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<th>Title</th>
<th>Quasi K-homology equivalences. II</th>
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<tr>
<td>Author(s)</td>
<td>Yosimura, Zen-ichi</td>
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0. Introduction

Let $E$ be an associative ring spectrum with unit, and $X, Y$ be $CW$-spectra. We say that $X$ is quasi $E_\ast$-equivalent to $Y$ if there exists a map $h: Y \to E \wedge X$ such that the composite $(\mu \wedge 1)(1 \wedge h): E \wedge Y \to E \wedge X$ is an equivalence where $\mu: E \wedge E \to E$ stands for the multiplication of $E$. In this case we write $X \sim Y$, and we call such a map $h: Y \to E \wedge X$ a quasi $E_\ast$-equivalence. We shall be concerned with the quasi $KO_\ast$- and $KU_\ast$-equivalences where $KO$ and $KU$ denote the real and complex $K$-spectrum respectively.

The conjugation $t$ on $KU$ gives rise to an involution $t_\ast$ on $KU_\ast X$ for any $CW$-spectrum $X$. Thus the $KU$-homology $KU_\ast X$ is regarded as a $\mathbb{Z}/2$-graded abelian group with involution. Note that there is an isomorphism between $KU_\ast X$ and $KU_\ast Y$ as $\mathbb{Z}/2$-graded abelian groups with involution if $X$ is quasi $KO_\ast$-equivalent to $Y$.

For any abelian group $G$ we denote by $SG$ the Moore spectrum of type $G$. Evidently $KU_0 SG \simeq G$ on which $t_\ast = 1$ and $KU_1 SG = 0$. Let us denote by $P$ and $Q$ the cofibers of the maps $\eta: \Sigma^1 \to \Sigma^0$ and $\eta^2: \Sigma^2 \to \Sigma^0$ respectively where $\eta: \Sigma^1 \to \Sigma^0$ is the stable Hopf map of order 2. It is well known that $KU_0 P \simeq \mathbb{Z} \oplus \mathbb{Z}$ on which $t_\ast = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $KU_1 P = 0$. On the other hand, $KU_0 Q \simeq \mathbb{Z}$ and $KU_1 Q \simeq \mathbb{Z}$ on both of which $t_\ast = 1$.

Let $H$ be a 2-torsion free abelian group which is written into a direct sum of cyclic groups. If the cyclic group $\mathbb{Z}/2$ acts on $H$, then $H$ admits a direct sum decomposition $H \simeq A \oplus B \oplus C \oplus C$ so that the involution $\rho$ behaves as

\[ \rho = 1 \text{ on } A, \quad \rho = -1 \text{ on } B \quad \text{and} \quad \rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ on } C \oplus C \]

respectively (see [6, Proposition 3.7] or [7]).

By observing these facts, Bousfield [6, Theorem 3.7] has proved the following satisfactory result.

**Theorem 1 (Bousfield).** Let $X$ be a $CW$-spectrum such that $KU_\ast X$ is a
direct sum of 2-torsion free cyclic groups. Then there exist abelian groups $A_i (0 \leq i \leq 7), C_j (0 \leq j \leq 1)$ and $G_k (0 \leq k \leq 3)$ so that $X$ is quasi $KO_n$-equivalent to the wedge sum $\vee (\Sigma^i SA_i) \vee (\Sigma^j P \wedge SC_j) \vee (\Sigma^{k+1} Q \wedge SG_k)$.

In [12, Theorems 1 and 2] or [9] a partial result of the above theorem was proved by a different method from Bousfield’s. In the forthcoming paper [15, Theorem 1] we will give a new proof of the above theorem by our method developed in [12, 13].

Let $H$ be a direct sum of 2-torsion free cyclic groups. If the cyclic group $\mathbb{Z}/2$ acts on the direct sum $H \oplus \mathbb{Z}/2m$, $m=2^i$, then its matrix representation is divided into one of the following types:

(0.2) \begin{align*}
\text{i)} \quad & \begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix} \\
\text{ii)} \quad & \begin{pmatrix} 0 & 0 \\ 0 & m+1 \end{pmatrix} \\
\text{iii)} \quad & \begin{pmatrix} \rho' & 0 \\ 0 & -1 \end{pmatrix} \\
\text{iv)} \quad & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & m \end{pmatrix} \\
\text{v)} \quad & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\
\text{vi)} \quad & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
\text{vii)} \quad & \begin{pmatrix} 0 & 0 \\ 1 & m \end{pmatrix}
\end{align*}

where $H \cong H' \oplus \mathbb{Z} \cong H'' \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\rho$, $\rho'$ or $\rho''$ is an involution on $H$, $H'$ or $H''$ respectively which is decomposed as in (0.1).

We denote by $M_{2m}, Q_{2m}, N_{2m}', R_{2m}', V_{2m}$ and $W_{2m}$ the cofibers of the maps

$i_\eta: \Sigma^1 \to \mathbb{Z}/2m$, $\eta_\eta: \Sigma^3 \to \mathbb{Z}/2m$, $\eta j: \Sigma^1 \mathbb{Z}/2m \to \Sigma^0$, $\eta j: \Sigma^1 \mathbb{Z}/2m \to \Sigma^0$, $i_\eta: \Sigma^1 \mathbb{Z}/2m \to \Sigma^0$ and $i_\eta + j_\eta: \Sigma^1 \mathbb{Z}/2m \to \Sigma^4$ respectively where $\eta: \Sigma^1 \to \mathbb{Z}/2m$ and $\eta: \Sigma^1 \mathbb{Z}/2m \to \Sigma^0$ stand for a coextension and an extension of $\eta$ satisfying $j_\eta = \eta$ and $\eta i = \eta$. In [12, Propositions 4.1, 4.2 and Corollary 4.6] we have investigated the $KU$- and $KO$-homologies of these elementary spectra.

We will moreover introduce some elementary spectra $M_{2m}', Q_{2m}', N_{2m}', R_{2m}'$ and $R_{2m}'$ constructed by the cofibers of the maps

$i_\eta \vee \eta_\eta: \Sigma^1 \vee \Sigma^3 \to \mathbb{Z}/2m$, $(\eta j, \eta): \Sigma^1 \mathbb{Z}/4m \to \Sigma^0 \vee \Sigma^0$, $j_\eta \eta: \Sigma^3 \mathbb{Z}/2m \to \Sigma^3 \vee \Sigma^3$ and $\eta_\eta j: \Sigma^3 \to R_{2m}'$

respectively where $\eta_\eta: \Sigma^3 \to R_{2m}'$ is a coextension of $\eta$ satisfying $j_\eta h R = \eta$. After studying the $KU$- and $KO$-homologies of these spectra with four cells (Propositions 1.2, 1.3, 2.3 and 2.4) we will prove the following result which is our main theorem in this note.

**Theorem 2.** Let $X$ be a CW-spectrum and $H$ be a direct sum of 2-torsion
free cyclic groups. Assume that $KU_0 X \simeq H \bigoplus Z/2m, m=2^s$, and $KU_1 X = 0$. Then there exist abelian groups $A_0, A_1, B_2, B_6$ and $C$ and a certain CW-spectrum $Y$ so that $X$ is quasi $KO$-equivalent to the wedge sum $SA_0 \bigvee \Sigma^2 SB_2 \bigvee \Sigma^4 SA_4 \bigvee \Sigma^6 SB_6 \bigvee (P \wedge SC) \cup Y$. Here $Y$ is taken to be one of the following elementary spectra $\Sigma^2 SZ/2m, \Sigma^2 V_2m, \Sigma^2 W_{2m}$ ($s \geq 2$), $\Sigma^2 M_{2m}, \Sigma^2 Q_{2m}, \Sigma^2 N_{2m}, \Sigma^2 R_{2m}, \Sigma^2 MQ_{2m}, \Sigma^2 NP_{4m}, \Sigma^2 NR_{4m}$ and $\Sigma^2 R_Q_{2m}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 1$.

In order to obtain our main theorem as a corollary we will give three theorems (Theorems 3.3, 4.2 and 4.4) in a slightly general form. The first theorem is established in the situation when the conjugation $t_*$ on $KU_0 X$ behaves as the types (0.2) ii) and v), and the second or the third theorem is done in the situation as the type (0.2) i) or the types (0.2) iii) and iv) respectively.

This paper is a continuation of [12] with the same title and we will use the same notations as in it.

1. Some elementary spectra $XY_{2m}$ and $XY_{2m}$ with four cells

1.1. For any map $f: Y \rightarrow X$ we denote by $C_f$ its cofiber. Thus $Y \rightarrow X \xrightarrow{i} Y$ is a cofiber sequence. The Moore spectrum $SZ/2m$ is obtained as the cofiber of multiplication by $2m$ on $\Sigma^0$. In this case the maps $i_{2m}: \Sigma^0 \rightarrow SZ/2m$ and $j_{2m}: SZ/2m \rightarrow \Sigma^1$ are often abbreviated to be $i$ and $j$ respectively. By applying Verdier's lemma (see [2]) we can easily show

**Lemma 1.1.** i) Given two maps $f: Y \rightarrow X$, $g: Z \rightarrow X$ the cofiber $C_{f \vee g}$ of the map $f \vee g: Y \vee Z \rightarrow X$ coincides with the cofiber $C_{i \vee g}$ of the composite $i \vee g: Z \rightarrow C_f$. In particular, the cofiber $C_{f \vee i}$ coincides with the wedge sum $C_f \vee \Sigma^1 Z$ if $g: Z \rightarrow X$ is factorized through $Y$ as $g = fh: Z \rightarrow Y \rightarrow X$ for some map $h$.

ii) Given two maps $f: X \rightarrow Y$, $g: X \rightarrow Z$ the cofiber $C_{(f,g)}$ of the map $(f,g): X \rightarrow Y \vee Z$ coincides with the cofiber $C_{i \vee f}$ of the composite $i \vee f: \Sigma^1 \rightarrow C_f$. In particular, the cofiber $C_{(f,g)}$ coincides with the wedge sum $C_f \vee Z$ if $g: X \rightarrow Z$ is factorized through $Y$ as $g = hf: X \rightarrow Y \rightarrow Z$ for some map $h$.

Let $\tilde{\eta}_{2m}: \Sigma^2 \rightarrow SZ/2m$ be a coextension of $\eta$ satisfying $j_{2m} \tilde{\eta}_{2m} = \eta$ and $\eta_{2m}: \Sigma^1 SZ/2m \rightarrow \Sigma^0$ an extension of $\eta$ satisfying $\eta_{2m} i_{2m} = \eta$ where $\eta: \Sigma^1 \rightarrow \Sigma^0$ denotes the stable Hopf map of order 2. The maps $\tilde{\eta}_{2m}$ and $\eta_{2m}$ are often abbreviated to be $\tilde{\eta}$ and $\eta$ respectively. After choosing these maps suitably there holds the following relation

$$\eta \wedge 1 = \tilde{\eta}_{2m} j_{2m} + i_{2m} \eta_{2m}: \Sigma^1 SZ/2m \rightarrow SZ/2m$$

(see [5, Lemma 7.2]).

Let us denote by $M_{2m}, N_{2m}, P_{2m}, Q_{2m}, R_{2m}, M_{2m}', N_{2m}', P_{2m}', Q_{2m}'$ and $R_{2m}'$ respectively the elementary spectra constructed by the following cofiber sequences as in [12, (4.1) and (4.2)]:
\[ \Sigma^1 \rightarrow \Sigma^2 \rightarrow \Sigma^3 \rightarrow \ldots \rightarrow \Sigma^n \rightarrow \Sigma^{n+1} \rightarrow \ldots \]

In [12, Propositions 4.1 and 4.2] we have calculated the $KU$- and $KO$-homologies of these elementary spectra with three cells.

Given two cofibers $X_{2m}, Y_{2m}$ of any maps $f: \Sigma^i \rightarrow \Sigma^j$, $g: \Sigma^i \rightarrow \Sigma^j$ (\(i \leq j\)) we denote by $XY_{2m}$ the cofiber of the maps $f \vee g: \Sigma^i \vee \Sigma^j \rightarrow \Sigma^k$. Dually we denote by $XY'_{2m}$ the cofiber of the map $(f, g): \Sigma^i \Sigma^j \rightarrow \Sigma^k$ for two cofibers $X_{2m}, Y_{2m}$ of any maps $f: \Sigma^i \Sigma^j \rightarrow \Sigma^k$. We will only deal with the CW-spectra $XY_{2m}$ and $XY'_{2m}$ when $X=M$ or $N$ and $Y=P, Q$ or $R$ as Lemma 1.1 may be applicable to the other cases. Note that

\[ MP_{2m} = \Sigma^2 D(MP_{2m}), \quad MQ_{2m} = \Sigma^2 D(MQ_{2m}), \quad MR_{2m} = \Sigma^2 D(MR_{2m}) \]
\[ NP_{2m} = \Sigma^2 D(NP_{2m}), \quad NQ_{2m} = \Sigma^2 D(NQ_{2m}), \quad NR_{2m} = \Sigma^2 D(NR_{2m}) \]

where $DW$ stands for the Spanier-Whitehead dual of $W$ (cf. [12, (4.3)]).

1.2. We will now compute the $KU$ homologies of the above mentioned spectra $W=XY_{2m}, XY'_{2m}$ with four cells, by making use of the results in [12, Proposition 4.1].

**Proposition 1.2.** The $KU$ homologies $KU_0 W, KU_1 W$ and the conjugation $t_*$ on them are given as follows:

<table>
<thead>
<tr>
<th>$W$</th>
<th>$MP_{2m}$</th>
<th>$MQ_{2m}$</th>
<th>$MR_{2m}$</th>
<th>$NP_{2m}$</th>
<th>$NQ_{2m}$</th>
<th>$NR_{2m}$</th>
</tr>
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<tbody>
<tr>
<td>$KU_0 W \cong Z \oplus Z/m \oplus Z \oplus Z/2m \oplus Z \oplus Z/2m \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/2m$</td>
<td>$Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z/2m$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_*$</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td></td>
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</tr>
<tr>
<td>$KU_1 W \cong Z \oplus Z/m \oplus Z \oplus Z/2m \oplus Z \oplus Z/2m \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/2m$</td>
<td>$Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z/2m$</td>
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</tr>
<tr>
<td>$t_*$</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad 1 \quad 1)</td>
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<table>
<thead>
<tr>
<th>$W$</th>
<th>$MP'_{2m}$</th>
<th>$MQ'_{2m}$</th>
<th>$MR'_{2m}$</th>
<th>$NP'_{2m}$</th>
<th>$NQ'_{2m}$</th>
<th>$NR'_{2m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$KU_0 W \cong Z \oplus Z/m \oplus Z \oplus Z/2m \oplus Z \oplus Z/2m \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/2m$</td>
<td>$Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z/2m$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_*$</td>
<td>((1 -1) \quad 0 \quad -1 \quad 0 \quad -1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad -1 \quad 0)</td>
<td>((1 0) \quad 0 \quad 1 \quad 0 \quad -1)</td>
<td>((1 0) \quad 0 \quad 1 \quad 0 \quad -1)</td>
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<tr>
<td>$KU_1 W \cong Z \oplus Z/m \oplus Z \oplus Z/2m \oplus Z \oplus Z/2m \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z \oplus Z/2m$</td>
<td>$Z/m \oplus Z \oplus Z/2m$</td>
<td>$Z/2m$</td>
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<tr>
<td>$t_*$</td>
<td>((1 -1) \quad 0 \quad -1 \quad 0 \quad -1)</td>
<td>((-1 0) \quad 1 \quad 0 \quad -1 \quad 0)</td>
<td>((1 0) \quad 0 \quad 1 \quad 0 \quad -1)</td>
<td>((1 0) \quad 0 \quad 1 \quad 0 \quad -1)</td>
<td></td>
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</table>
where the matrices behave as left action on abelian groups.

Proof. The $W = MP_{2m}$ case has been computed in [14, Proposition 1.2 i)]. We will investigate the behaviour of the conjugation $t_*$ on $KU_*$ $W$ only when $W = MQ_{2m}$, $NP_{2m}'$ and $NR_{2m}'$, the other cases being easy.

i) The $W = MQ_{2m}$ case: Consider the two commutative diagrams involving cofiber sequences. Evidently $KU_0MQ_{2m} = KU_0(\Sigma^2 \vee \Sigma^2) \oplus KU_0\Sigma^2/2m \approx \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m$ and $KU_1MQ_{2m} = 0$. In order to observe the behaviour of $t_*$ on $KU_0MQ_{2m}$ we use the three split short exact sequences $0 \to KU_0\Sigma^2/2m \to KU_0MQ_{2m} \to KU_0(\Sigma^2 \vee \Sigma^2) \to 0$, $0 \to KU_0\Sigma^2/2m \to KU_0MQ_{2m} \to KU_0\Sigma^2 \to 0$ and $0 \to KU_0\Sigma^2/2m \to KU_0MQ_{2m} \to KU_0\Sigma^2 \to 0$. Since [12, Proposition 4.1] says that $t_* = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ on $KU_0MQ_{2m} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m$, we can easily verify that $t_* = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ on $KU_0MQ_{2m} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m$ as desired.

ii) The $W = NP_{2m}'$ case: Consider the two commutative diagrams
involving cofiber sequences, where \( \iota_k: \Sigma^0 \to \Sigma^0 \vee \Sigma^0 \) and \( \pi_k: \Sigma^0 \vee \Sigma^0 \to \Sigma^0(k=1, 2) \) denote the \( k \)-th injection and projection respectively. We can easily see that the short exact sequence \( 0 \to \mathit{KU}_0 \Sigma^0 \to \mathit{KU}_0 \mathit{NP}'_{2m} \to \mathit{KU}_0 \mathit{P}'_{2m} \to 0 \) is split, by using the following commutative diagram

\[
\begin{array}{c}
\Sigma^0 \\ \downarrow \iota_1 \\
\Sigma^0 \vee \Sigma^0 \\ \downarrow \pi_1 \\
\Sigma^0 \vee \Sigma^0 \\ \downarrow \eta_2 \\
0
\end{array}
\begin{array}{cccc}
(\eta_2^j, \eta) & \Sigma^0 \\ \downarrow \iota_1 & \Sigma^0 \vee \Sigma^0 & N_{2m} & \Sigma^0 \vee \Sigma^0 \\ \downarrow \pi_1 & \Sigma^0 \vee \Sigma^0 & Q & \Sigma^0 \vee \Sigma^0 \\ \downarrow \eta_2 & \Sigma^0 \vee \Sigma^0 & \mathit{NP}'_{2m} & \Sigma^0 \vee \Sigma^0 \\ \downarrow \eta_2 & \Sigma^0 \vee \Sigma^0 & \mathit{P}'_{2m} & \Sigma^0 \vee \Sigma^0 \\
& & & & \mathit{KU}_0 \mathit{NP}'_{2m}
\end{array}
\]

with \( \pi_1 \iota_1 = 1 \). Thus \( \mathit{KU}_0 \mathit{NP}'_{2m} \cong \mathit{KU}_0 \Sigma^0 \vee \mathit{KU}_0 \mathit{P}'_{2m} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/m \) and \( \mathit{KU}_1 \mathit{NP}'_{2m} = 0 \). Since \( t_* = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \) on \( \mathit{KU}_0 \mathit{NP}'_{2m} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/m \) by means of [12, Proposition 4.1]), it follows immediately that \( t_* = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \) on \( \mathit{KU}_0 \mathit{NP}'_{2m} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/m \) as desired.

iii) The \( \mathit{W}=\mathit{NR}'_{2m} \) case: Use the commutative diagram

\[
\begin{array}{c}
\Sigma^0 \\ \downarrow \iota_1 \\
\Sigma^0 \vee \Sigma^0 \\ \downarrow \pi_1 \\
\Sigma^0 \vee \Sigma^0 \\ \downarrow \eta_2 \\
0
\end{array}
\begin{array}{cccc}
(\eta_2^j, \eta) & \Sigma^0 \\ \downarrow \iota_1 & \Sigma^0 \vee \Sigma^0 & N_{2m} & \Sigma^0 \vee \Sigma^0 \\ \downarrow \pi_1 & \Sigma^0 \vee \Sigma^0 & Q & \Sigma^0 \vee \Sigma^0 \\ \downarrow \eta_2 & \Sigma^0 \vee \Sigma^0 & \mathit{NR}'_{2m} & \Sigma^0 \vee \Sigma^0 \\ \downarrow \eta_2 & \Sigma^0 \vee \Sigma^0 & \mathit{P}'_{2m} & \Sigma^0 \vee \Sigma^0 \\
& & & & \mathit{KU}_0 \mathit{NP}'_{2m}
\end{array}
\]

involving cofiber sequences, in which the upper row becomes a cofiber sequence by means of Lemma 1.1 ii). Then we can easily see that the short exact sequence \( 0 \to \mathit{KU}_0 (\Sigma^2 \vee \Sigma^0) \to \mathit{KU}_0 \mathit{NR}'_{2m} \to \mathit{KU}_1 \mathit{SZ}/2m \to 0 \) is split, and \( \mathit{KU}_1 \mathit{NR}'_{2m} = 0 \). Hence it is immediate that \( \mathit{KU}_0 \mathit{NR}'_{2m} = \mathit{KU}_0 (\Sigma^2 \vee \Sigma^0) \oplus \mathit{KU}_0 \mathit{SZ}/2m \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/m \) on which \( t_* = \left( \begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array} \right) \).

We will next compute the \( \mathit{KO} \) homologies of the above mentioned spectra \( \mathit{W}=\mathit{XY}_{2m} \) and \( \mathit{XY}'_{2m} \), by making use of the results in [12, Proposition 4.2].

**Proposition 1.3.** The \( \mathit{KO} \) homologies \( \mathit{KO}, \mathit{W} \) are tabulated as follows:

<table>
<thead>
<tr>
<th>( i \equiv )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( i \equiv )</th>
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<th>( 1 )</th>
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<td>( \mathit{MR}'_{2m} )</td>
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</tbody>
</table>
Proof. We have computed $KO_*\mathbb{MP}_{2m}$ in [14, Proposition 1.2 ii)]. In the other cases we can similarly compute $KO_*\mathcal{W}$, by using the long exact sequences of $KO$ homologies induced by the cofiber sequences as appeared in the proof of Proposition 1.2. In computing $KO_*\mathcal{W}$ we may moreover apply the universal coefficient sequence $0 \to \text{Ext}(KO_{3-*} DW, Z) \to KO_*\mathcal{W} \to \text{Hom}(KO_{4-*} DW, Z) \to 0$ (see [11]) combined with (1.3).

2. Some elementary spectra $Y'X_{2m}$ with four cells

2.1. Let $X_{2m}$, $Y_{2m}$ denote the cofibers of maps $f: \Sigma^i \to SZ/2m$, $g: \Sigma^i SZ/2m \to \Sigma^0$ respectively. If the composite $g': \Sigma^{i+1} \to \Sigma^0$ is trivial, then there exists a coextension $h: \Sigma^{i+j+1} \to Y_{2m}$ of $f$ and an extension $k: \Sigma^i X_{2m} \to \Sigma^0$ of $g$ so that the following diagram is commutative

\[
\begin{array}{c}
\Sigma^i \\
\downarrow h \quad \downarrow f \quad \downarrow g \\
\Sigma^0 \to Y_{2m} \to \Sigma^{i+1}SZ/2m \to \Sigma^1 \\
\| \| \downarrow k \\
\Sigma^0 \to C_{h,k} \to \Sigma^{i+1}X_{2m} \to \Sigma^1 \\
\| \downarrow k' \|
\end{array}
\]

with four cofiber sequences. Here the maps $h$ and $k$ are dependent on each other so that their cofibers coincide. We will here choose suitable pairs $(h, k)$ to construct some elementary spectra $Y'X_{2m} = C_{h',k'}$.

There exist maps

\begin{equation}
(2.1) \quad k_M: M_{2m} \to \Sigma^i, \quad k_R: R_{2m} \to \Sigma^0, \quad k_Q: Q_{2m} \to \Sigma^0, \quad k_N: N_{2m} \to \Sigma^0
\end{equation}

\[h_M': \Sigma^i \to M_{2m}, \quad h_R': \Sigma^5 \to R_{2m}, \quad h_Q': \Sigma^5 \to Q_{2m}, \quad h_N': \Sigma^5 \to N_{2m}\]

such that $k_M i_M = j: SZ/2m \to \Sigma^i$, $k_R i_R = \eta j: SZ/2m \to \Sigma^0$, $k_Q i_Q = \eta j: \Sigma^i SZ/2m \to \Sigma^0$, $k_N i_N = \eta j: \Sigma^i SZ/2m \to \Sigma^0$, $j_k h_M = \eta j: \Sigma^3 \to SZ/2m$, $j_k h_Q = \eta j: \Sigma^3 \to SZ/2m$. Such maps $k_R$, $k_Q$, $h_R$, $h_Q$ and $\tilde{h}_N$ are uniquely chosen, and moreover the composites $\eta h_M$ and $h_M'\eta$ are determined uniquely although $k_M$ and $h_M'$ are not so.

Let $X_{2m}$, $Y_{2m}$ be the cofibers of maps $f: \Sigma^i \to SZ/2m$, $f: \Sigma^{i+1} \to SZ/2m$, and $Y'_{2m}$, $X'_{2m}$ the cofibers of maps $g: \Sigma^i SZ/2m \to \Sigma^i$, $g: \Sigma^{i+1}SZ/2m \to \Sigma^0$ respectively. Then there exist maps $\lambda_{X,Y}: X_{2m} \to Y_{2m}$, $\rho_{X,Y}: Y_{2m} \to X_{2m}$ and dually $\lambda_{X',Y}: X'_{2m} \to Y'_{2m}$, $\rho_{X',Y}: Y'_{2m} \to X'_{2m}$ related by the following commutative diagrams:

\[
\begin{array}{|c|c|c|c|c|}
\hline
NQ_{2m} & Z \oplus Z/2m & Z/2 & Z & NQ'_{2m} \\
\hline
NR_{2m} & Z/2m & Z \oplus Z/2 & Z/2 & Z \oplus Z/2 \\
\hline
NQ'_{2m} & Z \oplus Z/2m & Z/2 & Z/2 & Z \oplus Z/2 \\
\hline
\end{array}
\]
By composing the maps chosen in (2.1) with the above maps we set

\[ k_N = k_M \rho_{N,M} : \Sigma^1 \to N_2m \to \Sigma^1 \]
\[ h_N = \lambda_{M,N} h_M : \Sigma^2 \to N_2m \]
\[ k_R = k_Q \rho_{Q,R} : \Sigma^1 Q_2m \to \Sigma^0 \]
\[ h_R = \lambda_{Q,R} h_Q : \Sigma^2 \to R_2m \]
\[ k_P = k_Q \lambda_{P,Q} : \Sigma^1 P_2m \to \Sigma^0 \]
\[ h_P = \rho_{Q,P} h_Q : \Sigma^2 \to P_2m \]
\[ k_M = k_N \lambda_{M,N} : \Sigma^3 M_2m \to \Sigma^0 \]
\[ h_M = \rho_{M,N} h_N : \Sigma^2 \to M_2m . \]

These maps satisfy the following equalities respectively:

\[ \eta_k N = j, \quad k_Q Q = \eta_j j, \quad k_R i_R = \eta, \quad \lambda_{Q,P} = \eta \eta, \quad \eta^2 \eta = \eta^2 \eta, \]
\[ \eta \eta = \eta \eta. \]

Note that such maps \( k_Q, k_P, k_M, h_P, h_R \) and \( h_M \) are uniquely determined, and moreover the composites \( \eta^3 k_N \) and \( h \eta^2 \eta \) are so, too.

Using suitable pairs \( (h, k) \) consisting of maps chosen in (2.1) and (2.2), we can construct some elementary spectra \( Y'X_2m = C_{h,k} \) taken to be the cofiber of the two maps \( h, k \) as follows:

\[
\begin{array}{ccc}
Y'X_2m & h : \Sigma^{i+1} \to Y'_2m & k : \Sigma^1 X_2m \to \Sigma^0 \\
M'M_2m & k_M \eta : \Sigma^2 \to M'_2m & \eta k_M : M_2m \to \Sigma^0 \\
M'N_2m & h_M \eta : \Sigma^3 \to M'_2m & \eta h_M : N_2m \to \Sigma^0 \\
N'M_2m & \eta_M \eta : \Sigma^2 \to N'_2m & \eta k_M : M_2m \to \Sigma^0 \\
N'N_2m & \eta_M \eta^2 : \Sigma^4 \to N'_2m & \eta^2 k_M : N_2m \to \Sigma^0 \\
P'Q_2m & h_P \eta^2 : \Sigma^2 \to P'_2m & \eta h_P : Q_2m \to \Sigma^0 \\
P'R_2m & h_P \eta : \Sigma^3 \to R'_2m & \eta h_P : Q_2m \to \Sigma^0 \\
Q'P_2m & h_Q \eta : \Sigma^2 \to Q'_2m & \eta h_Q : P_2m \to \Sigma^0 \\
Q'Q_2m & h_Q \eta : \Sigma^2 \to Q'_2m & \eta h_Q : Q_2m \to \Sigma^0 \\
Q'R_2m & h_R \eta^2 : \Sigma^2 \to R'_2m & \eta h_R : Q_2m \to \Sigma^0 \\
R'P_2m & h_R \eta : \Sigma^3 \to R'_2m & \eta h_R : Q_2m \to \Sigma^0 \\
R'Q_2m & h_R \eta : \Sigma^3 \to R'_2m & \eta h_R : Q_2m \to \Sigma^0 \\
R'R_2m & h_R \eta^2 : \Sigma^2 \to R'_2m & \eta h_R : Q_2m \to \Sigma^0 \\
M'R_2m & h_R : \Sigma^2 \to M'_2m & \eta k_R : R_2m \to \Sigma^0 \\
N'Q_2m & h_R : \Sigma^2 \to N'_2m & k_Q : \Sigma^1 Q_2m \to \Sigma^0 \\
N'R_2m & h_R : \Sigma^3 \to N'_2m & k_Q : \Sigma^1 R_2m \to \Sigma^0 \\
Q'N_2m & h_Q : \Sigma^2 \to Q'_2m & \eta k_Q : N_2m \to \Sigma^0 \\
R'M_2m & h_Q : \Sigma^3 \to R'_2m & \eta k_Q : N_2m \to \Sigma^0 \\
R'N_2m & h_Q : \Sigma^3 \to R'_2m & \eta k_Q : N_2m \to \Sigma^0 \\
\end{array}
\]
For all of these elementary spectra we notice that

\[(2.5) \quad Y'X_{2m} = \Sigma^{i+j+2} \text{D}(X'Y_{2m})\]

where \(\text{D}W\) stands for the Spanier-Whitehead dual of \(W\).

2.2. Consider the cofiber sequence \(\Sigma_2 \to \Sigma_0 \to Q \to \Sigma_3\). Then the square \(\eta^2\) has a unique coextension \(\xi: \Sigma^2 \to Q\) and a unique extension \(\xi: \Sigma^2 Q \to \Sigma^0\) satisfying \(j_0 \xi = \eta^2\) and \(\xi i_0 = \eta^2\). Denote by \(QQ\) the cofiber of \(\xi\) which coincides with the cofiber of \(\xi\). Then we have

**Lemma 2.1.**

i) \(KU_0 QQ \approx Z \oplus Z\) on which \(t_* = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\), and \(KU_1 QQ \approx Z\) on which \(t_* = -1\).

ii) \(KO_i QQ \approx Z \oplus Z/2, Z/2, Z, Z, Z, 0, Z, Z\) according as \(i = 0, 1, \ldots, 7\).

**Proof.** Use the following commutative diagram

\[
\begin{array}{cccc}
\Sigma^0 & \to & \Sigma^0 \\
\downarrow & \downarrow & \downarrow \\
\Sigma^2 & \to & Q & \to & Q Q & \to & \Sigma^6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\Sigma^2 & \to & \Sigma^2 Q & \to & \Sigma^6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\Sigma^2 & \to & \Sigma^4 \\
\end{array}
\]

involving four cofiber sequences. Then it is obvious that \(KU_0 QQ \approx KU_0 \Sigma^4 \oplus KU_0 Q \approx Z \oplus Z\) and \(KU_1 QQ \approx KU_1 Q \approx Z\). Moreover \(KO_i QQ\) are easily computed except \(i = 0\) and 1. On the other hand, the Bott cofiber sequence induces two exact sequences \(0 \to KO_3 QQ \to KU_3 QQ \to KO_4 QQ \to 0\) and \(0 \to KU_3 QQ \to KO_4 QQ \to KO_5 QQ \to KU_6 QQ \to KO_7 QQ \to 0\). Since the above monomorphisms are both multiplications by 2 on \(Z\), we can also determine \(KO_i QQ\) \((i = 0, 1)\) immediately.

We next consider the commutative diagram

\[
\begin{array}{cccc}
0 & \to & KU_i Q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \leftarrow & KU_i QQ & \leftarrow & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & \to & KO_i QQ & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \leftarrow & KU_i \Sigma^6 & \leftarrow & 0 \\
\end{array}
\]

with exact diagonals. Here the two vertical arrows are both multiplications by 2 on \(Z\). As in [12, (2.3)] we can easily observe that \(t_* = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\) on \(KU_i QQ \approx KU_i \Sigma^4 \oplus KU_i Q \approx Z \oplus Z\) by replacing suitably the splitting of \(j_{QQ}^*\) if necessary.
On the other hand, it is obvious that \( t_* = -1 \) on \( KU_1 Q \cong KU_1 Q = \mathbb{Z} \).

Combining Lemma 2.1 with Theorem 1 we get

**Corollary 2.2.** \( Q \bigcirc Q \cong P \vee \Sigma^1 \)

Choose two maps \( \lambda_0 : Q_{2m} \to \Sigma^i Q \), \( \rho_0 : Q \to Q'_{2m} \) making the diagram below commutative

\[
\begin{array}{cccccc}
\Sigma^3 & \xrightarrow{\eta} & \Sigma^1 Q_{2m} & \to & Q_{2m} & \to & \Sigma^4 \\
\Sigma^3 & \xrightarrow{\eta} & \Sigma^1 Q & \to & \Sigma^4 Q & \to & \Sigma^3 Q_{2m} \\
\end{array}
\]

Then the following equalities hold:

\[(2.6)\quad \xi_0 \lambda_0 = k_0 : \Sigma^i Q_{2m} \to \Sigma^0 = \rho_0 : \Sigma^i Q \to Q'_{2m} .\]

2.3. We will now compute the \( KU \) homologies of the elementary spectra \( W = Y' X_{2m} \) with four cells mentioned in (2.4).

**Proposition 2.3.** The \( KU \) homologies \( KU_0 W, KU_1 W \) and the conjugation \( t_* \) on them are given as follows:

\[
\begin{array}{cccccc}
W & = & M'M_{2m} & M'N_{2m} & N'M_{2m} & N'N_{2m} & P'Q_{2m} \\
KU_0 W & \cong & Z & Z \oplus Z & Z \oplus Z \oplus Z/2m & Z \oplus Z/2m & Z \oplus Z \oplus Z/m \\
t_* & = & 1 & (1 \ 0) & (1 -1) & (1 -1) & (-1 0 0) \\
KU_1 W & \cong & Z \oplus Z/2m & Z/2m & 0 & Z & 0 \\
t_* & = & (-1 0) & (1 1) & 1 & 1 & 1 \\
KU_0 W & \cong & Z \oplus Z/m & Z \oplus Z & Z \oplus Z \oplus Z/m & Z \oplus Z \oplus Z/2m \\
t_* & = & (1 \ 0) & (1 -1) & 1 & (1 -1) & (1 0 0) \\
KU_1 W & \cong & Z & Z/m & Z \oplus Z/2m & Z/2m & Z & 0 \\
t_* & = & -1 & (0 -1) & -1 & -1 & -1 \\
KU_0 W & \cong & Z \oplus Z/2m & Z \oplus Z & Z \oplus Z \oplus Z/2m & Z \oplus Z \oplus Z/m & Z \oplus Z \oplus Z/2m \\
t_* & = & (1 \ 0) & (1 -1) & (1 -1) & (1 0 0) & (1 0 1) \\
KU_1 W & \cong & Z & Z/2m & 0 & Z & Z/2m & 0 & Z \\
t_* & = & 1 & 1 & -1 & -1 & -1
\end{array}
\]
where the matrices behave as left action on abelian groups.

Proof. By making use of [12, Propositions 4.1 and 4.2] we will investigate the behaviour of the conjugation $t_\star$ on $KU_*W$ when $W=N'M_{2m}$, $P'Q_{2m}$, $Q'P_{2m}$, $R'Q_{2m}$, $M'R_{2m}$, $N'Q_{2m}$, $Q'N_{2m}$ and $R'M_{2m}$, the other cases being easy. Denote by $t_W$ the conjugation $t_\star$ on $KU_*W$ for convenience sake.

i) The $W=N'M_{2m}$ case: Use the commutative diagram

\[
\begin{array}{ccc}
\Sigma^0 & \cong & \Sigma^0 \\
\Sigma^3 \cong & \Sigma^3 & \Sigma^3 \\
\Sigma^3 \cong & \Sigma^3 & \Sigma^3 \\
n_\Sigma \cong & n_\Sigma & n_\Sigma \\
\end{array}
\]

involving four cofiber sequences. Evidently $KU_0N'M_{2m} \cong KU_0 \Sigma^1 \oplus KU_0 N'_{2m} \cong Z \oplus Z \oplus Z/2m$ and $KU_1N'M_{2m}=0$. Set $t_{N'M_{2m}}=\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & -1 \end{pmatrix}$ on $KU_0N'M_{2m} \cong Z \oplus Z \oplus Z/2m$ for some integers $a$, $b$ because $t_{N'}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $KU_0 N'_{2m} \cong Z \oplus Z/2m$. Since $t_{M_{2m}}=\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ on $KU_{-2}M_{2m} \cong Z \oplus Z/2m$, we may take to be $b=1$. On the other hand, the equality $t_{N'M_{2m}}=1$ implies that $a=0$. Thus $t_{N'M_{2m}}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ as desired.

ii) The $W=P'Q_{2m}$ case: Use the commutative diagram

\[
\begin{array}{ccc}
\Sigma^0 & \cong & \Sigma^0 \\
\Sigma^3 \cong & \Sigma^3 & \Sigma^3 \\
\Sigma^5 \cong & \Sigma^5 & \Sigma^5 \\
n_\Sigma \cong & n_\Sigma & n_\Sigma \\
\end{array}
\]

involving four cofiber sequences. Evidently $KU_0P'Q_{2m} \cong KU_0 \Sigma^5 \oplus KU_0 P'_{2m} \cong Z \oplus Z \oplus Z/m$ and $KU_1P'Q_{2m}=0$. The induced homomorphism $j_{P'Q, Q_*}: KU_0 P'Q_{2m} \to KU_{-2}Q_{2m}$ may be expressed by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}: Z \oplus Z \oplus Z/m \to Z \oplus Z/2m$ since $j_{P'Q}: KU_0 P'_{2m} \to KU_{-2}SZ/2m$ is given by the row $(1 -2): Z \oplus Z/m \to Z/2m$. Set $t_{P'_{2m}}=\begin{pmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ b & 1 & -1 \end{pmatrix}$ on $KU_0 P'Q_{2m} \cong Z \oplus Z \oplus Z/m$ for some integers $a$, $b$. Recall that $t_Q=\begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix}$ on $KU_{-2}Q_{2m}$. Then the equality $j_{P'Q, Q_*} t_{P'Q}=\begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix}$ on $KU_{-2}Q_{2m}$. Then the equality $j_{P'Q, Q_*} t_{P'Q}=\begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix}$ on $KU_{-2}Q_{2m}$. Then the equality $j_{P'Q, Q_*} t_{P'Q}=$
implies that $a-2b \equiv m \mod 2m$, thus $a \equiv m \mod 2$. So we may take to be $(a, b) = (1, m+1/2)$ or $(0, m/2)$ according as $m$ is odd or even. Since the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ m+1/2 & 1 & -1 \end{pmatrix}$ is congruent to $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, the result is immediate.

iii) The $W=Q'P_{2m}$ case: Use the commutative diagram

\[
\begin{array}{cccc}
\Sigma^0 & \rightarrow & \Sigma^6 \\
\downarrow & & \downarrow \\
Q^2_{2m} & \rightarrow & Q'P_{2m} \rightarrow \Sigma^6 \\
\downarrow & & \downarrow j_{Q',P,P} \\
\Sigma^3 & \rightarrow & \Sigma^3 S^2m \rightarrow \Sigma^3 P_{2m} \rightarrow \Sigma^6 \\
\downarrow & & \downarrow k_P \\
\Sigma^1 & \rightarrow & \Sigma^1 \\
\end{array}
\]

involving four cofiber sequences. It follows immediately that $KU_0 Q'P_{2m} \cong KU_{-3} P_{2m} \oplus KU_0 \Sigma^6$ on which $t_* = (-1, 0, a, 1)$ for some integer $a$, and $KU_1 Q'P_{2m} \cong KU_{-3} P_{2m} \cong Z/m$ on which $t_* = -1$. We will show that the integer $a$ may be taken to be 1 or 0 according as $m$ is odd or even.

We will first compute the $KO$ homologies $KO_0 Q'P_{2m}$. By using the above commutative diagram it is easily checked that $KO_0 Q'P_{2m} \cong Z$, $KO_0 Q'P_{2m} \cong KO_0 Q'P_{2m} \cong Z/m$ and $KO_0 Q'P_{2m} \cong Z/m \oplus Z/2$. In order to determine the remainder $KO_0 Q'P_{2m}$ we consider the exact sequence $KO_0 Q'P_{2m} \rightarrow KU_0 Q'P_{2m} \rightarrow KO_0 Q'P_{2m} \rightarrow 0$ induced by the Bott cofiber sequence. Since there exists a short exact sequence $0 \rightarrow KO_0 Q'P_{2m} \rightarrow KU_0 Q'P_{2m} \rightarrow KO_0 Q'P_{2m} \rightarrow 0$, it is easily seen that $KO_0 Q'P_{2m} \cong Z/m \oplus Z/2$.

We next use the commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & KU_0 \Sigma^0 & \rightarrow & KO_0 Q'P_{2m} \\
\rightarrow & & \downarrow & \rightarrow & \downarrow \\
KU_0 Q'P_{2m} & \rightarrow & KO_0 Q'P_{2m} & \rightarrow & KU_{-3} P_{2m} \\
\rightarrow & & \downarrow & \rightarrow & \\
KO_0 Q'P_{2m} & \rightarrow & \epsilon_{0,P} & \rightarrow & KU_{-3} P_{2m} \\
\rightarrow & & \rightarrow & \rightarrow & \\
KO_0 Q'P_{2m} & \rightarrow & j_{Q',P,P} & \rightarrow & KU_{-3} P_{2m} \\
\rightarrow & & \rightarrow & \rightarrow & \\
0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

with exact diagonals. Here the left vertical arrow is just multiplication by 2 on $Z$, and the right one is multiplication by 2 or 1 on $Z$ according as $m$ is odd or even. By a parallel discussion to [12, (2.3)] it is easily observed that $a$ is odd or even according as $m$ is odd or even. Therefore we may take $a$ to be 1 or 0 according as $m$ is odd or even, by replacing suitably the splitting of $j_{Q',P,P}$ if necessary. Thus $t_{Q',P} = (-1, 0)$ or $(-1, 0)$ on $KU_0 Q'P_{2m} \cong Z \oplus Z$ according
as \( m \) is odd or even.

iv) The \( W=R'Q_{2m} \) case is shown similarly to the case i).

v) The \( W=N'Q_{2m} \) case: We have the following commutative diagram

\[
\begin{array}{ccc}
\Sigma^0 & \rightarrow & \Sigma^0 \\
\downarrow & & \downarrow \\
N'Q_{2m} & \rightarrow & \Sigma^1 \\
\downarrow & & \downarrow \\
Q Q & \rightarrow & \Sigma^1 \\
\downarrow & & \downarrow \\
\Sigma^1 & = & \Sigma^1 \\
\end{array}
\]

involving four cofiber sequences, because of (2.6). Evidently \( KU_0N'Q_{2m} \approx KU_2Z \oplus KU_3 \approx Z \oplus Z/2m \oplus Z \) and \( KU_1N'Q_{2m} = 0 \). Set \( t_{N'Q} = \begin{pmatrix} -1 & 0 & 0 \\ m & -1 & 0 \\ a & 0 & 1 \end{pmatrix} \) on \( KU_0N'Q_{2m} \approx Z \oplus Z/2m \oplus Z \) for some integer \( a \). Then the equality \( \lambda_{N'Q} t_{N'Q} = t_{Q Q} \lambda_{N'Q} \) implies that \( a = 1 \) because \( t_{Q Q} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \) on \( KU_0QQ \) by Lemma 2.1.

Since the matrix \( \begin{pmatrix} -1 & 0 & 0 \\ m & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \) is congruent to \( \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \), the result is immediate.

vi) The \( W=M'R_{2m} \) case: Consider the commutative diagram

\[
\begin{array}{ccc}
\Sigma^5 & \rightarrow & \Sigma^5 \\
\downarrow & & \downarrow \\
N'Q_{2m} & \rightarrow & \Sigma^6 \\
\downarrow & & \downarrow \\
M'R_{2m} & \rightarrow & \Sigma^6 \\
\downarrow & & \downarrow \\
\Sigma^1S2m & = & \Sigma^1S2m \\
\end{array}
\]

involving four cofiber sequences. Evidently \( KU_0M'R_{2m} \approx KU_2Z \oplus KU_3 \approx Z \oplus Z \) and \( KU_1M'R_{2m} \approx KU_4M'R_{2m} \approx Z/2m \). Since \( t_{M'R} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \) on \( KU_0M'R_{2m} \approx Z \oplus Z \). Hence the result follows.

vii) The \( W=Q'N_{2m} \) case: We have the following commutative diagram

\[
\begin{array}{ccc}
\Sigma^3 & = & \Sigma^3 \\
\downarrow & & \downarrow \\
Q Q & \rightarrow & \Sigma^5 \\
\downarrow & & \downarrow \\
\Sigma^4 & = & \Sigma^4 \\
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma^3 & \rightarrow & \Sigma^3 \\
\downarrow & & \downarrow \\
Q Q & \rightarrow & \Sigma^5 \\
\downarrow & & \downarrow \\
\Sigma^4 & = & \Sigma^4 \\
\end{array}
\]
involving four cofiber sequences, because of (2.6). Then it is easily obtained that $KU_0Q'N_{2m} \simeq KU_0Q \simeq \mathbb{Z} \oplus \mathbb{Z}$ on which $t_\ast = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, and $KU_1Q'N_{2m} \simeq KU_1Q'_2 \simeq \mathbb{Z}/2m$ on which $t_\ast = -1$.

viii) The $W = R'M_{2m}$ case: Consider the commutative diagram

\[
\begin{array}{cccc}
\Sigma^2\mathbb{Z}/2m \wedge P & \rightarrow & \Sigma^2\mathbb{Z}/2m \wedge P \\
\downarrow \Sigma^5 & & \downarrow \Sigma^5 \\
\Sigma^5 \rightarrow R'_2m & \rightarrow & R'M_{2m} & \rightarrow & \Sigma^6 \\
\downarrow \rho'_{R,Q} & & \downarrow \rho'_{R,M,Q'} & & \downarrow \\
\Sigma^5 \rightarrow Q'_2m & \rightarrow & Q'N_{2m} & \rightarrow & \Sigma^6 \\
\downarrow \Sigma^5 \wedge P & \rightarrow & \Sigma^5 \wedge P
\end{array}
\]

involving four cofiber sequences. Evidently $KU_0R'M_{2m} \simeq KU_0\Sigma^6 \oplus KU_0R'_2m \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m$ and $KU_1R'M_{2m} = 0$. Set $t_{R'M} = \begin{pmatrix} -1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$ on $KU_0R'M_{2m} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m$ for some integers $a, b$. Then the equality $\rho_{R'M,Q'}t_{R'M} = t_{Q'N}\rho_{R'M,Q'}$ implies that $a = 1$ because $t_{Q'N} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $KU_0Q'N_{2m} \simeq \mathbb{Z} \oplus \mathbb{Z}/2m$.

So the result follows immediately, since the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is always congruent to $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for any integer $b$.

### 2.4.
Finally we will compute the $KO$ homologies of the elementary spectra $W = Y'X_{2m}$ with four cells mentioned in (2.4).

**Proposition 2.4.** The $KO$ homologies $KO_\ast W$ are tabled as follows:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$M'M_{2m}$</th>
<th>$M'N_{2m}$</th>
<th>$N'M_{2m}$</th>
<th>$N'N_{2m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 4</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \oplus (\mathbb{Z}/2 \oplus \mathbb{Z}/m)$</td>
</tr>
<tr>
<td>1, 5</td>
<td>$\mathbb{Z}/4m$</td>
<td>$\mathbb{Z}/4m$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>2, 6</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4m$</td>
<td>$\mathbb{Z}/4m$</td>
</tr>
<tr>
<td>3, 7</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
</tr>
</tbody>
</table>

| | $Q'P_{2m}$ | $Q'Q_{2m}$ | $Q'R_{2m}$ | $R'P_{2m}$ | $R'Q_{2m}$ |
| | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/m$ | $\mathbb{Z} \oplus \mathbb{Z}/m$ | $\mathbb{Z} \oplus \mathbb{Z}/m$ |
| | $\mathbb{Z}/2 \oplus \mathbb{Z}/m$ | ($\ast)_m$ | ($\ast)_m$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ |
| | $\mathbb{Z}$ | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/m$ | ($\ast)_m$ | ($\ast)_m$ |
| | $\mathbb{Z}/m$ | $\mathbb{Z} \oplus \mathbb{Z}/m$ | $\mathbb{Z}/m$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}/2$ |
3. Elementary $\mathbb{Z}/2$-actions

3.1. Let $H$ be a direct sum of 2-torsion free cyclic groups. If the cyclic group $\mathbb{Z}/2$ of order 2 acts on the abelian group $H$, then there exists a direct sum decomposition $H \simeq A \oplus B \oplus C \oplus C$ with $C$ free on which the $\mathbb{Z}/2$-action $\rho_H$ is represented by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

(use [6, Propositions 3.7 and 3.8] or [7]).

If the cyclic group $\mathbb{Z}/2$ acts on the direct sum $H \oplus Z/2^{s+1}$, $s \geq 0$, then its matrix representation is written into one of the following types:

1. $\pm \begin{pmatrix} \rho_H & 0 \\ 0 & 1 \end{pmatrix}$
2. $\pm \begin{pmatrix} \rho_H & 0 \\ 0 & 2^{s+1} \end{pmatrix}$ ($s \geq 2$) on $H \oplus Z/2^{s+1}$
3. $\pm \begin{pmatrix} \rho_H' & 0 \\ 0 & 1 \end{pmatrix}$ on $H' \oplus Z \oplus Z/2^{s+1}$
4. $\pm \begin{pmatrix} \rho_H' & 0 \\ 0 & 1 \end{pmatrix}$ on $H'' \oplus Z \oplus Z \oplus Z/2^{s+1}$

where the matrices behave as left action on $H \oplus Z/2^{s+1}$ and $H \simeq H' \oplus Z \simeq H'' \oplus Z \oplus Z$.

A $\mathbb{Z}/2$-action $\rho$ on an abelian group $H$ is said to be elementary if the pair

<table>
<thead>
<tr>
<th>$i$</th>
<th>$MR'_2m$</th>
<th>$N'Q'_2m$</th>
<th>$N'R'_2m$</th>
<th>$Q'N'_2m$</th>
<th>$R'M'_2m$</th>
<th>$R'N'_2m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/4m$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/4m$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{Z}/4m$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4m \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4m$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/m$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/m$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/m$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{Z}/m$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/m$</td>
<td>$\mathbb{Z}/m$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}/2$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}/4m$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}/2$</td>
</tr>
</tbody>
</table>

in which $(\ast)_m$ stands for $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ according as $m$ is odd or even.
(H, ρ) is one of the following kinds of pairs (cf. [12, 5.1]):

\[(3.2)\]
\[
(A, 1), \quad (B, -1), \quad (C ⊕ C, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}), \quad (Z/8m, 4m+1),
\]
\[
(Z ⊕ Z/2m, ±\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), \quad (Z ⊕ Z/2m, ±\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}),
\]
\[
(Z ⊕ Z ⊕ Z/2m, ±\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & m & 1 \end{pmatrix}).
\]

We here deal with a CW-spectrum \(X\) such that the conjugation \(t_*\) on \(KU_0X\) is decomposed into a direct sum of the above elementary \(Z/2\)-actions, and \(KU_1X = 0\). Thus

\[(3.3)\]
\[
KU_0X = A ⊕ B ⊕ (C ⊕ C) ⊕ A' ⊕ B' ⊕ (D ⊕ D') ⊕ (E ⊕ E')
\]
\[⊕ (F ⊕ F') ⊕ (G ⊕ G') ⊕ (I ⊕ I ⊕ I') ⊕ (J ⊕ J ⊕ J')
\]
\[
\text{where each of the summands } A' \text{ and } B' \text{ is a direct sum of the forms } Z/8m, \text{ each of the summands } D ⊕ D', E ⊕ E', F ⊕ F' \text{ and } G ⊕ G' \text{ is a direct sum of the forms } Z ⊕ Z/2m, \text{ and each of the summands } I ⊕ I ⊕ I' \text{ and } J ⊕ J ⊕ J' \text{ is a direct sum of the form } Z ⊕ Z ⊕ Z/2m. \text{ Moreover the conjugation } t_* \text{ acts on each component of } KU_0X \text{ as follows:}
\]

\[(3.4)\]
\[
t_* = 1, \quad -1, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ on } A, \quad B, \quad C ⊕ C.
\]
\[
t_* = 4m+1, \quad 4m-1 \text{ on the component } Z/8m \text{ of } A', \quad B'.
\]
\[
t_* = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix} \text{ on the component } Z ⊕ Z/2m \text{ of } D ⊕ D', E ⊕ E', F ⊕ F', G ⊕ G'.
\]
\[
t_* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & m & 1 \\ 0 & -1 & 0 \\ 1 & m & -1 \end{pmatrix} \text{ on the component } Z ⊕ Z ⊕ Z/2m \text{ of } I ⊕ I ⊕ I', \quad J ⊕ J ⊕ J'.
\]

For any direct sum \(H = ⊕ Z/2m_i\) we denote by \(H(*)\) the direct sum \(⊕(*)_{m_i}\) where \(*_{m_i} = Z/4 \text{ or } Z/2 ⊕ Z/2\) according as \(m_i\) is odd or even. Moreover we write \(2H = ⊕ Z/m_i\) and \(1/2 H = ⊕ Z/4m_i\).

Let \(KC\) denote the self-conjugate \(K\)-spectrum, which is obtained as the fiber of the map \(1 - t: KU → KU\) (see [3]). Given a CW-spectrum \(X\) satisfying (3.3) with (3.4) we can easily compute its \(KC\) homology as in [12, Lemma 5.1].

**Lemma 3.1.** Assume that \(KU_1X = 0\).
Let us denote by $V_{2m}$ and $W_{4m}$ respectively the elementary spectra constructed by the following cofiber sequences:

\[\Sigma^1 \mathbb{Z}/2 \xrightarrow{i\eta} \mathbb{Z}/m \xrightarrow{i_Y} V_{2m} \xrightarrow{j_Y} \Sigma^2 \mathbb{Z}/2\]

\[\Sigma^1 \mathbb{Z}/2 \xrightarrow{i\eta + j_Y} \mathbb{Z}/2m \xrightarrow{i_W} W_{4m} \xrightarrow{j_W} \Sigma^2 \mathbb{Z}/2.\]

By observing [12, (5.4)] and Propositions 1.2 and 2.3 we here list up some of $CW$-spectra $X$ with a few cells such that $KU_0X$ contains only one 2-torsion cyclic group and $KU_1X=0$.

\[
\begin{array}{cccccccc}
X & V_{2m} & W_{4m} & M_{2m} & Q_{2m} & N_{2m} & R_{2m} \\
\hline
KU_0X & \simeq & \mathbb{Z}/2m & \mathbb{Z}/8m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m \\
t_* & = & 1 & 4m+1 & (-1,0) & (1,0) & (0,1) & (1,0) \\
\hline
KU_0X & \simeq & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m \\
t_* & = & \left(\begin{array}{c}
-1 \\
0 \\
1 \\
1
\end{array}\right) & \left(\begin{array}{c}
0 \\
1 \\
1 \\
1
\end{array}\right) & \left(\begin{array}{c}
0 \\
1 \\
1 \\
1
\end{array}\right) & \left(\begin{array}{c}
1 \\
0 \\
0 \\
1
\end{array}\right) & \left(\begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array}\right) & \left(\begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array}\right) \\
\hline
KU_0X & \simeq & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m & \mathbb{Z} \oplus \mathbb{Z}/2m \\
t_* & = & \left(\begin{array}{c}
0 \\
1 \\
1 \\
1
\end{array}\right) & \left(\begin{array}{c}
1 \\
0 \\
1 \\
1
\end{array}\right) & \left(\begin{array}{c}
-1 \\
0 \\
1 \\
1
\end{array}\right) & \left(\begin{array}{c}
1 \\
0 \\
0 \\
1
\end{array}\right) & \left(\begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array}\right) & \left(\begin{array}{c}
0 \\
1 \\
0 \\
1
\end{array}\right) \\
\hline
\end{array}
\]

We will write simply $Y_H=\bigvee_i Y_{2m_i}$ for any direct sum $H=\bigoplus_i \mathbb{Z}/2m_i$ when $Y=V$, $W$, $M$, $Q$ and so on.

3.2. For later use we will here study the induced homomorphism
\[ \varepsilon_{c^*} : KO_i X \to KC_i X \text{ when } X = Q_{2m}, N'_{2m}, R'_{2m}, NP'_{2m}, NR'_{2m} \text{ and } R'Q_{2m}. \]

**Lemma 3.2.** The induced homomorphisms \( \varepsilon_{c^*} : KO_i X \to KC_i X \) are represented by the following matrices \( M_i(X) \):

i) \[ M_0(Q_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2m \to \mathbb{Z} \oplus \mathbb{Z}/2m \]

\[ M_4(Q_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/m \to \mathbb{Z} \oplus \mathbb{Z}/2m \]

ii) \[ M_0(N'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \]

\[ M_4(N'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \]

iii) \[ M_0(R'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2m \to \mathbb{Z} \oplus \mathbb{Z}/2m \]

\[ M_4(R'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/m \to \mathbb{Z} \oplus \mathbb{Z}/2m \]

iv) \[ M_0(NP'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \]

\[ M_4(NP'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \]

v) \[ M_0(NR'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2m \to \mathbb{Z} \oplus \mathbb{Z}/2m \]

\[ M_2(NR'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \]

\[ M_4(NR'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2m \to \mathbb{Z} \oplus \mathbb{Z}/2m \]

\[ M_6(NR'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \]

vi) \[ M_0(R'Q_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/m \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m \]

\[ M_4(R'Q_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/m \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2m \]

where the matrices behave as left action.

Proof. i) The \( X = Q_{2m} \) case: Obviously \( \varepsilon_{c^*} : KO_0 Q_{2m} \to KC_0 Q_{2m} \) is an isomorphism, and moreover we have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & KO_5 \Sigma^4 & \to & KO_4 \Sigma^2 & \to & KO_3 Q_{2m} & \to & KO_3 \Sigma^2 & \to & KO_2 \Sigma^2 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & KU_5 \Sigma^2 & \to & KU_4 Q_{2m} & \to & KU_4 \Sigma^2 & \to & 0
\end{array}
\]

with exact rows. As is easily seen, the central arrow \( \varepsilon_{d^*} : KO_4 Q_{2m} \to KU_4 Q_{2m} \)
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is expressed as the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: $Z \oplus Z/m \to Z \oplus Z/2m$. The result is now immediate.

ii) The $X=N'_{2m}$ case: Using the commutative diagram

\[
\begin{array}{ccc}
KO_0 \Sigma^0 & \cong & KO_0 N'_{2m} \\
\downarrow & & \downarrow \\
0 & \to & KU_0 \Sigma^0 \to KU_0 N'_{2m} \to KU_0 SZ/2m \to 0 \\
\end{array}
\]

with a split exact row, it is easily checked that $M_i(N'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We next compare the two commutative diagrams

\[
\begin{array}{ccc}
KO_i \Sigma^0 & \to & KO_i N'_{2m} \\
\downarrow & & \downarrow \\
KO_i Q & \to & KO_i SZ/2m \\
\end{array}
\quad
\begin{array}{ccc}
KU_i \Sigma^0 & \to & KU_i N'_{2m} \\
\downarrow & & \downarrow \\
KU_i Q & \to & KU_i SZ/2m \\
\end{array}
\]

with exact diagonals. Since $KO_i N'_{2m} \cong KO_i Q \oplus KO_i \Sigma^0 \cong Z \oplus Z/2$ and $KU_i N'_{2m} \cong KU_i \Sigma^0 \oplus KU_0 SZ/2m \cong Z \oplus Z/2m$, the induced homomorphism $\epsilon_{\nu^*}: KO_i N'_{2m} \to KU_i N'_{2m}$ is expressed as the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: $Z \oplus Z/2 \to Z \oplus Z/2m$. Therefore it follows immediately that $M_i(N'_{2m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

iii) The $X=R'_{2m}$ case: Compare the two commutative diagrams

\[
\begin{array}{ccc}
KO_i \Sigma^0 & \to & KO_i P'_{2m} \\
\downarrow & & \downarrow \\
KO_i Q & \to & KO_i SZ/2m \\
\end{array}
\quad
\begin{array}{ccc}
KU_i \Sigma^0 & \to & KU_i P'_{2m} \\
\downarrow & & \downarrow \\
KU_i Q & \to & KU_i SZ/2m \\
\end{array}
\]

with exact diagonals, in dimensions $i=0$ and 4. Since $KO_i R'_{2m} \cong KO_i Q \oplus KO_{i-2} P'_{2m}$ and $KU_i R'_{2m} \cong KU_i \Sigma^0 \oplus KU_{i-4} SZ/2m$ for $i=0$ and 4, the induced homomorphism $\epsilon_{\nu^*}: KO_i R'_{2m} \to KU_i R'_{2m}$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ according as $i=0$ or 4. The result is now immediate.

iv) The $X=N P'_{4m}$ case: Use the following commutative diagram
with exact rows, in dimensions $i=0$ and 4. Then the result follows from ii) by a routine computation.

v) The $X=NR_2$ case: Use the following commutative diagrams

\[
\begin{array}{c}
KO_2 NR_2 \cong KO_0 R_2 \\
KO_0 NR_2 \cong KO_0 R_2 \\
KO_0 NR_2 \cong KO_0 N_2 \\
KO_0 NR_2 \cong KO_0 N_2 \\
KO_0 NR_2 \cong KO_0 N_2
\end{array}
\]

with exact rows. Then the result follows immediately from ii) and iii).

vi) The $X=R'Q_{2m}$ case is shown by a similar argument to the case iv) using the cofiber sequence $\Sigma^0 \to R'Q_{2m} \to \Sigma'Q_{2m} \to \Sigma'$ and the above result i).

3.3. As a special case of (3.3) we here deal with a $CW$-spectrum $X$ such that $KU_0 X$ has a direct sum decomposition

\[(3.7) \quad KU_0 X \cong A \oplus B \oplus (C \oplus C) \oplus A' \oplus B' \oplus (J \oplus J + J')\]

in which the conjugation $t_*$ acts on $KU_0 X$ as in (3.4). For such a $CW$-spectrum $X$ Lemma 2.1 ii) asserts that $KO_0 X \cong KO_0 X \cong (A \otimes Z/2) \oplus (B \otimes Z/2)$ and $KO_0 X \cong KO_0 X \cong (A \otimes Z/2) \oplus (B \otimes Z/2)$ under the assumption that $KU_0 X=0$. We will now show the first one of our main results.

**Theorem 3.3.** Let $X$ be a $CW$-spectrum such that $KU_0 X$ has a direct sum decomposition as (3.7) and $KU_i X=0$. Assume that $A$ and $B$ are both direct sums of 2-torsion free cyclic groups. Then there exist abelian groups $A_0$, $A_4$, $B_2$ and $B_6$ with $A_0 \oplus A_4 \approx A$, $B_2 \oplus B_6 \approx B$ so that $X$ is quasi $KO_0$-equivalent to the wedge sum $SA_0 \vee \Sigma^2 SB_2 \vee \Sigma^4 SA_4 \vee \Sigma^8 SB_6 \vee (P \land SC) \vee W'_n \vee \Sigma^2 W''_n \vee MQ'_n \vee \Sigma^2 MQ_{n'}$.

**Proof.** Consider the exact sequence

\[
KU_{j+2} X \xrightarrow{\varphi_j} KC_j X \xrightarrow{\psi_j} KO_{j+1} X \oplus KO_{j+2} X \to 0
\]

induced by the cofiber sequence $\Sigma^i KC \xrightarrow{(-\tau, \tau \pi^* \varepsilon_t \varepsilon_u)} KO \vee \Sigma^i KO \xrightarrow{\varepsilon_t \varepsilon_u} KU \xrightarrow{\varepsilon_t \varepsilon_u \pi^* \varepsilon_t \varepsilon_u} \Sigma^i KC$ when $j=0$ and 2. Since $KO_0 X \oplus KO_2 X \cong A \otimes Z/2$ and $KO_3 X \oplus KO_5 X \cong B \otimes Z/2$, we can choose direct sum decompositions $A \approx A_0 \oplus A_4$, $B \approx B_2 \oplus B_6$ with $A_4$, $B_6$ free so that $\psi_0(A_4) \approx A_4 \otimes Z/2 \approx KO_{i+1} X$, $\psi_2(B_{i+2}) \approx B_{i+2} \otimes Z/2 \approx$
$KO_{i+3}X$ for $i=0$ and 4.

Our proof will be established by the same method as in [12, Theorem 5.2] or [13, Theorem 2.5]. Abbreviate by $Y$ the desired wedge sum of nine elementary spectra. For each component $Y_H$ of the wedge sum $Y$ we choose a unique map $f_H: Y_H \to KU \wedge X$ whose induced homomorphism in $KU$ homologies is the canonical injection. Here $H$ is taken to be $A_0, A_1, B_2, B_6, C, A', B', I'$ or $f'$. Notice that there exists a map $g_H: Y_H \to KC \wedge X$ satisfying $(\zeta^A, 1)g_H = f_H$ for each $H$. We will find a map $h_H: Y_H \to KO \wedge X$ such that $(\epsilon_U, 1)h_H = f_H$ for each $H$, and then apply [12, Proposition 1.1] to show that the map $h = \vee H h_H: Y = \vee Y_H \to KO \wedge X$ becomes a quasi $KO_4$-equivalence. We will only find such maps $h_H$ in the cases $H = A_0, C, A'$ and $I'$, the other cases being done similarly.

i) The $H = A_0$ case: Consider the commutative diagram

$$0 \to \text{Ext}(A_0, KO_6X) \to [SA_0, \Sigma KO \wedge X] \xrightarrow{K_{KO}} \text{Hom}(A_0, KO_5X) \to 0$$

$$0 \to \text{Ext}(A_0, KO_7X) \to [SA_0, \Sigma KO \wedge X] \xrightarrow{K_{KO}} \text{Hom}(A_0, KO_6X) \to 0$$

with the universal coefficient sequences, in which the arrows $K_{KO}$ assign to any map $f$ its induced homomorphism of $KO$ homologies in dimension 0. Note that the induced homomorphism $K_{KO}((\tau \pi \zeta^C, 1)g_{A_0}): KO_6SA_0 \to KO_5X$ becomes trivial because $KO_5X = \psi(A_4)$. Then the composite $(\eta, 1)(\tau \pi \zeta^C, 1)g_{A_0} = (\epsilon_U, 1)f_{A_0}: \Sigma^2SA_0 \to KO \wedge X$ is in fact trivial because $\text{Ext}(A_0, KO_5X) = 0$. So we can find a desired map $h_{A_0}$.

ii) The $H = C$ case: Recall that $P$ is self-dual, thus $P = \Sigma^2DP$. Since $\eta, 1: \Sigma^2KO \wedge P \to KO \wedge P$ is trivial, it is easily seen that the composite $(\eta, 1)(\tau \pi \zeta^C, 1)g_C = (\epsilon_U, 1)f_C: P \wedge SC \to \Sigma^2KO \wedge X$ becomes trivial. So we can find a desired map $h_C$.

iii) The $H = A'$ case: Set $A' = \oplus \mathbb{Z}/2m_i$ and then write $2A' = \oplus \mathbb{Z}/4m_i$ and $A'' = \oplus \mathbb{Z}/2$. We will first find vertical arrows $h_0, h_1$ making the diagram below commutative

$$S(2A') \xrightarrow{i_W} W_{A'} \xrightarrow{f_W} \Sigma^2SA''$$

$$KO \wedge X \to KC \wedge X \to \Sigma^3KO \wedge X$$

$$KO \wedge X \to KU \wedge X \to \Sigma^2KO \wedge X$$

after replacing the map $g_{A'}$ with $(\zeta, 1)g_{A'} = f_{A'}$ suitably if necessary. The induced homomorphisms $K_{KO}(\tau \pi \zeta^C, 1)g_{A'}: KO_j W_{A'} \to KO_{j+5}X$ are trivial in dimensions $j = 0$ and 2 because $\psi(A') = \psi(A'A^*\mathbb{Z}/2) = 0$. So we get a map $h_0: \vee \Sigma \to \Sigma^2KO \wedge X$ such that $h_0f_j = (\tau \pi \zeta^C, 1)g_{A'}i_W: S(2A') \to \Sigma^3KO \wedge X$ and in addition
\((\eta,1)h_0=0\) where \(j_{2A}=\vee j_{4m_i}\); \(\vee SZ/4m_i \rightarrow \vee \Sigma^1\). Consequently the composite 
\((\eta,1)(\tau\pi\zeta^1,1)g_{A'}: S(2A') \rightarrow \Sigma^2KO \wedge X\) becomes trivial. Hence we can obtain desired maps \(h_0\) and \(h_1\) by applying [12, Lemma 1.3]. We will next find vertical maps \(k_0, k_1\) making the diagram below commutative

\[
\begin{array}{cccc}
M_{2A'} & k_{M,w} & W_{A'} & j'_{A'}j_w \\
\downarrow k_0 & \downarrow g_{A'} & \downarrow \eta_{A'} & \downarrow \eta_{A'} \\
KO \wedge X & \rightarrow & KC \wedge X & \rightarrow \Sigma^2KO \wedge X \\
\vdash & \eta_{A'} & \vdash & \eta_{A'} \\
KO \wedge X & \rightarrow & KU \wedge X & \rightarrow \Sigma^2KO \wedge X
\end{array}
\]

with \(j_{A''} = \vee j_2: \vee SZ/2 \rightarrow \vee \Sigma^1\), after replacing the map \(g_{A'}\) with \((\xi,1)g_{A'} = f_{A'}\) again if necessary. Notice that the composite \((\eta,1)i_{A''}j_{M}: M_{2A'} \rightarrow \Sigma^1SA''\) is trivial because \((\eta,1)i_{A''} = \vee (\rho_{4m_2}; i_{4m_2})\): \(\vee \Sigma^1 \rightarrow \vee SZ/2\) where \(\rho_{4m_2}: SZ/4m_i \rightarrow SZ/2\) denotes the associated map with the canonical epimorphism. Since \(j_wk_{M,w} = i_{A''}j_{M}: M_{2A'} \rightarrow \Sigma^2SA''\), the composite \((\eta,1)(\tau\pi\zeta^1,1)g_{A'}k_{M,w}: M_{2A'} \rightarrow \Sigma^2KO \wedge X\) coincides with the composite \((\xi,1)h_{A''}j_{M,1}\), which is trivial. So we can obtain desired maps \(k_0\) and \(k_1\) by applying [12, Lemma 1.3] again. However the composite \((\eta,1)i_{A''}j_{w}: W_{A'} \rightarrow \Sigma^2\) becomes trivial because \((\eta,1)i_{A''} = \vee (i_{4m_2}(n_z+\eta_{4m_2}j_2)): \vee SZ/2 \rightarrow \vee \Sigma^0\). Hence there exists a map \(h_{A'}: W_{A'} \rightarrow KO \wedge X\) with \((\xi,1)h_{A'} = f_{A'}\) as desired.

iv) The \(H=I'\) case: Setting \(I'=\oplus Z/2m_i\) we will find vertical maps \(h_0, h_1\) making the diagram below commutative

\[
\begin{array}{cccc}
SI' & i_{MQ} & MQ_{I'} & j_{MQ} \\
\downarrow h_0 & \downarrow g_{I'} & \downarrow h_1 & \downarrow h_1 \\
KO \wedge X & \rightarrow & KC \wedge X & \rightarrow \Sigma^2KO \wedge X \\
\vdash & \eta_{A'} & \vdash & \eta_{A'} \\
KO \wedge X & \rightarrow & KU \wedge X & \rightarrow \Sigma^2KO \wedge X
\end{array}
\]

after replacing the map \(g_{I'}\) with \((\xi,1)g_{I'} = f_{I'}\) suitably if necessary. The induced homomorphisms \(\bar{\rho}_{KO}(\tau\pi\zeta^1,1)g_{I'}: KO_{I'MQ_{I'}} \rightarrow KO_{I'MQ_{I'}}X\) are trivial in dimensions \(j=0\) and 2 because \(\rho_{I'}(I \oplus I') = 0 = \rho_{2}(I \oplus I' + Z/2)\). So we get a map \(h_0: \vee \Sigma^0 \rightarrow \Sigma^2KO \wedge X\) such that \(h_0j_{I'} = (\tau\pi\zeta^1,1)g_{I'}i_{MQ_{I'}}: SI' \rightarrow \Sigma^2KO \wedge X\) and in addition \((\eta,1)h_0=0\). Since the composite \((\eta,1)(\tau\pi\zeta^1,1)g_{I'}i_{MQ_{I'}}: SI' \rightarrow \Sigma^2KO \wedge X\) becomes trivial, we can obtain desired maps \(h_0\) and \(h_1\) by applying [12, Lemma 1.3]. Choose maps \(k_1: \Sigma^0 \rightarrow \Sigma^2KO \wedge X, k_{I}: \Sigma^0 \rightarrow KO \wedge X\) satisfying \(h_1 = \vee (k_{I'}\eta \vee k_{I'}: \Sigma^0 \rightarrow \Sigma^2KO \wedge X, and then set \(k = \vee (k_1\eta_{2m_1} + k_{I}j_{2m_1}): SI' \rightarrow \Sigma^1KO \wedge X. Notice that \(\eta,1)h_1 = k(\vee (i_{2m_1}\eta \vee \eta_{2m_1})): \Sigma^2 \rightarrow KO \wedge X\) because
\[ \tilde{k}(\sqrt{i_{2m}}) = \sqrt{k_i} \] and \[ \tilde{k}(\sqrt{g_{2m}}) = \sqrt{k'_i} \]. Hence the composite \((\eta, 1)h_{MQ}: MQ_{Y'} \to \Sigma^2 KO \wedge X\) becomes trivial. So there exists a map \(h_{Y'}: MQ_{Y'} \to KO \wedge X\) with \((\xi_{Y'}, 1)h_{Y'} = f_{Y'}\) as desired.

4. \(KU_0X\) containing only one 2-cyclic group \(Z/2^{t+1}\)

4.1. We first deal with a \(CW\)-spectrum \(X\) such that \(KU_0X\) has a direct sum decomposition

\[(4.1) \quad KU_0X \cong A \oplus B \oplus (C \oplus C) \oplus Z/2m\]

with \(A, B\) direct sums of 2-torsion free cyclic groups, and \(KU_1X = 0\). Here the conjugation \(t_\ast\) behaves on \(A, B\) and \(C \oplus C\) as in (3.4), and \(t_\ast = 1\) on the last factor \(Z/2m\). For such a \(CW\)-spectrum \(X\) we consider the exact sequence

\[KU_{j+2}X \overset{\psi_j}{\to} KJ_{j+1}X \oplus KO_{j+2}X \to 0\]

in dimensions \(j = 0\) and 2 as in the proof of Theorem 3.2. Recall that \(KC_0X \cong A \oplus C \oplus Z/2m\), \(KC_2X \cong B \oplus C \oplus Z/2\), \(KO_1X \oplus KO_5X \cong (A \otimes Z/2) \oplus Z/2\) and \(KO_3X \oplus KO_5X \cong (B \otimes Z/2) \oplus Z/2\).

Using the isomorphism \(\theta_\ast: (A \otimes Z/2) \oplus Z/2 \to KO_1X \oplus KO_5X\), we put \(\theta_\ast(0, 1) = (x, y) \in KO_1X \oplus KO_5X\). Then the pair \((x, y)\) is divided into the three types:

i) \(x \neq 0, y = 0\) ii) \(x = 0, y \neq 0\) iii) \(x \neq 0, y \neq 0\).

Corresponding to each type we can choose a direct sum decomposition of \(A\) as follows:

\[(4.2) \quad \text{i) } A \cong A_0 \oplus A_4 \text{ with } A_4 \text{ free so that } \psi_0(A_0 \otimes Z/2m) \cong (A_0 \otimes Z/2) \oplus Z/2 \langle x \rangle \\
\cong KO_1X \text{ and } \psi_0(A_4) \cong A_4 \otimes Z/2 \cong KO_5X.\]

\[\text{ii) } A \cong A_0 \oplus A_4 \text{ with } A_4 \text{ free so that } \psi_0(A_0) \cong A_0 \otimes Z/2 \cong KO_1X \text{ and } \psi_0(A_4) \cong (A_4 \otimes Z/2) \oplus Z/2 \langle y \rangle \cong KO_5X.\]

\[\text{iii) } A \cong A_0 \oplus A_4 \oplus Z \text{ with } A_4 \text{ free so that } \psi_0(A_0 \otimes Z/2m) \cong (A_0 \otimes Z/2) \oplus Z/2 \langle x \rangle \cong KO_1X, \psi_0(A_4 \otimes Z/2m) \cong (A_4 \otimes Z/2) \oplus Z/2 \langle y \rangle \cong KO_5X \text{ and } \psi_0(Z) \cong Z/2 \langle x \rangle.\]

Similarly we can choose a direct sum decomposition of \(B\) corresponding to each of the three types. Consequently we have

**Lemma 4.1.** Let \(X\) be a \(CW\)-spectrum satisfying (4.1).

i) \(KC_0X \cong A \oplus C \oplus Z/2m\) is decomposed into one of the following three types:

A1) \(KC_0X \cong A_0 \oplus A_4 \oplus C \oplus Z/2m\) so that \(KO_1X \cong (A_0 \oplus Z/2m) \otimes Z/2\), \(KO_5X \cong A_4 \otimes Z/2\) and both \(\tau_{Z/2}: KC_0X \to KO_1X\) and \((\tau \pi \tilde{c})_{Z/2}: KC_0X \to KO_5X\) are the canonical epimorphisms.

A2) \(KC_0X \cong A_0 \oplus A_4 \oplus C \oplus Z/2m\) so that \(KO_1X \cong A_0 \otimes Z/2\), \(KO_5X \cong (A_4 \otimes Z/2m) \)
and both $\tau_\ast\colon KC_0X\to KO_1X$ and $(\tau\pi C^{-1})_\ast\colon KC_0X\to KO_5X$ are the canonical epimorphisms.

A3) $KC_0X= A_0\oplus A_1\oplus Z\oplus C\oplus Z/2m$ so that $KO_0X= (A_0\oplus Z/2)\oplus Z/2$, $KO_2X= (A_1\oplus Z/2m)\oplus Z/2$ and $(\tau\pi C^{-1})_\ast\colon KC_0X\to KO_5X$ is the canonical epimorphism, but $\tau_\ast\colon KC_0X\to KO_1X$ is the epimorphism whose restriction to $Z\oplus Z/2m$ is given by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

ii) $KC_2X= B\oplus C\oplus Z/2$ is similarly decomposed into one of the three types:

B1) $KC_2X= B_2\oplus B_6\oplus C\oplus Z/2$ with $KO_0X= (B_2\oplus Z/2)\oplus Z/2$, $KO_2X= B_6\oplus Z/2$.

B2) $KC_2X= B_2\oplus B_6\oplus C\oplus Z/2$ with $KO_0X= B_2\oplus Z/2$, $KO_2X= (B_6\oplus Z/2)\oplus Z/2$.

B3) $KC_2X= B_2\oplus B_6\oplus Z\oplus C\oplus Z/2$ with $KO_3X= (B_2\oplus Z/2)\oplus Z/2$, $KO_5X= (B_6\oplus Z/2)\oplus Z/2$.

Here $\tau_\ast\colon KC_2X\to KO_3X$ and $(\tau\pi C^{-1})_\ast\colon KC_2X\to KO_5X$ are epimorphisms as given in A1), A2) and A3) respectively.

4.2. By making use of Lemma 4.1 we will now show the second one of our main results.

**Theorem 4.2.** Let $X$ be an $\text{OW}$-spectrum such that $KU_0X$ has a direct sum decomposition as (4.1) and $KU_X= 0$. Then there exist abelian groups $A_0$, $A_1$, $B_2$ and $B_6$ and a certain CW-spectrum $Y$ so that $X$ is quasi $KO_\ast$-equivalent to the wedge sum $SA_0\vee SB_2\vee SA_1\vee SB_6\vee (P\wedge SC)\vee Y$. Here $Y$ is taken to be one of the following elementary spectra $\Sigma^iSZ/2m$, $\Sigma^iV_{2m}$, $\Sigma^iN_{2m}$, $\Sigma^iR_{2m}$ and $NR_{2m}$ for $i=0, 4$.

Proof. Set $Y_{11}= SZ/2m$, $Y_{12}= \Sigma^iV_{2m}$, $Y_{13}= \Sigma^iN_{2m}$, $Y_{21}= V_{2m}$, $Y_{22}= \Sigma^iSZ/2m$, $Y_{23}= \Sigma^iN_{2m}$, $Y_{31}= \Sigma^iR_{2m}$, $Y_{32}= R_{2m}$ and $Y_{33}= NR_{2m}$. According to Lemma 4.1 $KC_0X$ and $KC_2X$ are respectively decomposed with the three types A1)-A3) and B1)-B3). We will prove that $X$ is quasi $KO_\ast$-equivalent to the wedge sum $SA_0\vee SB_2\vee SA_1\vee SB_6\vee (P\wedge SC)\vee Y_{ij}$ in each type $(Ai, Bj)$. In each type $(Ai, Bj)$ we choose a unique map $f_{ij}\colon Y_{ij}\to KU\wedge X$ whose induced homomorphism in $KU$ homologies is the canonical injection. Then there exists a map $g_{ij}\colon Y_{ij}\to KC\wedge X$ satisfying $(\xi, 1)g_{ij}= f_{ij}$. It is sufficient to find a map $h_{ij}\colon Y_{ij}\to KO\wedge X$ such that $(e_{U, 1})h_{ij}= f_{ij}$ for each pair $(Ai, Bj)$, because the other cases has been established in the proof of Theorem 3.3.

i) The $Y_{11}= SZ/2m$ case: Consider the commutative diagram

$$
\begin{array}{ccl}
0 \to \text{Ext}(Z/2m, KO_5X) & \to & [SZ/2m, \Sigma^iKO\wedge X] \xrightarrow{r_{KO}} \text{Hom}(Z/2m, KO_5X) \to 0 \\
\downarrow \eta_{**} & & \downarrow (\eta_{**, 1})_* \\
0 \to \text{Ext}(Z/2m, KO_7X) & \to & [SZ/2m, \Sigma^iKO\wedge X] \xrightarrow{r_{KO}} \text{Hom}(Z/2m, KO_7X) \to 0
\end{array}
$$

with the universal coefficient sequences. The induced homomorphisms $r_{KO}((\tau\pi C^{-1})_1g_{11})\colon KO_1SZ/2m\to KO_{i+5}X$ become trivial in dimensions $i=0$ and 2.
because of Lemma 4.1 A1) and B1). So it is easily verified that the composite

\((\eta, 1)g_{21}((\eta \pi c^{-1}, 1))f_{21}:SZ/2m \to \Sigma^2KO \wedge X\) is trivial. Hence we can find a desired map \(h_{11}\).

ii) The \(Y_{21} = V_{2m}\) case: We will first find vertical arrows \(h_0\) and \(h_1\), making the diagram below commutative

\[
\begin{array}{ccccc}
SZ/m & \overset{i_v}{\to} & V_{2m} & \overset{j_v}{\to} & \Sigma^2SZ/2 \\
\downarrow h_0 & & \downarrow g_{21} & & \downarrow h_1 \\
KO \wedge X & \to & KC \wedge X & \to & \Sigma^2KO \wedge X \\
\| & & \downarrow \pi, 1 & & \downarrow \eta, 1 \\
KO \wedge X & \to & KU \wedge X & \to & \Sigma^2KO \wedge X.
\end{array}
\]

The induced homomorphisms \(\bar{h}_{KO}(\eta \pi c^{-1}, 1)g_{21}) : KO_iV_{2m} \to KO_{i+5}X\) are trivial in dimensions \(i = 0\) and 2 because \(KO_0V_{2m} \cong Z/m, KC_0V_{2m} \cong Z/2m\) and \(KO_5X \cong \psi_0(B_0)\) by Lemma 4.1 B1). So we get a map \(h_0' : \Sigma^0 \to \Sigma^1KO \wedge X\) such that \(h_0'j_v = (\pi c^{-1}, 1)g_{21}i_v : SZ/m \to \Sigma^1KO \wedge X\) and in addition \((\eta, 1)h_0' = 0\) when \(m\) is even. Hence the composite \((\eta, 1)g_{21}i_v : SZ/m \to \Sigma^2KO \wedge X\) becomes trivial when \(m\) is even as well as odd. By applying [12, Lemma 1.3] we can obtain desired maps \(h_0\) and \(h_1\) after replacing the map \(g_{21}\) with \((\pi c^{-1}, 1)g_{21}j_v\) suitably if necessary.

Moreover we note that \(h_{1*} : KO_2SZ/2 \to KO_iX\) becomes trivial since the induced homomorphism \(\bar{h}_{KO}(\eta \pi c^{-1}, 1)g_{21}) : KO_iV_{2m} \to KO_{i+5}X\) is trivial by means of Lemma 4.1 A2). This implies that the composite \(h_1\theta_2 : \Sigma^1 \to KO \wedge X\) is trivial. Hence it follows that \((\eta, 1)h_1 = h_1i_2\theta_2 : SZ/2 \to KO \wedge X\) because \(\eta, 1 = \theta_2i_2 + i_2\theta_2 : \Sigma^1SZ/2 \to SZ/2\) by (1.1). When \(m\) is even, we see that \((\eta, 1)h_1 = h_1j_v = (\eta \pi c^{-1}, 1)g_{21}i_v : SZ/2 \to KO \wedge X\) where \(\rho_{m,2} : SZ/m \to SZ/2\) denotes the associated map with the canonical epimorphism. Hence it follows that the composite \((\eta, 1)h_{1j_v} : V_{2m} \to \Sigma^2KO \wedge X\) is trivial when \(m\) is even. When \(m\) is odd, \(h_{1*} : KO_2SZ/2 \to KO_iX\) becomes also trivial because \(h_{1j_v} = (\pi c^{-1}, 1)g_{21}\). Using the fact that \(h_{1*} : KO_2SZ/2 \to KO_{i+5}X\) are trivial in dimensions \(i = 0\) and 2, we can then verify that the composite \((\eta, 1)h_1 : SZ/2 \to KO \wedge X\) is trivial when \(m\) is odd. Consequently there exists a map \(h_{21} : V_{2m} \to KO \wedge X\) satisfying \((\eta \pi c^{-1}, 1)h_{21} = f_{21}\) for any \(m\).

iii) The \(Y_{32} = R_{2m}\) case: Note that the induced homomorphisms \(\bar{h}_{KO}(\eta \pi c^{-1}, 1)g_{32}) : KO_iR_{2m} \to KO_{i+7}X\) are trivial in dimensions \(i = 0, 4\) and 6 by means of Lemmas 3.2 iii) and 4.1 A3), B2). Then we can find vertical arrows \(h_0, h_1\), making the diagram below commutative

\[
\begin{array}{ccccc}
\Sigma^0 & \overset{i_r}{\to} & R_{2m} & \overset{j_r}{\to} & \Sigma^4SZ/2m \\
\downarrow h_0 & & \downarrow g_{32} & & \downarrow h_1 \\
KO \wedge X & \to & KC \wedge X & \to & \Sigma^2KO \wedge X \\
\| & & \downarrow \pi, 1 & & \downarrow \eta, 1 \\
KO \wedge X & \to & KU \wedge X & \to & \Sigma^2KO \wedge X.
\end{array}
\]

Moreover we can see that \(h_{1*} : KO_4SZ/2m \to KO_{i+7}X\) are trivial in dimensions
So we can verify that the composite $(\eta, 1)_{h_1}: \Sigma^2 SZ/2m \rightarrow KO \wedge X$ becomes trivial. Hence there exists a desired map $h_{2}$.

iv) The $Y_{22} = \Sigma^2 N'_{2m}$ case is shown similarly to the case iii), by means of Lemmas 3.2 ii) and 4.1 A2), B3) in place of Lemmas 3.2 iii) and 4.1 A3), B2).

v) The $Y_{33} = NR'_{2m}$ case: Note that the induced homomorphisms $r_{KO((\tau \pi_{c_1}), 1)_{g_{3m}}}: KO_{i-1}NR'_{2m} \rightarrow KO_{i-s}X$ are trivial in dimensions $i=0, 2, 4$ and 6, by means of Lemmas 3.2 v) and 4.1 A3), B3). Then we can find vertical arrows $h_0, h_1$ making the diagram below commutative

Moreover we can see that $h_{i}: KO_{i-1}SZ/2m \rightarrow KO_{i-s}X$ are trivial in dimensions $i=0, 2$. This implies that the composite $(\eta, 1)_{h_1}: \Sigma^2 SZ/2m \rightarrow KO \wedge X$ is trivial. The result is now immediate.

The other cases $Y_{22} = \Sigma^4 SZ/2m, Y_{12} = \Sigma^4 V_{2m}, Y_{31} = \Sigma^2 R_{2m}$ and $Y_{13} = \Sigma^2 N'_{2m}$ are evidently shown by parallel discussions to the above cases i), ii), iii) and iv) respectively.

4.3. We next deal with a $CW$-spectrum $X$ such that $KU_{0}X$ has a direct sum decomposition

(4.3)

i) $KU_{0}X \cong A \oplus B \oplus (C \oplus C) \oplus (Z \oplus Z/2m)$ or

\[ ii) \quad KU_{0}X \cong A \oplus B \oplus (C \oplus C) \oplus (Z \oplus Z/2m) \oplus (Z \oplus Z/2n) \]

with $A, B$ direct sums of 2-torsion free cyclic groups, and $KU_{1}X=0$. Here the conjugation $t_{\ast}$ behaves on $A, B$ and $C \oplus C$ as in (3.3), and moreover on $Z \oplus Z/2m, Z \oplus Z/2n$ as follows:

\[ t_{D} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad t_{E} = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \quad \text{on} \quad Z \oplus Z/2m , \]

\[ t_{E} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad t_{G} = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \quad \text{on} \quad Z \oplus Z/2n . \]

For such a $CW$-spectrum $X$ we recall that $KO_{0}X \oplus KO_{5}X \cong (A \otimes Z/2) \oplus Z/2$ and $KO_{0}X \oplus KO_{1}X \cong B \otimes Z/2$ or $=(B \otimes Z/2) \oplus Z/2$ in the case (4.3) i) or ii). By a parallel discussion to (4.2) we can show

**Lemma 4.3.** Let $X$ be a $CW$-spectrum satisfying (4.3).

i) When $t_{\ast}=t_{D}$ on $Z \oplus Z/2m$, $KU_{0}X \cong A \oplus C \oplus (Z \oplus Z/2) \oplus H$ with $H=0, Z/2n$ or $Z/2$ and it is decomposed into either of the following three types:
D1) \( KC_0 X \cong A_0 \oplus A_1 \oplus C \oplus (Z \oplus Z/2) \oplus H \) so that \( KO_1 X \cong (A_0 \oplus Z/2) \otimes Z/2 \), \( KO_0 X \cong A_1 \otimes Z/2 \) and both \( \tau_*: KC_0 X \to KO_1 X \) and \( (\pi \pi c^1)_*: KC_0 X \to KO_2 X \) are the canonical epimorphisms.

D2) \( KC_0 X \cong A_0 \oplus A_1 \oplus C \oplus (Z \oplus Z/2) \oplus H \) so that \( KO_1 X \cong A_0 \oplus Z/2 \), \( KO_0 X \cong (A_1 \oplus Z/2) \otimes Z/2 \) and both \( \tau_*: KC_0 X \to KO_1 X \) and \( (\pi \pi c^1)_*: KC_0 X \to KO_2 X \) are the canonical epimorphisms.

D3) \( KC_0 X \cong A_0 \oplus A_1 \oplus Z \oplus C \oplus (Z \oplus Z/2) \oplus H \) so that \( KO_1 X \cong (A_0 \oplus Z/2) \otimes Z/2 \), \( KO_0 X \cong (A_1 \oplus Z/2) \otimes Z/2 \) and both \( \tau_*: KC_0 X \to KO_1 X \) and \( (\pi \pi c^1)_*: KC_0 X \to KO_2 X \) is the canonical epimorphism, but \( \tau_*: KC_0 X \to KO_1 X \) is the epimorphism whose restriction to \( Z \oplus (Z \oplus Z/2) \) is given by the matrix \( \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \): \( Z \oplus Z \oplus Z \oplus Z \oplus Z \oplus Z \) is the epimorphism.

ii) When \( \tau_*=\tau \) on \( Z \oplus Z/2m \), \( KC_0 X \cong A \oplus C \oplus (Z \oplus Z/2m) \oplus H \) with \( H=0 \), \( Z/2n \) or \( Z/2 \) and it is decomposed similarly into one of the three types D4) and D5) corresponding to the above D1), D2) and D3).

iii) When \( \tau_*=\tau \) on \( Z \oplus Z/2n \), \( KC_2 X \cong B \oplus C \oplus H \oplus (Z \oplus Z/2) \) with \( H=Z/2m \) or \( Z/2 \) and it is also decomposed into one of the three types E1), E2) and E3) as the case i).

iv) When \( \tau_*=\tau \) on \( Z \oplus Z/2n \), \( KC_2 X \cong B \oplus C \oplus H \oplus (Z \oplus Z/2n) \) with \( H=Z/2m \) or \( Z/2 \) and it is also decomposed into one of the three types E4), E5) and E6) as the case ii).

4.4. By making use of Lemma 4.3 we will here show the third one of our main results.

**Theorem 4.4.** Let \( X \) be a CW-spectrum such that \( KU_0 X \) has a direct sum decomposition as (4.3) and \( KU_1 X=0 \). Then there exist abelian groups \( A_0, A_1, B_2 \) and \( B_6 \) and certain CW-spectra \( Y \) and \( Y' \) so that \( X \) is quasi \( KO_* \)-equivalent to the wedge sum \( SA_0 \vee B_2 \vee SC \vee TQ_2 \vee U_1 \vee Y \vee Y' \). Here \( Y \) is taken to be \( \Sigma^2 M_{2m}, \Sigma^2 Q_{2m}, \Sigma^3 N_{4m} \) or \( R'Q_{2m} \) for \( i=0, 4 \) and \( Y' \) to be \( \{pt\} \) in the (4.3) i) case and \( Y' \) to be \( M_{2m}, \Sigma^2 Q_{2m}, \Sigma^3 N_{4m} \) or \( R'Q_{2m} \) for \( i=0, 4 \) in the (4.3) ii) case.

**Proof.** Set \( Y_1 = \Sigma^2 M_{2m}, Y_2 = \Sigma^2 Q_{2m}, Y_3 = NP_{4m}, Y_4 = Q_{2m}, Y_5 = \Sigma Q_{2m}, Y_6 = R'Q_{2m} \) and then \( Y'_j = \Sigma^2 Y_j \) for \( 1 \leq j \leq 6 \). According to Lemma 4.3 \( KC_0 X \) is decomposed with the six types D1)-D6), and \( KC_2 X \) is decomposed with the six types E1)-E6) in the case (4.3) ii). We will prove that \( X \) is quasi \( KO_* \)-equivalent to the wedge sum \( SA_0 \vee \Sigma^2 SB_2 \vee \Sigma^4 SA_4 \vee \Sigma^6 SB_6 \vee (P \wedge SC) \vee Y_i \vee Y'_j \) in each type \( (Di, Ej) \). In each type \( Di \) we choose a unique map \( f_i: Y_i \to KU \wedge X \) whose induced homomorphism in \( KU \)-homologies is the canonical injection. Then there exists a map \( g_i: Y_i \to KC \wedge X \) satisfying \( (\xi, 1)g_i = f_i \). It is sufficient to find a map \( h_i: Y_i \to KO \wedge X \) such that \( (c_{v, r})h_i = f_i \) for each \( i \), the \( Y' = Y'_j \) case being similarly done.

i) The \( Y_2 = \Sigma^2 M_{2m} \) case: We will find vertical arrows \( h_0, h_1 \) making the
diagram below commutative

\[
\begin{array}{cccc}
\Sigma^2 S\Sigma^2/2m & \xrightarrow{i_M} & \Sigma^2 M_{2m} & \xrightarrow{j_M} & \Sigma^4 \\
\downarrow h_0 & & \downarrow g_2 & & \downarrow h_1 \\
KO \wedge X & \rightarrow & KC \wedge X & \rightarrow & \Sigma^3 KO \wedge X \\
\downarrow \varepsilon & & \downarrow \eta \gamma & & \downarrow \eta \lambda \\
KO \wedge X & \rightarrow & KU \wedge X & \rightarrow & \Sigma^2 KO \wedge X
\end{array}
\]

by replacing the map \(g_2\) with \((\varepsilon, 1)g_2 = f_2\) suitably if necessary. The induced homomorphisms \(\rho KO((\tau_2^1, \lambda)g_2): KO_i M_{2m} \rightarrow KO_{i+5} X\) become trivial in dimensions \(i=0, 2\) because of Lemma 4.3 D2) and E1)-E3). Hence it is easily seen that the composite \((\eta, 1)(\tau_2^1, \lambda)g_2: \Sigma^2 S\Sigma^2/2m \rightarrow KO \wedge X\) is trivial. So we get desired maps \(h_0, h_1\) by applying [12, Lemma 1.3]. However the map \(h_1: \Sigma^1 \rightarrow KO \wedge X\) has an extension \(\bar{h}_1: \Sigma^1 S\Sigma^2/2m \rightarrow KO \wedge X\) satisfying \(\bar{h}_1 i = h_1\). Since \((\eta, 1)h_1 = \bar{h}_1(\eta j): \Sigma^2 \rightarrow KO \wedge X\), the result is now immediate.

ii) The \(Y_3 = NP_{1m}\) case: Note that the induced homomorphisms \(\rho KO((\tau_2^1, \lambda)g_3): KO_i NP_{1m} \rightarrow KO_{i+5} X\) are trivial in dimensions \(i=0, 2\) and \(i=0, 4\), by means of Lemmas 3.2 iv) and 4.3 D3). Then we can find vertical arrows \(h_0, h_1\) making the diagram below commutative

\[
\begin{array}{cccc}
\Sigma^2 \Sigma^2/4m & \xrightarrow{i_P} & NP_{1m} & \xrightarrow{j_P} & \Sigma^2 S\Sigma^2/4m \\
\downarrow h_0 & & \downarrow g_2 & & \downarrow h_1 \\
KO \wedge X & \rightarrow & KC \wedge X & \rightarrow & \Sigma^3 KO \wedge X \\
\downarrow \varepsilon & & \downarrow \eta \gamma & & \downarrow \eta \lambda \\
KO \wedge X & \rightarrow & KU \wedge X & \rightarrow & \Sigma^2 KO \wedge X
\end{array}
\]

Moreover we notice that the composite \(h_1 \bar{h}_1: \Sigma^1 \rightarrow KO \wedge X\) becomes trivial because \(h_1 j_P = (\tau_2^1, \lambda)g_3\). Then it follows from (1.1) that \((\eta, 1)h_1 = h_1 j \eta = h_1 \pi_3(\eta j, \eta): \Sigma^2 \Sigma^2/4m \rightarrow KO \wedge X\) where \(\pi_2: \Sigma^2 \Sigma^2/4m \rightarrow \Sigma^2\) stands for the second projection. The result is now immediate.

iii) The \(Y_4 = Q_{2m}\) case: As in the case i) we can find vertical arrows \(h_0, h_1\) making the diagram below commutative

\[
\begin{array}{cccc}
S\Sigma^2/2m & \xrightarrow{i_Q} & Q_{2m} & \xrightarrow{j_Q} & \Sigma^4 \\
\downarrow h_0 & & \downarrow g_2 & & \downarrow h_1 \\
KO \wedge X & \rightarrow & KC \wedge X & \rightarrow & \Sigma^3 KO \wedge X \\
\downarrow \varepsilon & & \downarrow \eta \gamma & & \downarrow \eta \lambda \\
KO \wedge X & \rightarrow & KU \wedge X & \rightarrow & \Sigma^2 KO \wedge X
\end{array}
\]

since the induced homomorphisms \(\rho KO((\tau_2^1, \lambda)g_4): KO_i Q_{2m} \rightarrow KO_{i+5} X\) are trivial in dimensions \(i=0, 2\) by means of Lemma 4.3 D4) and E4)-E6). The map \(h_1: \Sigma^1 \rightarrow KO \wedge X\) is written as the composite \(h_1 = k_1 j(\eta j): \Sigma^2 \rightarrow KO \wedge X\). Hence we see that \((\eta, 1)h_1 = k_1 j(\eta j): \Sigma^2 \rightarrow KO \wedge X\) which implies our result immediately.

iv) The \(Y_6 = R'Q_{2m}\) case: We will find vertical arrows \(h_0, h_1\) making the
diagram below commutative

\[
\begin{array}{cccc}
R'_{2m} & \xrightarrow{i_{R',R'Q}} & R'Q_{2m} & \xrightarrow{j_{R'Q}} \Sigma^g \\
\downarrow h_0 & & \downarrow h_1 & \\
KO \wedge X & \longrightarrow & KC \wedge X & \longrightarrow \Sigma^g KO \wedge X \\
\downarrow \gamma & & \downarrow \eta & \\
KO \wedge X & \longrightarrow & KU \wedge X & \longrightarrow \Sigma^g KO \wedge X
\end{array}
\]

by replacing the map \(g_0\) with \((\xi,1)g_0 = f_0\) suitably if necessary. The induced homomorphisms \(\bar{\alpha}_{ko}(\tau \tau \xi^{-1}, 1)g_0) : KO(R'Q_{2m} \to KO_{i+5}X\) become trivial in dimensions \(i=0, 4\) and \(6\) by means of Lemmas 3.2 vi) and 4.1 D6), E4)-E6). Then we get a map \(h_0 : \Sigma^2 KO \wedge X\) such that \((\tau \tau \xi^{-1}, 1)g_0 h_0 = h_0 : R'_{2m} \to \Sigma^g KO \wedge X\) and in addition \((\eta,1)h_0 = 0\). So we obtain desired maps \(h_0\) and \(h_1\) by applying [12, Lemma 1.3]. Since there exists a map \(k_1 : \Sigma^4 KO \wedge X\) with \(k_1 h_0 = h_0\), it follows from (2.3) that \((\eta,1)h_1 = k_1 j j k_0 (h_0 h_0) ; \Sigma^g KO \wedge X\). The result is now immediate.

The other cases \(Y_i = \Sigma^g M_{2m}\) and \(Y_5 = \Sigma^4 Q_{2m}\) are evidently shown by parallel discussions to the cases i) and iii) respectively.

4.5. We will finally prove our main theorem as a corollary by putting Theorems 3.3, 4.2 and 4.4 together.

Proof of Theorem 2. Recall that the conjugation \(t_*\) on \(KU_X \cong H \oplus Z/2m, m=2^i\), is represented by one of the matrices given in (3.1) i)-v). If its matrix representation has the type i), we may apply Theorem 4.2 in order to observe that \(Y\) is taken to be one of the elementary spectra \(\Sigma^i SZ/2m, \Sigma^i V_{2m}, \Sigma^i N_{2m}, \Sigma^2 R'_{2m}\) and \(\Sigma^j NR'_{2m}\) for \(0 \leq i \leq 3\) and \(0 \leq j \leq 1\). If it has the type iii) or iv), we may apply Theorem 4.4 in order to observe that \(Y\) is taken to be one of the elementary spectra \(\Sigma^2 M_{2m}, \Sigma^2 Q_{2m}, \Sigma^2 NP'_{2m}\) and \(\Sigma^2 R'Q_{2m}\) for the above \(i, j\). If it has the type ii) or v), we may apply Theorem 3.3 in order to observe that \(Y\) is taken to be one of the elementary spectra \(\Sigma^2 W_{2m} (m=4n)\) and \(\Sigma^2 M Q_{2m}\) for the above \(j\).

Combining Theorem 2 with Propositions 1.2, 2.3 and 2.4, and then applying [12, Corollary 1.6] with (1.3) and (2.5) we obtain

**Corollary 4.5.**

i) \(N'M_{2m} \cong NP'_{4m}, N'Q_{2m} \cong P \vee \Sigma^g V_{2m}, R'M_{2m} \cong P \vee \Sigma^g V_{2m}, P'Q_{4m} \cong \Sigma^g M Q_{2m}\) and \(P'Q_{2m} \cong P \vee \Sigma^2 SZ/2m\) for \(n odd\).

ii) \(N'M_{2m} \cong \Sigma^1 NP'_{4m}, M'R_{2m} \cong P \vee \Sigma^g V_{2m}, Q'N_{2m} \cong P \vee \Sigma^3 V_{2m}\), \(Q'P_{4m} \cong MQ'_{2m}\) and \(Q'P_{2m} \cong P \vee \Sigma^2 SZ/2m\) for \(n odd\).

iii) \(M Q'_{2m} \cong \Sigma^1 M Q'_{2m}, N P'_{2m} \cong \Sigma^1 N P'_{2m}, N R'_{2m} \cong \Sigma^1 N R'_{2m}\) and \(R'Q_{2m} \cong \Sigma^1 R'Q_{2m}\).

iv) \(M Q'_{2m} \cong \Sigma^1 M Q'_{2m}, N P'_{2m} \cong \Sigma^1 N P'_{2m}, N R'_{2m} \cong \Sigma^1 N R'_{2m}\) and \(Q'R_{2m} \cong \Sigma^4 Q'R_{2m}\).
Remark. By applying [14, Theorem 2.6] we can observe that

\[(4.4) \quad M' M_{2mR^0} \oslash \Sigma^2 MP_{4m}, \quad MP_{2mR^0} \oslash \Sigma^2 MP_{2m} \quad \text{and} \quad MP_{2mR^0} \oslash \Sigma^2 MP_{2m}. \]

References


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