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Author(s)	Harada, Manabu; Ishii, Tadamasa
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## ON ENDOMORPHISM RINGS OF NOETHERIAN QUASI-INJECTIVE MODULES

MANABU HARADA AND TADAMASA ISHII

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Let  $R$  be a ring with identity element. One of the authors studied the endomorphism ring of projective right  $R$ -module  $P$  with chain conditions in [6] and showed that the ring is right artinian (resp. noetherian) if so is  $P$  as an  $R$ -module.

We shall consider its dual in this short note. Unfortunately, we could not give the complete dual of them.

Recently, many authors have studied structures of injective module  $Q$  and given many interesting results between ideals in  $R$  and  $S$ -submodules in  $Q$ , where  $S = \text{Hom}_R(Q, Q)$ . However, we shall study mainly, in this note, some properties between  $R$ -submodules and left ideals in  $S$ .

In the first section, we shall consider the above problem in an abelian  $C_3$ -category  $\mathcal{A}$  (see [10], Chap. III), and show that if  $A$  is a quasi-injective object in  $\mathcal{A}$  and  $A$  is noetherian (resp. artinian), then the endomorphism ring  $[A, A]$  of  $A$  is semi-primary (resp. left noetherian).

In the second section, we shall study conditions under which  $S = \text{Hom}_R(M, M)$  is left artinian, when  $M$  is a right  $R$ -quasi-injective noetherian module and shall give a condition that  $M$  gives us a Morita duality on categories of finitely generated right  $R$ - (resp. left  $S$ )-modules.

In this paper, we always assume that  $R$ -modules  $M$  are unitary and the ring of endomorphism of  $M$  operates from the left side.

After having completely settled this note, we have found J.W. Fisher's results in [5]. His Theorem 2 is contained in [6], Theorem 2. 8 and Theorem 3 coincides with our Theorem 1. Further, K. Motose obtained similar results in [12].

### 1. In cases of $C_3$ -abelian categories

Let  $\mathcal{A}$  be an abelian  $C_3$ -category (see [10], Chap. III). For any object  $A$  in  $\mathcal{A}$ , by  $S_A$  we denote the ring of morphisms of  $A$  to itself. Let  $B$  be a sub-object in  $\mathcal{A}$ . By  $l(B)$  we denote the left ideal in  $S_A$  whose elements consists of all  $s$  in

$S$  such that  $\text{Ker } s \supseteq B$ . We call  $l(B)$  the *left annihilator ideal* of  $B$ . Conversely, let  $T$  be a sub-set in  $S_A$ . By  $r(T)$  we denote  $\bigcap_{t \in T} \text{Ker } t$ . We call it an *annihilator sub-object* in  $A$ . We define the dual of idempotent sub-object in  $A$ , (cf. [6]). If  $r(I) = r(I^2)$  for a left ideal  $I$  in  $S_A$  then  $r(I)$  is called a *co-idempotent sub-object* in  $A$ . If the sub-objects in  $A$  satisfy the descending (resp. ascending) chain conditions, we say  $A$  is *artinian* (resp. *noetherian*).  $A$  is called a *quasi-injective*, if  $[A, A] \xrightarrow{[i, A]} [B, A]$  is surjective for any sub-object  $B$  and  $i : B \rightarrow A$  inclusion.

**Theorem 1.** *Let  $A$  be a quasi-injective object in the abelian  $C_3$ -category  $A$ . If  $A$  is noetherian with respect to annihilator sub-objects, then  $S = [A, A]$  is a semi-primary ring. (Dual of [6], Proposition 2. 4).*

In order to prove it we need some lemmas.

**Lemma 1.** *Let  $A$  be a quasi-injective object in  $A$  and  $I$  a left ideal in  $S_A$  such that  $lr(I) = I$ . Then  $lr(I + S_A x) = I + S_A x$  for any  $x$  in  $S_A$ . (Dual of [6], Proposition 2. 3, cf. [1], Lemma 1 in § 5 and [9], Theorem 2. 1).*

Proof. The proof is analogous to [9], Theorem 2. 1. It is clear that  $lr(I + Sx) \supseteq I + Sx$ , where  $S = S_A$ . Let  $y$  be in  $lr(I + Sx) = l(r(I) \cap r(x))$ . Then  $r(y) \supseteq r(I) \cap r(x)$  and hence, we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & r(I) \cap r(x) & \rightarrow & r(I) & \xrightarrow{x|r(I)} & xr(I) \rightarrow 0 \\
 & & \downarrow & & \downarrow i & & \downarrow \theta \\
 0 & \rightarrow & r(y) & \rightarrow & A & \xrightarrow{y} & yA \rightarrow 0
 \end{array}$$

where  $yA = \text{Im } y$  and  $xr(I) = \text{Im } (x|r(I))$ . Hence, we have a morphism  $\theta$  in  $[xr(I), yA]$  such that  $\theta x|r(I) = yi$  by [10], p. 23, Proposition 16. 5. Since  $A$  is quasi-injective,  $\theta$  is extended in an element  $s$  in  $S$ . Hence,  $y - sx \in lr(I) = I$ . Therefore,  $y \in I + Sx$ .

**Corollary.** *Let  $A$  be as above. Then  $lr(I) = I$  for any finitely generated left ideal  $I$  in  $S_A$ . (Dual of [6], Lemma 2.6 or [13], Theorem 1. 1).*

**Lemma 2.** *Let  $A$  be a quasi-injective object in  $A$ . If  $A$  satisfies the condition in the theorem, then every co-idempotent sub-object  $B (\neq A)$  of  $A$  is contained in a proper direct summand of  $A$ .*

Proof. The proof is a dual of [6], Proposition 2. 3. However, we shall give the proof for the sake of completeness. Let  $B = r(I) = r(I^2)$  for a left ideal  $I$  in  $S = [A, A]$ . From the assumption we can take a maximal sub-object  $C$  among  $C'$  such that  $A \cong C' \supseteq B$  and  $C' = r(I) = r(I^2)$ . Since  $I^2 \neq 0$ , we can choose  $x$  in  $I$  which has properties;  $lx \neq 0$  and  $r(x)$  is maximal among  $r(y)$  such that  $ly \neq 0$ ,  $y \in I$ . If  $lx = 0$ ,  $\text{Im } x \subseteq r(I^2) = r(I)$ , and  $lx = 0$ . Therefore, there exists  $y$  in  $I$

such that  $lyx \neq 0$ . Since  $r(yx) \supseteq r(x)$ ,  $r(yx) = r(x)$  by the maximality of  $r(x)$ . Hence,  $Syx = Sx$  by Lemma 1. Therefore, there exists  $a$  in  $I$  such that  $ax = x$ . If  $a$  is not idempotent, then  $0 \neq V = \{z \in I, zx = 0\} \subsetneq I$ . Further  $r(V) \supset \text{Im } x$  and  $r(I) \not\supset \text{Im } x$ . Hence,  $V$  is nilpotent by the maximality of  $C$ . Thus, we can find a non-zero idempotent  $e$  in  $I$ . Hence,  $r(I) \subseteq r(e) = \text{Im } (1 - e)$ .

*Proof of the theorem.* Since every direct summand of  $A$  is an annihilator object,  $A$  is a directsum of finite number of indecomposable objects. First we assume that  $A$  is indecomposable. Let  $I$  be a proper ideal in  $S$ . Then  $r(I^n) = r(I^{2^n})$  for some integer  $n$  by the assumption. Hence,  $r(I^n) = A$  by Lemma 2. Therefore,  $I$  is nilpotent, which implies that  $S$  is a semi-primary ring with unique maximal ideal. In general case, we can use the standard argument as in the proof of [6], Proposition 2. 4.

**Corollary.** *Let  $A$  be a quasi-injective and quasi-projective object in  $\mathcal{A}$ . If  $A$  is noetherian,  $S_A$  is right artinian.*

Proof.  $S_A$  is semi-primary by Theorem 1 and right noetherian by [6], Proposition 2, 7. Hence,  $S_A$  is right artinian.

**Proposition 1.** *Let  $A$  be a quasi-injective object in  $\mathcal{A}$ . Then the following statements are equivalent.*

- 1)  $S_A$  is left noetherian.
- 2)  $A$  is artinian with respect to annihilator sub-objects. (cf. [13] and [3]).

Proof. 1) $\rightarrow$ 2). It is clear. 2) $\rightarrow$ 1). The set of all finitely generated left ideals in  $S_A$  is noetherian from 2) and Lemma 1. Hence,  $S_A$  is left noetherian.

**Corollary.** *Let  $A$  be a quasi-injective object in  $\mathcal{A}$ . If  $A$  is artinian and noetherian with respect to annihilator sub-objects, then  $S_A$  is left artinian.*

Proof.  $S_A$  is semi-primary by Theorem 1 and left noetherian by Proposition 1. Therefore,  $S_A$  is left artinian.

## 2. In cases of modules

In this section, we assume that a ring  $R$  has the identity element and every right  $R$ -module is unitary.

**Proposition 2.** *Let  $M$  be a quasi-injective right  $R$ -module and  $S = \text{Hom}_R(M, M)$ . Then  $M$  is noetherian as a left  $S$ -module if and only if  $M$  is noetherian with respect to annihilator submodules for sub-sets in  $R$ .*

Proof. We assume the later condition in the proposition. Then  $R$  is artinian with respect to annihilator right ideals for sub-sets in  $M$ . Let  $T$  be an  $S$ -submodule in  $M$ . We take a minimal one  $r(T')$  among  $r(T^*)$ , where  $T^*$  runs

through all finitely generated  $S$ -submodules in  $T$ . Let  $t$  be any element in  $T$ . Then  $r(St+T')=r(T')$  by the minimality of  $T'$ . Hence,  $St+T'=T'$  by [9], Theorem 2. 1.

**Corollary 1.** *Let  $R$  be a right artinian ring and  $M$  a quasi-injective right  $R$ -module. Then  $M$  is a noetherian  $S$ -module. Furthermore, if  $M$  is artinian (or noetherian) as  $R$ -modules, then  $S$  is left artinian and  $M$  has a finite composition length as  $S$ -modules.*

*Proof.* The first part is clear. If  $M$  is artinian, then  $S$  is left noetherian by Proposition 1. Let  $J$  be the Jacobson radical of  $S$ . Then  $J^n M = J^{n+1} M$  for some  $n$ . Since  $M$  is  $S$ -noetherian,  $J^n M = 0$ . Hence,  $J^n = 0$  and  $S$  is semi-primary, since  $S/J$  is a regular ring in the sense of Von Neumann, (see [4]). Therefore,  $S$  is left artinian. The last part is clear from the above and the first part.

**Corollary 2.** ([2]). *Let  $R$  be a right noetherian and self-injective as a right  $R$ -module. Then  $R$  is left and right artinian (QF-ring).*

*Proof.*  $R$  is a projective injective right  $R$ -module. Hence,  $R$  is right artinian by Corollary to Theorem 1. Therefore,  $R$  is left artinian by the above corollary.

According to Azumaya [1], we define a *weakly distinguished*  $R$ -module  $T$  as follows: for any  $R$ -submodules  $T_1 \supset T_2$  in  $T$  such that  $T_1/T_2$  is  $R$ -irreducible,  $\text{Hom}_R(T_1/T_2, T) \neq 0$ . It is clear that if  $T$  is an  $R$ -cogenerator, then  $T$  is weakly distinguished. Furthermore, if  $T$  is quasi-injective,  $T$  is weakly distinguished if and only if  $l(T_1) \not\subseteq l(T_2)$  for any  $R$ -submodules  $T_1 \supsetneq T_2$  or equivalently,  $rl(T') = T'$  for any  $R$ -submodule  $T'$  of  $T$ , (cf. [1], Proposition 6).

**Lemma 3.** *Let  $M$  be a right  $R$ -quasi-injective and noetherian with respect to annihilator  $R$ -submodules for sub-sets in  $S$ , where  $S = \text{Hom}_R(M, M)$ . We assume that  $S$  satisfies a condition: for any left ideals  $I$  and  $I'$  in  $S$*

$$(*) \quad r(I \cap I') = r(I) + r(I').$$

*Then  $S$  is left artinian.*

*Proof.* Since  $S$  is semi-primary,  $S$  contains the non-zero left socle  $T$ , say  $T = \sum_i \oplus I_i$ , where  $I_i$ 's are minimal left ideals. Put  $L_i = \sum_{j>i} \oplus I_j$ . Then  $L_1 \supset L_2 \supset L_3 \supset \dots$  and  $r(L_1) \subseteq r(L_2) \subseteq r(L_3) \subseteq \dots$ . Hence,  $r(L_n) = r(L_{n+1})$  for some  $n$  by the assumption. We assume  $L_n \neq 0$ . Then  $L_n = I_n \oplus L_{n+1}$ ,  $r(L_n) \subseteq r(I_n)$  and  $M = r(I_n \cap L_{n+1}) = r(I_n) + r(L_{n+1})$ , which is a contradiction. Hence,  $T = \sum_{i=1}^m \oplus I_i$ . Put  $M_1 = r(T)$ , Since  $T$  is finitely generated,  $l(M_1) = T$  by Lemma 1. Further-

more,  $T$  is a two-sided ideal and hence,  $M_1$  is a left  $S$ -module, which implies  $M_1$  is a quasi-injective  $R$ -module by [9], Theorem 1.2. Put  $S_1 = \text{Hom}_R(M_1, M_1)$ . Then we have a natural epimorphism  $\varphi$  of  $S$  to  $S_1$  with  $\text{Ker } \varphi = I(M_1) = T$  and hence,  $M_1$  is noetherian with respect to annihilator  $R$ -submodules. Put  $T_1$  the left socle of  $S_1$ , say  $T_1 = \sum_i \oplus \bar{I}_i \supseteq$  where  $I_i \supseteq T$  and  $\bar{I}_i = I_i/T$  is irreducible. Then  $M_1 = r(T) = r(I_{1n} \cap L_{1n}) = r(I_{1n}) + r(L_{1n})$ . Hence, we know from the same argument in the above that  $T_1 = \sum_{i=1}^m \oplus \bar{I}_i$ . Repeating this we have a series of ideals  $S \supset T_n \supset T_{n-1} \supset \dots \supset T_1 \supset 0$  such that  $T_i/T_{i-1}$  is the left socle of  $S/T_{i-1}$  which has a finite composition length. Now  $S$  is semi-primary and hence,  $N^{n-i} \subseteq T_i$ , where  $N^{n-1} \neq 0$  and  $N^n = 0$ . Therefore,  $S$  is left artinian.

**Theorem 2.** *Let  $M$  be  $R$ -weakly distinguished and quasi-injective and  $S = \text{Hom}_R(M, M)$ . Then the following two conditions are equivalent.*

- 1)  $S$  is left noetherian.
- 2)  $M$  is artinian as an  $R$ -module.

And 1) or 2) implies

- 3)  $M$  is  $S$ -injective.

Furthermore, if  $M$  is noetherian with respect to annihilator submodules for sub-sets in  $S$ , then 3) implies 1) and 2) and  $S$  is left artinian and  $M$  is  $R$ -noetherian.

Proof. 1)  $\rightarrow$  2). It is clear from the remark before Lemma 3. 2)  $\rightarrow$  1). It is clear from Proposition 1. 1)  $\rightarrow$  3). We assume that  $S$  is left noetherian. Let  $I_1, I_2$  be left ideals in  $S$ . Then  $r(I_1) + r(I_2) = lr(I_1) \cap lr(I_2) = I_1 \cap I_2$  by Corollary to Lemma 1. Hence,  $r(I_1) + r(I_2) = r(I_1 \cap I_2)$  by the above remark. Now, we shall show by the induction on the number of generators of left ideals in  $S$  that  $M$  satisfies the Bear's condition, (it is essentially due to [8]). Let  $I = Sx$ , Then  $l(xM) = l_S(x) = \{y \in S, yx = 0\}$  and  $r(l_S(x)) \supseteq xM$ . Hence,  $r(l_S(x)) = xM$ . Let  $f$  be an element in  $\text{Hom}_S(I, M)$ , then  $f(x) \in r(l_S(x)) = xM$ . Hence, there exists  $m$  in  $M$  such that  $f(x) = xm$ . Let  $I = \sum_{i=1}^n Sx_i$  and  $I_1 = \sum_{i=1}^{n-1} Sx_i$ . From an exact sequence:  $0 \rightarrow I_1 \rightarrow I \rightarrow I/I_1 \xrightarrow{\varphi} Sx_n/(Sx_n \cap I_1) \rightarrow 0$ , we have the exact sequence:  $\text{Hom}_S(I_1, M) \leftarrow \text{Hom}_S(I, M) \leftarrow \text{Hom}_S(I/I_1, M) \xrightarrow{[\varphi, M]} \text{Hom}_S(Sx_n/(Sx_n \cap I_1), M) \leftarrow 0$ . Let  $f$  be in  $\text{Hom}_S(I, M)$ . Then there exists  $m$  in  $M$  such that  $f(x) = xm$  for  $x \in I_1$  by the hypothesis of the induction. We define an element  $f_m$  in  $\text{Hom}_S(I, M)$  by setting  $f_m(x) = xm$  for  $x \in I$ . Then  $g = f - f_m \in \text{Hom}_S(I/I_1, M)$ . Since  $\text{Hom}_S(Sx_n, M) \leftarrow \text{Hom}_S(Sx_n/Sx_n \cap I_1, M)$  is monomorphic, there exists  $m'$  in  $M$  such that  $g(\varphi^{-1}(s\bar{x}_n)) = sx_n m'$ , where  $s\bar{x}_n$  means a residue class of  $sx_n$  in  $Sx_n/(Sx_n \cap I_1)$ . Hence,  $m' \in r(Sx_n \cap I_1) = r(Sx_n) + r(I_1)$ . Let  $m' = m_1 + m_2$ ,  $m_1 \in r(x_n)$ ,  $m_2 \in r(I_1)$  and define  $f_{m_2}$  as above. Then for any  $x = x_1 + x_2$  in  $I$  ( $x_1 \in I_1, x_2 \in Sx_n$ ),  $g(x) = g(\varphi^{-1}(x_2)) = x_2 m' = x_2 m_2 = f_{m_2}(x)$ . Therefore,  $f = f_{m+m_2}$ . 3)  $\rightarrow$  1).

Let  $M$  be  $S$ -injective and  $I_1, I_2$  be left ideals in  $S$ . Then we have an exact sequence:  $0 \leftarrow \text{Hom}_S(S/(I_1 \cap I_2), M) \leftarrow \text{Hom}_S(S/I_1, M) \oplus \text{Hom}_S(S/I_2, M)$ , which means that  $M$  satisfies the condition (\*). Hence,  $S$  is left artinian from Lemma 3. The last part is clear, since  $S$  is artinian by Theorem 1.

**Corollary.** *Let  $M$  and  $S$  be as in Theorem 2. If  $M$  is  $R$ -artinian, then any  $S$ - $R$  bi-submodule  $N$  of  $M$  is  $S/l(N)$ -injective.*

*Proof.* Let  $N$  be an  $S$ - $R$  submodule of  $M$ . Then  $N = r_l(N)$  and  $l(N)$  is a two-sided ideal in  $S$ . Put  $\bar{S} = S/l(N)$ . Then  $\bar{S} = \text{Hom}_R(N, N)$  and  $N$  satisfies the same conditions as  $M$  by [9], Theorem 1.1. Hence,  $N$  is  $\bar{S}$ -injective by Theorem 2.

**Theorem 3.** *Let  $M$  be an  $S$ - $R$  bi-module such that  $\text{Hom}_R(M, M) = S$  and  $\text{Hom}_S(M, M) = R$ . Furthermore, we assume that  $M$  is  $S$ - and  $R$ -injective, respectively. Then the following two statements are equivalent.*

- 1)  $M$  is  $R$ -noetherian,
- 2)  $S$  is left artinian.

*And 1) or 2) implies that  $M$  is  $R$ -artinian. Thus, if  $M$  is  $R$ - and  $S$ -noetherian or if  $R$  and  $S$  are right and left artinian, respectively, then  $M$  gives us a duality between the category of finitely generated right  $R$ -modules and the category of finitely generated left  $S$ -modules in the sense of Morita.*

*Proof.* 1)  $\rightarrow$  2). Since  $S$  satisfies the condition (\*) of Lemma 3,  $S$  is left artinian. 2)  $\rightarrow$  1). It is obtained by Corollary to Proposition 2. Now, we assume 1) or 2). Let  $T$  be an  $R$ -submodule, then  $T = r_l(T)$  by [9], Theorem 2.1. Hence,  $M$  is  $R$ -artinian, since  $S$  is left noetherian. The last part is clear from [11], Theorem 6.3, v.

**REMARK.** Let  $M$  and  $S$  be as the first half in Theorem 2. Then the injectivity of  $M$  as an  $S$ -module does not imply the fact that  $S$  is left noetherian. Furthermore, if  $R$  is commutative, then a fact that  $M$  is  $R$ -noetherian implies that  $M$  is  $R$ -artinian, (see Proposition 2 in [7]). However, the converse is not true in general.

Finally, we shall give an example of injective noetherian but not artinian modules. Let  $K$  be a field and  $I = Z^+ \cup \alpha$  the set of indices, where  $Z^+$  is the set of positive integers. Let  $R$  be the ring of upper tri-angular matrices over  $K$  with indices  $I$ , ( $\alpha$  is the last index and  $\alpha$ -column consists of all column finite). Let  $e_{i,j}$  be matrix units in  $R$  and put  $M = e_{11}R$ . Then  $M \approx \text{Hom}_K(Re_{\alpha\alpha}, Ke_{1\alpha})$ . Hence,  $M$  is  $R$ -injective. It is clear that  $M$  is  $R$ -noetherian but not  $R$ -artinian.

OSAKA CITY UNIVERSITY  
KINKI UNIVERSITY

## References

- [1] G. Azumaya: *A duality theory for injective modules*, Amer J. Math. **81** (1959), 249–278.
- [2] S. Eilenberg and T. Nakayama: *On the dimension of modules and algebras II*, Nagoya Math. J. **9** (1956), 1–16.
- [3] C. Faith: *Rings with ascending condition on annihilators*, Nagoya Math. J. **27**–1 (1966), 179–191.
- [4] ———: *Lectures on Injective Modules and Quotient Rings*, Lecture Notes in Math. **49**, Springer, Heidelberg, 1967.
- [5] J.W. Fisher: *Endomorphism of modules*, Notices Amer. Math. Soc. **18** (no. 4), (1971).
- [6] M. Harada: *On semi-simple abelian categories*, Osaka J. Math. **7** (1970), 89–95.
- [7] ———: *On quasi-injective modules with a chain condition over a commutative ring*, to appear.
- [8] M. Ikeda and T. Nakayama: *On some characteristic properties of quasi-Frobenius and regular rings*, Proc. Amer. Math. Soc. **5** (1954), 15–19.
- [9] R.E. Johnson and E. Wong: *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.
- [10] B. Mitchell: *Theory of Categories*, Academic Press, New York and London, 1965.
- [11] K. Morita: *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A. **6** (1958), 83–142.
- [12] K. Motose: *Note on the endomorphism ring*, J. Fac. Sci. Shinshu Univ. **6** (1971), 35–36.
- [13] A. Rosenberg and D. Zelinsky: *Annihilator*, Portugal Math. **20** (1961), 53–65.



