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Osaka University
THE FUNDAMENTAL GROUP OF THE SMOOTH
PART OF A LOG FANO VARIETY

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(Received March 22, 1994)

Introduction

Let $X$ be a normal projective variety over the complex number field $\mathbb{C}$. We call $X$ a Fano variety if $X$ is $\mathbb{Q}$-Gorenstein and the anti-canonical divisor $-K_X$ is ample. A Fano variety $X$ is said to be a log Fano variety if $X$ has only log terminal singularities (cf. [6]). A Fano variety $X$ is called a canonical Fano variety if $X$ has only canonical singularities (cf. [6]). The Cartier index $c(X)$ is the smallest positive integer such that $c(X)K_X$ is a Cartier divisor. The Fano index, denoted by $r(X)$, is the largest positive rational number such that $-K_X \sim_q r(X)H$ ($\mathbb{Q}$-linear equivalence) for a Cartier divisor $H$.

This note consists of two sections. In §1, we shall consider canonical Fano 3-folds and prove the following:

**Theorem 1.** Let $X$ be a canonical Fano 3-fold. Let $X^0 := X - \text{Sing } X$ be the smooth part of $X$. Assume that $X$ has only isolated singularities. Then we have:

1. Suppose the Fano index $r(X)$ is 1. Then $\pi_1(X^0) = \mathbb{Z}/c(X)\mathbb{Z}$ (cf. Remark 1.1 in §1).
2. Suppose that the canonical divisor $K_X$ is a Cartier divisor. Then $X^0$ is simply connected.

**Remark.** (1) The assumption that $X$ has only isolated singularities is used to prove Lemma 1.3 in §1.

(2) Using the same proof (see §1) one can show that $\pi_1(X^0) = \mathbb{Z}/c(X)\mathbb{Z}$ when $X$ is a log Fano variety of Fano index one and with only isolated singularities because even in this case the $\mathbb{Z}/c(X)\mathbb{Z}$-covering $Y$ constructed in §1 has only isolated canonical singularities.

In [19], we shall give a universal bound for $c(X)$. Under the much stronger condition that $X$ has only terminal and cyclic quotient singularities, T. Sano
proved that \(c(X) \leq 2\) (cf. \([15]\)).

In §2, we shall consider n-dimensional log Fano varieties and prove the following:

**Theorem 2.** Let \(X\) be a log Fano variety of Fano index \(r(X) > \dim X - 2\). Let \(X^0 := X - \text{Sing } X\) be the smooth part of \(X\). Then we have:

1. The fundamental group \(\pi_1(X^0)\) of \(X^0\) is a finite group.
2. Suppose that \(X\) has only canonical singularities. Then \(\pi_1(X^0)\) is an abelian group of order \(\leq 9\). (See \([9]\) for the unique canonical Fano surface \(X\) with \(|\pi_1(X^0)| = 9\).)
3. Suppose that \(r(X) \geq \dim X - 1\). Then \(\pi_1(X^0)\) is a finite abelian group generated by two elements, and has order \(\leq 9\).
4. Suppose \(r(X) > \dim X - 1\). Then the smooth part \(X^0\) of \(X\) is simply connected.

To prove Theorem 2, we manage to reduce to the dimension two case. In the dimension two case, Theorem 2(1) was proved in our joint works \([3, 4]\) (see \([18]\), and also \([2]\) for a differential geometric proof), and Theorem 2(2) proved in \([8, 9]\). In this reduction process, in order to apply the crucial theorem due to Alexeev, we need the hypothesis about the Fano index \(r(X)\). We hope this hypothesis can be eventually dropped.

Note that in Theorem 2(2) the hypothesis “\(X\) has only canonical singularities” is necessary. Indeed, in \([17]\) we have a lot of examples of log Fano surfaces \(X\) with exactly one triple point and several double points but with a non-abelian \(\pi_1(X^0)\).

We want to remove the condition about the Fano index \(r(X)\) in the above theorems and raise the following question:

Let \(X\) be a Fano variety. Let \(X^0 := X - \text{Sing } X\) be the smooth part of \(X\). Suppose that \(X\) has only log-terminal (or canonical, or terminal) singularities. Is \(\pi_1(X^0)\) a finite group?

In general, it is not true that the smooth part of a rational variety with only log-terminal singularities has finite fundamental group. Just consider the example \((\mathbb{P}^1 \times E) / \tau\) in \([3, \S 1.15]\) where \(E\) is an elliptic curve and \(\tau\) is an involution acting on both \(\mathbb{P}^1\) and \(E\) non-trivially and diagonally. So, the ampleness of \(-K_X\) is an essential condition.

This work was started during the author’s stay in Max-Planck-Institut in Bonn. The author thanks very much to the institute and Professor F. Hirzebruch for the hospitality and financial support. The author also thanks the referee for very careful reading.
1. Proof of Theorem 1

We shall first show that Theorem 1(1) follows from Theorem 1(2). Let \( X \) be as in Theorem 1(1). We have a linear equivalence \(-cK_x \sim cH\) where \( c = c(X) \) is the Cartier index and \( H \) is a primitive Cartier divisor. Here we say that a Cartier divisor \( H \) on \( X \) is primitive if \( O(H) \) is not divisible in the Picard group of \( X \). Let

\[ Y := \text{Spec } \bigoplus_{i=0}^{c-1} O_X(i(K_x + H)). \]

Then the natural \( \mathbb{Z}/c\mathbb{Z} \)-covering morphism \( f : Y \to X \) is etale over the smooth part \( X^o := X - \text{Sing } X \) of \( X \). Moreover, \(-K_Y = -f^*K_x \sim f^*H\). So, \( Y \) is a (Gorenstein) Fano variety with only canonical singularities. Thus, the hypotheses in Theorem 1(2) are satisfied by \( Y \). Note that \( \pi_1(Y^o) = \pi_1(f^{-1}(X^o)) \) because \( Y^o - f^{-1}(X^o) \) has codimension \( \geq 2 \) in the smooth 3-fold \( Y^o := Y - \text{Sing } Y \). So, Theorem 1(1) will follow from Theorem 1(2). Moreover, we have

**Remark 1.1.** \( f^{-1}(X^o) \) is the universal covering of \( X^o \).

Now we shall prove Theorem 1(2). Let \( X \) be as in Theorem 1(2). We can write \(-K_x = rH\) where \( r = r(X) \in \mathbb{Z}_{>0} \) is the Fano index and \( H \) an ample Cartier divisor. Let \( S \subsetneq -K_x \) be a general member. By [12], we know that \( S \) is a K3-surface possibly with rational double singularities. Let \( \sigma : T \to S \) be a minimal resolution of singularities. Then \( T \) is a K3-surface. The first assertion of the following Lemma 1.2 is from [13] or [10, Theorem 5] and the second is a consequence of the first.

**Lemma 1.2.** Let \( T \) be a K3-surface defined over \( \mathbb{C} \). Let \( L \) be a nonzero numerically effective divisor on \( T \). Then we have:

1. \( |L| \) has base points if and only if there exist irreducible curves \( E, \Gamma \), and an integer \( k \geq 2 \) such that \( L \sim kE + \Gamma \), \( (E^2) = 0 \), \( (\Gamma^2) = -2 \), \( E . \Gamma = 1 \). In this case, every member of \( |\Gamma| \) is of the form \( E' + \Gamma \), where \( E' \) is a sum of \( k \) effective divisors \( E_1, \cdots, E_k \) such that \( E_i \sim E \) for all \( i \); in particular, there is an elliptic fibration \( \varphi : T \to \mathbb{P}^1 \) such that \( E \) is a fiber and \( \Gamma \) is a cross-section.
2. \( |sL| \) is base point free for all \( s \geq 2 \).

We need the following lemma which is proved in [16, Theorem 0.5].

**Lemma 1.3.** (1) The singular locus \( \text{Sing } S \) of a general member \( S \subsetneq -K_x \) contains \( S \cap \text{Sing } X \).

(2) If \( r(X) > 1 \) then \( -K_x \) is base point free. Hence a general member \( S \subsetneq -K_x \) is disjoint from \( \text{Sing } X \).

(3) Let \( Bs|-K_x| \) be the base locus. If \( \dim Bs|-K_x| = 1 \) then \( Bs|-K_x| \) is
disjoint from Sing X. Hence a general member \( S \subseteq |-K_X| \) is disjoint from Sing X.

(4) If \( \dim Bs| -K_X| = 0 \) then \( P := Bs| -K_X| \) is a single point and \( P \) is a rational double point of \( S \) of Dynkin type \( A_1 \). So, \( S \cap \text{Sing} X = \emptyset \) or \( \{ P \} \). (Indeed, \( S \cap \text{Sing} X = \{ P \} \) (cf. [16]).

Proof. (1) follows from the condition that the divisor \( S \) is a Cartier divisor on \( X \).

(2) and (3) are proved in [16]. Moreover, we have \( Bs| -K_X| = Bs| -K_X|_{sl} \).

(4) By Lemma 1.2 and by \( \dim Bs| (-K_X)_{sl} | = \dim Bs| -K_X | = 0 \), we see that \( |\sigma^*((-K_X)_{sl})| = |kE| \). Here \( E \) is a fiber of an elliptic fibration \( \varphi : T \to P^1 \), \( \Gamma \) is a cross-section of \( \varphi \) and \( P := \sigma(\Gamma) \) is a singularity on \( S \) with \( \sigma^{-1}(P) = \Gamma \). So, \( P \) is a rational double point of \( S \) of Dynkin type \( A_1 \). We have also \( Bs| -K_X| = \{ P \} \). (4) is proved.

Since the Kodaira D-dimension \( \kappa(X, S) \geq 2 \), the following natural homomorphism is surjective (see Theorem 2.4 below and the arguments after Theorem 2.4)

\[ \pi_i(S - S \cap \text{Sing} X) \to \pi_i(X^o). \]

To finish the proof of Theorem 1(2), we need to prove that \( \pi_1(S - S \cap \text{Sing} X) = (1) \). If \( S \cap \text{Sing} X = \emptyset \), then \( \pi_1(S) = (1) \) and we are done because \( S \) is a K3-surface possibly with rational double singularities. Suppose \( P := S \cap \text{Sing} X \neq \emptyset \), then by Lemma 1.3, \( P \) is a single point and a rational double point of \( S \) of Dynkin type \( A_1 \). Using the notations in Lemmas 1.2 and 1.3, we have \( S \cap \text{Sing} X = S - \{ P \} = T' - \Gamma \). Here \( T' \) is the resolution of the singularity \( P \) on \( S \).

Without loss of generality, we may assume that \( T' = T \) and it suffices to prove the following lemma in order to finish the proof of Theorem 1(2).

**Lemma 1.4.** Let \( T \) be a K3-surface. Let \( \varphi : T \to P^1 \) be an elliptic fibration. Let \( \Gamma \) be a \((-2)\)-curve which is a cross-section of \( \varphi \). Then \( T - \Gamma \) is simply connected.

Proof. Consider the long cohomology exact sequence:

\[ H^2(T, Z) \to H^2(\Gamma, Z) \to H^3(T, \Gamma, Z) \to H^3(T, Z). \]

Since \( T \) is simply connected we have \( H^3(T, Z) = 0 \). Let \( E \) be a fiber of \( \varphi \). Then \( E.\Gamma = 1 \). So the map \( H^2(T, Z) \to H^2(\Gamma, Z) \) takes \( E \) to the generator. Thus, by the duality we have \( H_0(T - \Gamma, Z) = 0 \).

To prove Lemma 1.4, we have only to show that \( \pi_1(T - \Gamma) \) is abelian. Let \( E_i \) be a singular fiber of \( \varphi \). Since the blowing-up of smooth points in the open surface \( T - \Gamma \) does not affect the fundamental group of this open surface, one may assume that \( E_i \) is simple normal crossing and hence \( E_i \) consists of \( P^1 \)'s. Clearly, \( \kappa(T, \)}
Thus, to finish the proof of Lemma 1.4, we have only to prove that $\pi_1((U_1 - \Gamma \cap U_1) \cup (U_2 - \Gamma))$ is abelian. Applying Van-Kampen Theorem, we see easily that $\pi_1(U_1 - \Gamma \cap U_1) \cup (U_2 - \Gamma)) = \pi_1(U_1 - \Gamma \cap U_1)$. Indeed, $\pi_1(U_2 - \Gamma)$ is generated by a loop $\sigma$ around $\Gamma$, and we may assume that $\sigma$ is taken from $(U_1 - \Gamma \cap U_1) \cap (U_2 - \Gamma)$. We may even assume that $\sigma$ is a loop in $E_1 - \Gamma \cap E_1$ around the point $\Gamma \cap E_1$. So, $\sigma$ is contractible in $U_1 - \Gamma \cap U_1$ because $E_1 - \Gamma \cap E_1$ is a union of one $A^1$ and several $P^1$'s.

Now the proof of Lemma 1.4 is reduced to the proof that $\pi_1(U_1 - \Gamma \cap U_1)$ is abelian. Note that $\pi_1(U_1 - \Gamma \cap U_1) = \pi_1(E_1 - \text{the smooth point } \Gamma \cap E_1)$ because $E_1$ is a strong deformation retract of $U_1$, and $\pi_1(E_1 - \text{the smooth point } \Gamma \cap E_1)$ is abelian because $E_1$ is a divisor with simple normal crossing whose irreducible components are all isomorphic to $P^1$ and $E_1$ meets $\Gamma$ transversally in a single point. Note that $\pi_1(E_1)$ is equal to (1) when $E_1$ is a tree of $P^1$'s, and equal to $Z$ when $E_1$ is a simple loop of $P^1$'s. So $\pi_1(U_1 - \Gamma \cap U_1)$ is abelian.

The proof of Lemma 1.4 is completed. This proof does not work when $\Gamma$ is a multiple section.

2. Proof of Theorem 2

Let $X$ be a log Fano variety of dimension $d(d \geq 2)$ satisfying the hypothesis in Theorem 2. Write $-K_X - q\gamma H$ where $r$ is a positive rational number such that $r > d - 2$ and $H$ is an ample Cartier divisor. This is possible by the hypothesis in Theorem 2. We need the following:

**Theorem 2.1** (cf. Theorem 0.5 in [1]). Let $X$ be a log Fano variety. With the above notations and assumptions, then $|H|$ is non-empty and base component free, and a general member $X_{d-1}$ in $|H|$ is a normal projective variety with only log terminal singularities.

By the adjunction formula, we have $-K_x - q(r-1)H|H|$. Hence when $d \geq 3$, $X_{d-1}$ is a log Fano variety by the definition and its Fano index $r(X_{d-1}) \geq r-1 > \dim X_{d-1}$. Applying Alexeev's Theorem $(d-1)$-times, we get a ladder below such that the assertions in the following lemma hold true:

$$(X_d = X, H_d = H), (X_{d-1}, H_{d-1}), \cdots, (X_1, H_1).$$
Lemma 2.2. (1) We have \( X_i \subset |H_{i+1}|, \ H_i = H_{i+1}|X_i \) and \(-K_{X_i} \sim_q (r + i - d)H_i\) for all \( i \geq 1 \). Moreover, \( X_i \)'s \((i \geq 2)\) are log Fano varieties of Fano index \( r(X_i) > \dim X_i - 2 \).

(2) If \( r = d - 1 \), then \( X_1 \) is a nonsingular elliptic curve.

(3) If \( r > d - 1 \), then \( X_1 \) is a nonsingular rational curve.

The following lemma is, though easy, crucial in order to reduce to the dimension two case.

Lemma 2.3. Let \( X \) be a log Fano variety of dimension \( d \) \((d \geq 2)\). With the notations and assumptions at the beginning of the section, we have:

(1) \(|H|\) has at most isolated base points and the base locus \( Bs|H|\) is contained in the smooth part \( X^o \) of \( X \).

(2) The singular locus \( \text{Sing} X_{d-1} \) of \( X_{d-1} \) contains \( X_{d-1} \cap \text{Sing} X \).

(3) If \( X \) has only canonical singularity then so does \( X_{d-1} \).

Proof. Note that (2) follows from the condition that \( X_{d-1} \) is a Cartier divisor on \( X \). The assertion (3) follows from (1) and the definition of canonical singularity.

Now we shall prove (1). Consider the following exact sequence:

\[
0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_{X_{d-1}}(H) \to 0.
\]

By Kawamata's vanishing theorem (cf. [6, Theorem 1-2-5]), we have \( H^1(X, \mathcal{O}_X) = 0 \) and hence we have a surjection:

\[
(*) \quad H^0(X, \mathcal{O}_X(H)) \to H^0(X_{d-1}, \mathcal{O}_{X_{d-1}}(H)).
\]

By the result (*) we have \( Bs|H| = Bs|H_{d-1}| \) where \( H_{d-1} := H_{X_{d-1}} \). So if \( |H_{d-1}| \) has at most isolated base points then the same is true for \(|H|\). If a point \( P \) in \( Bs|H_{d-1}| \) is a smooth point on \( X_{d-1} \) then \( P \) is also a smooth point on \( X \) because \( X_{d-1} \) is a Cartier divisor on \( X \). Thus we are reduced to prove a statement similar to (1) for \( X_{d-1} \) (cf. Lemma 2.2). By the same argument, we can reduce to prove (1) for \( X_2 \). So to prove (1), we may assume that \( \dim X = 2 \).

By Alexeev's Theorem, we may assume that \( H \) is normal. Hence \( H \) is nonsingular because \( \dim H = 1 \). This, together with the condition that \( H \) is a Cartier divisor on \( X \), implies that \( H \) is contained in the smooth part of \( X \). Hence follows the second part of (1). Thus (1) follows because \(|H|\) is base component free by Alexeev's Theorem.

Now we shall apply the following:

Theorem 2.4 (cf. [11, Cor. 2.3]). Let \( \tilde{X} \) be a nonsingular projective variety. Let \( \tilde{H} \) be a divisor on \( \tilde{X} \) such that the Kodaira D-dimension \( \kappa(\tilde{X}, \tilde{H}) \geq 2 \). Let \( \Delta \) be a Zariski-closed proper subset. Let \( U \) be any open tubular neighborhood of
Then the following natural homomorphism is a surjection:
\[ \pi_i(U - \Delta \cap U) \rightarrow \pi_i(\bar{X} - \Delta). \]

Let \( f : \bar{X} \rightarrow X \) be a resolution of singularities such that \( f^*X_{d-1} + \Delta \) is a normal crossing divisor. Here \( \Delta \) is the exceptional divisor of \( f \). By Lemma 2.3(2), \( X_{d-1} := X_{d-1} - \text{Sing } X_{d-1} \) is a Zariski-open subset of \( X_{d-1} - \text{Sing } X = f^*X_{d-1} - \Delta \). So one has a surjective homomorphism
\[ \pi_i(X_{d-1}) \rightarrow \pi_i(f^*X_{d-1} - \Delta). \]

Applying Nori's Theorem to \( \tilde{H} = f^*X_{d-1} \), one obtains a surjective homomorphism:
\[ \pi_i(f^*X_{d-1} - \Delta) \rightarrow \pi_i(X^o). \]

Combining these two surjections, we have proved the following Lemma 2.5 when \( i = d - 1 \). Applying the same arguments several times, one can prove Lemma 2.5 for all \( i \geq 1 \).

**Lemma 2.5.** The natural homomorphism \( \pi_i(X^o) \rightarrow \pi_i(X^o_{i+1}) \) is surjective for all \( i \geq 1 \).

Now Theorem 2(1) follows from Lemma 2.5 because it is true in the dimension two case by [3, 4]. Theorem 2(2) follows from Lemma 2.5 and Lemma 2.3(3) since it is true in the dimension two case by [8, 9].

Theorem 2(4) follows from Lemma 2.5 and Lemma 2.2(3) because now \( X_i = X_1 \approx P^1 \) and \( \pi_i(X^o) = (1) \).

In view of Lemma 2.5 and Theorem 2(4), it suffices to prove Theorem 2(3) in the case where \( d = \dim X = 2 \) and \( r(X) = d - 1 = 1 \). It is easy to see that \( H_1(X^o, Z) \) is finite (cf. e.g. [18, Lemma 1.3]). By Lemma 2.5, \( \pi_i(X^o) \) is a surjective image of \( \pi_1(X_1) = Z \times Z \) because \( X^o_1 = X_1 \) is a nonsingular elliptic curve by Lemma 2.2(2). Theorem 2(3) is proved via [9] for \( X_2 \) is Gorenstein now by Th. 2.1.

We have proved Theorem 2 stated in the Introduction. Actually, the proofs for (3) and (4) of Theorem 2 are self-contained. In other words, we obtained a simpler proof for the result in [3, 4] when \( r(X) \geq \dim X - 1 = 1 \).

**References**


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