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DIFFERENTIAL RELATIONS OF THETA FUNCTIONS

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0. Introduction

There are many applications of modular functions or modular forms in mathematics. The ring structure of modular forms is deepy studied from algebraic viewpoint. But from analytic viewpoint, there are only a few results on modular forms.

In 1881, Halphen solved a nonlinear differential system

$$\begin{aligned}\frac{d(u_1+u_2)}{dx}&=u_1u_2,\\ \frac{d(u_2+u_3)}{dx}&=u_2u_3,\\ \frac{d(u_3+u_1)}{dx}&=u_3u_1,\end{aligned}\tag{0.1}$$

by theta constants ([3]). He deduced the system above from the addition formula for Weierstrass' ζ-function. In 1910, Chazy considered a third-order nonlinear differential equation

$$y'''=2y'y''-3(y')^2,$$

for $y=u_1+u_2+u_3$, in his study of Painlevé type equations of third order ([2]).

Recently it is studied that a reduction of the self-dual Yang-Mills equations is related to Halphen's system ([1]). We can consider modular functions as special solutions for the self-dual Yang-Mills equations. It seems interesting to study relations between 'Integrable systems', such as the self-dual Yang-Mills equations or self-dual metrics, and modular forms.

In this paper we will study Halphen's system (0.1). Halphen showed that

(1) A solution of (0.1) is given by logarithmic derivatives of theta constants.

(2) (0.1) is $SL(2,C)$-invariant.

(3) The action of the subgroup $SL(2,Z)$ in $SL(2,C)$ induces the perrmutation

between u_1, u_2 and u_3.
He also considered generalized equation of (0.1), whose solutions are written by hypergeometric functions. These equations are also $SL(2, \boldsymbol{C})$-invariant.

When we study automorphic forms by nonlinear differential equations, at first westudy differential equations which are $SL(2, \boldsymbol{C})$-invariant, next we should consider the action of discrete subgroups. In Proposition 1.2, we will give a condition of $SL(2, \boldsymbol{C})$-invariance for special type of equations. It seems difficult to treat action of discrete groups in the term of analysis, hence we will study examples of equations whose solutions are given by modular forms.

Halphen's equation is solved by modular forms of level 2. As is shown by Halphen, generic solutions are written by theta constants. In secton 2, we will determine the whole solution space of (0.1). If u_1, u_2 and u_3 are distinct to each other, solutions are given by theta constants as Halphen solved. In the proof, $SL(2, \boldsymbol{C})$-invariance of (0.1) plays key role. If one u_j is coincident with the other, solutions are given by rational functions. Hence Halphen's stsyem (0.1) may be considered as the fundamental equations which dominates theta constants.

We can use (0.1) as a defining equation of modular forms of modular forms of level 2. In section 3, we will study theta functions using Halphen's equation. We can start only from Halphen's equation and the heat equation in order to study theta functions form analytic vieqpoint. Other property such as addition formulae can be deduced from (0.1) and the heat equation.

Chazy's equation is solved by modular forms of level 1. Although Chazy's equation is also $SL(2, \boldsymbol{C})$-invariant, it is not the type in Proposition 1.2. In section 4, we will study rèlation between differential rings which are defined by Halphen's equation and by Chazy's euqation. Halphen's equation is a Galois extension of Chazy's equation with Galois group S_3. The study of differential equations for higher levels are problems in the future.

Historically speaking, it is Jacobi who first study a differential equation which is satisfied by theta constants ([5]). Jacobi's equation is founded again by Ehrenpreis ([8]). It is difficult to find differential equations which dominate algebraic functions because there are too many relations between their derivatives. But for transcendental functions, there are few relations between their derivatives, and there may exist differential equations which dominate the transcendental functions. In section 5, we will show the relation between Haplhen's equation and Jacobi's equation.

Since the natural boundary of theta constants is real axis, generic solutions of (0.1) have a natural boundary. Hence (0.1) does not have Painlevé property. Most of equations appeared in mathematical physics have Paimlevé property and Halphen's equation is exceptional. It may be interesting to study more general dynamical system of Halphen's type in order to understand the meaning of Painlivé property.

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1. Halphen's equation

It is customary to consider the following Theta-functions due to Jacobi (c.f. [6], [7]).

$$(1.1)\qquad \begin{aligned} \vartheta_1(z,\tau)&=i\sum_{n=-\infty}^{\infty}(-)^n e^{(n-1/2)^2\pi i\tau}e^{(2n-1)\pi iz},\\ \vartheta_2(z,\tau)&=\sum_{n=-\infty}^{\infty}e^{(n-1/2)^2\pi i\tau}e^{(2n-1)\pi iz},\\ \vartheta_3(z,\tau)&=\sum_{n=-\infty}^{\infty}e^{n^2\pi i\tau}e^{2n\pi iz},\\ \vartheta_4(z,\tau)&=\sum_{n=-\infty}^{\infty}(-)^n e^{n^2\pi i\tau}e^{2n\pi iz},\end{aligned}$$

where $z\in\boldsymbol{C}, \Im\tau>0$. We shall sometimes write $\vartheta_j(z)$ instead of $\vartheta_j(z,\tau)$, when the period τ is fixed. $\vartheta_1(z)$ is an odd function of z, and the other ϑ-functions are even.

From the definition, it follows immediately that

$$(1.2)\qquad \frac{\partial^2\vartheta_j(z,\tau)}{\partial z^2}=4\pi i\frac{\partial\vartheta_j(z,\tau)}{\partial\tau},$$

for $j=1,2,3,4$.

We can express products of ϑ-functions with arguments $x+y$ and $x-y$ in terms of those with arguments x and y. These formulae are an analogy of addition formulae for the trigonometric functions. The addition formulae for the products of two distinct even ϑ-functions are given by

$$(1.3)\qquad \vartheta_2(x+y)\vartheta_3(x-y)\vartheta_2\vartheta_3=\vartheta_2(x)\vartheta_3(x)\vartheta_2(y)\vartheta_3(y)-\vartheta_1(x)\vartheta_4(x)\vartheta_1(y)\vartheta_4(y),$$

$$(1.4)\qquad \vartheta_4(x+y)\vartheta_2(x-y)\vartheta_2\vartheta_2=\vartheta_4(x)\vartheta_2(x)\vartheta_4(y)\vartheta_2(y)+\vartheta_1(x)\vartheta_3(x)\vartheta_1(y)\vartheta_3(y),$$

$$(1.5)\qquad \vartheta_3(x+y)\vartheta_4(x-y)\vartheta_3\vartheta_4=\vartheta_3(x)\vartheta_4(x)\vartheta_3(y)\vartheta_4(y)-\vartheta_1(x)\vartheta_2(x)\vartheta_1(y)\vartheta_2(y),$$

where $\vartheta_j=\vartheta(0,\tau)$. Although there are ten additon formulae for four ϑ-functions, other seven formulae are deduced from the heat equation (1.2) and addition formulae $(1.3\sim 5)$ (See Corollary in section 3).

Now we forget the definition of ϑ-functions (1.1). We shall study srbitary solutions of functional equations (1.2) and $(1.3\sim 5)$. We use notation θ_j instead of ϑ_j and ϑ_j's shall be used only for the original Jacobi Theta-funcions. We note that θ_j's are functions on some domains in $\boldsymbol{C}$ although ϑ_j's are defined on $H=\{\tau\in\boldsymbol{C};\Im\tau>0\}$.

Assume that $\theta_j\neq 0$ for $j=2,3,4$.

Set $x=y=0$ in (1.3). Then we have $\theta_1(0)^2\theta_4(0)^2=0$. By the assumption $\theta_4\neq 0$, we obtain $\theta_1(0,\tau)=0$. Therefore $\theta_1(z,\tau)$ is an odd function of z by (1.2).

Set $x=0$ in (1.3). Then we have

$$\theta_2(y)\theta_3(-y)\theta_2\theta_3=\theta_2(y)\theta_3(y)\theta_2\theta_3.$$

By the assumption $\theta_2\neq 0$, $\theta_3(z)$ is an even function of z. In the same way, $\theta_2(z)$ and $\theta_4(z)$ are even functions of z.

Differentiate (1.3) with both x and y twice, and set $x=y=0$. Then we have

$$(\theta_2^{(4)}\theta_3-2\theta_2^{(2)}\theta_3^{(2)}+\theta_2\theta_3^{(4)})\theta_2\theta_3=(\theta_2^{(2)}\theta_3+\theta_2\theta_3^{(2)})^2, \tag{1.6}$$

where $\theta_j^{(n)}=\frac{\partial^n}{\partial z^n}\theta_j(0,\tau)$. In the same way, we have

$$(\theta_4^{(4)}\theta_2-2\theta_4^{(2)}\theta_2^{(2)}+\theta_4\theta_2^{(4)})\theta_4\theta_2=(\theta_4^{(2)}\theta_2+\theta_4\theta_2^{(2)})^2, \tag{1.7}$$

$$(\theta_3^{(4)}\theta_4-2\theta_3^{(2)}\theta_4^{(2)}+\theta_3\theta_4^{(4)})\theta_3\theta_4=(\theta_3^{(2)}\theta_4+\theta_3\theta_4^{(2)})^2. \tag{1.8}$$

We set

$$X_j(\tau)=2\frac{\partial}{\partial\tau}\log\theta_j(0,\tau), \tag{1.9}$$

for $j=2,3,4$. By the heat equation, we have

$$X_j=\frac{2}{4\pi i}\frac{\theta_j^{(2)}}{\theta_j},$$

$$\frac{dX_j}{d\tau}=\frac{2}{(4\pi i)^2}\left(\frac{\theta_j^{(4)}}{\theta_j}-\left(\frac{\theta_j^{(2)}}{\theta_j}\right)^2\right).$$

Therefore we obtain Halphen's equuation ([3]) by (1.6~8):

$$\begin{aligned}
&\frac{dX_2}{d\tau}+\frac{dX_3}{d\tau}=2X_2X_3,\\
&\frac{dX_4}{d\tau}+\frac{dX_2}{d\tau}=2X_4X_2,\\
&\frac{dX_3}{d\tau}+\frac{dX_4}{d\tau}=2X_3X_4.
\end{aligned} \tag{1.10}$$

Set $u_j=2X_j$. Then we have the original equation of Halphen in the introduction.

(1.10) is equivalent to the following system:

$$\frac{dX_2}{d\tau}=X_2X_3+X_2X_4-X_3X_4$$

$$\frac{dX_3}{d\tau} = X_2X_3 - X_2X_4 + X_3X_4$$

$$\frac{dX_4}{d\tau} = -X_2X_3 + X_2X_4 + X_3X_4.$$

Thus Halphen's equation is holonomic with rank three.

For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C})$, we denote

$$A \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

From now on, we will consider the $SL(2, \boldsymbol{C})$-action

$$\tilde{X}_j(\tau) = \frac{1}{(c\tau + d)^2} X_j(A \cdot \tau) - \frac{c}{c\tau + d}. \tag{1.11}$$

Proposition 1.1. *If a triplet* $(X_2(\tau), X_3(\tau), X_4(\tau))$ *is a solution of* (1.10), *then the new triplet* $(\tilde{X}_2(\tau), \tilde{X}_3(\tau), \tilde{X}_4(\tau))$ *defined by*

$$\tilde{X}_j(\tau) = 2\frac{\partial}{\partial \tau}\log\left(\theta_j\left(0, \frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-1/2}\right)$$

is also a solution for for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C})$.

Proof. We have $\dfrac{d}{d\tau} A \cdot \tau = \dfrac{1}{(c\tau + d)^2}$. Therefore we have

$$\frac{d}{d\tau}\tilde{X}_j = \frac{1}{(c\tau + d)^4}\frac{dX_j}{d\tau}(A \cdot \tau) - \frac{2c}{(c\tau + d)^3}X_j(A \cdot \tau) + \frac{c^2}{(c\tau + d)^2},$$

and

$$\tilde{X}_j\tilde{X}_k = \frac{1}{(c\tau + d)^4}X_j(A \cdot \tau)X_k(A \cdot \tau) - \frac{c}{(c\tau + d)^3}(X_j(A \cdot \tau) + X_k(A \cdot \tau)) + \frac{c^2}{(c\tau + d)^2}.$$

Comparing two equations above, we obtain Proposition 1.1. □

From now on, we forget theta functions, and we will consider differential equations which have $SL(2, \boldsymbol{C})$-invariance. We will consider only the following type of equations:

$$
(1.12) \qquad \begin{aligned}
\frac{dX}{dt} &= F_1(X, Y, Z) \\
\frac{dY}{dt} &= F_2(X, Y, Z) \\
\frac{dZ}{dt} &= F_3(X, Y, Z),
\end{aligned}
$$

where F_j is a quadric polynomial of X, Y, and Z for $j=1,2,3$.

We will call (1.12) $SL(2, \boldsymbol{C})$-invariant if and only if for any solution $X(t)$, $Y(t)$ and $Z(t)$ for (1.12), $\tilde{X}(t)$, $\tilde{Y}(t)$ and $\tilde{Z}(t)$ are also solutions of (1.12). We will decide the condition for $SL(2, \boldsymbol{C})$-invariance for the equations of type (1.12).

Proposition 1.2. (1.12) *is* $SL(2, \boldsymbol{C})$*-invariant if and only if* (1.12) *is written in the following form*:

$$
\begin{aligned}
\frac{dX}{dt} &= X^2 + a_{11}(X-Y)^2 + a_{12}(Y-Z)^2 + a_{13}(Z-X)^2, \\
\frac{dY}{dt} &= Y^2 + a_{21}(X-Y)^2 + a_{22}(Y-Z)^2 + a_{23}(Z-X)^2, \\
\frac{dZ}{dt} &= Z^2 + a_{31}(X-Y)^2 + a_{32}(Y-Z)^2 + a_{33}(Z-X)^2,
\end{aligned}
$$

for some constants $a_{jk} \in \boldsymbol{C}$ $(j,k=1,2,3)$.

Proof. In the same way as proof on Proposition 1.1, we have

$$
\frac{d}{dt}\tilde{X}(t) - \tilde{X}(t)^2 = \frac{1}{(ct+d)^4}\left(\frac{d}{dt}X(A\cdot t) - X(A\cdot t)^2\right).
$$

Therefore, if we set

$$
G_1 = F_1 - X^2, \qquad G_2 = F_2 - Y^2, \qquad G_3 = F_3 - Z^2,
$$

we obtain

$$
G_j(\tilde{X}(t), \tilde{Y}(t), \tilde{Z}(t)) = \frac{1}{(ct+d)^4} G_j(X(A\cdot t), Y(A\cdot t), Z(A\cdot t)).
$$

Because each G_j is quadric,

$$
G_j(X(A\cdot t) - c(ct+d), Y(A\cdot t) - c(ct+d), Z(A\cdot t) - c(ct+d)) = G_j(X(A\cdot t), Y(A\cdot t), Z(A\cdot t)).
$$

If we set $\tau=A\cdot t$, we get $ct+d=\dfrac{1}{-c\tau+a}$. Hence

$$G_j\left(X(\tau)+\frac{c}{c\tau-a},\ Y(\tau)+\frac{c}{c\tau-a},Z(\tau)+\frac{c}{c\tau-a}\right)=G_j(X(\tau),Y(\tau),Z(\tau)).$$

Since a and c are any complex numbers, we obtain

$$\frac{\partial}{\partial X}G_j+\frac{\partial}{\partial Y}G_j+\frac{\partial}{\partial Z}G_j=0.$$

Because G_j is a quadric polynomial, there exist some constants $a_{jk}\in\boldsymbol{C}$ such that

$$G_j=a_{j1}(X-Y)^2+a_{j2}(Y-Z)^2+a_{j3}(Z-X)^2,$$

for $j=1,2,3$.

The converse is evident. □

REMARK. Halphen's equation is equivalent to the following equations.

$$\frac{dX_2}{d\tau}=X_2^2-\frac{1}{2}(X_2-X_3)^2+\frac{1}{2}(X_3-X_4)^2-\frac{1}{2}(X_4-X_2)^2,$$

$$\frac{dX_3}{d\tau}=X_3^2-\frac{1}{2}(X_2-X_3)^2-\frac{1}{2}(X_3-X_4)^2+\frac{1}{2}(X_4-X_2)^2,$$

$$\frac{dX_4}{d\tau}=X_4^2+\frac{1}{2}(X_2-X_3)^2-\frac{1}{2}(X_3-X_4)^2-\frac{1}{2}(X_4-X_2)^2,$$

In [4], Halphen studied generalized equations of (1.10) which are $SL(2,\boldsymbol{C})$-invariant. Solutions of these equations are written by Gauss' hypergeometric functions.

2. Solution space of Halphen's equation

In this section, we will study the whole solution of (1.10).

Since Halphen's equation is a nonlinear equation of rank three, three-parameter triplet

$$X_j=2\frac{\partial}{\partial\tau}\log\left(\vartheta_j\left(0,\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}\right)\qquad(j=2,3,4)$$

gives a general solution of (1.10). But there may exist special solutions of (1.10). We will consider the initial value problem of (1.10), and find holomorphic solutions for any intial values.

For 'generic' initial values, logarithmic derivatives of theta constants give solutions of (1.10):

Theorem 2.1. *Assume that holomorphic functions X_2, X_3, and X_4 satisfy* (1.10) *near $z \in \boldsymbol{C}$ and that*

$$X_2(z)=k_2, \qquad X_3(z)=k_3, \qquad X_4(z)=k_4$$

for any complex numbers k_2, k_3, k_4 distinct to each other. Then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C})$ such that

$$X_j(\tau)=2\frac{\partial}{\partial\tau}\log\left(\vartheta_j\left(0,\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}\right)$$

for $j=2,3,4$.

Proof. Let $2\omega_1$ and $2\omega_3$ be the elliptic integrals

$$2\omega_1=\int_\alpha \frac{du}{\sqrt{4(u+2\pi ik_2)(u+2\pi ik_3)(u+2\pi ik_4)}},$$

$$2\omega_3=\int_\beta \frac{du}{\sqrt{4(u+2\pi ik_2)(u+2\pi ik_3)(u+2\pi ik_4)}},$$

where the cycles α and β are shown in Fig.1. Since the intersection number $\alpha\cdot\beta$ is equal to $+1$, we have $\Im\left(\frac{\omega_3}{\omega_1}\right)>0$.

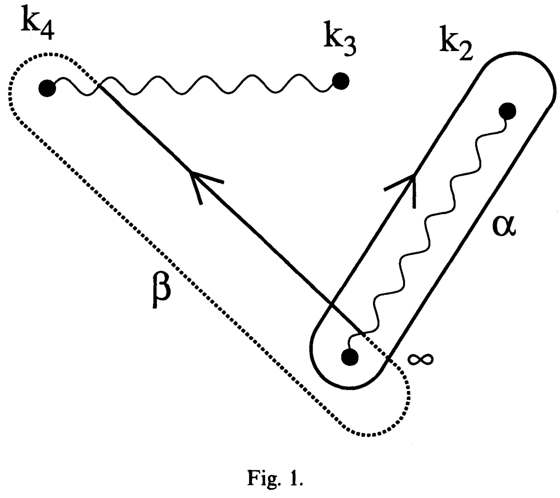

Fig. 1.

The Weierstrass $\mathscr{P}$-function is the meromorphic fundtion

$$\mathscr{P}(z)=\frac{1}{z^2}+\sum_{\substack{m,n\in\boldsymbol{Z}\\(m,n)\neq(0,0)}}\left(\frac{1}{(z-2m\omega_1-2n\omega_3)^2}-\frac{1}{(2m\omega_1+2n\omega_3)^2}\right).$$

Consider the functions

$$x_j(\tau)=2\frac{\partial}{\partial\tau}\log\vartheta_j(0,\tau), \tag{2.1}$$

for $j=2,3,4$. Then the values $e_1=\mathscr{P}(\omega_1)$, $e_2=\mathscr{P}(-\omega_1-\omega_3)$ and $e_3=\mathscr{P}(\omega_3)$ are given by

$$\begin{aligned}
e_1&=-\frac{2\pi i}{(2\omega_1)^2}\left(x_2(\tau)-\frac{1}{3}(x_2(\tau)+x_3(\tau)+x_4(\tau))\right),\\
e_2&=-\frac{2\pi i}{(2\omega_1)^2}\left(x_3(\tau)-\frac{1}{3}(x_2(\tau)+x_3(\tau)+x_4(\tau))\right),\\
e_3&=-\frac{2\pi i}{(2\omega_1)^2}\left(x_4(\tau)-\frac{1}{3}(x_2(\tau)+x_3(\tau)+x_4(\tau))\right),
\end{aligned} \tag{2.2}$$

where $\tau=\dfrac{\omega_3}{\omega_1}$ (cf. [6]).

If we replace the parameter k_j by $k_j-\dfrac{1}{3}(k_2+k_3+k_4)$ for each $j=2,3,4$, the periods $2\omega_1$ and $2\omega_2$ are not changed. Hence we have

$$\begin{aligned}
e_1&=-2\pi i\left(k_2-\frac{1}{3}(k_2+k_3+k_4)\right),\\
e_2&=-2\pi i\left(k_3-\frac{1}{3}(k_2+k_3+k_4)\right),\\
e_3&=-2\pi i\left(k_4-\frac{1}{3}(k_2+k_3+k_4)\right).
\end{aligned} \tag{2.3}$$

Choose a matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\boldsymbol{C})$ such that

$$k_2+k_3+k_4=\frac{1}{(2\omega_1)^2}\left(x_2\left(\frac{\omega_3}{\omega_1}\right)+x_3\left(\frac{\omega_3}{\omega_1}\right)+x_4\left(\frac{\omega_3}{\omega_1}\right)\right)-\frac{3c}{2\omega_1}. \tag{2.4}$$

Because of $ad-bc=1$, we have

$$a=\frac{2c\omega_3+1}{2\omega_1},$$

$$b=2\omega_3-az,$$

$$c=\frac{1}{6\omega_1}\left(x_2\left(\frac{\omega_3}{\omega_1}\right)+x_3\left(\frac{\omega_3}{\omega_1}\right)+x_4\left(\frac{\omega_3}{\omega_1}\right)\right)-\frac{2\omega_1}{3}(k_2+k_3+k_4),$$

$$d=2\omega_1-cz.$$

By (2.3) and (2.4) we obtain

$$k_j=\frac{1}{(cz+d)^2}x_j\left(\frac{az+b}{cz+d}\right)-\frac{c}{cz+d}$$

for $j=2,3,4$.

Because the Cauchy problem of (1.10) has a unique solution, the triplet

$$\begin{aligned}X_j(\tau)&=\frac{1}{(c\tau+d)^2}x_j\left(\frac{a\tau+b}{c\tau+d}\right)-\frac{c}{c\tau+d}\\&=2\frac{\partial}{\partial\tau}\log\left(\vartheta_j\left(0,\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}\right).\end{aligned}$$

is a solution of (1.10), and $X_j(z)=k_j$ for $j=2,3,4$. □

REMARK. The choice of α and β is not unique. If α' and β' are new cycles, there exists $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in\Gamma_2$ such that

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

where

$$\Gamma_2=\left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in SL(2,\boldsymbol{Z});\ p\equiv 1, q\equiv 0, r\equiv 0, s\equiv 1 \pmod 2\right\}.$$

Let ω_1' and ω_2' be the elliptic integrals over the cycle α' and β', respectively. Then we have

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Replacing ω_j by ω_j', we get another matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\in SL(2,\boldsymbol{C})$. Then we have

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore $\Gamma_2 \backslash SL(2, \boldsymbol{C})$ is a dense set in the solution space of (1.10).

If $k_3 = k_4$, we have $X_3(\tau) = X_4(\tau)$ identically. By Halphen's equation, $\dfrac{dX_3}{d\tau} = X_3^2$. Therefore we obtain

$$X_3(\tau) = X_4(\tau) = -\frac{c}{c\tau + d},$$

for $(c:d) \in \boldsymbol{P}^1(\boldsymbol{C})$. In this case

$$X_2(\tau) = -\frac{c}{c\tau + d} + \frac{a}{(c\tau + d)^2}, \tag{2.5}$$

for some constant a, which is considered as a section of the line bundle of degree 2 on $\boldsymbol{P}^1(\boldsymbol{C})$.

The following proposition can be shown by direct calculations:

Proposition 2.2. *Let $\theta_j(z,\tau)$ satisfy the heat equation for $j = 2,3,4$ and let the triplet $(X_2(\tau), X_3(\tau), X_4(\tau))$ defined by* (1.9) *be a solution of* (1.10). *Then the triplet $(\tilde{\theta}_2(z,\tau), \tilde{\theta}_3(z,\tau), \tilde{\theta}_4(z,\tau))$ defined by*

$$\tilde{\theta}_j(z,\tau) = \theta_j\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-1/2} \exp\left(-\frac{c\pi i z^2}{c\tau + d}\right)$$

also satisfy (1.2) *and the triplet $(\tilde{X}_2(\tau), \tilde{X}_3(\tau), \tilde{X}_4(\tau))$ defined by*

$$\tilde{X}_j(\tau) = 2\frac{\partial}{\partial\tau}\tilde{\theta}_j(0,\tau)$$

is a solution of (1.10).

Given $A \in \Gamma_2$, we have for a suitable ζ, an 8th root of 1, that

$$\vartheta_j(A \cdot \tau) = \zeta(c\tau + d)^{1/2} \exp\left(\frac{c\pi i z^2}{c\tau + d}\right) \vartheta_j(\tau).$$

Hence the action of $SL(2, \boldsymbol{Z})$ in Proposition 2.1 is compatible with the action of Γ_2 up to the constant multiplication.

In the case of $X_3(\tau)=X_4(\tau)=-\dfrac{c}{c\tau+d}$, we have

$$\theta_3=\theta_4=\frac{1}{\sqrt{c\tau+d}}.$$

By the heat equation (1.2), we obtain

$$\theta_3(z,\tau)=\theta_4(z,\tau)=\frac{1}{\sqrt{c\tau+d}}\exp\left(-\frac{c\pi iz^2}{c\tau+d}\right),$$

which is a well-known heat kernel. If we take $\theta_j(z,\tau)\equiv 1$ in Proposition 2.2, which satisfies both addition formulae and the heat equation, we get the heat kernel. Moreover, by (2.5) we have

$$\theta_2=\frac{1}{\sqrt{c\tau+d}}\exp\left(-\frac{a}{2c(c\tau+d)}\right).$$

By the heat equation (1.2), we obtain

$$\begin{aligned}\theta_2(z,\tau)=&\frac{1}{2\sqrt{c\tau+d}}\left\{\exp\left(-\frac{c\pi i}{c\tau+d}\left(\tau+\sqrt{\frac{a}{2\pi i}}\cdot\frac{1}{c}\right)^2\right)\right.\\&\left.+\exp\left(-\frac{c\pi i}{c\tau+d}\left(\tau-\sqrt{\frac{a}{2\pi i}}\cdot\frac{1}{c}\right)^2\right)\right\}\\=&\frac{1}{\sqrt{c\tau+d}}\exp\left(-\frac{a}{2c(c\tau+d)}-\frac{c\pi iz^2}{c\tau+d}\right)\cosh\left(\frac{\sqrt{2a\pi iz}}{c\tau+d}\right).\end{aligned}$$

3. Theta relations

In section 1 we derived a nonlinear differential equation for theta constants. In this section we derive other relations for theta constants only from addition formulae and the heat equation.

Differentiate (1.3) with respect to x and y and substitute $x=y=0$. Then we have

$$(\theta_2^{(2)}\theta_3-\theta_2\theta_3^{(2)})\theta_2\theta_3=-(\theta'_1\theta_4)^2. \tag{3.1}$$

Differentiate (1.3) with respect to x three times and with respect to y once, and substitute $x=y=0$. Then we have

$$(\theta_2^{(4)}\theta_3-\theta_2\theta_3^{(4)})\theta_2\theta_3=-(\theta'_1\theta_4)(\theta'''_1\theta_4+3\theta'_1\theta_4^{(2)}) \tag{3.2}$$

$$=-(\theta'_1\theta_4)^2\left(\frac{\theta'''_1}{\theta'_1}+3\frac{\theta_4^{(2)}}{\theta_4}\right).$$

Applying (3.1) in (3.2), we obtain

$$\left(2\frac{dX_2}{d\tau}-2\frac{dX_3}{d\tau}+X_2^2-X_3^2\right)=(X_2-X_3)(X_1+3X_4),$$

where $X_1=2\frac{\theta'''_1}{\theta'_1}$. By (1.10), we have $\frac{dX_2}{d\tau}-\frac{dX_3}{d\tau}=2X_0(X_2-X_3)$. Hence, if $X_2\neq X_3$, we have

$$X_1=X_2+X_3+X_4. \tag{3.3}$$

If $X_1=X_3$, the left hand side of (3.1) is zero. Therefore $\theta'_1=0$. Because $\theta_1(z,\tau)$ satisfies (1.2), by the initial condition $\theta_1=0$ and $\theta'_1=0$, θ_1 is identically zero. In this case X_2, X_3 and X_4 are rational functions of τ.

Integrating (3.3) by τ,

$$\theta'_1=p\theta_2\theta_3\theta_4, \tag{3.4}$$

for a suitable constant p. In the case $\theta_j=\vartheta_j$, $p=\pi$ (Jacobi's derivative formula).

In the same way as (3.1), we have

$$\begin{aligned}(\theta_4^{(2)}\theta_2-\theta_4\theta_2^{(2)})\theta_4\theta_2&=(\theta'_1\theta_3)^2,\\(\theta_3^{(2)}\theta_4-\theta_3\theta_4^{(2)})\theta_3\theta_4&=-(\theta'_1\theta_4)^2\end{aligned} \tag{3.5}$$

Substitute (3.4) into (3.1) and (3.5),

$$\begin{aligned}X_2-X_3&=-p^2\theta_4^4,\\X_4-X_2&=p^2\theta_3^4,\\X_3-X_4&=-p^2\theta_2^4.\end{aligned} \tag{3.6}$$

Thus we obtain

$$\theta_3^4=\theta_2^4+\theta_3^4. \tag{3.7}$$

This is a famous identity for the theta constants.

Lemma 3.1. *Let a set* $(\theta_1(z),\theta_2(z),\theta_3(z),\theta_4(z))$ *satisfies addition formulae* (1.3 ∼ 5). *If the set* $(\tilde{\theta}_1(z),\tilde{\theta}_2(z),\tilde{\theta}_3(z),\tilde{\theta}_4(z))$ *defined by* $\tilde{\theta}_j(z)=A_j\theta_j(z)$ *also satisfies* (1.3 ∼ 5) *for some constants* A_j, *then*

$$A_2^2=a_2A_1^2,$$
$$A_3^2=a_3A_1^2, \tag{3.8}$$
$$A_4^2=a_4A_1^2,$$

where $a_j=\pm 1$, *and either all of* a_j *are* $+1$ *or one* a_j *is* $+1$ *and others are* -1.

Proof. If θ_j's are replaced by $\tilde{\theta}_j'$s, the constant p in (3.4) should be replaced by

$$\tilde{p}=\frac{A_2A_3A_4}{A_1}p.$$

If θ_j's are replaced by $\tilde{\theta}_j'$s in (3.6), p should be replaced by $\tilde{p}$ and X_j's are the same as before. Hence we have

$$A_2^2A_3^2=A_1^2A_4^2,$$
$$A_4^2A_2^2=A_1^2A_3^2,$$
$$A_3^2A_4^2=A_1^2A_2^2.$$

By this relation we obtain (3.8). □

Theorem 3.2. *Let a set* $(\theta_1(z,\tau),\theta_2(z,\tau),\theta_3(z,\tau),\theta_4(z,\tau))$ *satisfy addition formulae* $(1.3\sim 5)$ *and the heat equation* (1.2). *Assume that* $\theta_j\neq 0$ *for* $j=2,3,4$ *and* $\theta_1'\neq 0$. *Then*

$$\theta_j(z,\tau)=A_j\vartheta_j\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}\exp\left(-\frac{\pi i z^2}{c\tau+d}\right),$$

where the constants A_j *satisfy* (3.8).

Proof. Since $\theta_1'\neq 0$, X_2, X_3 and X_4 are distinct to eact other. Because the triplet $(X_2(\tau),X_3(\tau),X_4(\tau))$ satisfies Halphen's equation, θ_j are expressed as

$$\theta_j(z,\tau)=A_j\vartheta_j\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}\exp\left(-\frac{\pi i z^2}{c\tau+d}\right),$$

by Proposition 2.1.

By (3.3), we have

$$X_1(\tau)=2\frac{\partial}{\partial\tau}\log\left(\vartheta_1'\left(0,\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-3/2}\right).$$

Thus by the heat equation again, $\theta_1(z,\tau)$ is expressed as in the Theorem. By Lemma 3.1, we get Theorem 3.2. □

REMARK. By the form of the addition formulae, the signature of A_j's cannot be determined.

By Theorem 3.2, we get other type of addition formulae for $\theta_j(z)$ than (1.3 5).

Corollary. *Let a set* $\theta_j(z,\tau)$ *satisfy* (1.3$\sim$5) *and* (1.2). *Then* $\theta_j(z)$ *satisfies the following addition formulae*:

$$\begin{aligned}
\theta_4(x+y)\theta_4(x-y)\theta_4^2 &= \theta_4(x)^2\theta_4(y)^2 - \theta_1(x)^2\theta_1(y)^2 \\
&= \theta_3(x)^2\theta_3(y)^2 - \theta_2(x)^2\theta_2(y)^2, \\
\theta_2(x+y)\theta_2(x-y)\theta_2^2 &= \theta_2(x)^2\theta_2(y)^2 - \theta_1(x)^2\theta_1(y)^2 \\
&= \theta_3(x)^2\theta_3(y)^2 - \theta_4(x)^2\theta_4(y)^2, \\
\theta_3(x+y)\theta_3(x-y)\theta_3^2 &= \theta_3(x)^2\theta_3(y)^2 - \theta_1(x)^2\theta_1(y)^2 \\
&= \theta_4(x)^2\theta_4(y)^2 - \theta_2(x)^2\theta_2(y)^2.
\end{aligned}$$

$$\begin{aligned}
\theta_1(x+y)\theta_1(x-y)\theta_4^2 &= \theta_1(x)^2\theta_4(y)^2 - \theta_4(x)^2\theta_1(y)^2 \\
&= \theta_3(x)^2\theta_2(y)^2 - \theta_2(x)^2\theta_3(y)^2, \\
\theta_1(x+y)\theta_1(x-y)\theta_3^2 &= \theta_1(x)^2\theta_3(y)^2 - \theta_3(x)^2\theta_1(y)^2 \\
&= \theta_4(x)^2\theta_2(y)^2 - \theta_2(x)^2\theta_4(y)^2, \\
\theta_1(x+y)\theta_1(x-y)\theta_2^2 &= \theta_1(x)^2\theta_2(y)^2 - \theta_2(x)^2\theta_1(y)^2 \\
&= \theta_4(x)^2\theta_3(y)^2 - \theta_3(x)^2\theta_4(y)^2.
\end{aligned}$$

$$\begin{aligned}
\theta_1(x+y)\theta_2(x-y)\theta_3\theta_4 &= \theta_1(x)\theta_2(x)\theta_3(y)\theta_4(y) + \theta_3(x)\theta_4(x)\theta_1(y)\theta_2(y), \\
\theta_1(x+y)\theta_3(x-y)\theta_2\theta_4 &= \theta_1(x)\theta_3(x)\theta_2(y)\theta_4(y) + \theta_2(x)\theta_4(x)\theta_1(y)\theta_3(y), \\
\theta_1(x+y)\theta_4(x-y)\theta_2\theta_3 &= \theta_1(x)\theta_4(x)\theta_2(y)\theta_3(y) + \theta_2(x)\theta_3(x)\theta_1(y)\theta_4(y).
\end{aligned}$$

Proof. By Theorem 3.2, θ_j's are written by ϑ_j's. Hence the addition formulae above are obtained from the usual addition formulae for Jacobi Theta functions. Comparing both hands in addition formulae, we see that there is no problem on the signature of A_j's. □

4. Chazy's equation and ring structure

In this section we study three functions $x_2(\tau)$, $x_3(\tau)$ and $x_4(\tau)$ (see (2.1)). We will show that x_2, x_3 and x_4 are algebraically independent.

Consider the fundamental symmetric functions of x_2, x_3 and x_4:

$$h_1 = x_2 + x_3 + x_4,$$
$$h_2 = x_2 x_3 + x_2 x_4 + x_3 x_4,$$
$$h_3 = x_2 x_3 x_4.$$

By transformation formulae of theta constants, we have

$$h_1\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 h_1(\tau) + 3c(c\tau+d),$$
$$h_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^4 h_2(\tau) + 2c(c\tau+d)^3 h_1(\tau) + 3c^2(c\tau+d)^2,$$
$$h_3\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^6 h_3(\tau) + c(c\tau+d)^5 h_2(\tau) + c^2(c\tau+d)^4 h_1(\tau) + c^3(c\tau+d)^3,$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z})$. Set

$$\begin{aligned}\tilde{h}_2 &= h_2 - \frac{1}{3}h_1^2 \\ &= (x_2 - \frac{1}{3}h_1)(x_3 - \frac{1}{3}h_1) + (x_2 - \frac{1}{3}h_1)(x_4 - \frac{1}{3}h_1) + (x_3 - \frac{1}{3}h_1)(x_4 - \frac{1}{3}h_1),\end{aligned}$$

$$\begin{aligned}\tilde{h}_3 &= h_3 - \frac{1}{3}h_1 h_2 + \frac{2}{27}h_1^3 \\ &= (x_2 - \frac{1}{3}h_1)(x_3 - \frac{1}{3}h_1)(x_4 - \frac{1}{3}h_1).\end{aligned}$$

Then $\tilde{h}_2$ and $\tilde{h}_3$ are modular forms of weight 4 and 6 for $SL(2, \boldsymbol{Z})$, respectively. By (2.2), we have

$$\begin{aligned}\frac{1}{(2\omega_1)^4}\tilde{h}_2 &= -\frac{1}{4\pi^2}(e_1 e_2 + e_2 e_3 + e_3 e_1) \\ &= \frac{15}{4\pi^2} \sum_{\substack{m,n \in \boldsymbol{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^4},\end{aligned}$$

$$\begin{aligned}\frac{1}{(2\omega_1)^6}\tilde{h}_3 &= -\frac{i}{8\pi^3} e_1 e_2 e_3 \\ &= -\frac{35i}{8\pi^3} \sum_{\substack{m,n \in \boldsymbol{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^6}.\end{aligned}$$

It is known that $\tilde{h}_2$ and $\tilde{h}_3$ are algebraically independent.

By (1.10), h_1, h_2 and h_3 satisfy the following equations:

$$
\begin{aligned}
h_1' &= h_2, \\
h_2' &= 6h_3, \\
h_3' &= 4h_1h_3 - h_2^2,
\end{aligned}
\tag{4.1}
$$

where $'$ means a differentiation by τ. We can rewrite (4.1) to a single equation with h_1:

$$
h_1''' = 4h_1'h_1'' - 6(h_1')^2. \tag{4.2}
$$

If we set $y = 2h_1$, we get Chazy's equation([2]).

Proposition 4.1. *Let $h_1(\tau)$ be any solution of the equation* (4.2). *If*

$$
D = 972h_1''^3 + (24h_1^3 - 108h_1h_1')h_1'' + 4h_1'^2 - (h_1h_1')^2
$$

is not zero, there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{C})$ such that

$$
h_1(\tau) = 2\frac{\partial}{\partial\tau}\log\left(\vartheta_1'\left(0, \frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-3/2}\right).
$$

And if $D = 0$, there exist $(c:d) \in \boldsymbol{P}^1(\boldsymbol{C})$ and $a \in \mathcal{O}(2)$ such that

$$
h_1(\tau) = -\frac{3c}{c\tau+d} + \frac{a}{(c\tau+d)^2}.
$$

Proof. D is the discriminant of the following equation:

$$
t^3 - h_1t^2 + h_1't - 6h_1'' = 0.
$$

By Theorem 2.1, if there ecists $\tau_0 \in \boldsymbol{C}$ such that $D(\tau_0) = 0$, D is identically zero. If $D \neq 0$, X_2, X_3 and X_4 are distinct. By Theorem 2.1, $h_1 = X_2 + X_3 + X_4$ is written in the form of the theorem. If $D = 0$, X_2, X_3 and X_4 are rational functions, which concludes the last part of the theorem. □

REMARK. In the first part of Proposition 4.1, the choice of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unique up to the left action of $SL(2, \boldsymbol{Z})$. Therefore the solution space of (4.2) is

$$
(SL(2, \boldsymbol{Z})\backslash SL(2, \boldsymbol{C})) \cup \mathcal{O}(2).
$$

Theorem 4.2. *The functions h_1, h_2 and h_3 are algebraically independent.*

Proof. It is sufficient to prove that h_1, $\tilde{h}_2$ and $\tilde{h}_3$ are algebraically independent. Assume that there exists a polynomial

$$F(h_1,\tilde{h}_2,\tilde{h}_3)=0.$$

Let the weight of h_j be $2j$ for each $j=1,2,3$. Decompose the polynommial F into weighted homogeneous polynomials:

$$F=\sum_{j=0}^{n} F_{2j}(h_1,\tilde{h}_2,\tilde{h}_3),$$

where F_{2j} is a polynomial of weight $2j$. By the transformation formula we have

$$F\left(h_1\left(\frac{a\tau+b}{c\tau+d}\right),\tilde{h}_2\left(\frac{a\tau+b}{c\tau+d}\right),\tilde{h}_3\left(\frac{a\tau+b}{c\tau+d}\right)\right)$$
$$=\sum_{j=0}^{n}(c\tau+d)^{2j}F_{2j}\left(h_1(\tau)+\frac{3}{c\tau+d},\tilde{h}_2(\tau),\tilde{h}_3(\tau)\right).$$

Sending d to ∞, we obtain $F_{2n}(h_1,\tilde{h}_2,\tilde{h}_3)=0$. Hence we may assume F is a weighted homogeneous polynomial.

By the transformation formula again, we have

$$F\left(h_1(\tau)+\frac{3c}{c\tau+d},\tilde{h}_2(\tau),\tilde{h}_3(\tau)\right)=0. \tag{4.3}$$

Fixing τ, we consider the algebraic equation

$$F(t,\tilde{h}_2(\tau),\tilde{h}_3(\tau))=0. \tag{4.4}$$

If $\dfrac{\partial F}{\partial h_1}\neq 0$, the equation (4.4) has infinitely many zeros $t=h_1(\tau)+\dfrac{3c}{c\tau+d}$ by (4.3). Hence $F\equiv 0$. If $\dfrac{\partial F}{\partial h_1}=0$, $F\equiv 0$ because $\tilde{h}_2$ and $\tilde{h}_3$ are algebraically independent.

We have proved that there exists no algebraic relation between h_1, $\tilde{h}_2$ and $\tilde{h}_3$. □

Theorem 4.3. *The functions x_2, x_3 and x_4 are algebraically independent. All of differential relations are generated by Halphen's equation* (1.10).

Proof. For the fundamental symmetric functions h_1, h_2 and h_3 are algebraically independent, x_2, x_3 and x_4 are also algebraically independent.

The last part is obvious, because (1.10) is holonomic. $\square$

5. Halphen's system and Jacobi's equation

In 1847, Jacobi showed that if $y=\theta_2, \theta_3$ or θ_4, y satisfies the following third order equation ([5]):

$$(5.1)\qquad (y^2y'''-15yy'y''+30y'^3)^2+32(yy''-3y'^2)^3=-\pi^2y^{10}(yy''-3y'^2)^2.$$

In this section we will see a relation between Jacobi's equation and Halphen's system.

We can rewrite Halphen's equation into a single equation of third order:

$$(5.2)\qquad -6X_2^2X_2'^2+2X_2'^3+4X_2^3X_2''-X_2''^2-X_2^2X_2'''+X_2'X_2'''=0.$$

Lemma 5.1. *Halphen's equation* (1.10) *is equivalent to* (5.2) *in the category of algebraic functions of* X_2.

Proof. (1.10) is symmetric with respect to X_3 and X_4. We set

$$Y=X_3+X_4, \quad Z=X_3X_4.$$

Then from (1.10) we get the following system:

$$(5.3)\qquad \begin{aligned} X_2' &= X_2Y-Z \\ Y' &= 2Z \\ Z' &= YZ-X_2Y^2+4X_2Z. \end{aligned}$$

In order to obtain (1.10) from (5.3), we have to solve the algebraic equation $t^2-Yt+z=0$. We obtain from (5.3) that

$$(5.4)\qquad \begin{aligned} X_2' &= X_2Y-Z \\ X_2'' &= 2X_2'Y-2X_2Z=2Y(X_2Y-Z)-2X_2Z \\ X_2''' &= 2X_2^2Y^2+4X_2Y^3-8X_2^2Z-4X_2YZ-4Y^2Z-2Z^2. \end{aligned}$$

Equations (5.4) satisfy (5.2). Conversely, setting

$$Y=\frac{X_2''-2X_2X_2'}{2(X_2'-X_2^2)}$$

$$Z=\frac{X_2X_2''-2X_2'^2}{2(X_2'-X_2^2)},$$

then we obtain (5.3) from (5.2). Hence nonlinear differential system (5.3) is equivalent to the single equation (5.2) in the category of rational functions. □

Now we will derive Jacobi's equation from (5.2). First, we rewrite (5.2) in terms of θ_2 instead of X_2. Setting $X_2=2(\log y)'$, we have

$$\begin{aligned}(5.5)\qquad &18y'^6-18yy'^4y''-24y^2y'^2y''^2+y^3y''^3+24y^2y'^3y^{(3)}\\&+2y^3y'y''y^{(3)}-y^4y^{(3)2}-3y^3y'^2y^{(4)}+y^4y''y^{(4)}=0.\end{aligned}$$

(5.1) is not homogeneous with respect to y, and it includes a constant π. Since the addition formulae, which are our starting point, are homogeneous equations, we cannot determine the constant π from Halphen's system. Hence we should differentiate (5.1) in order to eliminate π. Then we obtain 4th order homogeneous equation, which is equivalent to (5.5).

The left Hand side of (5.5) can be rewritten as follows:

LHS of (5.5)=

$$\frac{y^{11}(yy''-3y'^2)^3}{2(y^2y^{(3)}-15yy'y''+30y'^3)}\frac{d}{dx}\left(\frac{(y^2y^{(3)}-15yy'y''+30y'^3)^2+32(yy''-3y'^2)^3}{y^{10}(yy''-3y'^2)^2}\right).$$

Thus Jacobi's equation is deduced from Halphen's equation.

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