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Author(s)	Harada, Manabu; Ishii, Tadamasa
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Osaka University

ON ENDOMORPHISM RINGS OF NOETHERIAN QUASI-INJECTIVE MODULES

MANABU HARADA AND TADAMASA ISHII

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Let R be a ring with identity element. One of the authors studied the endomorphism ring of projective right R -module P with chain conditions in [6] and showed that the ring is right artinian (resp. noetherian) if so is P as an R -module.

We shall consider its dual in this short note. Unfortunately, we could not give the complete dual of them.

Recently, many authors have studied structures of injective module Q and given many interesting results between ideals in R and S -submodules in Q , where $S = \text{Hom}_R(Q, Q)$. However, we shall study mainly, in this note, some properties between R -submodules and left ideals in S .

In the first section, we shall consider the above problem in an abelian C_3 -category \mathcal{A} (see [10], Chap. III), and show that if A is a quasi-injective object in \mathcal{A} and A is noetherian (resp. artinian), then the endomorphism ring $[A, A]$ of A is semi-primary (resp. left noetherian).

In the second section, we shall study conditions under which $S = \text{Hom}_R(M, M)$ is left artinian, when M is a right R -quasi-injective noetherian module and shall give a condition that M gives us a Morita duality on categories of finitely generated right R - (resp. left S)-modules.

In this paper, we always assume that R -modules M are unitary and the ring of endomorphism of M operates from the left side.

After having completely settled this note, we have found J.W. Fisher's results in [5]. His Theorem 2 is contained in [6], Theorem 2. 8 and Theorem 3 coincides with our Theorem 1. Further, K. Motose obtained similar results in [12].

1. In cases of C_3 -abelian categories

Let \mathcal{A} be an abelian C_3 -category (see [10], Chap. III). For any object A in \mathcal{A} , by S_A we denote the ring of morphisms of A to itself. Let B be a sub-object in \mathcal{A} . By $l(B)$ we denote the left ideal in S_A whose elements consists of all s in

S such that $\text{Ker } s \supseteq B$. We call $l(B)$ the *left annihilator ideal* of B . Conversely, let T be a sub-set in S_A . By $r(T)$ we denote $\bigcap_{t \in T} \text{Ker } t$. We call it an *annihilator sub-object* in A . We define the dual of idempotent sub-object in A , (cf. [6]). If $r(I) = r(I^2)$ for a left ideal I in S_A then $r(I)$ is called a *co-idempotent sub-object* in A . If the sub-objects in A satisfy the descending (resp. ascending) chain conditions, we say A is *artinian* (resp. *noetherian*). A is called a *quasi-injective*, if $[A, A] \xrightarrow{[i, A]} [B, A]$ is surjective for any sub-object B and $i : B \rightarrow A$ inclusion.

Theorem 1. *Let A be a quasi-injective object in the abelian C_3 -category A . If A is noetherian with respect to annihilator sub-objects, then $S = [A, A]$ is a semi-primary ring. (Dual of [6], Proposition 2. 4).*

In order to prove it we need some lemmas.

Lemma 1. *Let A be a quasi-injective object in A and I a left ideal in S_A such that $lr(I) = I$. Then $lr(I + S_A x) = I + S_A x$ for any x in S_A . (Dual of [6], Proposition 2. 3, cf. [1], Lemma 1 in § 5 and [9], Theorem 2. 1).*

Proof. The proof is analogous to [9], Theorem 2. 1. It is clear that $lr(I + Sx) \supseteq I + Sx$, where $S = S_A$. Let y be in $lr(I + Sx) = l(r(I) \cap r(x))$. Then $r(y) \supseteq r(I) \cap r(x)$ and hence, we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & r(I) \cap r(x) & \rightarrow & r(I) & \xrightarrow{x|r(I)} & xr(I) \rightarrow 0 \\
 & & \downarrow & & \downarrow i & & \downarrow \theta \\
 0 & \rightarrow & r(y) & \rightarrow & A & \xrightarrow{y} & yA \rightarrow 0
 \end{array}$$

where $yA = \text{Im } y$ and $xr(I) = \text{Im } (x|r(I))$. Hence, we have a morphism θ in $[xr(I), yA]$ such that $\theta x|r(I) = yi$ by [10], p. 23, Proposition 16. 5. Since A is quasi-injective, θ is extended in an element s in S . Hence, $y - sx \in lr(I) = I$. Therefore, $y \in I + Sx$.

Corollary. *Let A be as above. Then $lr(I) = I$ for any finitely generated left ideal I in S_A . (Dual of [6], Lemma 2.6 or [13], Theorem 1. 1).*

Lemma 2. *Let A be a quasi-injective object in A . If A satisfies the condition in the theorem, then every co-idempotent sub-object $B (\neq A)$ of A is contained in a proper direct summand of A .*

Proof. The proof is a dual of [6], Proposition 2. 3. However, we shall give the proof for the sake of completeness. Let $B = r(I) = r(I^2)$ for a left ideal I in $S = [A, A]$. From the assumption we can take a maximal sub-object C among C' such that $A \cong C' \supseteq B$ and $C' = r(I) = r(I^2)$. Since $I^2 \neq 0$, we can choose x in I which has properties; $lx \neq 0$ and $r(x)$ is maximal among $r(y)$ such that $ly \neq 0$, $y \in I$. If $lx = 0$, $\text{Im } x \subseteq r(I^2) = r(I)$, and $lx = 0$. Therefore, there exists y in I

such that $lyx \neq 0$. Since $r(yx) \supseteq r(x)$, $r(yx) = r(x)$ by the maximality of $r(x)$. Hence, $Syx = Sx$ by Lemma 1. Therefore, there exists a in I such that $ax = x$. If a is not idempotent, then $0 \neq V = \{z \in I, zx = 0\} \subsetneq I$. Further $r(V) \supset \text{Im } x$ and $r(I) \not\supset \text{Im } x$. Hence, V is nilpotent by the maximality of C . Thus, we can find a non-zero idempotent e in I . Hence, $r(I) \subseteq r(e) = \text{Im}(1 - e)$.

Proof of the theorem. Since every direct summand of A is an annihilator object, A is a directsum of finite number of indecomposable objects. First we assume that A is indecomposable. Let I be a proper ideal in S . Then $r(I^n) = r(I^{2^n})$ for some integer n by the assumption. Hence, $r(I^n) = A$ by Lemma 2. Therefore, I is nilpotent, which implies that S is a semi-primary ring with unique maximal ideal. In general case, we can use the standard argument as in the proof of [6], Proposition 2. 4.

Corollary. *Let A be a quasi-injective and quasi-projective object in \mathcal{A} . If A is noetherian, S_A is right artinian.*

Proof. S_A is semi-primary by Theorem 1 and right noetherian by [6], Proposition 2, 7. Hence, S_A is right artinian.

Proposition 1. *Let A be a quasi-injective object in \mathcal{A} . Then the following statements are equivalent.*

- 1) S_A is left noetherian.
- 2) A is artinian with respect to annihilator sub-objects. (cf. [13] and [3]).

Proof. 1) \rightarrow 2). It is clear. 2) \rightarrow 1). The set of all finitely generated left ideals in S_A is noetherian from 2) and Lemma 1. Hence, S_A is left noetherian.

Corollary. *Let A be a quasi-injective object in \mathcal{A} . If A is artinian and noetherian with respect to annihilator sub-objects, then S_A is left artinian.*

Proof. S_A is semi-primary by Theorem 1 and left noetherian by Proposition 1. Therefore, S_A is left artinian.

2. In cases of modules

In this section, we assume that a ring R has the identity element and every right R -module is unitary.

Proposition 2. *Let M be a quasi-injective right R -module and $S = \text{Hom}_R(M, M)$. Then M is noetherian as a left S -module if and only if M is noetherian with respect to annihilator submodules for sub-sets in R .*

Proof. We assume the later condition in the proposition. Then R is artinian with respect to annihilator right ideals for sub-sets in M . Let T be an S -submodule in M . We take a minimal one $r(T')$ among $r(T^*)$, where T^* runs

through all finitely generated S -submodules in T . Let t be any element in T . Then $r(St+T')=r(T')$ by the minimality of T' . Hence, $St+T'=T'$ by [9], Theorem 2. 1.

Corollary 1. *Let R be a right artinian ring and M a quasi-injective right R -module. Then M is a noetherian S -module. Furthermore, if M is artinian (or noetherian) as R -modules, then S is left artinian and M has a finite composition length as S -modules.*

Proof. The first part is clear. If M is artinian, then S is left noetherian by Proposition 1. Let J be the Jacobson radical of S . Then $J^n M = J^{n+1} M$ for some n . Since M is S -noetherian, $J^n M = 0$. Hence, $J^n = 0$ and S is semi-primary, since S/J is a regular ring in the sense of Von Neumann, (see [4]). Therefore, S is left artinian. The last part is clear from the above and the first part.

Corollary 2. ([2]). *Let R be a right noetherian and self-injective as a right R -module. Then R is left and right artinian (QF-ring).*

Proof. R is a projective injective right R -module. Hence, R is right artinian by Corollary to Theorem 1. Therefore, R is left artinian by the above corollary.

According to Azumaya [1], we define a *weakly distinguished* R -module T as follows: for any R -submodules $T_1 \supset T_2$ in T such that T_1/T_2 is R -irreducible, $\text{Hom}_R(T_1/T_2, T) \neq 0$. It is clear that if T is an R -cogenerator, then T is weakly distinguished. Furthermore, if T is quasi-injective, T is weakly distinguished if and only if $l(T_1) \not\subseteq l(T_2)$ for any R -submodules $T_1 \supsetneq T_2$ or equivalently, $rl(T') = T'$ for any R -submodule T' of T , (cf. [1], Proposition 6).

Lemma 3. *Let M be a right R -quasi-injective and noetherian with respect to annihilator R -submodules for sub-sets in S , where $S = \text{Hom}_R(M, M)$. We assume that S satisfies a condition: for any left ideals I and I' in S*

$$(*) \quad r(I \cap I') = r(I) + r(I').$$

Then S is left artinian.

Proof. Since S is semi-primary, S contains the non-zero left socle T , say $T = \sum_i \oplus I_i$, where I_i 's are minimal left ideals. Put $L_i = \sum_{j>i} \oplus I_j$. Then $L_1 \supset L_2 \supset L_3 \supset \dots$ and $r(L_1) \subseteq r(L_2) \subseteq r(L_3) \subseteq \dots$. Hence, $r(L_n) = r(L_{n+1})$ for some n by the assumption. We assume $L_n \neq 0$. Then $L_n = I_n \oplus L_{n+1}$, $r(L_n) \subseteq r(I_n)$ and $M = r(I_n \cap L_{n+1}) = r(I_n) + r(L_{n+1})$, which is a contradiction. Hence, $T = \sum_{i=1}^m \oplus I_i$. Put $M_1 = r(T)$, Since T is finitely generated, $l(M_1) = T$ by Lemma 1. Further-

more, T is a two-sided ideal and hence, M_1 is a left S -module, which implies M_1 is a quasi-injective R -module by [9], Theorem 1.2. Put $S_1 = \text{Hom}_R(M_1, M_1)$. Then we have a natural epimorphism φ of S to S_1 with $\text{Ker } \varphi = I(M_1) = T$ and hence, M_1 is noetherian with respect to annihilator R -submodules. Put T_1 the left socle of S_1 , say $T_1 = \sum_i \oplus \bar{I}_i \supseteq$ where $I_i \supseteq T$ and $\bar{I}_i = I_i/T$ is irreducible. Then $M_1 = r(T) = r(I_{1n} \cap L_{1n}) = r(I_{1n}) + r(L_{1n})$. Hence, we know from the same argument in the above that $T_1 = \sum_{i=1}^m \oplus \bar{I}_i$. Repeating this we have a series of ideals $S \supset T_n \supset T_{n-1} \supset \dots \supset T_1 \supset 0$ such that T_i/T_{i-1} is the left socle of S/T_{i-1} which has a finite composition length. Now S is semi-primary and hence, $N^{n-i} \subseteq T_i$, where $N^{n-1} \neq 0$ and $N^n = 0$. Therefore, S is left artinian.

Theorem 2. *Let M be R -weakly distinguished and quasi-injective and $S = \text{Hom}_R(M, M)$. Then the following two conditions are equivalent.*

- 1) S is left noetherian.
- 2) M is artinian as an R -module.

And 1) or 2) implies

- 3) M is S -injective.

Furthermore, if M is noetherian with respect to annihilator submodules for sub-sets in S , then 3) implies 1) and 2) and S is left artinian and M is R -noetherian.

Proof. 1)→2). It is clear from the remark before Lemma 3. 2)→1). It is clear from Proposition 1. 1)→3). We assume that S is left noetherian. Let I_1, I_2 be left ideals in S . Then $r(I_1) + r(I_2) = lr(I_1) \cap lr(I_2) = I_1 \cap I_2$ by Corollary to Lemma 1. Hence, $r(I_1) + r(I_2) = r(I_1 \cap I_2)$ by the above remark. Now, we shall show by the induction on the number of generators of left ideals in S that M satisfies the Bear's condition, (it is essentially due to [8]). Let $I = Sx$, Then $l(xM) = l_S(x) = \{y \in S, yx = 0\}$ and $r(l_S(x)) \supseteq xM$. Hence, $r(l_S(x)) = xM$. Let f be an element in $\text{Hom}_S(I, M)$, then $f(x) \in r(l_S(x)) = xM$. Hence, there exists m in M such that $f(x) = xm$. Let $I = \sum_{i=1}^n Sx_i$ and $I_1 = \sum_{i=1}^{n-1} Sx_i$. From an exact sequence: $0 \rightarrow I_1 \rightarrow I \rightarrow I/I_1 \xrightarrow{\varphi} Sx_n/(Sx_n \cap I_1) \rightarrow 0$, we have the exact sequence: $\text{Hom}_S(I_1, M) \leftarrow \text{Hom}_S(I, M) \leftarrow \text{Hom}_S(I/I_1, M) \xrightarrow{[\varphi, M]} \text{Hom}_S(Sx_n/(Sx_n \cap I_1), M) \leftarrow 0$. Let f be in $\text{Hom}_S(I, M)$. Then there exists m in M such that $f(x) = xm$ for $x \in I_1$ by the hypothesis of the induction. We define an element f_m in $\text{Hom}_S(I, M)$ by setting $f_m(x) = xm$ for $x \in I$. Then $g = f - f_m \in \text{Hom}_S(I/I_1, M)$. Since $\text{Hom}_S(Sx_n, M) \leftarrow \text{Hom}_S(Sx_n/Sx_n \cap I_1, M)$ is monomorphic, there exists m' in M such that $g(\varphi^{-1}(\bar{sx}_n)) = sx_n m'$, where \bar{sx}_n means a residue class of sx_n in $Sx_n/(Sx_n \cap I_1)$. Hence, $m' \in r(Sx_n \cap I_1) = r(Sx_n) + r(I_1)$. Let $m' = m_1 + m_2$, $m_1 \in r(x_n)$, $m_2 \in r(I_1)$ and define f_{m_2} as above. Then for any $x = x_1 + x_2$ in I ($x_1 \in I_1, x_2 \in Sx_n$), $g(x) = g(\varphi^{-1}(x_2)) = x_2 m' = x_2 m_2 = f_{m_2}(x)$. Therefore, $f = f_{m+m_2}$. 3)→1).

Let M be S -injective and I_1, I_2 be left ideals in S . Then we have an exact sequence: $0 \leftarrow \text{Hom}_S(S/(I_1 \cap I_2), M) \leftarrow \text{Hom}_S(S/I_1, M) \oplus \text{Hom}_S(S/I_2, M)$, which means that M satisfies the condition (*). Hence, S is left artinian from Lemma 3. The last part is clear, since S is artinian by Theorem 1.

Corollary. *Let M and S be as in Theorem 2. If M is R -artinian, then any S - R bi-submodule N of M is $S/l(N)$ -injective.*

Proof. Let N be an S - R submodule of M . Then $N = r_l(N)$ and $l(N)$ is a two-sided ideal in S . Put $\bar{S} = S/l(N)$. Then $\bar{S} = \text{Hom}_R(N, N)$ and N satisfies the same conditions as M by [9], Theorem 1.1. Hence, N is \bar{S} -injective by Theorem 2.

Theorem 3. *Let M be an S - R bi-module such that $\text{Hom}_R(M, M) = S$ and $\text{Hom}_S(M, M) = R$. Furthermore, we assume that M is S - and R -injective, respectively. Then the following two statements are equivalent.*

- 1) M is R -noetherian,
- 2) S is left artinian.

And 1) or 2) implies that M is R -artinian. Thus, if M is R - and S -noetherian or if R and S are right and left artinian, respectively, then M gives us a duality between the category of finitely generated right R -modules and the category of finitely generated left S -modules in the sense of Morita.

Proof. 1) \rightarrow 2). Since S satisfies the condition (*) of Lemma 3, S is left artinian. 2) \rightarrow 1). It is obtained by Corollary to Proposition 2. Now, we assume 1) or 2). Let T be an R -submodule, then $T = r_l(T)$ by [9], Theorem 2.1. Hence, M is R -artinian, since S is left noetherian. The last part is clear from [11], Theorem 6.3, v.

REMARK. Let M and S be as the first half in Theorem 2. Then the injectivity of M as an S -module does not imply the fact that S is left noetherian. Furthermore, if R is commutative, then a fact that M is R -noetherian implies that M is R -artinian, (see Proposition 2 in [7]). However, the converse is not true in general.

Finally, we shall give an example of injective noetherian but not artinian modules. Let K be a field and $I = Z^+ \cup \alpha$ the set of indices, where Z^+ is the set of positive integers. Let R be the ring of upper tri-angular matrices over K with indices I , (α is the last index and α -column consists of all column finite). Let $e_{i,j}$ be matrix units in R and put $M = e_{11}R$. Then $M \approx \text{Hom}_K(Re_{\alpha\alpha}, Ke_{1\alpha})$. Hence, M is R -injective. It is clear that M is R -noetherian but not R -artinian.

OSAKA CITY UNIVERSITY
KINKI UNIVERSITY

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