed algebra is $\mathbb{H}$, then* Theorem 1 *is true.*

Proof. By Remark 1, either $a \circ b = ab$ or $= ba$ for all $a, b \in \mathbb{H}$.

CASE 1. $a \circ b = ba, \forall a, b$.

Then by (36), $|\mathbf{q}^*(X, Y, Z)| = |X(ZY) - (XZ)Y| = 0$ by the associativity of $\mathbb{H}$. So,

$$\mathbf{q}^* = 0 = X(Y \circ Z) - Y \circ (XZ).$$

The hypersurface is of the type constructed by Ferus, Karcher and Münzner by Section 5.1.

CASE 2. $a \circ b = ab, \forall a, b$.

Let X, Y be mutually orthogonal and purely imaginary. We set $Z = XY$. Then the first, third and fourth identities of Corollary 7 imply $\mathbf{q}^*(X, Y, Z)$ is perpendicular to X, Y, Z; therefore, $\mathbf{q}^*(X, Y, Z)$ is parallel to e_0. Let $\mathbf{q}^*(X, Y, Z) = -2c|X|^2|Y|^2 e_0$ for some constant c. By identity (36) we obtain the identity $|\mathbf{q}^*(X, Y, Z)| = 2|X|^2|Y|^2$; we see therefore $c = \pm 1$. Thus,

$$\mathbf{q}^*(X, Y, Z) = -2c|X|^2|Y|^2 e_0 = 2cZZ = c(XY - YX)Z.$$

Meanwhile,

$$\mathbf{q}^*(X, Y, e_0) = R(X, Y) = (XY - YX)e_0.$$

Corollary 7 also yields

$$\mathbf{q}^*(X, Y, X) = 2|X|^2 = (XY - YX)X,$$
$$\mathbf{q}^*(X, Y, Y) = -2|Y|^2 X = (XY - YX)Y.$$

Putting all these together, we arrive at

$$\mathbf{q}^*(X, Y, W) = (XY - YX)W, \quad \text{or}$$
$$\mathbf{q}^*(X, Y, W) = (XY - YX)W - \langle W, XY - YX\rangle e_0, \tag{48}$$

where $c = 1$ for the first equation and $c = -1$ for the second. Although we have derived the formulae assuming that X and Y are perpendicular, the same formulae remain true for any two imaginary X and Y since $\mathbf{q}^*(U, V, W)$ is skew-symmetric in U, V.

If $c = 1$, then

$$\mathbf{q}^*(X, Y, W) = X(Y \circ W) - Y \circ (XW).$$

So the hypersurface is of the type constructed by Ferus, Karcher and Münzner by Section 5.1. It satisfies (33)

$$\langle X(Y \circ W) - Y \circ (XW), XW\rangle = 0. \tag{49}$$

We show $c = -1$ is impossible. Assume otherwise. Then since such an isoparametric hypersurface must also satisfy (33), we would conclude

$$\begin{aligned} 0 &= \langle \mathbf{q}^*(X, Y, W), XW\rangle \\ &= \langle X(Y \circ W) - Y \circ (XW), XW\rangle - \langle W, XY - YX\rangle e_0, XW\rangle \\ &= \langle W, XY - YX\rangle\langle W, X\rangle \neq 0 \end{aligned}$$

by (49). This is a contradiction. □

To finish Theorem 1 in the octonion case, we break it into two cases.

CASE 1. $a \circ b = ab, \ \forall a, b.$

Identity (36) shows that $|\mathbf{q}^*(X, X, Z)| = 0, \ \forall X, Z \in \mathbb{O}$, so that $\mathbf{q}^*(X, Y, Z)$ is skew-symmetric in $X, Y, \ \forall X, Y \in \mathbb{O}$.

Let $X, Y \neq 0$ be perpendicular and purely imaginary and W be in the orthogonal complement of $\mathcal{A}$, the quaternion algebra generated by X and Y. We know by (37) and (38) that $\mathbf{q}^*(X, Y, W) = \pm 2((XY)W)$, if X, Y and XY are all perpendicular to e, and the same formula holds if $XY = e$, where the signs might not be related a priori in the two cases. We assume first that the signs are identical. Namely,

$$\mathbf{q}^*(X, Y, W) = 2c((XY)W),$$

where $c = 1$ or $c = -1$ for all W perpendicular to $\mathcal{A}$. If $c = 1$, then

$$\mathbf{q}^*(X, Y, W) = (XY - YX)W,$$

which remains true for any two purely imaginary X and Y not necessarily perpendicular to each other, as $\mathbf{q}^*$ is skew-symmetric in X, Y. It follows that

$$\mathbf{q}^*(X, Y, Z) = (XY - YX)Z$$

for any $Z \in \mathbb{O}$, as it is also true for $Z \in \mathcal{A}$ by Corollary 8, where we use (43) and (44) to see that $\mathbf{q}^*(X, Y, Z) \in \mathcal{A}$ for $Z \in \mathcal{A}$. This is the isoparametric hypersurface constructed by Ozeki and Takeuchi.

If $c = -1$, then

$$\mathbf{q}^*(X, Y, W) = -2(XY)W = X(YW) - Y(XW),$$

so that there holds

$$\mathbf{q}^*(X, Y, Z) = X(YZ) - Y(XZ) = X(Y \circ Z) - Y \circ (XZ)$$

for any $X, Y, Z \in \mathbb{O}$, as it is true for $Z \in \mathcal{A}$ by Corollary 8. These are the isoparametric hypersurfaces constructed by Ferus, Karcher and Münzner.

We need to remove the case when $\mathbf{q}^*(X, Y, W) = 2((XY)W)$ if X, Y, and XY are all perpendicular to e, whereas $\mathbf{q}^*(X, Y, W) = -2((XY)W)$ when $XY = e$. Assuming this is the case. Then Corollary 8 implies

$$\mathbf{q}^*(X, Y, W) = (XY - YX)W + h(X, Y, W),$$

where $h(X, Y, W) = -4eW^{\perp}$ if $XY = e$. As seen in Corollary 8, the existence of an isoparametric hypersurface with such a $\mathbf{q}^*$ would imply

$$\langle h(X, Y, W), XW\rangle = \langle \mathbf{q}^*(X, Y, W) - (XY - YX)W, XW\rangle = 0.$$

But then if we set $e = e_4$ and $W = e_2$, we get

$$\langle h(X, Y, W), XW\rangle \neq 0.$$

This is a contradiction.

CASE 2. $a \circ b = ba$, $\forall a, b$.

Note that again $|\mathbf{q}^*(U, U, Z)| = |U(ZU) - (UZ)U| = 0$, $\forall U, Z \in \mathbb{O}$, so that $\mathbf{q}^*$ is skew-symmetric in the first two slots.

If $c = 1$, then

$$\mathbf{q}^*(X, Y, W) = 2(XY)W = X(WY) - (XW)Y,$$

so that

$$\mathbf{q}^*(X, Y, Z) = X(ZY) - (XZ)Y = X(Y \circ Z) - Y \circ (XZ)$$

for any $X, Y, Z \in \mathbb{O}$, as $\mathbf{q}^* = 0$ on $\mathcal{A}$.

If $c = -1$, then $\mathbf{q}^*$ only differs from the previous case by a negative sign. Changing X, Y, Z to $-X, -Y, -Z$ converts it to the previous case.

This completes the classification of Theorem 1.

REMARK 6. In the octonion case, the two isoparametric hypersurfaces with $\mathbf{q}^* = X(Y \circ Z) - Y \circ (XZ)$ constructed by Ferus, Karcher and Münzner are of Condition B at $x^* \in M_-$. In contrast, the hypersurface with $\mathbf{q}^* = (XY - YX)Z$ is not of Condition B at x^*; however, it is of both Conditions A and B at $x \in M_+$ constructed by Ozeki and Takeuchi.

In the quaternionic case, however, $(XY - YX)Z = X(YZ) - Y(XZ)$, so that we have only two different such isoparametric hypersurfaces, where the example of Ozeki and Takeuchi of multiplicities (3,4) of Conditions A and B at $x \in M_+$ is also of Condition B at $x^* \in M_-$. The other isoparametric hypersurface is of Condition B at $x^* \in M_-$ with $q^* = X(ZY) - (XZ)Y = 0$; it is the homogeneous example of multiplicities (4,3).

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