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## ON BORDISM INVARIANCE OF AN OBSTRUCTION TO TOPOLOGICAL EMBEDDINGS

To the memory of Maria Helena Derigi (Lene)

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### 1. Introduction

Let  $f: M \rightarrow N$  be a continuous map of a smooth closed  $m$ -dimensional manifold into a smooth  $n$ -dimensional manifold with  $k = n - m > 0$ . In [2] we have defined a primary obstruction  $\theta_1(f) \in H_{m-k}(M; \mathbb{Z}_2)$  to the existence of a homotopy between  $f$  and a topological embedding. This homology class is represented by the closure of the self-intersection set of a generic smooth map (in the sense of Ronga [9]) homotopic to  $f$  and it has been shown that it is a homotopy invariant (for a precise definition of  $\theta_1(f)$ , see §2). Thus, if  $f$  is homotopic to a topological embedding (not necessarily locally flat), then  $\theta_1(f)$  necessarily vanishes. (Nevertheless, we warn the reader that the vanishing of this primary obstruction does not necessarily imply the existence of a homotopy between  $f$  and a topological embedding.)

In this paper, we study the bordism invariance of the primary obstruction  $\theta_1(f)$ , which is a homotopy invariant. Here, two continuous maps  $f$  and  $g: M \rightarrow N$  of a closed  $m$ -dimensional manifold  $M$  into a manifold  $N$  are said to be *bordant*, if there exist a compact (unoriented)  $(m+1)$ -dimensional manifold  $W$  with  $\partial W$  the disjoint union of two copies  $M_1$  and  $M_2$  of  $M$  and a continuous map  $F: W \rightarrow N$  (called a *bordism* between  $f$  and  $g$ ) with  $F|_{M_1} = f$  and  $F|_{M_2} = g$  (see [4] for the terminology). Note that, here, the domains of  $f$  and  $g$  are the same manifold.

Our main result of this paper is the following.

**Theorem 1.1.** *Let  $f$  and  $g: M \rightarrow N$  be continuous maps of a smooth closed  $m$ -dimensional manifold into a smooth  $n$ -dimensional manifold with  $k = n - m > 0$ . Suppose that  $H^{m-k}(M; \mathbb{Z}_2)$  is generated by the elements of the form  $w_{i_1}(M) \cup \cdots \cup w_{i_s}(M)$  with  $i_1 + \cdots + i_s = m - k$ , where  $w_j(M)$  denotes the  $j$ -th Stiefel-Whitney class of  $M$ . Then if  $f$  and  $g$  are bordant, then  $\theta_1(f) = \theta_1(g)$ .*

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This paper is organized as follows. In §2, we recall the definition of the primary obstruction  $\theta_1(f)$  and also recall some related results. Theorem 1.1 will be proved in §3. As an application of our theorem, we show, in §4, that no constant map of a real, complex or quaternionic projective space into a smooth manifold can be bordant to a topological embedding under some dimension assumptions. As a corollary, we show that no null-homologous continuous map of the projective plane into an arbitrary 3-manifold can be a topological embedding. In §4 we also give explicit examples which show that the primary obstruction is not invariant under bordism in general.

Throughout the paper, all homology and cohomology groups have  $\mathbb{Z}_2$  coefficients unless otherwise indicated.

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## 2. Primary obstruction and related results

Let  $f: M \rightarrow N$  be a continuous map of an  $m$ -dimensional manifold  $M$  into an  $n$ -dimensional manifold  $N$ . We suppose that  $k = n - m > 0$  and that the map  $f$  is proper. For the moment, we assume no differentiability of  $M$ ,  $N$  or  $f$ . Let the stable normal bundle  $f^*TN \oplus \nu_M$  of  $f$  be denoted by  $\nu_f$ , where  $\nu_M$  is the stable normal bundle of the manifold  $M$ . Then we denote by  $w_k(f) (\in H^k(M))$  the  $k$ -th Stiefel-Whitney class of the stable vector bundle  $\nu_f$ . Note that this is a homotopy invariant; i.e., if  $f$  and  $g: M \rightarrow N$  are homotopic, then  $w_k(f) = w_k(g)$ . This is easily seen, since  $w_k(f)$  is the degree  $k$  term of  $f^*w(N) \cup \bar{w}(M)$ , where  $w(N)$  is the total Stiefel-Whitney class of  $N$  and  $\bar{w}(M)$  is the total dual Stiefel-Whitney class of  $M$ .

For the proper continuous map  $f: M \rightarrow N$ , we define  $v(f) \in H^k(M)$  to be the image of the fundamental class  $[M] \in H_m^c(M)$  by the composite

$$H_m^c(M) \xrightarrow{f_*} H_m^c(N) \xrightarrow{D_N^{-1}} H^k(N) \xrightarrow{f^*} H^k(M),$$

where  $H_*^c$  denotes the (singular) homology of the compatible family with respect to the compact subsets ([11, Chapter 6, Section 3]), and  $D_N$  denotes the Poincaré duality isomorphism. By the definition, it is easy to see that  $v(f)$  is a homotopy invariant (when  $M$  is not compact, the homotopy should be through proper maps).

We note that when  $M$  and  $N$  are smooth and  $f$  is an immersion, the above definitions of  $w_k(f)$  and  $v(f)$  coincide with those of  $w_k(\nu_f)$  and  $v_k(f)$  respectively given in [1]. See also [6] and [5, Proposition 4.1]. Furthermore, we recall the reader that there are some alternative definitions for  $v(f)$  (see [2, §2]).

**DEFINITION 2.1.** For a proper continuous map  $f: M \rightarrow N$ , we define the homology class  $\theta_1(f) \in H_{m-k}^c(M)$  by  $\theta_1(f) = D_M(v(f) - w_k(f))$ , where  $D_M: H^k(M) \rightarrow H_{m-k}^c(M)$  is the Poincaré duality isomorphism. Note that this is a homotopy invariant of  $f$ .

As has been seen in [2, §5], we know that, if  $M$  and  $N$  are smooth manifolds,  $M$  is closed, and  $f$  is a topological embedding, then  $\theta_1(f)$  necessarily vanishes.

The reason why we use the homology class  $\theta_1(f)$  instead of its Poincaré dual is that, if  $f$  is a generic smooth map, then the homology class  $\theta_1(f)$  is exactly the homology class represented by the closure of the self-intersection set of  $f$  (see [9, 2]).

**EXAMPLE 2.2.** Let  $f: K \rightarrow S^3$  be a continuous map, where  $K$  is the Klein bottle. Then it is easy to see that  $\theta_1(f)$  does not vanish, since  $w_1(f)$  does not vanish while  $v(f)$  vanishes. Hence  $f$  is not homotopic to a topological embedding.

In some cases we have the vanishing of the obstruction  $\theta_1(f)$ . The following proposition has been proved in [2].

**Proposition 2.3.** *Let  $f: M \rightarrow N$  be a proper continuous map of a smooth  $m$ -dimensional manifold into a smooth  $n$ -dimensional manifold with  $k = n - m > 0$ . If  $f_*: H_{m-k}^c(M) \rightarrow H_{m-k}^c(N)$  is injective, then  $\theta_1(f) = 0$  in  $H_{m-k}^c(M)$ .*

We note that, since  $\theta_1(f)$  depends only on the neighborhood of  $f(M)$  (see [2]), the above proposition is valid also when  $f_*$  on the  $(m-k)$ -th homology is injective after a sequence of surgeries performed in  $N - f(M)$ .

The obstruction  $\theta_1(f)$  plays an important role in the recognition problem of smooth embeddings. In fact, in [2, 3], the authors have proved that, if  $f: M \rightarrow N$  is a differentiable map which is generic for the double points in the sense of Ronga [9] and if  $M$  is closed, then  $f$  is a differentiable embedding if and only if the  $(m-k+1)$ -th Betti numbers (with respect to Čech homology) of  $M$  and  $f(M)$  coincide and  $\theta_1(f)$  vanishes.

### 3. Proof of Theorem 1.1

**Proof of Theorem 1.1.** Since  $f$  and  $g$  are bordant, we have

$$\begin{aligned} & \langle f^*(\alpha) \cup w_{i_1}(M) \cup \cdots \cup w_{i_s}(M), [M] \rangle \\ &= \langle g^*(\alpha) \cup w_{i_1}(M) \cup \cdots \cup w_{i_s}(M), [M] \rangle \end{aligned}$$

for all  $\alpha \in H^k(N)$  and for all  $(i_1, \dots, i_s)$  with  $i_1 + \cdots + i_s = m - k$ , where  $[M] \in H_m(M)$  is the fundamental class of  $M$  (see [4]). This implies that

$$\begin{aligned} & \langle f^*(\alpha), (w_{i_1}(M) \cup \cdots \cup w_{i_s}(M)) \cap [M] \rangle \\ &= \langle g^*(\alpha), (w_{i_1}(M) \cup \cdots \cup w_{i_s}(M)) \cap [M] \rangle. \end{aligned}$$

Hence by our hypothesis, we see that  $f^*(\alpha) = g^*(\alpha)$  for all  $\alpha \in H^k(N)$ . Then, this together with the fact that  $f_*[M] = g_*[M]$  implies that  $v(f) = v(g)$ .

Now let  $F: W \rightarrow N$  be a bordism between  $f$  and  $g$  as in the paragraph just before the statement of Theorem 1.1 in §1. Note that, since  $T(\partial W) \oplus \varepsilon^1 = TW|_{\partial W}$  for the trivial line bundle  $\varepsilon^1$  over  $\partial W$ , we see that  $(v_F|_{\partial W}) \oplus \varepsilon^1 = v_{(f \cup g)}$ , where  $v$  denotes the stable normal bundle. Hence we have  $w(\partial W) = i^*(w(W))$  and  $w(v_{(f \cup g)}) = i^*(w(v_F))$ , where  $w$  denotes the total Stiefel-Whitney class and  $i: \partial W \rightarrow W$  denotes the inclusion map. Then we have

$$\begin{aligned} & \langle w_k(v_{(f \cup g)}) \cup w_{i_1}(\partial W) \cup \cdots \cup w_{i_s}(\partial W), [\partial W] \rangle \\ &= \langle i^*(w_k(v_F) \cup w_{i_1}(W) \cup \cdots \cup w_{i_s}(W)), [\partial W] \rangle \\ &= \langle w_k(v_F) \cup w_{i_1}(W) \cup \cdots \cup w_{i_s}(W), i_*[\partial W] \rangle \\ &= 0 \end{aligned}$$

for every  $(i_1, \dots, i_s)$  with  $i_1 + \cdots + i_s = m - k$ . This implies that

$$\begin{aligned} & \langle w_k(v_f) \cup w_{i_1}(M) \cup \cdots \cup w_{i_s}(M), [M] \rangle \\ &= \langle w_k(v_g) \cup w_{i_1}(M) \cup \cdots \cup w_{i_s}(M), [M] \rangle. \end{aligned}$$

This together with our assumption shows that  $w_k(f) = w_k(v_f) = w_k(v_g) = w_k(g)$ . This completes the proof.

Note that the hypothesis of Theorem 1.1 is always satisfied when  $m = k$ . Another example for which the hypothesis is satisfied will be given in the next section.

#### 4. Application and examples

In the following,  $K$  will denote the field  $R$  of real numbers, the field  $C$  of complex numbers, or the field  $H$  of quaternions, and  $d$  will denote 1, 2 or 4 respectively.

**Proposition 4.1.** *Let  $f: KP^m \rightarrow N$  be a constant map of the  $m$ -dimensional  $K$ -projective space ( $\dim KP^m = dm$ ) into a smooth  $n$ -dimensional manifold with  $m$  even and  $k = n - dm > 0$ . If the  $k$ -th dual Stiefel-Whitney class  $\bar{w}_k(KP^m)$  does not vanish, then  $f$  is not bordant to a topological embedding of  $KP^m$ .*

**Proof.** It is known that, if  $k$  is not a multiple of  $d$ , then  $H^{dm-k}(KP^m)$  vanishes

and when  $m$  is even and  $k$  is a multiple of  $d$ ,  $H^{dm-k}(KP^m)$  is generated by  $w_d(KP^m)^{m-(k/d)}$  (see [7], for example). Thus the obstruction  $\theta_1(f)$  is a bordism invariant by Theorem 1.1. Furthermore, it is easy to see that, in our case,  $\theta_1(f)$  coincides with the Poincaré dual of  $\bar{w}_k(KP^m)$ . Thus, if this class does not vanish, then  $f$  is not bordant to a topological embedding of  $KP^m$ . This completes the proof.

For example, if  $m=2^r$  for some positive integer  $r$ , then  $\bar{w}_{d(m-1)}(KP^m) \neq 0$  (see [7, p.49]) and we see that a constant map  $KP^m \rightarrow N$  is not bordant to a topological embedding of  $KP^m$  for any manifold  $N$  with  $\dim N = d(2m-1)$ .

Since the 2-dimensional unoriented bordism group of a manifold  $N$  is isomorphic to the direct sum of  $H_2(N)$  and the usual 2-dimensional unoriented cobordism group  $\mathcal{N}_2$  ([4, (8.3) Theorem]), a continuous map  $f: RP^2 \rightarrow N$  is bordant to a constant map if and only if  $f_*[RP^2] = 0$  in  $H_2(N)$ . Furthermore, it is known that every topological 3-manifold admits a smooth structure. Thus we have the following.

**Corollary 4.2.** *Let  $f: RP^2 \rightarrow N$  be a continuous map of the projective plane into a topological 3-manifold  $N$ . If  $f_*[RP^2] = 0$  in  $H_2(N)$ , then  $f$  is not a topological embedding.*

Note that a topological embedding of a surface into a 3-manifold is not necessarily locally flat and can be very complicated. For example, see [8, Chapter 3]. Note also that there even exist topological embeddings which are nowhere locally flat ([10, Remark 3.3]).

Now we give examples which show that the primary obstruction  $\theta_1(f)$  is not invariant under bordism in general. In fact, neither  $v(f)$  nor  $w_k(f)$  is a bordism invariant in general.

**EXAMPLE 4.3.** Consider smooth immersions with normal crossings  $f$  and  $g: K \rightarrow S^1 \times S^2$  whose images are as in Figure 1, where  $K$  is the Klein bottle, and in the figure we have regarded  $S^1 \times S^2$  as  $[0,1] \times S^2$  with  $\{0\} \times S^2$  and  $\{1\} \times S^2$  identified. The maps  $f$  and  $g$  are bordant, since  $f_*[K] = g_*[K]$  in  $H_2(S^1 \times S^2)$  and the bordism group  $\mathcal{N}_2(S^1 \times S^2)$  is naturally isomorphic to  $H_2(S^1 \times S^2) \oplus \mathcal{N}_2$  (see [4]). However, we see that  $v(f) \neq v(g)$ , while  $w_1(f) = w_1(g)$ . The latter is easy to see, since  $S^1 \times S^2$  is orientable. The former follows from the latter together with the fact that  $D_K(v(f) - w_1(f)) = \theta_1(f) = j_*[\overline{M(f)}] \in H_1(K)$  is non-zero, where  $M(f) = \{x \in K: f^{-1}(f(x)) \neq \{x\}\}$  is the self-intersection set of  $f$  and  $j: M(f) \rightarrow K$  is the inclusion map ([9]), while the corresponding class vanishes for  $g$ . Note that  $H^1(K)$  is not generated by  $w_1(K)$  and our Theorem 1.1 cannot be applied.

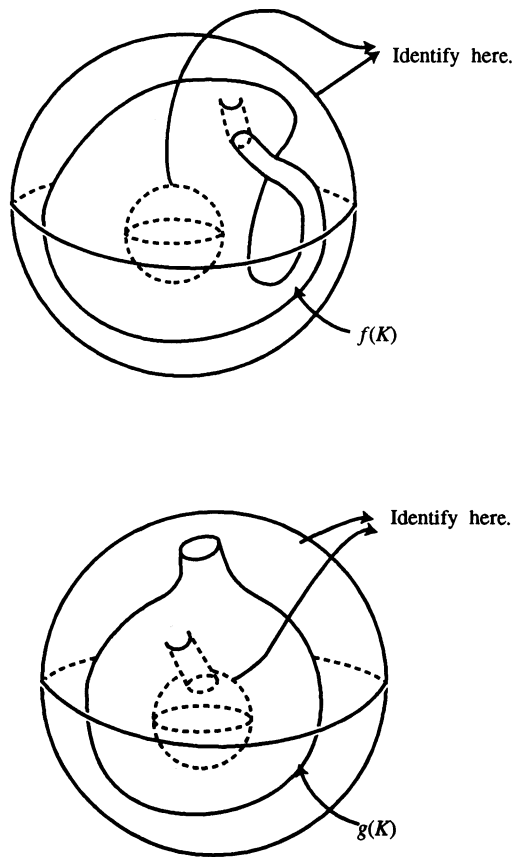


Fig. 1.

EXAMPLE 4.4. Another example is given as follows. Consider an embedding  $f: K \rightarrow V$  whose image is parallel to the boundary of  $V$  and a constant map  $g: K \rightarrow V$ , where  $V$  is the solid Klein bottle. Note that  $f$  and  $g$  are bordant. Furthermore, we see that  $w_1(f) \neq w_1(g)$ , while  $v(f) = v(g) = 0$ .

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