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QUASI K-HOMOLOGY EQUIVALENCES, II

Dedicated to Professor Junzo Tao on his sixtieth birthday

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0. Introduction

Let E be an associative ring spectrum with unit, and X, Y be CW-spectra. We say that X is *quasi E_*-equivalent to* Y if there exists a map $h\colon Y\to E\wedge X$ such that the composite $(\mu_\wedge 1)(1_\wedge h)\colon E\wedge Y\to E\wedge X$ is an equivalence where $\mu\colon E\wedge E\to E$ stands for the multiplication of E. In this case we write $X\underset{E}{\sim}Y$, and we call such a map $h\colon Y\to E\wedge X$ *a quasi E_*-equivalence*. We shall be concerned with the quasi KO_*- and KU_*-equivalences where KO and KU denote the real and complex K-spectrum respectively.

The conjugation t on KU gives rise to an involution t_* on KU_*X for any CW-spectrum X. Thus the KU-homology KU_*X is regarded as a $Z/2$-graded abelian group with involution. Note that there is an isomorphism between KU_*X and KU_*Y as $Z/2$-graded abelian groups with involution if X is quasi KO_*-equivalent to Y.

For any abelian group G we denote by SG the Moore spectrum of type G. Evidently $KU_0SG\cong G$ on which $t_*=1$ and $KU_1SG=0$. Let us denote by P and Q the cofibers of the maps $\eta\colon\Sigma^1\to\Sigma^0$ and $\eta^2\colon\Sigma^2\to\Sigma^0$ respectively where $\eta\colon\Sigma^1\to\Sigma^0$ is the stable Hopf map of order 2. It is well known that $KU_0P\cong Z\oplus Z$ on which $t_*=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $KU_1P=0$. On the other hand, $KU_0Q\cong Z$ and $KU_{-1}Q\cong Z$ on both of which $t_*=1$.

Let H be a 2-torsion free abelian group which is written into a direct sum of cyclic groups. If the cyclic group $Z/2$ acts on H, then H admits a direct sum decomposition $H\cong A\oplus B\oplus C\oplus C$ so that the involution ρ behaves as

$$(0.1)\qquad \rho=1 \text{ on } A,\quad \rho=-1 \text{ on } B\quad\text{and}\quad \rho=\begin{pmatrix}0&1\\1&0\end{pmatrix} \text{ on } C\oplus C$$

respectively (see [6, Proposition 3.7] or [7]).

By observing these facts, Bousfield [6, Theorem 3.7] has proved the following satisfactory result.

Theorem 1 (*Bousfield*). *Let X be a CW-spectrum such that KU_*X is a* *direct sum of 2-torsion free cyclic groups. Then there exist abelian groups* A_i $(0 \leqq i \leqq 7)$, C_j $(0 \leqq j \leqq 1)$ *and* G_k $(0 \leqq k \leqq 3)$ *so that* X *is quasi* KO_**-equivalent to the wedge sum* $\bigvee_i (\Sigma^i SA_i) \vee \bigvee_j (\Sigma^j P \wedge SC_j) \bigvee_k \vee (\Sigma^{k+1} Q \wedge SG_k)$.

In [12, Theorems 1 and 2] or [9] a partial result of the above theorem was proved by a different method from Bousfield's. In the forthcoming paper [15, Theorem 1] we will give a new proof of the above theorem by our method developed in [12, 13].

Let H be a direct sum of 2-torsion free cyclic groups. If the cyclic group $Z/2$ acts on the direct sum $H \oplus Z/2m$, $m=2^s$, then its matrix representation is divided into one of the following types:

$$(0.2) \quad \text{i)} \ \pm\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{ii)} \ \pm\begin{pmatrix} \rho & 0 \\ 0 & m+1 \end{pmatrix} \ (s \geqq 2) \qquad \text{on} \quad H \oplus Z/2m\,,$$

$$\text{iii)} \ \pm\begin{pmatrix} \rho' & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{iv)} \ \pm\begin{pmatrix} \rho' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix} \qquad \text{on} \quad H' \oplus Z \oplus Z/2m\,,$$

$$\text{v)} \ \pm\begin{pmatrix} \rho'' & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & m & 1 \end{pmatrix} \qquad \text{on} \quad H'' \oplus Z \oplus Z \oplus Z/2m$$

where $H \cong H' \oplus Z \cong H'' \oplus Z \oplus Z$ and ρ, ρ' or ρ'' is an involution on H, H' or H'' respectively which is decomposed as in (0.1).

We denote by M_{2m}, Q_{2m}, N'_{2m}, R'_{2m}, V_{2m} and W_{8m} the cofibers of the maps

$$i\eta\colon \Sigma^1 \to SZ/2m\,, \quad \tilde{\eta}\eta\colon \Sigma^3 \to SZ/2m\,, \quad \eta^2 j\colon \Sigma^1 SZ/2m \to \Sigma^0\,,$$
$$\eta^2\bar{\eta}\colon \Sigma^3 SZ/2m \to \Sigma^0\,, \quad i\bar{\eta}\colon \Sigma^1 SZ/2 \to SZ/m \quad \text{and} \quad i\bar{\eta} + \tilde{\eta} j\colon \Sigma^1 SZ/2 \to SZ/4m$$

respectively where $\tilde{\eta}\colon \Sigma^2 \to SZ/2m$ and $\bar{\eta}\colon \Sigma^1 SZ/2m \to \Sigma^0$ stand for a coextension and an extension of η satisfying $j\tilde{\eta} = \eta$ and $\bar{\eta} i = \eta$. In [12, Propositions 4.1, 4.2 and Corollary 4.6] we have investigated the KU- and KO-homologies of these elementary spectra.

We will moreover introduce some elementary spectra MQ_{2m}, NP'_{4m}, NR'_{2m} and $R'Q_{2m}$ constructed by the cofibers of the maps

$$i\eta \vee \tilde{\eta}\eta\colon \Sigma^1 \vee \Sigma^3 \to SZ/2m\,, \qquad (\eta^2 j, \bar{\eta})\colon \Sigma^1 SZ/4m \to \Sigma^0 \vee \Sigma^0\,,$$
$$(\eta^2 j, \eta^2 \bar{\eta})\colon \Sigma^3 SZ/2m \to \Sigma^2 \vee \Sigma^0 \quad \text{and} \quad \tilde{h}_R \eta\colon \Sigma^7 \to R'_{2m}$$

respectively where $\tilde{h}_R\colon \Sigma^6 \to R'_{2m}$ is a coextension of $\tilde{\eta}$ satisfying $j'_R \tilde{h}_R = \tilde{\eta}$. After studying the KU- and KO-homologies of these spectra with four cells (Propositions 1.2, 1.3, 2.3 and 2.4) we will prove the following result which is our main theorem in this note.

Theorem 2. *Let* X *be a* CW*-spectrum and* H *be a direct sum of* 2*-torsion* *free cyclic groups. Assume that* $KU_0X \cong H \oplus Z/2m$, $m=2^s$, *and* $KU_1X=0$. *Then there exist abelian groups* A_0, A_4, B_2, B_6 *and* C *and a certain CW-spectrum* Y *so that* X *is quasi* KO_**-equivalent to the wedge sum* $SA_0 \vee \Sigma^2 SB_2 \vee \Sigma^4 SA_4 \vee \Sigma^6 SB_6 \vee (P \wedge SC) \vee Y$. *Here* Y *is taken to be one of the following elementary spectra* $\Sigma^{2i} SZ/2m$, $\Sigma^{2i} V_{2m}$, $\Sigma^{2i} W_{2m}$ $(s \geqq 2)$, $\Sigma^{2i} M_{2m}$, $\Sigma^{2i} Q_{2m}$, $\Sigma^{2i} N'_{2m}$, $\Sigma^{2i} R'_{2m}$, $\Sigma^{2j} MQ_{2m}$, $\Sigma^{2j} NP'_{4m}$, $\Sigma^{2j} NR'_{2m}$ *and* $\Sigma^{2j} R'Q_{2m}$ *for* $0 \leqq i \leqq 3$ *and* $0 \leqq j \leqq 1$.

In order to obtain our main theorem as a corollary we will give three theorems (Theorems 3.3, 4.2 and 4.4) in a slightly general form. The first theorem is established in the situation when the conjugation t_* on KU_0X behaves as the types (0.2) ii) and v), and the second or the third theorem is done in the situation as the type (0.2) i) or the types (0.2) iii) and iv) respectively.

This paper is a continuation of [12] with the same title and we will use the same notations as in it.

1. Some elementary spectra XY_{2m} and XY'_{2m} with four cells

1.1. For any map $f\colon Y \to X$ we denote by C_f its cofiber. Thus $Y \xrightarrow{f} X \xrightarrow{i_f} C_f \xrightarrow{j_f} \Sigma^1 Y$ is a cofiber sequence. The Moore spectrum $SZ/2m$ is obtained as the cofiber of multiplication by $2m$ on Σ^0. In this case the maps $i_{2m}\colon \Sigma^0 \to SZ/2m$ and $j_{2m}\colon SZ/2m \to \Sigma^1$ are often abbreviated to be i and j respectively. By applying Verdier's lemma (see [2]) we can easily show

Lemma 1.1. i) *Given two maps* $f\colon Y \to X$, $g\colon Z \to X$ *the cofiber* $C_{f \vee g}$ *of the map* $f \vee g\colon Y \vee Z \to X$ *coincides with the cofiber* $C_{i_f g}$ *of the composite* $i_f g\colon Z \to C_f$. *In particular, the cofiber* $C_{f \vee g}$ *coincides with the wedge sum* $C_f \vee \Sigma^1 Z$ *if* $g\colon Z \to X$ *is factorized through* Y *as* $g=fh\colon Z \to Y \to X$ *for some map* h.

ii) *Given two maps* $f\colon X \to Y$, $g\colon X \to Z$ *the cofiber* $C_{(f,g)}$ *of the map* $(f,g)\colon X \to Y \vee Z$ *coincides with the cofiber* C_{gj_f} *of the composite* $gj_f\colon \Sigma^{-1} C_f \to Z$. *In particular, the cofiber* $C_{(f,g)}$ *coincides with the wedge sum* $C_f \vee Z$ *if* $g\colon X \to Z$ *is factorized through* Y *as* $g=hf\colon X \to Y \to Z$ *for some map* h.

Let $\tilde{\eta}_{2m}\colon \Sigma^2 \to SZ/2m$ be a coextension of η satisfying $j_{2m}\tilde{\eta}_{2m}=\eta$ and $\bar{\eta}_{2m}\colon \Sigma^1 SZ/2m \to \Sigma^0$ an extension of η satisfying $\bar{\eta}_{2m} i_{2m}=\eta$ where $\eta\colon \Sigma^1 \to \Sigma^0$ denotes the stable Hopf map of order 2. The maps $\tilde{\eta}_{2m}$ and $\bar{\eta}_{2m}$ are often abbreviated to be $\tilde{\eta}$ and $\bar{\eta}$ respectively. After choosing these maps suitably there holds the following relation

$$(1.1) \qquad \eta_\wedge 1 = \tilde{\eta}_{2m} j_{2m} + i_{2m} \bar{\eta}_{2m}\colon \Sigma^1 SZ/2m \to SZ/2m$$

(see [5, Lemma 7.2]).

Let us denote by M_{2m}, N_{2m}, P_{2m}, Q_{2m}, R_{2m}, M'_{2m}, N'_{2m}, P'_{2m}, Q'_{2m} and R'_{2m} respectively the elementary spectra constructed by the following cofiber sequences as in [12, (4.1) and (4.2)]:

$$
(1.2)\qquad
\begin{array}{ll}
\Sigma^1 \xrightarrow{i\eta} SZ/2m \xrightarrow{i_M} M_{2m} \xrightarrow{j_M} \Sigma^2 & SZ/2m \xrightarrow{\eta j} \Sigma^0 \xrightarrow{i'_M} M'_{2m} \xrightarrow{j'_M} \Sigma^1 SZ/2m \\
\Sigma^2 \xrightarrow{i\eta^2} SZ/2m \xrightarrow{i_N} N_{2m} \xrightarrow{j_N} \Sigma^3 & \Sigma^1 SZ/2m \xrightarrow{\eta^2 j} \Sigma^0 \xrightarrow{i'_N} N'_{2m} \xrightarrow{j'_N} \Sigma^2 SZ/2m \\
\Sigma^2 \xrightarrow{\tilde{\eta}} SZ/2m \xrightarrow{i_P} P_{2m} \xrightarrow{j_P} \Sigma^3 & \Sigma^1 SZ/2m \xrightarrow{\bar{\eta}} \Sigma^0 \xrightarrow{i'_P} P'_{2m} \xrightarrow{j'_P} \Sigma^2 SZ/2m \\
\Sigma^3 \xrightarrow{\tilde{\eta}\eta} SZ/2m \xrightarrow{i_Q} Q_{2m} \xrightarrow{j_Q} \Sigma^4 & \Sigma^2 SZ/2m \xrightarrow{\eta\bar{\eta}} \Sigma^0 \xrightarrow{i'_Q} Q'_{2m} \xrightarrow{j'_Q} \Sigma^3 SZ/2m \\
\Sigma^4 \xrightarrow{\tilde{\eta}\eta^2} ZS/2m \xrightarrow{i_R} R_{2m} \xrightarrow{j_R} \Sigma^5 & \Sigma^3 SZ/2m \xrightarrow{\eta^2\bar{\eta}} \Sigma^0 \xrightarrow{i'_R} R'_{2m} \xrightarrow{j'_R} \Sigma^4 SZ/2m
\end{array}
$$

In [12, Propositions 4.1 and 4.2] we have calculated the KU- and KO-homologies of these elementary spectra with three cells.

Given two cofibers X_{2m}, Y_{2m} of any maps $f\colon \Sigma^i \to SZ/2m$, $g\colon \Sigma^j \to SZ/2m$ ($i \leqq j$) we denote by XY_{2m} the cofiber of the maps $f \vee g\colon \Sigma^i \vee \Sigma^j \to SZ/2m$. Dually we denote by XY'_{2m} the cofiber of the map $(f, g)\colon \Sigma^j SZ/2m \to \Sigma^{j-i} \vee \Sigma^0$ for two cofibers X'_{2m}, Y'_{2m} of any maps $f\colon \Sigma^i SZ/2m \to \Sigma^0$, $g\colon \Sigma^j SZ/2m \to \Sigma^0$ ($i \leqq j$). We will only deal with the CW-spectra XY_{2m} and XY'_{2m} when $X=M$ or N and $Y=P$, Q or R as Lemma 1.1 may be applicable to the other cases. Note that

$$
(1.3)\qquad
\begin{array}{lll}
MP_{2m} = \Sigma^3 D(MP'_{2m})\,, & MQ_{2m} = \Sigma^4 D(MQ'_{2m})\,, & MR_{2m} = \Sigma^5 D(MR'_{2m}) \\
NP_{2m} = \Sigma^3 D(NP'_{2m})\,, & NQ_{2m} = \Sigma^4 D(NQ'_{2m})\,, & NR_{2m} = \Sigma^5 D(NR'_{2m})
\end{array}
$$

where DW stands for the Spanier-Whitehead dual of W (cf. [12, (4.3)]).

1.2. We will now compute the KU homologies of the above mentioned spectra $W = XY_{2m}$, XY'_{2m} with four cells, by making use of the results in [12, Proposition 4.1].

Proposition 1.2. *The KU homologies KU_0W, KU_1W and the conjugation t_* on them are given as follows:*

$W =$	MP_{2m}	MQ_{2m}	MR_{2m}	NP_{2m}	NQ_{2m}	NR_{2m}
$KU_0W \cong$	$Z \oplus Z/m$	$Z \oplus Z \oplus Z/2m$	$Z \oplus Z/2m$	Z/m	$Z \oplus Z/2m$	$Z/2m$
$t_* =$	$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & m & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 0 & 1 \\ m & 1 \end{pmatrix}$	1
$KU_1W \cong$	Z	0	Z	$Z \oplus Z$	Z	$Z \oplus Z$
$t_* =$	-1		1	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$W =$	MP'_{2m}	MQ'_{2m}	MR'_{2m}	NP'_{2m}	NQ'_{2m}	NR'_{2m}
$KU_0W \cong$	$Z \oplus Z/m$	$Z \oplus Z$	$Z \oplus Z/2m$	$Z \oplus Z \oplus Z/m$	Z	$Z \oplus Z \oplus Z/2m$
$t_* =$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	1	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$KU_1W \cong$	Z	$Z/2m$	Z	0	$Z \oplus Z/2m$	0
$t_* =$	1	-1	-1		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	

where the matrices behave as left action on abelian groups.

Proof. The $W=MP_{2m}$ case has been computed in [14, Proposition 1.2 i)]. We will investigate the behaviour of the conjugation t_* on KU_*W only when $W=MQ_{2m}$, NP'_{2m} and NR'_{2m}, the other cases being easy.

i) The $W=MQ_{2m}$ case: Consider the two commutative diagrams

$$\begin{tikzcd}
 & & \Sigma^3 \arrow[r, equal] \arrow[d, "i_M\tilde{\eta}\eta"'] & \Sigma^3 \arrow[d, "0"] \\
\Sigma^1 \arrow[r, "i\eta"] \arrow[d] & SZ/2m \arrow[r] \arrow[d, equal] & M_{2m} \arrow[r] \arrow[d] & \Sigma^2 \arrow[d] \\
\Sigma^1\vee\Sigma^3 \arrow[r, "i\eta\vee\tilde{\eta}\eta"'] & SZ/2m \arrow[r] & MQ_{2m} \arrow[r] \arrow[d] & \Sigma^2\vee\Sigma^4 \arrow[d] \\
 & & \Sigma^4 \arrow[r, equal] & \Sigma^4
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \Sigma^1 \arrow[r, equal] \arrow[d, "i_Q i\eta"'] & \Sigma^1 \arrow[d, "0"] \\
\Sigma^3 \arrow[r, "\tilde{\eta}\eta"] \arrow[d] & SZ/2m \arrow[r] \arrow[d, equal] & Q_{2m} \arrow[r] \arrow[d] & \Sigma^4 \arrow[d] \\
\Sigma^1\vee\Sigma^3 \arrow[r, "i\eta\vee\tilde{\eta}\eta"'] & SZ/2m \arrow[r] & MQ_{2m} \arrow[r] \arrow[d] & \Sigma^2\vee\Sigma^4 \arrow[d] \\
 & & \Sigma^2 \arrow[r, equal] & \Sigma^2
\end{tikzcd}$$

involving cofiber sequences. Evidently $KU_0MQ_{2m}\cong KU_0(\Sigma^2\vee\Sigma^4)\oplus KU_0SZ/2m \cong Z\oplus Z\oplus Z/2m$ and $KU_1MQ_{2m}=0$. In order to observe the behaviour of t_* on KU_0MQ_{2m} we use the three split short exact sequences $0\to KU_0SZ/2m\to KU_0MQ_{2m}\to KU_0(\Sigma^2\vee\Sigma^4)\to 0$, $0\to KU_0M_{2m}\to KU_0MQ_{2m}\to KU_0\Sigma^4\to 0$ and $0\to KU_0Q_{2m}\to KU_0MQ_{2m}\to KU_0\Sigma^2\to 0$. Since [12, Proposition 4.1] says that $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ on $KU_0M_{2m}\cong Z\oplus Z/2m$ and $t_*=\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}$ on $KU_0Q_{2m}\cong Z\oplus Z/2m$, we can easily verify that $t_*=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 1 & m & 1\end{pmatrix}$ on $KU_0MQ_{2m}\cong Z\oplus Z\oplus Z/2m$ as desired.

ii) The $W=NP'_{2m}$ case: Consider the two commutative diagrams

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d, "\iota_2"] & \Sigma^0 \arrow[d] & \\
\Sigma^1SZ/2m \arrow[r, "{(\eta^2j,\ \bar{\eta})}"] \arrow[d, equal] & \Sigma^0\vee\Sigma^0 \arrow[r] \arrow[d, "\pi_1"] & NP'_{2m} \arrow[r] \arrow[d] & \Sigma^2SZ/2m \arrow[d, equal] \\
\Sigma^1SZ/2m \arrow[r, "\eta^2j"'] & \Sigma^0 \arrow[r] \arrow[d, "0"] & N'_{2m} \arrow[r] \arrow[d, "\bar{\eta}j'_N"] & \Sigma^2SZ/2m \\
 & \Sigma^1 \arrow[r, equal] & \Sigma^1 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d, "\iota_1"] & \Sigma^0 \arrow[d] & \\
\Sigma^1SZ/2m \arrow[r, "{(\eta^2j,\ \bar{\eta})}"] \arrow[d, equal] & \Sigma^0\vee\Sigma^0 \arrow[r] \arrow[d, "\pi_2"] & NP'_{2m} \arrow[r] \arrow[d] & \Sigma^2SZ/2m \arrow[d, equal] \\
\Sigma^1SZ/2m \arrow[r, "\bar{\eta}"'] & \Sigma^0 \arrow[r] \arrow[d, "0"] & P'_{2m} \arrow[r] \arrow[d, "\eta^2jj'_P"] & \Sigma^2SZ/2m \\
 & \Sigma^1 \arrow[r, equal] & \Sigma^1 &
\end{tikzcd}$$

involving cofiber sequnces, where ι_k: $\Sigma^0\to\Sigma^0\vee\Sigma^0$ and π_k: $\Sigma^0\vee\Sigma^0\to\Sigma^0 (k=1, 2)$ denote the k-th injection and projection respectively. We can easily see that the short exact sequence $0\to KU_0\Sigma^0\to KU_0NP'_{2m}\to KU_0P'_{2m}\to 0$ is split, by using the following commutative diagram

$$\begin{array}{ccccccc}
 & & \Sigma^0 & = & \Sigma^0 & & \\
 & & \downarrow \iota_1 & & \downarrow & & \\
\Sigma^1 SZ/2m & \xrightarrow{(\eta^2 j,\, \bar{\eta})} & \Sigma^0\vee\Sigma^0 & \to & NP'_{2m} & \to & \Sigma^2 SZ/2m \\
\| & & \downarrow \pi_1 & & \downarrow & & \| \\
\Sigma^1 SZ/2m & \longrightarrow & \Sigma^0 & \to & N'_{2m} & \to & \Sigma^2 SZ/2m \\
\downarrow j & & \| & & \downarrow & & \downarrow j \\
\Sigma^2 & \xrightarrow[\eta^2]{} & \Sigma^0 & \to & Q & \to & \Sigma^3
\end{array}$$

with $\pi_1\iota_1=1$. Thus $KU_0NP'_{2m}\cong KU_0\Sigma^0\oplus KU_0P'_{2m}\cong Z\oplus Z\oplus Z/m$ and $KU_1NP'_{2m}=0$. Since $t_*=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ on $KU_0P'_{2m}\cong Z\oplus Z/m$ by means of [12, Proposition 4.1]), it follows immediately that $t_*=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & -1\end{pmatrix}$ on $KU_0NP'_{2m}\cong Z\oplus Z\oplus Z/m$ as desired.

iii) The $W=NR'_{2m}$ case: Use the commutative diagram

$$\begin{array}{ccccccc}
 & & \Sigma^1 Q\vee Q & = & \Sigma^1 Q\vee Q & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^3 SZ/2m & \xrightarrow{(j,\, \bar{\eta})} & \Sigma^4\vee\Sigma^2 & \to & P & \to & \Sigma^4 SZ/2m \\
\| & & \downarrow \eta^2\vee\eta^2 & & \downarrow & & \| \\
\Sigma^3 SZ/2m & \xrightarrow[(\eta^2 j,\, \eta^2\bar{\eta})]{} & \Sigma^2\vee\Sigma^0 & \to & NR'_{2m} & \to & \Sigma^4 SZ/2m \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma^2 Q\vee Q & = & \Sigma^2 Q\vee Q & &
\end{array}$$

involving cofiber sequences, in which the upper row becomes a cofiber sequence by means of Lemma 1.1 ii). Then we can easily see that the short exact sequence $0\to KU_0(\Sigma^2\vee\Sigma^0)\to KU_0NR'_{2m}\to KU_0\Sigma^4 SZ/2m\to 0$ is split, and $KU_1NR'_{2m}=0$. Hence it is immediate that $KU_0NR'_{2m}\cong KU_0(\Sigma^2\vee\Sigma^0)\oplus KU_0\Sigma^4 SZ/2m\cong Z\oplus Z\oplus Z/2m$ on which $t_*=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

We will next compute the KO homologies of the above mentioned spectra $W=XY_{2m}$ and XY'_{2m}, by making use of the results in [12, Proposition 4.2].

Proposition 1.3. *The KO homologies KO_iW are tabled as follows*:

$i\equiv$	0	1	2	3	$i\equiv$	0	1	2	3
MP_{2m}	$Z/2m$	0	Z	Z	MP'_{2m}	Z	Z	$Z/2m$	0
MQ_{2m}	$Z\oplus Z/2m$	0	$Z\oplus Z/2$	0	MQ'_{2m}	Z	$Z/2$	Z	$Z/2m$
MR_{2m}	$Z/2m$	Z	$Z\oplus Z/2$	$Z/2$	MR'_{2m}	$Z\oplus Z/2m$	$Z/2$	$Z/2$	Z
NP_{2m}	$Z/2m$	$Z/2$	0	$Z\oplus Z$	NP'_{2m}	$Z\oplus Z$	$Z/2$	$Z/2m$	0

NQ_{2m}	$Z\oplus Z/2m$	$Z/2$	$Z/2$	Z	NQ'_{2m}	Z	$Z\oplus Z/2$	$Z/2$	$Z/2m$
NR_{2m}	$Z/2m$	$Z\oplus Z/2$	$Z/2$	$Z\oplus Z/2$	NR'_{2m}	$Z\oplus Z/2m$	$Z/2$	$Z\oplus Z/2$	$Z/2$

in which $\equiv$ *stands for the congruence modulo* 4.

Proof. We have computed KO_*MP_{2m} in [14, Proposition 1.2 ii)]. In the other cases we can similarly compute KO_*W, by using the long exact sequences of KO homologies induced by the cofiber seqeunces as appeared in the proof of Proposition 1.2. In computing KO_*W we may moreover apply the universal coefficient sequence $0\to \mathrm{Ext}(KO_{3-*}DW, Z)\to KO_* W\to \mathrm{Hom}(KO_{4-*}DW, Z)\to 0$ (see [11]) combined with (1.3).

2. Some elementary spectra $Y'X_{2m}$ with four cells

2.1. Let X_{2m}, Y'_{2m} denote the cofibers of maps $f\colon \Sigma^i\to SZ/2m$, $g\colon \Sigma^j SZ/2m\to\Sigma^0$ respectively. If the composite $gf\colon \Sigma^{i+j}\to\Sigma^0$ is trivial, then there exists a coextension $h\colon \Sigma^{i+j+1}\to Y'_{2m}$ of f and an extension $k\colon \Sigma^j X_{2m}\to\Sigma^0$ of g so that the following diagram is commutative

$$\begin{array}{ccccccc}
 & & \Sigma^{i+j+1} & = & \Sigma^{i+j+1} & & \\
 & & \downarrow h & & \downarrow f & & \\
\Sigma^0 & \to & Y'_{2m} & \to & \Sigma^{j+1}SZ/2m & \xrightarrow{g} & \Sigma^1 \\
\| & & \downarrow & & \downarrow & k & \| \\
\Sigma^0 & \to & C_{h,k} & \to & \Sigma^{j+1}X_{2m} & \to & \Sigma^1 \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma^{i+j+2} & = & \Sigma^{i+j+2} & &
\end{array}$$

with four cofiber sequences. Here the maps h and k are dependent on each other so that their cofibers coincide. We will here choose suitable pairs (h, k) to construct some elementary spectra $Y'X_{2m}=C_{h,k}$.

There exist maps

$$\begin{aligned}
(2.1)\quad & k_M\colon M_{2m}\to\Sigma^0,\quad k_R\colon R_{2m}\to\Sigma^0,\quad \bar{k}_Q\colon \Sigma^1 Q_{2m}\to\Sigma^0,\quad \bar{k}_N\colon \Sigma^2 N_{2m}\to\Sigma^0 \\
& h'_M\colon \Sigma^1\to M'_{2m},\quad h'_R\colon \Sigma^5\to R'_{2m},\quad \tilde{h}_Q\colon \Sigma^5\to Q'_{2m},\quad \tilde{h}_N\colon \Sigma^5\to N'_{2m}
\end{aligned}$$

such that $k_M i_M=j\colon SZ/2m\to\Sigma^1$, $k_R i_R=\eta j\colon SZ/2m\to\Sigma^0$, $\bar{k}_Q i_Q=\bar{\eta}\colon \Sigma^1 SZ/2m\to\Sigma^0$, $\bar{k}_N i_N=\eta\bar{\eta}\colon \Sigma^2 SZ/2m\to\Sigma^0$, $j'_M h'_M=i\colon \Sigma^0\to SZ2/m$, $j'_R h'_R=i\eta\colon \Sigma^1\to SZ/2m$, $j'_Q\tilde{h}_Q=\tilde{\eta}\colon \Sigma^2\to SZ/2m$ and $j'_N\tilde{h}_N=\tilde{\eta}\eta\colon \Sigma^3\to SZ/2m$. Such maps k_R, $\bar{k}_Q$, $\bar{k}_N$, h'_R, $\tilde{h}_Q$ and $\tilde{h}_N$ are uniquely chosen, and moreover the composites ηk_M and $h'_M\eta$ are also determined uniquely although k_M and h'_M are not so.

Let X_{2m}, Y_{2m} be the cofibers of maps $f\colon \Sigma^i\to SZ/2m, f\eta\colon \Sigma^{i+1}\to SZ/2m$, and Y'_{2m}, X'_{2m} the cofibers of maps $g\colon \Sigma^j SZ/2m\to\Sigma^0$, $\eta g\colon \Sigma^{j+1}SZ/2m\to\Sigma^0$ respectively. Then there exist maps $\lambda_{X,Y}\colon \Sigma^1 X_{2m}\to Y_{2m}$, $\rho_{Y,X}\colon Y_{2m}\to X_{2m}$ and dually $\lambda'_{X,Y}\colon \Sigma^1 Y'_{2m}\to X'_{2m}$, $\rho'_{X,Y}\colon X'_{2m}\to Y'_{2m}$ related by the following commutative diagrams:

$$\begin{tikzcd}
\Sigma^{i+1} \arrow[r, "f"] \arrow[d, equal] & \Sigma^1 SZ/2m \arrow[r] \arrow[d, "\eta"] & \Sigma^1 X_{2m} \arrow[r] \arrow[d, "\lambda_{X,Y}"] & \Sigma^{i+2} \arrow[d, equal] \\
\Sigma^{i+1} \arrow[r, "f\eta"] \arrow[d, "\eta"] & SZ/2m \arrow[r] \arrow[d, equal] & Y_{2m} \arrow[r] \arrow[d, "\rho_{Y,X}"] & \Sigma^{i+2} \arrow[d, "\eta"] \\
\Sigma^{i} \arrow[r, "f"'] & SZ/2m \arrow[r] & X_{2m} \arrow[r] & \Sigma^{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{j+1} SZ/2m \arrow[r, "g"] \arrow[d, equal] & \Sigma^1 \arrow[r] \arrow[d, "\eta"] & \Sigma^1 Y'_{2m} \arrow[r] \arrow[d, "\lambda'_{Y,X}"] & \Sigma^{j+2} SZ/2m \arrow[d, equal] \\
\Sigma^{j+1} SZ/2m \arrow[r, "\eta g"] \arrow[d, "\eta"] & \Sigma^0 \arrow[r] \arrow[d, equal] & X'_{2m} \arrow[r] \arrow[d, "\rho'_{X,Y}"] & \Sigma^{j+2} SZ/2m \arrow[d, "\eta"] \\
\Sigma^{j} SZ/2m \arrow[r, "g"'] & \Sigma^0 \arrow[r] & Y'_{2m} \arrow[r] & \Sigma^{j} SZ/2m
\end{tikzcd}$$

By composing the maps chosen in (2.1) with the above maps we set

$$(2.2)\qquad \begin{array}{ll}
k_N = k_M \rho_{N,M}\colon N_{2m} \to \Sigma^1 & h'_N = \lambda'_{M,N} h'_M\colon \Sigma^2 \to N'_{2m} \\
k_Q = k_R \lambda_{Q,R}\colon \Sigma^1 Q_{2m} \to \Sigma^0 & h'_Q = \rho'_{R,Q} h'_R\colon \Sigma^5 \to Q'_{2m} \\
\bar{k}_R = \bar{k}_Q \rho_{R,Q}\colon \Sigma^1 R_{2m} \to \Sigma^0 & \tilde{h}_R = \lambda'_{Q,R} \tilde{h}_Q\colon \Sigma^6 \to R'_{2m} \\
\bar{k}_P = \bar{k}_Q \lambda_{P,Q}\colon \Sigma^2 P_{2m} \to \Sigma^0 & \tilde{h}_P = \rho'_{Q,P} \tilde{h}_Q\colon \Sigma^5 \to P'_{2m} \\
\bar{k}_M = \bar{k}_N \lambda_{M,N}\colon \Sigma^3 M_{2m} \to \Sigma^0 & \tilde{h}_M = \rho'_{N,M} \tilde{h}_N\colon \Sigma^5 \to M'_{2m}\,.
\end{array}$$

These maps satisfy the following equalities respectively:

$$(2.3)\qquad \begin{array}{l}
k_N i_N = j\,, \quad k_Q i_Q = \eta^2 j\,, \quad \bar{k}_R i_R = \bar{\eta}\,, \quad \bar{k}_P i_P = \eta\bar{\eta}\,, \quad \bar{k}_M i_M = \eta^2\bar{\eta}\,, \\
j'_N h'_N = i\,, \quad j'_Q h'_Q = i\eta^2\,, \quad j'_R \tilde{h}_R = \tilde{\eta}\,, \quad j'_P \tilde{h}_P = \tilde{\eta}\eta\,, \quad j'_M \tilde{h}_M = \tilde{\eta}\eta^2\,.
\end{array}$$

Note that such maps k_Q, $\bar{k}_P$, $\bar{k}_M$, h'_Q, $\tilde{h}_P$ and $\tilde{h}_M$ are uniquely determined, and moreover the composites $\eta^2 k_N$ and $h'_N \eta^2$ are so, too.

Using suitable pairs (h, k) consisting of maps chosen in (2.1) and (2.2), we can construct some elementary spectra $Y'X_{2m} = C_{h,k}$ taken to be the cofiber of the two maps h, k as follows:

(2.4)

$Y'X_{2m}$	$h : \Sigma^{i+j+1} \to Y'_{2m}$	$k : \Sigma^j X_{2m} \to \Sigma^0$
$M'M_{2m}$	$h'_M\eta : \Sigma^2 \to M'_{2m}$	$\eta k_M : M_{2m} \to \Sigma^0$
$M'N_{2m}$	$h'_M\eta^2 : \Sigma^3 \to M'_{2m}$	$\eta k_N : N_{2m} \to \Sigma^0$
$N'M_{2m}$	$h'_N\eta : \Sigma^3 \to N'_{2m}$	$\eta^2 k_M : \Sigma^1 M_{2m} \to \Sigma^0$
$N'N_{2m}$	$h'_N\eta^2 : \Sigma^4 \to N'_{2m}$	$\eta^2 k_N : \Sigma^1 N_{2m} \to \Sigma^0$
$P'Q_{2m}$	$\tilde{h}_P : \Sigma^5 \to P'_{2m}$	$\bar{k}_Q : \Sigma^1 Q_{2m} \to \Sigma^0$
$P'R_{2m}$	$\tilde{h}_P\eta : \Sigma^6 \to P'_{2m}$	$\bar{k}_R : \Sigma^1 R_{2m} \to \Sigma^0$
$Q'P_{2m}$	$\tilde{h}_Q : \Sigma^5 \to Q'_{2m}$	$\bar{k}_P : \Sigma^2 P_{2m} \to \Sigma^0$
$Q'Q_{2m}$	$\tilde{h}_Q\eta : \Sigma^6 \to Q'_{2m}$	$\eta\bar{k}_Q : \Sigma^2 Q_{2m} \to \Sigma^0$
$Q'R_{2m}$	$\tilde{h}_Q\eta^2 : \Sigma^7 \to Q'_{2m}$	$\eta\bar{k}_R : \Sigma^2 R_{2m} \to \Sigma^0$
$R'P_{2m}$	$\tilde{h}_R : \Sigma^6 \to R'_{2m}$	$\eta\bar{k}_P : \Sigma^3 P_{2m} \to \Sigma^0$
$R'Q_{2m}$	$\tilde{h}_R\eta : \Sigma^7 \to R'_{2m}$	$\eta^2\bar{k}_Q : \Sigma^3 Q_{2m} \to \Sigma^0$
$R'R_{2m}$	$\tilde{h}_R\eta^2 : \Sigma^8 \to R'_{2m}$	$\eta^2\bar{k}_R : \Sigma^3 R_{2m} \to \Sigma^0$
$M'R_{2m}$	$\tilde{h}_M : \Sigma^5 \to M'_{2m}$	$k_R : R_{2m} \to \Sigma^0$
$N'Q_{2m}$	$\tilde{h}_N : \Sigma^5 \to N'_{2m}$	$k_Q : \Sigma^1 Q_{2m} \to \Sigma^0$
$N'R_{2m}$	$\tilde{h}_N\eta : \Sigma^6 \to N'_{2m}$	$\eta k_R : \Sigma^1 R_{2m} \to \Sigma^0$
$Q'N_{2m}$	$h'_Q : \Sigma^5 \to Q'_{2m}$	$\bar{k}_N : \Sigma^2 N_{2m} \to \Sigma_0$
$R'M_{2m}$	$h'_R : \Sigma^5 \to R'_{2m}$	$\bar{k}_M : \Sigma^3 M_{2m} \to \Sigma^0$
$R'N_{2m}$	$h'_R\eta : \Sigma^6 \to R'_{2m}$	$\eta\bar{k}_N : \Sigma^3 N_{2m} \to \Sigma^0$

For all of these elementary spectra we notice that

$$Y'X_{2m} = \Sigma^{i+j+2}D(X'Y_{2m}) \tag{2.5}$$

where DW stands for the Spanier-Whitehead dual of W.

2.2. Consider the cofiber sequence $\Sigma^2 \xrightarrow{\eta^2} \Sigma^0 \xrightarrow{i_Q} Q \xrightarrow{j_Q} \Sigma^3$. Then the square η^2 has a unique coextension $\tilde{\xi}\colon \Sigma^5 \to Q$ and a unique extension $\xi\colon \Sigma^2 Q \to \Sigma^0$ satisfying $j_Q\tilde{\xi}=\eta^2$ and $\xi i_Q=\eta^2$. Denote by QQ the cofiber of $\tilde{\xi}$ which coincides with the cofiber of ξ. Then we have

Lemma 2.1. i) *$KU_0QQ \cong Z\oplus Z$ on which $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$, and $KU_1QQ\cong Z$ on which $t_*=-1$.*

ii) *$KO_iQQ\cong Z\oplus Z/2$, $Z/2$, Z, Z, Z, 0, Z, Z according as $i=0, 1, \cdots, 7$.*

Proof. Use the following commutative diagram

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d] & \Sigma^0 \arrow[d] & \\
\Sigma^5 \arrow[r, "\tilde{\xi}"] \arrow[d, equal] & Q \arrow[r] \arrow[d] & QQ \arrow[r, "j_{QQ}"] \arrow[d] & \Sigma^6 \arrow[d, equal] \\
\Sigma^5 \arrow[r] \arrow[dr, "\eta^2"'] & \Sigma^3 \arrow[r] \arrow[d, "\eta^2"] & \Sigma^3 Q \arrow[r] \arrow[d, "\xi"] & \Sigma^6 \\
 & \Sigma^1 \arrow[r] & \Sigma^1 &
\end{tikzcd}$$

involving four cofiber sequences. Then it is obvious that $KU_0QQ\cong KU_0\Sigma^6\oplus KU_0Q\cong Z\oplus Z$ and $KU_1QQ\cong KU_1Q\cong Z$. Moreover KO_iQQ are easily computed except $i=0$ and 1. On the other hand, the Bott cofiber sequence induces two exact sequences $0\to KO_3QQ\to KU_3QQ\to KO_1QQ\to 0$ and $0\to KU_1QQ\to KO_7QQ\to KO_0QQ\to KU_0QQ\to KO_6QQ\to 0$. Since the above monomorphisms are both multiplications by 2 on Z, we can also determine KO_iQQ ($i=0$, 1) immediately.

We next consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
 & KU_4Q \arrow[d] \arrow[r] & KU_4QQ \arrow[dl, "\varepsilon_{0*}"] \arrow[dr, "j_{QQ*}"'] & KO_6QQ \arrow[l] \arrow[d] \arrow[ur, leftarrow] & \\
 & KO_4QQ \arrow[dl] & & KU_4\Sigma^6 \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

with exact diagonals. Here the two vertical arrows are both multiplications by 2 on Z. As in [12, (2.3)] we can easily observe that $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ on $KU_4QQ\cong KU_4\Sigma^6\oplus KU_4Q\cong Z\oplus Z$ by replacing suitably the splitting of j_{QQ*} if necessary.

On the other hand, it is obvious that $t_*=-1$ on $KU_1QQ\cong KU_1Q\cong Z$.

Combining Lemma 2.1 with Theorem 1 we get

Corollary 2.2. $QQ\underset{KO}{\simeq}P\vee\Sigma^7$

Choose two maps $\lambda_Q\colon Q_{2m}\to\Sigma^1Q$, $\rho_Q\colon Q\to Q'_{2m}$ making the diagram below commutative

$$\begin{array}{ccccccc} \Sigma^3 & \xrightarrow{\tilde{\eta}\eta} & SZ/2m & \to & Q_{2m} & \to & \Sigma^4 \\ \| & & \downarrow j & & \downarrow \lambda_Q & & \| \\ \Sigma^3 & \xrightarrow[\eta^2]{} & \Sigma^1 & \to & \Sigma^1Q & \to & \Sigma^4 \end{array} \qquad \begin{array}{ccccccc} \Sigma^2 & \xrightarrow{\eta^2} & \Sigma^0 & \to & Q & \to & \Sigma^3 \\ \downarrow i & & \| & & \downarrow \rho_Q & & \downarrow \\ \Sigma^2SZ/2m & \xrightarrow[\eta\bar{\eta}]{} & \Sigma^0 & \to & Q'_{2m} & \to & \Sigma^3SZ/2m\,. \end{array}$$

Then the following equalities hold:

$$\xi\lambda_Q=k_Q\colon\Sigma^1Q_{2m}\to\Sigma^0\,,\quad \rho_Q\tilde{\xi}=h'_Q\colon\Sigma^5\to Q'_{2m}\,. \tag{2.6}$$

2.3. We will now compute the KU homologies of the elementary spectra $W=Y'X_{2m}$ with four cells mentioned in (2.4).

Proposition 2.3. *The KU homologies KU_0W, KU_1W and the conjugation t_* on them are given as follows:*

W =	$M'M_{2m}$	$M'N_{2m}$	$N'M_{2m}$	$N'N_{2m}$	$P'Q_{2m}$
$KU_0W\cong$	Z	$Z\oplus Z$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z/2m$	$Z\oplus Z\oplus Z/m$
t_* =	1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\1&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\1&1&0\\0&0&-1\end{pmatrix}$ m : odd, $\begin{pmatrix}-1&0&0\\0&1&0\\m/2&1&-1\end{pmatrix}$ m : even
$KU_1W\cong$	$Z\oplus Z/2m$	$Z/2m$	0	Z	0
t_* =	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	1		1	

W =	$P'R_{2m}$	$Q'P_{2m}$	$Q'Q_{2m}$	$Q'R_{2m}$	$R'P_{2m}$	$R'Q_{2m}$
$KU_0W\cong$	$Z\oplus Z/m$	$Z\oplus Z$	Z	$Z\oplus Z$	$Z\oplus Z/m$	$Z\oplus Z\oplus Z/2m$
t_* =	$\begin{pmatrix}1&0\\1&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$ m : odd, $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ m : even	1	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\m&0&1\end{pmatrix}$
$KU_1W\cong$	Z	Z/m	$Z\oplus Z/2m$	$Z/2m$	Z	0
t_* =	-1	-1	$\begin{pmatrix}-1&0\\m&-1\end{pmatrix}$	-1	-1	

W =	$R'R_{2m}$	$M'R_{2m}$	$N'Q_{2m}$	$N'R_{2m}$	$Q'N_{2m}$	$R'M_{2m}$	$R'N_{2m}$
$KU_0W\cong$	$Z\oplus Z/2m$	$Z\oplus Z$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z/2m$	$Z\oplus Z$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z/2m$
t_* =	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\1&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\1&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$KU_1W\cong$	Z	$Z/2m$	0	Z	$Z/2m$	0	Z
t_* =	1	1		-1	-1		-1

where the matrices behave as left action on abelian groups.

Proof. By making use of [12, Propositions 4.1 and 4.2] we will investigate the behaviour of the conjugation t_* on KU_*W when $W=N'M_{2m}$, $P'Q_{2m}$, $Q'P_{2m}$, $R'Q_{2m}$, $M'R_{2m}$, $N'Q_{2m}$, $Q'N_{2m}$ and $R'M_{2m}$, the other cases being easy. Denote by t_W the conjugation t_* on KU_*W for convenience sake.

i) The $W=N'M_{2m}$ case: Use the commutative diagram

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d] & \Sigma^0 \arrow[d] & \\
\Sigma^3 \arrow[r, "h'_N\eta"] \arrow[d, equal] & N'_{2m} \arrow[r] \arrow[d] & N'M_{2m} \arrow[r] \arrow[d] & \Sigma^4 \arrow[d, equal] \\
\Sigma^3 \arrow[r, "i\eta"'] & \Sigma^2SZ/2m \arrow[r] \arrow[d, "\eta^2 j"] & \Sigma^2M_{2m} \arrow[r] \arrow[d, "\eta^2 k_M"] & \Sigma^4 \\
 & \Sigma^1 \arrow[r, equal] & \Sigma^1 &
\end{tikzcd}$$

involving four cofiber sequences. Evidently $KU_0N'M_{2m}\cong KU_0\Sigma^4\oplus KU_0N'_{2m}\cong Z\oplus Z\oplus Z/2m$ and $KU_1N'M_{2m}=0$. Set $t_{N'M}=\begin{pmatrix}1 & 0 & 0\\ a & 1 & 0\\ b & 0 & -1\end{pmatrix}$ on $KU_0N'M_{2m}\cong Z\oplus Z\oplus Z/2m$ for some integers a, b because $t_{N'}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ on $KU_0N'_{2m}\cong Z\oplus Z/2m$. Since $t_M=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ on $KU_{-2}M_{2m}\cong Z\oplus Z/2m$, we may take to be $b=1$. On the other hand, the equality $t^2_{N'M}=1$ implies that $a=0$. Thus $t_{N'M}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & -1\end{pmatrix}$ as desired.

ii) The $W=P'Q_{2m}$ case: Use the commutative diagram

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d] & \Sigma^0 \arrow[d] & \\
\Sigma^5 \arrow[r, "\tilde{h}_P"] \arrow[d, equal] & P'_{2m} \arrow[r] \arrow[d, "j'_P"] & P'Q_{2m} \arrow[r] \arrow[d, "j_{P'Q,Q}"] & \Sigma^6 \arrow[d, equal] \\
\Sigma^5 \arrow[r, "\tilde{\eta}\eta"'] & \Sigma^2SZ/2m \arrow[r] \arrow[d, "\bar{\eta}"] & \Sigma^2Q_{2m} \arrow[r] \arrow[d, "\bar{k}_Q"] & \Sigma^6 \\
 & \Sigma^1 \arrow[r, equal] & \Sigma^1 &
\end{tikzcd}$$

involving four cofiber sequences. Evidently $KU_0P'Q_{2m}\cong KU_0\Sigma^6\oplus KU_0P'_{2m}\cong Z\oplus Z\oplus Z/m$ and $KU_1P'Q_{2m}=0$. The induced homomorphism $j_{P'Q,Q*}\colon KU_0P'Q_{2m}\to KU_{-2}Q_{2m}$ may be expressed by the matrix $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -2\end{pmatrix}\colon Z\oplus Z\oplus Z/m\to Z\oplus Z/2m$ since $j'_{P*}\colon KU_0P'_{2m}\to KU_{-2}SZ/2m$ is given by the row $(1\ -2)\colon Z\oplus Z/m\to Z/2m$. Set $t_{P'Q}=\begin{pmatrix}-1 & 0 & 0\\ a & 1 & 0\\ b & 1 & -1\end{pmatrix}$ on $KU_0P'Q_{2m}\cong Z\oplus Z\oplus Z/m$ for some integers a, b. Recall that $t_Q=\begin{pmatrix}-1 & 0\\ m & -1\end{pmatrix}$ on $KU_{-2}Q_{2m}$. Then the equality $j_{P'Q,Q*}t_{P'Q}=$

$t_Q j_{P'Q,Q*}$ implies that $a-2b\equiv m \bmod 2m$, thus $a\equiv m \bmod 2$. So we may take to be $(a, b)=(1, m+1/2)$ or $(0, m/2)$ according as m is odd or even. Since the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ m+1/2 & 1 & -1 \end{pmatrix}$ is congruent to $\begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, the result is immediate.

iii) The $W=Q'P_{2m}$ case: Use the commutative diagram

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r] \arrow[d] & \Sigma^0 \arrow[d] & \\
\Sigma^5 \arrow[r, "\tilde{h}_Q"] \arrow[d, equal] & Q'_{2m} \arrow[r] \arrow[d] & Q'P_{2m} \arrow[r] \arrow[d, "j_{Q'P,P}"] & \Sigma^6 \arrow[d, equal] \\
\Sigma^5 \arrow[r, "\tilde{\eta}"'] & \Sigma^3 SZ/2m \arrow[r] \arrow[d, "\eta\bar{\eta}"] & \Sigma^3 P_{2m} \arrow[r] \arrow[d, "\bar{k}_P"] & \Sigma^6 \\
 & \Sigma^1 \arrow[r, equal] & \Sigma^1 &
\end{tikzcd}$$

involving four cofiber sequences. It follows immediately that $KU_0Q'P_{2m}\cong KU_{-3}P_{2m}\oplus KU_0\Sigma^0$ on which $t_*=\begin{pmatrix} -1 & 0 \\ a & 1 \end{pmatrix}$ for some integer a, and $KU_1Q'P_{2m}\cong KU_{-2}P_{2m}\cong Z/m$ on which $t_*=-1$. We will show that the integer a may be taken to be 1 or 0 according as m is odd or even.

We will first compute the KO homologies $KO_iQ'P_{2m}$. By using the above commutative diagram it is easily checked that $KO_{2j}Q'P_{2m}\cong Z$, $KO_3Q'P_{2m}\cong KO_7Q'P_{2m}\cong Z/m$ and $KO_5Q'P_{2m}\cong Z/m\otimes Z/2$. In order to determine the remainder $KO_1Q'P_{2m}$ we consider the exact sequence $KO_3Q'P_{2m}\to KU_3Q'P_{2m}\to KO_1Q'P_{2m}\to 0$ induced by the Bott cofiber sequence. Since there exists a short exact sequence $0\to KO_3Q'_{2m}\to KU_3Q'_{2m}\to KO_1Q'_{2m}\to 0$, it is easily seen that $KO_1Q'P_{2m}\cong Z/m\otimes Z/2$.

We next use the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & 0 \arrow[dl] & \\
 & & KU_0\Sigma^0 \arrow[dr] \arrow[dd] & & KO_2Q'P_{2m} \arrow[dl] \arrow[dd] & & \\
 & & & KU_0Q'P_{2m} \arrow[dl, "\varepsilon_{0*}"'] \arrow[dr, "j_{Q'P,P*}"] & & & \\
 & & KO_0Q'P_{2m} \arrow[dl] & & KU_{-3}P_{2m} \arrow[dr] & & \\
 & KO_1Q'P_{2m} \arrow[dl] & & & & 0 & \\
0 & & & & & &
\end{tikzcd}$$

with exact diagonals. Here the left vertical arrow is just multiplication by 2 on Z, and the right one is multiplication by 2 or 1 on Z according as m is odd or even. By a parallel discussion to [12, (2.3)] it is easily observed that a is odd or even according as m is odd or even. Therefore we may take a to be 1 or 0 according as m is odd or even, by replacing suitably the splitting of $j_{Q'P,P*}$ if necessary. Thus $t_{Q'P}=\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ on $KU_0Q'P_{2m}\cong Z\oplus Z$ according as m is odd or even.

iv) The $W=R'Q_{2m}$ case is shown similarly to the case i).

v) The $W=N'Q_{2m}$ case: We have the following commutative diagram

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d] & \Sigma^0 \arrow[d] & \\
\Sigma^0 \arrow[r] \arrow[d, equal] & N'Q_{2m} \arrow[r] \arrow[d, "\lambda_{N'Q}"] & \Sigma^2 Q_{2m} \arrow[r, "k_Q"] \arrow[d, "\lambda_Q"] & \Sigma^1 \arrow[d, equal] \\
\Sigma^0 \arrow[r] & QQ \arrow[r] \arrow[d] & \Sigma^3 Q \arrow[r, "\xi"'] \arrow[d] & \Sigma^1 \\
 & \Sigma^1 \arrow[r, equal] & \Sigma^1 &
\end{tikzcd}$$

involving four cofiber sequences, because of (2.6). Evidently $KU_0 N'Q_{2m} \cong KU_{-2}Q_{2m} \oplus KU_0 \Sigma^0 \cong Z \oplus Z/2m \oplus Z$ and $KU_1 N'Q_{2m}=0$. Set $t_{N'Q}=\begin{pmatrix} -1 & 0 & 0 \\ m & -1 & 0 \\ a & 0 & 1 \end{pmatrix}$ on $KU_0 N'Q_{2m} \cong Z \oplus Z/2m \oplus Z$ for some integer a. Then the equality $\lambda_{N'Q*} t_{N'Q} = t_{QQ} \lambda_{N'Q*}$ implies that $a=1$ because $t_{QQ}=\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ on $KU_0 QQ$ by Lemma 2.1. Since the matrix $\begin{pmatrix} -1 & 0 & 0 \\ m & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is congruent to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, the result is immediate.

vi) The $W=M'R_{2m}$ case: Consider the commutative diagram

$$\begin{tikzcd}
 & SZ/2m \wedge P \arrow[r, equal] \arrow[d] & SZ/2m \wedge P \arrow[d] & \\
\Sigma^5 \arrow[r, "\tilde{h}_N"] \arrow[d, equal] & N'_{2m} \arrow[r] \arrow[d, "\rho'_{N,M}"] & N'Q_{2m} \arrow[r] \arrow[d] & \Sigma^6 \arrow[d, equal] \\
\Sigma^5 \arrow[r, "\tilde{h}_M"'] & M'_{2m} \arrow[r] \arrow[d] & M'R_{2m} \arrow[r] \arrow[d] & \Sigma^6 \\
 & \Sigma^1 SZ/2m \wedge P \arrow[r, equal] & \Sigma^1 SZ/2m \wedge P &
\end{tikzcd}$$

involving four cofiber sequences. Evidently $KU_0 M'R_{2m} \cong KU_0 \Sigma^6 \oplus KU_0 M'_{2m} \cong Z \oplus Z$ and $KU_1 M'R_{2m} \cong KU_1 M'_{2m} \cong Z/2m$. Since $t_{N'Q} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ on $KU_0 N'Q_{2m} \cong Z \oplus Z \oplus Z/2m$, it is easily seen that $t_{M'R} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ on $KU_0 M'R_{2m} \cong Z \oplus Z$. Hence the result follows.

vii) The $W=Q'N_{2m}$ case: We have the following commutative diagram

$$\begin{tikzcd}
 & \Sigma^3 \arrow[r, equal] \arrow[d] & \Sigma^3 \arrow[d] & \\
\Sigma^3 \arrow[r, "\xi"] \arrow[d, equal] & Q \arrow[r] \arrow[d, "\rho_Q"] & QQ \arrow[r] \arrow[d, "\rho_{Q'N}"] & \Sigma^6 \arrow[d, equal] \\
\Sigma^3 \arrow[r, "h'_Q"'] & Q'_{2m} \arrow[r] \arrow[d] & Q'N_{2m} \arrow[r] \arrow[d] & \Sigma^6 \\
 & \Sigma^4 \arrow[r, equal] & \Sigma^4 &
\end{tikzcd}$$

involving four cofiber sequences, because of (2.6). Then it is easily obtained that $KU_0Q'N_{2m}\cong KU_0QQ\cong Z\oplus Z$ on which $t_*=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$, and $KU_1Q'N_{2m}\cong KU_1Q'_{2m}\cong Z/2m$ on which $t_*=-1$.

viii) The $W=R'M_{2m}$ case: Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \Sigma^2SZ/2m\wedge P & = & \Sigma^2SZ/2m\wedge P & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^5 & \xrightarrow{h'_R} & R'_{2m} & \rightarrow & R'M_{2m} & \rightarrow & \Sigma^6 \\
\| & & \downarrow \rho'_{R,Q} & & \downarrow \rho_{R'M,Q'N} & & \| \\
\Sigma^5 & \xrightarrow[h'_Q]{} & Q'_{2m} & \rightarrow & Q'N_{2m} & \rightarrow & \Sigma^6 \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma^3SZ/2m\wedge P & = & \Sigma^3SZ/2m\wedge P & &
\end{array}$$

involving four cofiber sequences. Evidently $KU_0R'M_{2m}\cong KU_0\Sigma^6\oplus KU_0R'_{2m}\cong Z\oplus Z\oplus Z/2m$ and $KU_1R'M_{2m}=0$. Set $t_{R'M}=\begin{pmatrix}-1 & 0 & 0\\ a & 1 & 0\\ b & 0 & 1\end{pmatrix}$ on $KU_0R'M_{2m}\cong Z\oplus Z\oplus Z/2m$ for some integers a, b. Then the equality $\rho_{R'M,Q'N*}t_{R'M}=t_{Q'N}\rho_{R'M,Q'N*}$ implies that $a=1$ because $t_{Q'N}=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ on $KU_0Q'N_{2m}\cong Z\oplus Z/2m$. So the result follows immediately, since the matrix $\begin{pmatrix}-1 & 0 & 0\\ 1 & 1 & 0\\ b & 0 & 1\end{pmatrix}$ is always congruent to $\begin{pmatrix}-1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ for any integer b.

2.4. Finally we will compute the KO homologies of the elementary spectra $W=Y'X_{2m}$ with four cells mentioned in (2.4).

Proposition 2.4. *The KO homologies KO_iW are tabled as follows:*

i	$M'M_{2m}$	$M'N_{2m}$	$N'M_{2m}$	$N'N_{2m}$	$P'Q_{2m}$	$P'R_{2m}$
0, 4	Z	$Z\oplus Z$	$Z\oplus Z$	Z	$Z\oplus(Z/2\otimes Z/m)$	$Z\oplus(Z/2\otimes Z/m)$
1, 5	$Z/4m$	$Z/4m$	$Z/2$	$Z\oplus Z/2$	0	$Z/2$
2, 6	0	$Z/2$	$Z/4m$	$Z/4m$	$Z\oplus Z/m$	Z/m
3, 7	Z	0	0	$Z/2$	0	Z
i	$Q'P_{2m}$	$Q'Q_{2m}$	$Q'R_{2m}$	$R'P_{2m}$	$R'Q_{2m}$	$R'R_{2m}$
0, 4	Z	Z	$Z\oplus Z$	$Z\oplus Z/m$	$Z\oplus Z\oplus Z/m$	$Z\oplus Z/m$
1, 5	$Z/2\otimes Z/m$	$(*)_m$	$(*)_m$	$Z/2$	$Z/2$	$Z\oplus Z/2$
2, 6	Z	0	$Z/2$	$Z/2\otimes Z/m$	$(*)_m$	$(*)_m$
3, 7	Z/m	$Z\oplus Z/m$	Z/m	Z	0	$Z/2$

i	MR'_{2m}	$N'Q_{2m}$	$N'R_{2m}$	$Q'N_{2m}$	$R'M_{2m}$	$R'N_{2m}$
0	$Z\oplus Z/2$	$Z\oplus Z/2$	$Z\oplus Z/2$	$Z\oplus Z/2$	$Z\oplus Z/4m$	$Z\oplus Z/4m$
1	$Z/4m$	$Z/2$	$Z/2\oplus Z/2$	$Z/2$	$Z/2$	$Z/2\oplus Z/2$
2	$Z\oplus Z/2$	$Z\oplus Z/4m$	$Z/4m$	Z	$Z\oplus Z/2$	$Z/2$
3	$Z/2$	$Z/2$	$Z\oplus Z/2$	Z/m	0	Z
4	Z	$Z\oplus Z/2$	$Z\oplus Z/2$	Z	$Z\oplus Z/m$	$Z\oplus Z/m$
5	Z/m	0	$Z/2$	$Z/2$	0	$Z/2$
6	Z	$Z\oplus Z/m$	Z/m	$Z\oplus Z/2$	$Z\oplus Z/2$	$Z/2$
7	$Z/2$	0	Z	$Z/4m$	$Z/2$	$Z\oplus Z/2$

in which $(*)_m$ *stands for* $Z/4$ *or* $Z/2\oplus Z/2$ *according as m is odd or even.*

Proof. We have computed $KO_*Q'P_{2m}$ in the proof of Proposition 2.3. In the other cases we can similarly compute by using the long exact sequences of KO homologies induced by the cofiber sequences as appeared in the proof of Proposition 2.3. We may also apply the universal coefficient sequence combined with (2.5) as in the proof of Proposition 1.3.

3. Elementary Z/2-actions

3.1. Let H be a direct sum of 2-torsion free cyclic groups. If the cyclic group $Z/2$ of order 2 acts on the abelian group H, then there exists a direct sum decomposition $H\cong A\oplus B\oplus C\oplus C$ with C free on which the $Z/2$-action ρ_H is represented by the matrix $\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$ (use [6, Propositions 3.7 and 3.8] or [7]).

If the cyclic group $Z/2$ acts on the direct sum $H\oplus Z/2^{s+1}$, $s\geqq 0$, then its matrix representation is written into one of the following types:

$$(3.1)\quad \text{i)}\ \pm\begin{pmatrix}\rho_H&0\\0&1\end{pmatrix}\qquad \text{ii)}\ \pm\begin{pmatrix}\rho_H&0\\0&2^s+1\end{pmatrix}\ (s\geqq 2)\ \text{on}\ H\oplus Z/2^{s+1}$$

$$\text{iii)}\ \pm\begin{pmatrix}\rho_{H'}&0&0\\0&-1&0\\0&1&1\end{pmatrix}\quad \text{iv)}\ \pm\begin{pmatrix}\rho_{H'}&0&0\\0&1&0\\0&2^s&1\end{pmatrix}\qquad \text{on}\ H'\oplus Z\oplus Z/2^{s+1}$$

$$\text{v)}\ \pm\begin{pmatrix}\rho_{H'}&0&0&0\\0&-1&0&0\\0&0&1&0\\0&1&2^s&1\end{pmatrix}\qquad \text{on}\ H''\oplus Z\oplus Z\oplus Z/2^{s+1}$$

where the matrices behave as left action on $H\oplus Z/2^{s+1}$ and $H\cong H'\oplus Z\cong H''\oplus Z\oplus Z$.

A $Z/2$-action ρ on an abelian group H is said to be elementary if the pair

(H, ρ) is one of the following kinds of pairs (cf. [12, 5.1]):

$$(3.2) \qquad (A, 1), \quad (B, -1), \quad \left(C\oplus C, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad (Z/8m, 4m\pm 1),$$
$$\left(Z\oplus Z/2m, \pm\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\right), \quad \left(Z\oplus Z/2m, \pm\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}\right),$$
$$\left(Z\oplus Z\oplus Z/2m, \pm\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & m & 1 \end{pmatrix}\right).$$

We here deal with a CW-spectrum X such that the conjugation t_* on KU_0X is decomposed into a direct sum of the above elementary $Z/2$-actions, and $KU_1X=0$. Thus

$$(3.3) \qquad KU_0X \cong A\oplus B\oplus(C\oplus C)\oplus A'\oplus B'\oplus(D\oplus D')\oplus(E\oplus E')$$
$$\oplus(F\oplus F')\oplus(G\oplus G')\oplus(I\oplus I\oplus I')\oplus(J\oplus J\oplus J')$$

where each of the summands A' and B' is a direct sum of the forms $Z/8m$, each of the summands $D\oplus D'$, $E\oplus E'$, $F\oplus F'$ and $G\oplus G'$ is a direct sum of the forms $Z\oplus Z/2m$, and each of the summands $I\oplus I\oplus I'$ and $J\oplus J\oplus J'$ is a direct sum of the form $Z\oplus Z\oplus Z/2m$. Moreover the conjugation t_* acts on each component of KU_0X as follows:

$$(3.4) \quad t_* = 1, \quad -1, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ on } A, \quad B, \quad C\oplus C.$$

$t_* = 4m+1$, $4m-1$ on the component $Z/8m$ of A', B'.

$t_* = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix}$ on the component $Z\oplus Z/2m$ of $D\oplus D'$, $E\oplus E'$, $F\oplus F'$, $G\oplus G'$.

$t_* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & m & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & m & -1 \end{pmatrix}$ on the component $Z\oplus Z\oplus Z/2m$ of $I\oplus I\oplus I'$, $J\oplus J\oplus J'$.

For any direct sum $H=\bigoplus_i Z/2m_i$ we denote by $H(*)$ the direct sum $\bigoplus_i (*)_{m_i}$ where $(*)_{m_i}=Z/4$ or $Z/2\oplus Z/2$ according as m_i is odd or even. Moreover we write $2H=\bigoplus_i Z/m_i$ and $1/2\, H=\bigoplus_i Z/4m_i$.

Let KC denote the self-conjugate K-spectrum, which is obtained as the fiber of the map $1-t$: $KU\to KU$ (see [3]). Given a CW-spectrum X satisfying (3.3) with (3.4) we can easily compute its KC homology as in [12, Lemma 5.1].

Lemma 3.1. *Assume that* $KU_1X=0$.

$$
\begin{aligned}
\text{i)}\quad KC_0X &\cong A\oplus(B*Z/2)\oplus C\oplus(2A')\oplus(B'*Z/2)\oplus(D\oplus D'*Z/2)\oplus E'\\
&\quad\oplus(F\oplus F')\oplus(G'*Z/2)\oplus(I\oplus I')\oplus(J\oplus J'*Z/2)\\
KC_1X &\cong (A\otimes Z/2)\oplus B\oplus C\oplus(A'\otimes Z/2)\oplus(2B')\oplus(1/2\,D')\oplus E\\
&\quad\oplus F'(*)\oplus(G\oplus 2G')\oplus(I\oplus I'\otimes Z/2)\otimes(J\oplus J')\\
KC_2X &\cong (A*Z/2)\oplus B\oplus C\oplus(A'*Z/2)\oplus(2B')\oplus D'\oplus(E\oplus E'*Z/2)\\
&\quad\oplus(F'*Z/2)\oplus(G\oplus G')\oplus(I\oplus I'*Z/2)\oplus(J\oplus J')\\
KC_3X &\cong A\oplus(B\otimes Z/2)\oplus C\oplus(2A')\oplus(B'\otimes Z/2)\oplus D\oplus(1/2\,E')\\
&\quad\oplus(F\oplus 2F')\oplus G'(*)\oplus(I\oplus I')\oplus(J\oplus J'\otimes Z/2)\\
\text{ii)}\quad KO_1X\oplus KO_5X &\cong (A\otimes Z/2)\oplus(B*Z/2)\oplus(D'*Z/2)\oplus(F'\otimes Z/2)\\
KO_3X\oplus KO_7X &\cong (A*Z/2)\oplus(B\otimes Z/2)\oplus(E'*Z/2)\oplus(G'\otimes Z/2)
\end{aligned}
$$

Let us denote by V_{2m} and W_{4m} respectively the elementary spectra constructed by the following cofiber sequences:

(3.5)
$$
\begin{aligned}
&\Sigma^1 SZ/2 \xrightarrow{i\eta} SZ/m \xrightarrow{i_V} V_{2m} \xrightarrow{j_V} \Sigma^2 SZ/2\\
&\Sigma^1 SZ/2 \xrightarrow{i\eta+\tilde{\eta}j} SZ/2m \xrightarrow{i_W} W_{4m} \xrightarrow{j_W} \Sigma^2 SZ/2\,.
\end{aligned}
$$

By observing [12, (5.4)] and Propositions 1.2 and 2.3 we here list up some of CW-spectra X with a few cells such that KU_0X contains only one 2-torsion cyclic group and $KU_1X=0$.

(3.6)

$X =$	V_{2m}	W_{8m}	M_{2m}	Q_{2m}	N'_{2m}	R'_{2m}
$KU_0X \cong$	$Z/2m$	$Z/8m$	$Z\oplus Z/2m$	$Z\oplus Z/2m$	$Z\oplus Z/2m$	$Z\oplus Z/2m$
$t_* =$	1	$4m+1$	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\m&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$X =$	MQ_{2m}	NP'_{4m}	NR'_{2m}	$N'M_{2m}$
$KU_0X \cong$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z\oplus Z/2m$
$t_* =$	$\begin{pmatrix}-1&0&0\\0&1&0\\1&m&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&1&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&1&-1\end{pmatrix}$

$X =$	$P'Q_{2m}$	$R'Q_{2m}$	$N'Q_{2m}$	$R'M_{2m}$
$KU_0X \cong$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z\oplus Z/2m$	$Z\oplus Z\oplus Z/2m$
$t_* =$	$\begin{pmatrix}1&0&0\\0&-1&0\\1&m&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&m&1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\1&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0&0\\1&1&0\\0&0&1\end{pmatrix}$

We will write simply $Y_H=\bigvee_i Y_{2m_i}$ for any direct sum $H=\bigoplus_i Z/2m_i$ when $Y=V$, W, M, Q and so on.

3.2. For later use we will here study the induced homomorphism

ε_{C*}: $KO_iX \to KC_iX$ when $X=Q_{2m}$, N'_{2m}, R'_{2m}, NP'_{4m}, NR'_{2m} and $R'Q_{2m}$.

Lemma 3.2. *The induced homomorphisms* ε_{C*}: $KO_iX \to KC_iX$ *are represented by the following matrices* $M_i(X)$:

i) $M_0(Q_{2m}) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$: $Z \oplus Z/2m \to Z \oplus Z/2m$

$M_4(Q_{2m}) = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$: $Z \oplus Z/m \to Z \oplus Z/2m$

ii) $M_0(N'_{2m}) = \begin{pmatrix}1\\ 0\end{pmatrix}$: $Z \to Z \oplus Z/2$

$M_4(N'_{2m}) = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$: $Z \oplus Z/2 \to Z \oplus Z/2$

iii) $M_0(R'_{2m}) = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$: $Z \oplus Z/2m \to Z \oplus Z/2m$

$M_4(R'_{2m}) = \begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$: $Z \oplus Z/m \to Z \oplus Z/2m$

iv) $M_0(NP'_{4m}) = \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$: $Z \oplus Z \to Z \oplus Z \oplus Z/2$

$M_4(NP'_{4m}) = \begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}$: $Z \oplus Z \to Z \oplus Z \oplus Z/2$

v) $M_0(NR'_{2m}) = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$: $Z \oplus Z/2m \to Z \oplus Z/2m$

$M_2(NR'_{2m}) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$: $Z \oplus Z/2 \to Z \oplus Z/2$

$M_4(NR'_{2m}) = \begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$: $Z \oplus Z/2m \to Z \oplus Z/2m$

$M_6(NR'_{2m}) = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$: $Z \oplus Z/2 \to Z \oplus Z/2$

vi) $M_0(R'Q_{2m}) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$: $Z \oplus Z \oplus Z/m \to Z \oplus Z \oplus Z/2m$

$M_4(R'Q_{2m}) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 2\end{pmatrix}$: $Z \oplus Z \oplus Z/m \to Z \oplus Z \oplus Z/2m$

where the matrices behave as left action.

Proof. i) The $X=Q_{2m}$ case: Obviously ε_{C*}: $KO_0Q_{2m} \to KC_0Q_{2m}$ is an isomorphism, and moreover we have the following commutative diagram

$$\begin{array}{ccccccccccc} 0 \to KO_5\Sigma^4 \to & KO_4SZ/2m & \to & KO_4Q_{2m} & \to & KO_4\Sigma^4 & \to KO_3SZ/2m \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & KU_4SZ/2m & \to & KU_4Q_{2m} & \to & KU_4\Sigma^4 & \to 0 \end{array}$$

with exact rows. As is easily seen, the central arrow ε_{U*}: $KO_4Q_{2m} \to KU_4Q_{2m}$

is expressed as the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: $Z\oplus Z/m \to Z\oplus Z/2m$. The result is now immediate.

ii) The $X=N'_{2m}$ case: Using the commutative diagram

$$\begin{tikzcd}
 & KO_0\Sigma^0 \arrow[r, "\cong"] \arrow[d, "\cong"] & KO_0N'_{2m} \arrow[d] & & \\
0 \arrow[r] & KU_0\Sigma^0 \arrow[r] & KU_0N'_{2m} \arrow[r] & KU_6SZ/2m \arrow[r] & 0
\end{tikzcd}$$

with a split exact row, it is easily checked that $M_0(N'_{2m})=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We next compare the two commutative diagrams

$$\begin{tikzcd}[column sep=small]
0 \arrow[dr] & & & & 0 \arrow[dl] \\
 & KO_4\Sigma^0 \arrow[dr] \arrow[dd] & & KO_4\Sigma^2 \arrow[dl] \arrow[dd] & \\
 & & KO_4N'_{2m} \arrow[dl] \arrow[dr] & & \\
 & KO_4Q \arrow[dl] & & KO_2SZ/2m \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
0 \arrow[dr] & & & & 0 \arrow[dl] \\
 & KU_4\Sigma^0 \arrow[dr] \arrow[dd] & & KU_4\Sigma^2 \arrow[dl] \arrow[dd] & \\
 & & KU_4N'_{2m} \arrow[dl] \arrow[dr] & & \\
 & KU_4Q \arrow[dl] & & KU_2SZ/2m \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

with exact diagonals. Since $KO_4N'_{2m}\cong KO_4Q\oplus KO_4\Sigma^2\cong Z\oplus Z/2$ and $KU_4N'_{2m}\cong KU_4\Sigma^0\oplus KU_2SZ/2m\cong Z\oplus Z/2m$, the induced homomorphism ε_{U*}: $KO_4N'_{2m}\to KU_4N'_{2m}$ is expressed as the matrix $\begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix}$: $Z\oplus Z/2 \to Z\oplus Z/2m$. Therefore it follows immediately that $M_4(N'_{2m})=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

iii) The $X=R'_{2m}$ case: Compare the two commutative diagrams

$$\begin{tikzcd}[column sep=small]
0 \arrow[dr] & & & & 0 \arrow[dl] \\
 & KO_i\Sigma^0 \arrow[dr] \arrow[dd] & & KO_{i-2}P'_{2m} \arrow[dl] \arrow[dd] & \\
 & & KO_iR'_{2m} \arrow[dl] \arrow[dr] & & \\
 & KO_iQ \arrow[dl] & & KO_{i-4}SZ/2m \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
0 \arrow[dr] & & & & 0 \arrow[dl] \\
 & KU_i\Sigma^0 \arrow[dr] \arrow[dd] & & KU_{i-2}P'_{2m} \arrow[dl] \arrow[dd] & \\
 & & KU_iR'_{2m} \arrow[dl] \arrow[dr] & & \\
 & KU_iQ \arrow[dl] & & KU_{i-4}SZ/2m \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

with exact diagonals, in dimensions $i=0$ and 4. Since $KO_iR'_{2m}\cong KO_iQ\oplus KO_{i-2}P'_{2m}$ and $KU_iR'_{2m}\cong KU_i\Sigma^0\oplus KU_{i-4}SZ/2m$ for $i=0$ and 4, the induced homomorphism ε_{U*}: $KO_iR'_{2m}\to KU_iR'_{2m}$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ according as $i=0$ or 4. The result is now immediate.

iv) The $X=NP'_{4m}$ case: Use the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & KO_i\Sigma^0 & \to & KO_iNP'_{4m} & \to & KO_iN'_{4m} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & KC_i\Sigma^0 & \to & KC_iNP'_{4m} & \to & KC_iN'_{4m} & \to & 0 \end{array}$$

with exact rows, in dimensions $i=0$ and 4. Then the result follows from ii) by a routine computation.

v) The $X=NR'_{2m}$ case: Use the following commutative diagrams

$$\begin{array}{ccc} KO_0NR'_{2m} & \xrightarrow{\cong} & KO_0R'_{2m} \\ \downarrow & & \downarrow \\ KC_0NR'_{2m} & \xrightarrow{\cong} & KC_0R'_{2m} \end{array} \qquad \begin{array}{ccccccccc} 0 & \to & KO_4\Sigma^2 & \to & KO_4NR'_{2m} & \to & KO_4R'_{2m} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & KC_4NR'_{2m} & \xrightarrow{\cong} & KC_4R'_{2m} & & \end{array}$$

$$\begin{array}{ccc} KO_6NR'_{2m} & \xrightarrow{\cong} & KO_4N'_{2m} \\ \downarrow & & \downarrow \\ KC_6NR'_{2m} & \xrightarrow{\cong} & KC_4N'_{2m} \end{array} \qquad \begin{array}{ccccccccc} 0 & \to & KO_2\Sigma^0 & \to & KO_2NR'_{2m} & \to & KO_0N'_{2m} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & KC_2NR'_{2m} & \xrightarrow{\cong} & KC_0N'_{2m} & & \end{array}$$

with exact rows. Then the result follows immediately from ii) and iii).

vi) The $X=R'Q_{2m}$ case is shown by a similar argument to the case iv) using the cofiber sequence $\Sigma^0 \to R'Q_{2m} \to \Sigma^4 Q_{2m} \xrightarrow{\eta^2\bar{k}_Q} \Sigma^1$ and the above result i).

3.3. As a special case of (3.3) we here deal with a CW-spectrum X such that KU_0X has a direct sum decomposition

$$(3.7) \qquad KU_0X \cong A \oplus B \oplus (C \oplus C) \oplus A' \oplus B' \oplus (I \oplus I \oplus I') \oplus (J \oplus J \oplus J')$$

in which the conjugation t_* acts on KU_0X as in (3.4). For such a CW-spectrum X Lemma 2.1 ii) asserts that $KO_1X \oplus KO_5X \cong (A \otimes Z/2) \oplus (B * Z/2)$ and $KO_3X \oplus KO_7X \cong (A * Z/2) \oplus (B \otimes Z/2)$ under the assumption that $KU_1X=0$. We will now show the first one of our main results.

Theorem 3.3. *Let X be a CW-spectrum such that KU_0X has a direct sum decomposition as* (3.7) *and $KU_1X=0$. Assume that A and B are both direct sums of 2-torsion free cyclic groups. Then there exist abelian groups A_0, A_4, B_2 and B_6 with $A_0 \oplus A_4 \cong A$, $B_2 \oplus B_6 \cong B$ so that X is quasi KO_*-equivalent to the wedge sum $SA_0 \vee \Sigma^2 SB_2 \vee \Sigma^4 SA_4 \vee \Sigma^6 SB_6 \vee (P \wedge SC) \vee W_{A'} \vee \Sigma^2 W_{B'} \vee MQ_{I'} \vee \Sigma^2 MQ_{J'}$.*

Proof. Consider the exact sequence

$$KU_{j+2}X \xrightarrow{\varphi_j} KC_jX \xrightarrow{\psi_j} KO_{j+1}X \oplus KO_{j+5}X \to 0$$

induced by the cofiber sequence $\Sigma^1 KC \xrightarrow{(-\tau, \tau\pi\bar{c}^{-1})} KO \vee \Sigma^4 KO \xrightarrow{\varepsilon_U \vee \pi_U^2 \varepsilon_U} KU \xrightarrow{\varepsilon_C \varepsilon_O \pi_U^{-1}} \Sigma^2 KC$ when $j=0$ and 2. Since $KO_1X \oplus KO_5X \cong A \otimes Z/2$ and $KO_3X \oplus KO_7X \cong B \otimes Z/2$, we can choose direct sum decompositions $A \cong A_0 \oplus A_4$, $B \cong B_2 \oplus B_6$ with A_4, B_6 free so that $\psi_0(A_i) \cong A_i \otimes Z/2 \cong KO_{i+1}X$, $\psi_2(B_{i+2}) \cong B_{i+2} \otimes Z/2 \cong$

$KO_{i+3}X$ for $i=0$ and 4.

Our proof will be established by the same method as in [12, Theorem 5.2] or [13, Theorem 2.5]. Abbreviate by Y the desired wedge sum of nine elementary spectra. For each component Y_H of the wedge sum Y we choose a unique map f_H: $Y_H \to KU\wedge X$ whose induced homomorphism in KU homologies is the canonical injection. Here H is taken to be A_0, A_4, B_2, B_6, C, A', B', I' or J'. Notice that there exists a map g_H: $Y_H \to KC\wedge X$ satisfying $(\zeta_\wedge 1)g_H=f_H$ for each H. We will find a map h_H: $Y_H \to KO\wedge X$ such that $(\varepsilon_{U\wedge}1)h_H=f_H$ for each H, and then apply [12, Proposition 1.1] to show that the map $h=\bigvee_H h_H$: $Y=\bigvee_H Y_H \to KO\wedge X$ becomes a quasi KO_*-equivalence. We will only find such maps h_H in the cases $H=A_0$, C, A' and I', the other cases being done similarly.

i) The $H=A_0$ case: Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(A_0, KO_6X) & \to & [SA_0, \Sigma^3 KO\wedge X] & \overset{\tilde{\kappa}_{KO}}{\to} & \mathrm{Hom}(A_0, KO_5X) & \to 0 \\
& \downarrow \eta_{**} & & \downarrow (\eta_\wedge 1)_* & & \downarrow \eta_{**} & \\
0 \to & \mathrm{Ext}(A_0, KO_7X) & \to & [SA_0, \Sigma^2 KO\wedge X] & \underset{\tilde{\kappa}_{KO}}{\to} & \mathrm{Hom}(A_0, KO_6X) & \to 0
\end{array}$$

with the universal coefficient sequences, in which the arrows $\tilde{\kappa}_{KO}$ assign to any map f its induced homomorphism of KO homologies in dimension 0. Note that the induced homomorphism $\tilde{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_{A_0})$: $KO_0SA_0 \to KO_5X$ becomes trivial because $KO_5X \cong \psi_0(A_4)$. Then the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{A_0}=(\varepsilon_O\pi\bar{U}^1{}_\wedge 1)f_{A_0}$: $\Sigma^2 SA_0 \to KO\wedge X$ is in fact trivial becasuse $\mathrm{Ext}(A_0, KO_7X)=0$. So we can find a desired map h_{A_0}.

ii) The $H=C$ case: Recall that P is self dual, thus $P=\Sigma^2 DP$. Since $\eta_\wedge 1$: $\Sigma^1 KO\wedge P \to KO\wedge P$ is trivial, it is easily seen that the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_C=(\varepsilon_O\pi\bar{U}^1{}_\wedge 1)f_C$: $P\wedge SC \to \Sigma^2 KO\wedge X$ becomes trivial. So we can find a desired map h_C.

iii) The $H=A'$ case: Set $A'=\bigoplus_i Z/2m_i$, and then write $2A'=\bigoplus_i Z/4m_i$ and $A''=\bigoplus_i Z/2$. We will first find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc}
S(2A') & \overset{i_W}{\to} & W_{A'} & \overset{j_W}{\to} & \Sigma^2 SA'' \\
\downarrow h_0 & & \downarrow g_{A'} & & \downarrow h_1 \\
KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\
\| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\
KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X
\end{array}$$

after replacing the map $g_{A'}$ with $(\zeta_\wedge 1)g_{A'}=f_{A'}$ suitably if necessary. The induced homomorphisms $\tilde{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_{A'})$: $KO_jW_{A'} \to KO_{j+5}X$ are trivial in dimensions $j=0$ and 2 because $\psi_0(2A')=0=\psi_2(A'*Z/2)$. So we get a map h'_0: $\bigvee_i \Sigma^0 \to \Sigma^2 KO\wedge X$ such that $h'_0 j_{2A'}=(\tau\pi\bar{c}^1{}_\wedge 1)g_{A'}i_W$: $S(2A') \to \Sigma^3 KO\wedge X$ and in addition

$(\eta_\wedge 1)h_0'=0$ where $j_{2A'}=\bigvee_i j_{4m_i}\colon \bigvee_i SZ/4m_i \to \bigvee_i \Sigma^1$. Consequently the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{A'}i_W\colon S(2A')\to\Sigma^2 KO\wedge X$ becomes trivial. Hence we can obtain desired maps h_0 and h_1 by applying [12, Lemma 1.3].

We will next find vertical maps k_0, k_1 making the diagram below commutative

$$\begin{array}{ccccc} M_{2A'} & \xrightarrow{k_{M,W}} & W_{A'} & \xrightarrow{j_{A''}j_W} & \bigvee_i \Sigma^3 \\ \downarrow k_0 & & \downarrow g_{A'} & & \downarrow k_1 \\ KO\wedge X & \longrightarrow & KC\wedge X & \longrightarrow & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \longrightarrow & KU\wedge X & \longrightarrow & \Sigma^2 KO\wedge X \end{array}$$

with $j_{A''}=\bigvee_i j_2\colon \bigvee_i SZ/2\to\bigvee_i \Sigma^1$, after replacing the map $g_{A'}$ with $(\zeta_\wedge 1)g_{A'}=f_{A'}$ again if necessary. Notice that the composite $(\eta_\wedge 1)i_{A''}j_M\colon M_{2A'}\to\Sigma^1 SA''$ is trivial because $(\eta_\wedge 1)i_{A''}=\bigvee_i(\rho_{4m_i,2}i_{4m_i}\eta)\colon \bigvee_i\Sigma^1\to\bigvee_i SZ/2$ where $\rho_{4m_i,2}\colon SZ/4m_i\to SZ/2$ denotes the associated map with the canonical epimorphism. Since $j_W k_{M,W}=i_{A''}j_M\colon M_{2A'}\to\Sigma^2 SA''$, the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{A'}k_{M,W}\colon M_{2A'}\to \Sigma^2 KO\wedge X$ coincides with the composite $(\eta_\wedge 1)h_1 i_{A''}j_M$, which is trivial. So we can obtain desired maps k_0 and k_1 by applying [12, Lemma 1.3] again. However the composite $(\eta_\wedge 1)j_{A''}j_W\colon W_{A'}\to\bigvee_i\Sigma^2$ becomes trivial because $(\eta_\wedge 1)j_{A''}=\bigvee_i(j_{4m_i}(i_{4m_i}\bar{\eta}_2+\tilde{\eta}_{4m_i}j_2))\colon \bigvee_i SZ/2\to\bigvee_i\Sigma^0$. Hence there exists a map $h_{A'}\colon W_{A'}\to KO\wedge X$ with $(\varepsilon_{U\wedge}1)h_W=f_W$ as desired.

iv) The $H=I'$ case: Setting $I'=\bigoplus_i Z/2m_i$ we will find vertical maps h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} SI' & \xrightarrow{i_{MQ}} & MQ_{I'} & \xrightarrow{j_{MQ}} & \bigvee_i(\Sigma^2\vee\Sigma^4) \\ \downarrow h_0 & & \downarrow g_{I'} & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X \end{array}$$

after replacing the map $g_{I'}$ with $(\zeta_\wedge 1)g_{I'}=f_{I'}$ suitably if necessary. The induced homomorphisms $\kappa_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_{I'})\colon KO_j MQ_{I'}\to KO_{i+5}X$ are trivial in dimensions $j=0$ and 2 because $\psi_0(I\oplus I')=0=\psi_2(I\oplus I'*Z/2)$. So we get a map $h_0'\colon \bigvee_i\Sigma^0\to\Sigma^2 KO\wedge X$ such that $h_0'j_{I'}=(\tau\pi\bar{c}^1{}_\wedge 1)g_{I'}i_{MQ}\colon SI'\to\Sigma^3 KO\wedge X$ and in addition $(\eta_\wedge 1)h_0'=0$. Since the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{I'}i_{MQ}\colon SI'\to\Sigma^2 KO\wedge X$ becomes trivial, we can obtain desired maps h_0 and h_1 by applying [12, Lemma 1.3].

Choose maps $k_i'\colon \Sigma^0\to\Sigma^2 KO\wedge X$, $k_i''\colon \Sigma^0\to KO\wedge X$ satisfying $h_1=\bigvee_i(k_i'\eta\vee k_i''\eta)\colon \bigvee_i(\Sigma^0\vee\Sigma^2)\to\Sigma^1 KO\wedge X$, and then set $\bar{k}=\bigvee_i(k_i'\bar{\eta}_{2m_i}+k_i''j_{2m_i})\colon SI'\to\Sigma^1 KO\wedge X$. Notice that $(\eta_\wedge 1)h_1=\bar{k}(\bigvee_i(i_{2m_i}\eta\vee\tilde{\eta}_{2m_i}\eta))\colon \bigvee_i(\Sigma^0\vee\Sigma^2)\to KO\wedge X$ because

$\bar{k}(\bigvee_i i_{2m_i}\eta)=\bigvee_i k_i'\eta^2$ and $\bar{k}(\bigvee_i \tilde{\eta}_{2m_i}\eta)=\bigvee_i k_i''\eta^2$. Hence the composite $(\eta_\wedge 1)h_1 j_{MQ}$: $MQ_{I'}\to\Sigma^2 KO\wedge X$ becomes trivial. So there exists a map $h_{I'}$: $MQ_{I'}\to KO\wedge X$ with $(\varepsilon_{U\wedge}1)h_{I'}=f_{I'}$ as desired.

4. KU_0X containing only one 2-cyclic group $Z/2^{s+1}$

4.1. We first deal with a CW-spectrum X such that KU_0X has a direct sum decomposition

$$KU_0X\cong A\oplus B\oplus(C\oplus C)\oplus Z/2m \tag{4.1}$$

with A, B direct sums of 2-torsion free cyclic groups, and $KU_1X=0$. Here the conjugation t_* behaves on A, B and $C\oplus C$ as in (3.4), and $t_*=1$ on the last factor $Z/2m$. For such a CW-spectrum X we consider the exact sequence

$$KU_{j+2}X\xrightarrow{\varphi_j}KC_jX\xrightarrow{\psi_j}KO_{j+1}X\oplus KO_{j+5}X\to 0$$

in dimensions $j=0$ and 2 as in the proof of Theorem 3.2. Recall that $KC_0X\cong A\oplus C\oplus Z/2m$, $KC_2X\cong B\oplus C\oplus Z/2$, $KO_1X\oplus KO_5X\cong(A\otimes Z/2)\oplus Z/2$ and $KO_3X\oplus KO_7X\cong(B\otimes Z/2)\oplus Z/2$.

Using the isomorphism θ_0: $(A\otimes Z/2)\oplus Z/2\to KO_1X\oplus KO_5X$, we put $\theta_0(0,1)=(x,y)\in KO_1X\oplus KO_5X$. Then the pair (x,y) is divided into the three types:

i) $x\neq 0$, $y=0$ ii) $x=0$, $y\neq 0$ iii) $x\neq 0$, $y\neq 0$.

Corresponding to each type we can choose a direct sum decomposition of A as follows:

(4.2) i) $A\cong A_0\oplus A_4$ with A_4 free so that $\psi_0(A_0\oplus Z/2m)\cong(A_0\otimes Z/2)\oplus Z/2\langle x\rangle\cong KO_1X$ and $\psi_0(A_4)\cong A_4\otimes Z/2\cong KO_5X$.

ii) $A\cong A_0\oplus A_4$ with A_4 free so that $\psi_0(A_0)\cong A_0\otimes Z/2\cong KO_1X$ and $\psi_0(A_4\oplus Z/2m)\cong(A_4\otimes Z/2)\oplus Z/2\langle y\rangle\cong KO_5X$.

iii) $A\cong A_0\oplus A_4\oplus Z$ with A_4 free so that $\psi_0(A_0\oplus Z/2m)\cong(A_0\otimes Z/2)\oplus Z/2\langle x\rangle\cong KO_1X$, $\psi_0(A_4\oplus Z/2m)\cong(A_4\otimes Z/2)\oplus Z/2\langle y\rangle\cong KO_5X$ and $\psi_0(Z)\cong Z/2\langle x\rangle$.

Similarly we can choose a direct sum decomposition of B corresponding to each of the three types. Consequently we have

Lemma 4.1. *Let X be a CW-spectrum satisfying* (4.1).

i) *$KC_0X\cong A\oplus C\oplus Z/2m$ is decomposed into one of the following three types*:

A1) *$KC_0X\cong A_0\oplus A_4\oplus C\oplus Z/2m$ so that $KO_1X\cong(A_0\oplus Z/2m)\otimes Z/2$, $KO_5X\cong A_4\otimes Z/2$ and both τ_*: $KC_0X\to KO_1X$ and $(\tau\pi\bar{c}^{-1})_*$: $KC_0X\to KO_5X$ are the canonical epimorphisms.*

A2) *$KC_0X\cong A_0\oplus A_4\oplus C\oplus Z/2m$ so that $KO_1X\cong A_0\otimes Z/2$, $KO_5X\cong(A_4\oplus Z/2m)$*

$\otimes Z/2$ *and both* τ_*: $KC_0X \to KO_1X$ *and* $(\tau\pi_c^{-1})_*$: $KC_0X \to KO_5X$ *are the canonical epimorphisms.*

A3) $KC_0X \cong A_0 \oplus A_4 \oplus Z \oplus C \oplus Z/2m$ *so that* $KO_1X \cong (A_0 \otimes Z/2) \oplus Z/2$, $KO_5X \cong (A_4 \oplus Z/2m) \otimes Z/2$ *and* $(\tau\pi_c^{-1})_*$: $KC_0X \to KO_5X$ *is the canonical epimorphism, but* τ_*: $KC_0X \to KO_1X$ *is the epimorphism whose restriction to* $Z \oplus Z/2m$ *is given by the matrix* $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$: $Z \oplus Z/2m \to (A_0 \otimes Z/2) \oplus Z/2$.

ii) $KC_2X \cong B \oplus C \oplus Z/2$ *is similarly decomposed into one of the three types*:

B1) $KC_2X \cong B_2 \oplus B_6 \oplus C \oplus Z/2$ *with* $KO_3X \cong (B_2 \oplus Z/2) \otimes Z/2$, $KO_7X \cong B_6 \otimes Z/2$.

B2) $KC_2X \cong B_2 \oplus B_6 \oplus C \oplus Z/2$ *with* $KO_3X \cong B_2 \otimes Z/2$, $KO_7X \cong (B_6 \oplus Z/2) \otimes Z/2$.

B3) $KC_2X \cong B_2 \oplus B_6 \oplus Z \oplus C \oplus Z/2$ *with* $KO_3X \cong (B_2 \otimes Z/2) \oplus Z/2$, $KO_7X \cong (B_6 \oplus Z/2) \otimes Z/2$.

Here τ_*: $KC_2X \to KO_3X$ *and* $(\tau\pi_c^{-1})_*$: $KC_2X \to KO_7X$ *are epimorphisms as given in* A1), A2) *and* A3) *respectively.*

4.2. By making use of Lemma 4.1 we will now show the second one of our main results.

Theorem 4.2. *Let X be a CW-spectrum such that KU_0X has a direct sum decomposition as* (4.1) *and* $KU_1X=0$. *Then there exist abelian groups A_0, A_4, B_2 and B_6 and a certain CW-spectrum Y so that X is quasi KO_*-equivalent to the wedge sum* $SA_0 \vee \Sigma^2SB_2 \vee \Sigma^4SA_4 \vee \Sigma^6SB_6 \vee (P \wedge SC) \vee Y$. *Here Y is taken to be one of the following elementary spectra* $\Sigma^iSZ/2m$, Σ^iV_{2m}, $\Sigma^{2+i}N'_{2m}$, $\Sigma^iR'_{2m}$ *and* NR'_{2m} *for* $i=0, 4$.

Proof. Set $Y_{11}=SZ/2m$, $Y_{12}=\Sigma^4V_{2m}$, $Y_{13}=\Sigma^6N'_{2m}$, $Y_{21}=V_{2m}$, $Y_{22}=\Sigma^4SZ/2m$, $Y_{23}=\Sigma^2N'_{2m}$, $Y_{31}=\Sigma^4R'_{2m}$, $Y_{32}=R'_{2m}$ and $Y_{33}=NR'_{2m}$. According to Lemma 4.1 KC_0X and KC_2X are respectively decomposed with the three types A1)-A3) and B1)-B3). We will prove that X is quasi KO_*-equivalent to the wedge sum $SA_0 \vee \Sigma^2SB_2 \vee \Sigma^4SA_4 \vee \Sigma^6SB_6 \vee (P_\wedge SC) \vee Y_{ij}$ in each type (Ai, Bj). In each type (Ai, Bj) we choose a unique map f_{ij}: $Y_{ij} \to KU \wedge X$ whose induced homomorphism in KU homologies is the canonical injection. Then there exists a map g_{ij}: $Y_{ij} \to KC \wedge X$ satisfying $(\zeta_\wedge 1)g_{ij}=f_{ij}$. It is sufficient to find a map h_{ij}: $Y_{ij} \to KO \wedge X$ such that $(\varepsilon_{U\wedge}1)h_{ij}=f_{ij}$ for each pair (Ai, Bj), because the other cases has been established in the proof of Theorem 3.3.

i) The $Y_{11}=SZ/2m$ case: Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Ext}(Z/2m, KO_6X) & \to & [SZ/2m, \Sigma^3KO \wedge X] & \overset{\tilde{\kappa}_{KO}}{\to} & \mathrm{Hom}(Z/2m, KO_5X) & \to 0 \\
& \downarrow \eta_{**} & & \downarrow (\eta_\wedge 1)_* & & \downarrow \eta_{**} & \\
0 \to & \mathrm{Ext}(Z/2m, KO_7X) & \to & [SZ/2m, \Sigma^2KO \wedge X] & \underset{\tilde{\kappa}_{KO}}{\to} & \mathrm{Hom}(Z/2m, KO_6X) & \to 0
\end{array}$$

with the universal coefficient sequences. The induced homomorphisms $\tilde{\kappa}_{KO}((\tau\pi_c^{-1}{}_\wedge 1)g_{11})$: $KO_iSZ/2m \to KO_{i+5}X$ become trivial in dimensions $i=0$ and 2 because of Lemma 4.1 A1) and B1). So it is easily verified that the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{11}=(\varepsilon_O\pi\bar{u}^1{}_\wedge 1)f_{11}$: $SZ/2m\to\Sigma^2KO\wedge X$ is trivial. Hence we can find a desired map h_{11}.

ii) The $Y_{21}=V_{2m}$ case: We will first find vertical arrows h_0 and h_1 making the diagram below commutative

$$\begin{array}{ccccc} SZ/m & \xrightarrow{i_V} & V_{2m} & \xrightarrow{j_V} & \Sigma^2SZ/2 \\ \downarrow h_0 & & \downarrow g_{21} & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2KO\wedge X. \end{array}$$

The induced homomorphisms $\bar{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_{21})$: $KO_iV_{2m}\to KO_{i+5}X$ are trivial in dimensions $i=0$ and 2 because $KO_0V_{2m}\cong Z/m$, $KC_0V_{2m}\cong Z/2m$ and $KO_7X\cong\psi_0(B_6)$ by Lemma 4.1 B1). So we get a map h_0': $\Sigma^0\to\Sigma^2KO\wedge X$ such that $h_0'j_m=(\tau\pi\bar{c}^1{}_\wedge 1)g_{21}i_V$: $SZ/m\to\Sigma^1KO\wedge X$ and in addition $(\eta_\wedge 1)h_0'=0$ when m is even. Hence the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{21}i_V$: $SZ/m\to\Sigma^2KO\wedge X$ becomes trivial when m is even as well as odd. By applying [12, Lemma 1.3] we can obtain desired maps h_0 and h_1 after replacing the map g_{21} with $(\zeta_\wedge 1)g_{21}=f_{21}$ suitably if necessary.

Moreover we note that h_{1*}: $KO_2SZ/2\to KO_1X$ becomes trivial since the induced homomorphism $\bar{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_{21})$: $KO_4V_{2m}\to KO_1X$ is also trivial by means of Lemma 4.1 A2). This implies that the composite $h_1\tilde{\eta}_2$: $\Sigma^1\to KO\wedge X$ is trivial. Hence it follows that $(\eta_\wedge 1)h_1=h_1i_2\bar{\eta}_2$: $SZ/2\to KO\wedge X$ because $\eta_\wedge 1=\tilde{\eta}_2j_2+i_2\bar{\eta}_2$: $\Sigma^1SZ/2\to SZ/2$ by (1.1). When m is even, we see that $(\eta_\wedge 1)h_1=h_1\rho_{m,2}i_m\bar{\eta}_2$: $SZ/2\to KO\wedge X$ where $\rho_{m,2}$: $SZ/m\to SZ/2$ denotes the associated map with the canonical epimorphism. Hence it follows that the composite $(\eta_\wedge 1)h_1j_V$: $V_{2m}\to\Sigma^2KO\wedge X$ is trivial when m is even. When m is odd, h_{1*}: $KO_0SZ/2\to KO_7X$ becomes also trivial because $h_1j_V=(\tau\pi\bar{c}^1{}_\wedge 1)g_{21}$. Using the fact that h_{1*}: $KO_iSZ/2\to KO_{i+7}X$ are trivial in dimensions $i=0$ and 2, we can then verify that the composite $(\eta_\wedge 1)h_1$: $SZ/2\to KO\wedge X$ is trivial when m is odd. Conseqently there exists a map h_{21}: $V_{2m}\to KO\wedge X$ satisfying $(\varepsilon_O\pi\bar{u}^1{}_\wedge 1)h_{21}=f_{21}$ for any m.

iii) The $Y_{32}=R'_{2m}$ case: Note that the induced homomorphisms $\bar{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_{32})$: $KO_iR'_{2m}\to KO_{i+5}X$ are trivial in dimensions $i=0$, 4 and 6 by means of Lemmas 3.2 iii) and 4.1 A3), B2). Then we can find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} \Sigma^0 & \xrightarrow{i'_R} & R'_{2m} & \xrightarrow{j'_R} & \Sigma^4SZ/2m \\ \downarrow h_0 & & \downarrow g_{32} & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2KO\wedge X. \end{array}$$

Moreover we can see that h_{1*}: $KO_iSZ/2m\to KO_{i+1}X$ are trivial in dimensions

$i=0$ and 2 because $h_1 j'_R=(\tau\pi\bar{c}^{-1}{}_\wedge 1)g_{32}$. So we can verify that the composite $(\eta_\wedge 1)h_1$: $\Sigma^2 SZ/2m\to KO\wedge X$ becomes trivial. Hence there exists a desired map h_{32}.

iv) The $Y_{23}=\Sigma^2 N'_{2m}$ case is shown similarly to the case iii), by means of Lemmas 3.2 ii) and 4.1 A2), B3) in place of Lemmas 3.2 iii) and 4.1 A3), B2).

v) The $Y_{33}=NR'_{2m}$ case: Note that the induced homomorphisms $\kappa_{KO}((\tau\pi\bar{c}^{-1}{}_\wedge 1)g_{33})$: $KO_i NR'_{2m}\to KO_{i+5}X$ are trivial in dimensions $i=0$, 2, 4 and 6, by means of Lemmas 3.2 v) and 4.1 A3), B3). Then we can find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc}
\Sigma^2\vee\Sigma^0 & \xrightarrow{i'_{NR}} & NR'_{2m} & \xrightarrow{j'_{NR}} & \Sigma^4 SZ/2m \\
\downarrow h_0 & & \downarrow g_{33} & & \downarrow h_1 \\
KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\
\| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\
KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X.
\end{array}$$

Moreover we can see that h_{1*}: $KO_i SZ/2m\to KO_{i+1}X$ are trivial in dimensions $i=0$, 2. This implies that the composite $(\eta_\wedge 1)h_1$: $\Sigma^2 SZ/2m\to KO\wedge X$ is trivial. The result is now immediate.

The other cases $Y_{22}=\Sigma^4 SZ/2m$, $Y_{12}=\Sigma^4 V_{2m}$, $Y_{31}=\Sigma^4 R'_{2m}$ and $Y_{13}=\Sigma^6 N'_{2m}$ are evidently shown by parallel discussions to the above cases i), ii), iii) and iv) respectively.

4.3. We next deal with a CW-spectrum X such that $KU_0 X$ has a direct sum decomposition

(4.3) i) $KU_0 X\cong A\oplus B\oplus(C\oplus C)\oplus(Z\oplus Z/2m)$ or
ii) $KU_0 X\cong A\oplus B\oplus(C\oplus C)\oplus(Z\oplus Z/2m)\oplus(Z\oplus Z/2n)$

with A, B direct sums of 2-torsion free cyclic groups, and $KU_1 X=0$. Here the conjugation t_* behaves on A, B and $C\oplus C$ as in (3.3), and moreover on $Z\oplus Z/2m$, $Z\oplus Z/2n$ as follows:

$$t_D=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix} \quad\text{or}\quad t_F=\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix} \qquad \text{on} \quad Z\oplus Z/2m\,,$$
$$t_E=\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix} \quad\text{or}\quad t_G=\begin{pmatrix}-1 & 0\\ n & -1\end{pmatrix} \quad \text{on} \quad Z\oplus Z/2n\,.$$

For such a CW-spectrum X we recall that $KO_1 X\oplus KO_5 X\cong(A\otimes Z/2)\oplus Z/2$ and $KO_3 X\oplus KO_7 X\cong B\otimes Z/2$ or $\cong(B\otimes Z/2)\oplus Z/2$ in the case (4.3) i) or ii). By a parallel discussion to (4.2) we can show

Lemma 4.3. *Let X be a CW-spectrum satisfying* (4.3).

i) *When $t_*=t_D$ on $Z\oplus Z/2m$, $KC_0 X\cong A\oplus C\oplus(Z\oplus Z/2)\oplus H$ with $H=0$, $Z/2n$ or $Z/2$ and it is decomposed into either of the following three types*:

D1) *$KC_0X \cong A_0 \oplus A_4 \oplus C \oplus (Z \oplus Z/2) \oplus H$ so that $KO_1X \cong (A_0 \oplus Z/2) \otimes Z/2$, $KO_5X \cong A_4 \otimes Z/2$ and both $\tau_*\colon KC_0X \to KO_1X$ and $(\tau\pi\bar{c}^1)_*\colon KC_0X \to KO_5X$ are the canonical epimorphisms.*

D2) *$KC_0X \cong A_0 \oplus A_4 \oplus C \oplus (Z \oplus Z/2) \oplus H$ so that $KO_1X \cong A_0 \otimes Z/2$, $KO_5X \cong (A_4 \oplus Z/2) \otimes Z/2$ and both $\tau_*\colon KC_0X \to KO_1X$ and $(\tau\pi\bar{c}^1)_*\colon KC_0X \to KO_5X$ are the canonical epimorphisms.*

D3) *$KC_0X \cong A_0 \oplus A_4 \oplus Z \oplus C \oplus (Z \oplus Z/2) \oplus H$ so that $KO_1X \cong (A_0 \otimes Z/2) \oplus Z/2$, $KO_5X \cong (A_4 \oplus Z/2) \otimes Z/2$ and $(\tau\pi\bar{c}^1)_*\colon KC_0X \to KO_5X$ is the canonical epimorphism, but $\tau_*\colon KC_0X \to KO_1X$ is the epimorphism whose restriction to $Z \oplus (Z \oplus Z/2)$ is given by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\colon Z \oplus Z \oplus Z/2 \to (A_0 \otimes Z/2) \oplus Z/2$.*

ii) *When $t_* = t_F$ on $Z \oplus Z/2m$, $KC_0X \cong A \oplus C \oplus (Z \oplus Z/2m) \oplus H$ with $H = 0$, $Z/2n$ or $Z/2$ and it is decomposed similarly into one of the three types D4), D5) and D6) corresponding to the above D1), D2) and D3).*

iii) *When $t_* = t_E$ on $Z \oplus Z/2n$, $KC_2X \cong B \oplus C \oplus H \oplus (Z \oplus Z/2)$ with $H = Z/2m$ or $Z/2$ and it is also decomposed into one of the three types E1), E2) and E3) as the case* i).

iv) *When $t_* = t_G$ on $Z \oplus Z/2n$, $KC_2X \cong B \oplus C \oplus H \oplus (Z \oplus Z/2n)$ with $H = Z/2m$ or $Z/2$ and it is also decomposed into one of the three types E4), E5) and E6) as the case* ii).

4.4. By making use of Lemma 4.3 we will here show the third one of our main results.

Theorem 4.4. *Let X be a CW-spectrum such that KU_0X has a direct sum decomposition as* (4.3) *and $KU_1X = 0$. Then there exist abelain groups A_0, A_4, B_2 and B_6 and certain CW-spectra Y and Y' so that X is quasi KO_*-equivalent to the wedge sum $SA_0 \vee \Sigma^2 SB_2 \vee \Sigma^4 SA_4 \vee \Sigma^6 SB_6 \vee Y \vee Y'$. Here Y is taken to be $\Sigma^{2+i} M_{2m}$, $\Sigma^i Q_{2m}$, NP'_{4m} or $R'Q_{2m}$ for $i = 0, 4$ and Y' to be $\{pt\}$ in the* (4.3) i) *case and Y' to be $\Sigma^i M_{2n}$, $\Sigma^{2+i} Q_{2n}$, $\Sigma^2 NP'_{4n}$ or $\Sigma^2 R'Q_{2n}$ for $i = 0, 4$ in the* (4.3) ii) *case.*

Proof. Set $Y_1 = \Sigma^6 M_{2m}$, $Y_2 = \Sigma^2 M_{2m}$, $Y_3 = NP'_{4m}$, $Y_4 = Q_{2m}$, $Y_5 = \Sigma^4 Q_{2m}$, $Y_6 = R'Q_{2m}$ and then $Y'_j = \Sigma^2 Y_j$ for $1 \leqq j \leqq 6$. According to Lemma 4.3 KC_0X is decomposed with the six types D1)-D6), and KC_2X is decomposed with the six types E1)-E6) in the case (4.3) ii). We will prove that X is quasi KO_*-equivalent to the wedge sum $SA_0 \vee \Sigma^2 SB_2 \vee \Sigma^4 SA_4 \vee \Sigma^6 SB_6 \vee (P \wedge SC) \vee Y_i \vee Y'_j$ in each type (Di, Ej). In each type Di) we choose a unique map $f_i\colon Y_i \to KU \wedge X$ whose induced homomorphism in KU-homologies is the canonical injection. Then there exists a map $g_i\colon Y_i \to KC \wedge X$ satisfying $(\zeta_\wedge 1) g_i = f_i$. It is sufficient to find a map $h_i\colon Y_i \to KO \wedge X$ such that $(\varepsilon_{U\wedge} 1) h_i = f_i$ for each i, the $Y' = Y'_j$ case being similarly done.

i) The $Y_2 = \Sigma^2 M_{2m}$ case: We will find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} \Sigma^2 SZ/2m & \xrightarrow{i_M} & \Sigma^2 M_{2m} & \xrightarrow{j_M} & \Sigma^4 \\ \downarrow h_0 & & \downarrow g_2 & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X \end{array}$$

by replacing the map g_2 with $(\zeta_\wedge 1)g_2=f_2$ suitably if necessary. The induced homomorphisms $\tilde{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_2)$: $KO_iM_{2m}\to KO_{i+7}X$ become trivial in dimensions $i=0$, 2 because of Lemma 4.3 D2) and E1)-E3). Hence it is easily seen that the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_2 i_M$: $\Sigma^2 SZ/2m\to KO\wedge X$ is trivial. So we get desired maps h_0, h_1 by applying [12, Lemma 1.3]. However the map h_1: $\Sigma^1\to KO\wedge X$ has an extension $\bar{h}_1$: $\Sigma^1 SZ/2m\to KO\wedge X$ satisfying $\bar{h}_1 i=h_1$. Since $(\eta_\wedge 1)h_1=\bar{h}_1(i\eta)$: $\Sigma^2\to KO\wedge X$, the result is now immediate.

ii) The $Y_3=NP'_{4m}$ case: Note that the induced homomorphisms $\tilde{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_3$: $KO_iNP'_{4m}\to KO_{i+5}X$ are trivial in dimensions $i=0$ and 4, by means of Lemmas 3.2 iv) and 4.3 D3). Then we can find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} \Sigma^0\vee\Sigma^0 & \xrightarrow{i'_{NP}} & NP'_{4m} & \xrightarrow{j'_{NP}} & \Sigma^2 SZ/4m \\ \downarrow h_0 & & \downarrow g_3 & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X. \end{array}$$

Moreover we notice that the composite $h_1\tilde{\eta}$: $\Sigma^1\to KO\wedge X$ becomes trivial because $h_1 j'_{NP}=(\tau\pi\bar{c}^1{}_\wedge 1)g_3$. Then it follows from (1.1) that $(\eta_\wedge 1)h_1=h_1 i\bar{\eta}=h_1 i\pi_2(\eta^2 j, \bar{\eta})$: $SZ/4m\to KO\wedge X$ where π_2: $\Sigma^0\vee\Sigma^0\to\Sigma^0$ stands for the second projection. The result is now immediate.

iii) The $Y_4=Q_{2m}$ case: As in the case i) we can find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} SZ/2m & \xrightarrow{i_Q} & Q_{2m} & \xrightarrow{j_Q} & \Sigma^4 \\ \downarrow h_0 & & \downarrow g_4 & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X \end{array}$$

since the induced homomorphisms $\tilde{\kappa}_{KO}((\tau\pi\bar{c}^1{}_\wedge 1)g_4)$: $KO_iQ_{2m}\to KO_{i+5}X$ are trivial in dimensions $i=0$, 2 by means of Lemma 4.3 D4) and E4)-E6). The map h_1: $\Sigma^1\to KO\wedge X$ is written as the composite $h_1=k_1\eta$ for some map k_1: $\Sigma^0\to KO\wedge X$. Hence we see that $(\eta_\wedge 1)h_1=k_1 j(\tilde{\eta}\eta)$: $\Sigma^2\to KO\wedge X$ which implies our result immediately.

iv) The $Y_6=R'Q_{2m}$ case: We will find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc}
R'_{2m} & \xrightarrow{i_{R',R'Q}} & R'Q_{2m} & \xrightarrow{j_{R'Q}} & \Sigma^8 \\
\downarrow h_0 & & \downarrow g_6 & & \downarrow h_1 \\
KO\wedge X & \longrightarrow & KC\wedge X & \longrightarrow & \Sigma^3 KO\wedge X \\
\| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\
KO\wedge X & \longrightarrow & KU\wedge X & \longrightarrow & \Sigma^2 KO\wedge X
\end{array}$$

by replacing the map g_6 with $(\zeta_\wedge 1)g_6=f_6$ suitably if necessary. The induced homomorphisms $\tilde{\kappa}_{KO}((\tau\pi\bar{c}^{1}{}_\wedge 1)g_6)$: $KO_iR'Q_{2m}\rightarrow KO_{i+5}X$ become trivial in dimensions $i=0$, 4 and 6 by means of Lemmas 3.2 vi) and 4.1 D6), E4)-E6). Then we get a map h'_0: $\Sigma^2\rightarrow KO\wedge X$ such that $(\tau\pi\bar{c}^{1}{}_\wedge 1)g_6i_{R',R'Q}=h'_0jj_{R'}$: $R'_{2m}\rightarrow\Sigma^3KO\wedge X$ and in addition $(\eta_\wedge 1)h'_0=0$. So we obtain desired maps h_0 and h_1 by applying [12, Lemma 1.3]. Since there exists a map k_1: $\Sigma^4\rightarrow KO\wedge X$ with $k_1\eta=h_1$, it follows from (2.3) that $(\eta_\wedge 1)h_1=k_1jj'_R(\tilde{h}_R\eta)$: $\Sigma^6\rightarrow KO\wedge X$. The result is now immediate.

The other cases $Y_1=\Sigma^6M_{2m}$ and $Y_5=\Sigma^4Q_{2m}$ are evidently shown by parallel discussions to the cases i) and iii) respectively.

4.5. We will finally prove our main theorem as a corollary by putting Theorems 3.3, 4.2 and 4.4 together.

Proof of Theorem 2. Recall that the conjugation t_* on $KU_0X\cong H\oplus Z/2m$, $m=2^s$, is represented by one of the matrices given in (3.1) i)-v). If its matrix representation has the type i), we may apply Theorem 4.2 in order to observe that Y is taken to be one of the elementary spectra $\Sigma^{2i}SZ/2m$, $\Sigma^{2i}V_{2m}$, $\Sigma^{2i}N'_{2m}$, $\Sigma^{2i}R'_{2m}$ and $\Sigma^{2j}NR'_{2m}$ for $0\leqq i\leqq 3$ and $0\leqq j\leqq 1$. If it has the type iii) or iv), we may apply Theorem 4.4 in order to observe that Y is taken to be one of the elementary spectra $\Sigma^{2i}M_{2m}$, $\Sigma^{2i}Q_{2m}$, $\Sigma^{2j}NP'_{4m}$ and $\Sigma^{2j}R'Q_{2m}$ for the above i, j. If it has the type ii) or v), we may apply Theorem 3.3 in order to observe that Y is taken to be one of the elementary spectra $\Sigma^{2j}W_{2m}$ $(m=4n)$ and $\Sigma^{2j}MQ_{2m}$ for the above j.

Combining Theorem 2 with Propositions 1.2, 2.3 and 2.4, and then applying [12, Corollary 1.6] with (1.3) and (2.5) we obtain

Corollary 4.5. i) *$N'M_{2m}\underset{KO}{\simeq}NP'_{4m}$, $N'Q_{2m}\underset{KO}{\simeq}P\vee\Sigma^6V_{2m}$, $R'M_{2m}\underset{KO}{\simeq}P\vee\Sigma^4V_{2m}$, $P'Q_{4m}\underset{KO}{\simeq}\Sigma^2MQ_{2m}$ and $P'Q_{2n}\underset{KO}{\simeq}P\vee\Sigma^2SZ/n$ for n odd.*

ii) *$M'N_{2m}\underset{KO}{\simeq}\Sigma^1NP_{4m}$, $M'R_{2m}\underset{KO}{\simeq}P\vee\Sigma^5V_{2m}$, $Q'N_{2m}\underset{KO}{\simeq}P\vee\Sigma^3V_{2m}$, $Q'P_{4m}\underset{KO}{\simeq}MQ'_{2m}$ and $Q'P_{2n}\underset{KO}{\simeq}P\vee\Sigma^3SZ/n$ for n odd.*

iii) *$MQ_{2m}\underset{KO}{\simeq}\Sigma^4MQ_{2m}$, $NP'_{2m}\underset{KO}{\simeq}\Sigma^4NP'_{2m}$, $NR'_{2m}\underset{KO}{\simeq}\Sigma^4NR'_{2m}$ and $R'Q_{2m}\underset{KO}{\simeq}\Sigma^4R'Q_{2m}$.*

iv) *$MQ'_{2m}\underset{KO}{\simeq}\Sigma^4MQ'_{2m}$, $NP_{2m}\underset{KO}{\simeq}\Sigma^4NP_{2m}$, $NR_{2m}\underset{KO}{\simeq}\Sigma^4NR_{2m}$ and $Q'R_{2m}\underset{KO}{\simeq}\Sigma^4Q'R_{2m}$.*

REMARK. By applying [14, Theorem 2.6] we can observe that

$$(4.4) \qquad M'M_{2m} \underset{KO}{\simeq} \Sigma^1 MP_{4m}\,, \quad MP_{2m} \underset{KO}{\simeq} \Sigma^4 MP_{2m} \quad \textit{and} \quad MP'_{2m} \underset{KO}{\simeq} \Sigma^4 MP'_{2m}\,.$$

References

[1] J.F. Adams: *Vector fields on spheres*, Ann. of Math **75** (1962), 603–622.

[2] J.F. Adams: Stable homotopy and generalized homology, Univ. of Chicago, 1974.

[3] D.W. Anderson: *A new cohomology theory*, Thesis, Univ. of California, Berkeley, 1964.

[4] D.W. Anderson: *Universal coefficient theorems for K-theory*, mimeographed notes, Berkeley.

[5] S. Araki and H. Toda: *Multiplicative structures in mod q cohomology theories, I and II*, Osaka J. Math. **2** (1965), 71–115 and **3** (1966), 81–120.

[6] A.K. Bousfield: *A classification of K-local spectra*, preprint.

[7] C.W. Curtis and I. Reiner: Methods of representation theory with applications to finite groups and orders, Pure and Applied Math. A Wiley-Interscience series of texts, monographs and tracts, 1981.

[8] A.G. Kurosh: The theory of groups, I. Chelsea Publ. Co., New York, 1955.

[9] M. Mimura, S. Oka and M. Yasuo: *K-theory and the homotopy types of some function spaces*, in preparation.

[10] E.H. Spanier: *Function spaces and duality*, Ann. of Math. **70** (1959), 338–378.

[11] Z. Yosimura: *Universal coefficient sequences for cohomology theories of CW-spectra, I and II*, Osaka J. Math. **12** (1975), 305–323 and **16** (1979), 201–217.

[12] Z. Yosimura: *Quasi K-homology equivalences, I*, Osaka J. Math. **27** (1990), 465–498.

[13] Z. Yosimura: *The quasi KO-homology types of the real projective spaces*, Proc. Int. Conf. at Kinosaki, Springer-Verlag, **1418** (1990), 156–174.

[14] Z. Yosimura: *The quasi KO-homology types of the stunted real projective spaces*, J. Math. Soc. Japan **42** (1990), 445–466.

[15] Z. Yosimura: *Quasi KO_*-equivalences; Wood spectra and Anderson spectra*, Math. J. Okayama Univ., to appear.

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