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THE FUNDAMENTAL GROUP OF THE SMOOTH PART OF A LOG FANO VARIETY

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Introduction

Let X be a normal projective variety over the complex number field C . We call X a *Fano variety* if X is \mathbb{Q} -Gorenstein and the anti-canonical divisor $-K_X$ is ample. A Fano variety X is said to be a *log Fano variety* if X has only log terminal singularities (cf. [6]). A Fano variety X is called a *canonical Fano variety* if X has only canonical singularities (cf. [6]). The *Cartier index* $c(X)$ is the smallest positive integer such that $c(X)K_X$ is a Cartier divisor. The *Fano index*, denoted by $r(X)$, is the largest positive rational number such that $-K_X \sim_{\mathbb{Q}} r(X)H$ (\mathbb{Q} -linear equivalence) for a Cartier divisor H .

This note consists of two sections. In §1, we shall consider canonical Fano 3-folds and prove the following :

Theorem 1. *Let X be a canonical Fano 3-fold. Let $X^o := X - \text{Sing } X$ be the smooth part of X . Assume that X has only isolated singularities. Then we have :*

- (1) *Suppose the Fano index $r(X)$ is 1. Then $\pi_1(X^o) = \mathbb{Z}/c(X)\mathbb{Z}$ (cf. Remark 1.1 in §1).*
- (2) *Suppose that the canonical divisor K_X is a Cartier divisor. Then X^o is simply connected.*

REMARK. (1) The assumption that X has only isolated singularities is used to prove Lemma 1.3 in §1.

(2) Using the same proof (see §1) one can show that $\pi_1(X^o) = \mathbb{Z}/c(X)\mathbb{Z}$ when X is a log Fano variety of Fano index one and with only isolated singularities because even in this case the $\mathbb{Z}/c(X)\mathbb{Z}$ -covering Y constructed in §1 has only isolated canonical singularities.

In [19], we shall give a universal bound for $c(X)$. Under the much stronger condition that X has only terminal and cyclic quotient singularities, T. Sano

proved that $c(X) \leq 2$ (cf. [15]).

In §2, we shall consider n -dimensional log Fano varieties and prove the following :

Theorem 2. *Let X be a log Fano variety of Fano index $r(X) > \dim X - 2$. Let $X^o := X - \text{Sing } X$ be the smooth part of X . Then we have :*

- (1) *The fundamental group $\pi_1(X^o)$ of X^o is a finite group.*
- (2) *Suppose that X has only canonical singularities. Then $\pi_1(X^o)$ is an abelian group of order ≤ 9 . (See [9] for the unique canonical Fano surface X with $|\pi_1(X^o)| = 9$.)*
- (3) *Suppose that $r(X) \geq \dim X - 1$. Then $\pi_1(X^o)$ is a finite abelian group generated by two elements, and has order ≤ 9 .*
- (4) *Suppose $r(X) > \dim X - 1$. Then the smooth part X^o of X is simply connected.*

To prove Theorem 2, we manage to reduce to the dimension two case. In the dimension two case, Theorem 2(1) was proved in our joint works [3, 4] (see [18], and also [2] for a differential geometric proof), and Theorem 2(2) proved in [8, 9]. In this reduction process, in order to apply the crucial theorem due to Alexeev, we need the hypothesis about the Fano index $r(X)$. We hope this hypothesis can be eventually dropped.

Note that in Theorem 2(2) the hypothesis “ X has only canonical singularities” is necessary. Indeed, in [17] we have a lot of examples of log Fano surfaces X with exactly one triple point and several double points but with a non-abelian $\pi_1(X^o)$.

We want to remove the condition about the Fano index $r(X)$ in the above theorems and raise the following question :

Let X be a Fano variety. Let $X^o := X - \text{Sing } X$ be the smooth part of X . Suppose that X has only log-terminal (or canonical, or terminal) singularities. Is $\pi_1(X^o)$ a finite group ?

In general, it is not true that the smooth part of a rational variety with only log-terminal singularities has finite fundamental group. Just consider the example $(\mathbf{P}^1 \times E)/\tau$ in [3, §1.15] where E is an elliptic curve and τ is an involution acting on both \mathbf{P}^1 and E non-trivially and diagonally. So, the ampleness of $-K_X$ is an essential condition.

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1. Proof of Theorem 1

We shall first show that Theorem 1(1) follows from Theorem 1(2). Let X be as in Theorem 1(1). We have a linear equivalence $-cK_X \sim cH$ where $c=c(X)$ is the Cartier index and H is a primitive Cartier divisor. Here we say that a Cartier divisor H on X is primitive if $\mathcal{O}(H)$ is not divisible in the Picard group of X . Let

$$Y := \text{Spec} \bigoplus_{i=0}^{c-1} \mathcal{O}_X(i(K_X + H)).$$

Then the natural $\mathbb{Z}/c\mathbb{Z}$ -covering morphism $f: Y \rightarrow X$ is etale over the smooth part $X^o := X - \text{Sing } X$ of X . Moreover, $-K_Y = -f^*K_X \sim f^*H$. So, Y is a (Gorenstein) Fano variety with only canonical singularities. Thus, the hypotheses in Theorem 1(2) are satisfied by Y . Note that $\pi_1(Y^o) = \pi_1(f^{-1}(X^o))$ because $Y^o - f^{-1}(X^o)$ has codimension ≥ 2 in the smooth 3-fold $Y^o := Y - \text{Sing } Y$. So, Theorem 1(1) will follow from Theorem 1(2). Moreover, we have

REMARK 1.1. $f^{-1}(X^o)$ is the universal covering of X^o .

Now we shall prove Theorem 1(2). Let X be as in Theorem 1(2). We can write $-K_X = rH$ where $r=r(X) \in \mathbb{Z}_{>0}$ is the Fano index and H an ample Cartier divisor. Let $S \in |-K_X|$ be a general member. By [12], we know that S is a K3-surface possibly with rational double singularities. Let $\sigma: T \rightarrow S$ be a minimal resolution of singularities. Then T is a K3-surface. The first assertion of the following Lemma 1.2 is from [13] or [10, Theorem 5] and the second is a consequence of the first.

Lemma 1.2. *Let T be a K3-surface defined over \mathbb{C} . Let L be a nonzero numerically effective divisor on T . Then we have :*

- (1) *$|L|$ has base points if and only if there exist irreducible curves E, Γ , and an integer $k \geq 2$ such that $L \sim kE + \Gamma$, $(E^2) = 0$, $(\Gamma^2) = -2$, $E \cdot \Gamma = 1$. In this case, every member of $|\Gamma|$ is of the form $E' + \Gamma$, where E' is a sum of k effective divisors E_1, \dots, E_k such that $E_i \sim E$ for all i ; in particular, there is an elliptic fibration $\varphi: T \rightarrow \mathbb{P}^1$ such that E is a fiber and Γ is a cross-section.*
- (2) *$|sL|$ is base point free for all $s \geq 2$.*

We need the following lemma which is proved in [16, Theorem 0.5].

Lemma 1.3. (1) *The singular locus $\text{Sing } S$ of a general member $S \in |-K_X|$ contains $S \cap \text{Sing } X$.*

(2) *If $r(X) > 1$ then $|-K_X|$ is base point free. Hence a general member $S \in |-K_X|$ is disjoint from $\text{Sing } X$.*

(3) *Let $\text{Bs}|-K_X|$ be the base locus. If $\dim \text{Bs}|-K_X| = 1$ then $\text{Bs}|-K_X|$ is*

disjoint from $\text{Sing } X$. Hence a general member $S \in |-K_X|$ is disjoint from $\text{Sing } X$.

(4) If $\dim \text{Bs}|-K_X|=0$ then $P := \text{Bs}|-K_X|$ is a single point and P is a rational double point of S of Dynkin type A_1 . So, $S \cap \text{Sing } X = \emptyset$ or $\{P\}$. (Indeed, $S \cap \text{Sing } X = \{P\}$ (cf. [16])).

Proof. (1) follows from the condition that the divisor S is a Cartier divisor on X .

(2) and (3) are proved in [16]. Moreover, we have $\text{Bs}|-K_X| = \text{Bs}|(-K_X)_{|S}|$.

(4) By Lemma 1.2 and by $\dim \text{Bs}|(-K_X)_{|S}| (= \dim \text{Bs}|-K_X|) = 0$, we see that $|\sigma^*((-K_X)_{|S})| = |kE| + \Gamma$. Here E is a fiber of an elliptic fibration $\varphi: T \rightarrow \mathbf{P}^1$, Γ is a cross-section of φ and $P := \sigma(\Gamma)$ is a singularity on S with $\sigma^{-1}(P) = \Gamma$. So, P is a rational double point of S of Dynkin type A_1 . We have also $\text{Bs}|-K_X| = \{P\}$. (4) is proved.

Since the Kodaira D -dimension $\kappa(X, S) \geq 2$, the following natural homomorphism is surjective (see Theorem 2.4 below and the arguments after Theorem 2.4)

$$\pi_1(S - S \cap \text{Sing } X) \rightarrow \pi_1(X^o).$$

To finish the proof of Theorem 1(2), we need to prove that $\pi_1(S - S \cap \text{Sing } X) = (1)$. If $S \cap \text{Sing } X = \emptyset$, then $\pi_1(S) = (1)$ and we are done because S is a K3-surface possibly with rational double singularities. Suppose $P := S \cap \text{Sing } X \neq \emptyset$, then by Lemma 1.3, P is a single point and a rational double point of S of Dynkin type A_1 . Using the notations in Lemmas 1.2 and 1.3, we have $S - S \cap \text{Sing } X = S - \{P\} = T' - \Gamma$. Here T' is the resolution of the singularity P on S . Without loss of generality, we may assume that $T' = T$ and it suffices to prove the following lemma in order to finish the proof of Theorem 1(2).

Lemma 1.4. *Let T be a K3-surface. Let $\varphi: T \rightarrow \mathbf{P}^1$ be an elliptic fibration. Let Γ be a (-2) -curve which is a cross-section of φ . Then $T - \Gamma$ is simply connected.*

Proof. Consider the long cohomology exact sequence :

$$H^2(T, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^3(T, \Gamma; \mathbf{Z}) \rightarrow H^3(T, \mathbf{Z}).$$

Since T is simply connected we have $H^3(T, \mathbf{Z}) = 0$. Let E be a fiber of φ . Then $E \cdot \Gamma = 1$. So the map $H^2(T, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{Z})$ takes E to the generator. Thus, by the duality we have $H_1(T - \Gamma, \mathbf{Z}) = (0)$.

To prove Lemma 1.4, we have only to show that $\pi_1(T - \Gamma)$ is abelian. Let E_1 be a singular fiber of φ . Since the blowing-up of smooth points in the open surface $T - \Gamma$ does not affect the fundamental group of this open surface, one may assume that E_1 is simple normal crossing and hence E_1 consists of \mathbf{P}^1 's. Clearly, $\kappa(T,$

$E_1 + \Gamma = 2$. So, we can apply [11, Cor. 2.3, see Theorem 2.4 below]. Let U_1 , U_2 be (open) tubular neighborhoods of sufficiently small radii in T of the curves E_1 , Γ , respectively. Then $U_1 \cup U_2$ is an open tubular neighborhood of $E_1 + \Gamma$. Hence the following natural homomorphism is surjective :

$$\pi_1((U_1 - \Gamma \cap U_1) \cup (U_2 - \Gamma)) \rightarrow \pi_1(T - \Gamma).$$

Thus, to finish the proof of Lemma 1.4, we have only to prove that $\pi_1((U_1 - \Gamma \cap U_1) \cup (U_2 - \Gamma))$ is abelian. Applying Van-Kampen Theorem, we see easily that $\pi_1((U_1 - \Gamma \cap U_1) \cup (U_2 - \Gamma)) = \pi_1(U_1 - \Gamma \cap U_1)$. Indeed, $\pi_1(U_2 - \Gamma)$ is generated by a loop σ around Γ , and we may assume that σ is taken from $(U_1 - \Gamma \cap U_1) \cap (U_2 - \Gamma)$. We may even assume that σ is a loop in $E_1 - \Gamma \cap E_1$ around the point $\Gamma \cap E_1$. So, σ is contractible in $U_1 - \Gamma \cap U_1$ because $E_1 - \Gamma \cap E_1$ is a union of one A^1 and several P^1 's.

Now the proof of Lemma 1.4 is reduced to the proof that $\pi_1(U_1 - \Gamma \cap U_1)$ is abelian. Note that $\pi_1(U_1 - \Gamma \cap U_1) = \pi_1(E_1 - \text{the smooth point } \Gamma \cap E_1 \text{ of } E_1)$ because E_1 is a strong deformation retract of U_1 , and $\pi_1(E_1 - \text{the smooth point } \Gamma \cap E_1 \text{ of } E_1) = \pi_1(E_1)$ because E_1 is a divisor with simple normal crossing whose irreducible components are all isomorphic to P^1 and E_1 meets Γ transversally in a single point. Note that $\pi_1(E_1)$ is equal to (1) when E_1 is a tree of P^1 's, and equal to Z when E_1 is a simple loop of P^1 's. So $\pi_1(U_1 - \Gamma \cap U_1)$ is abelian.

The proof of Lemma 1.4 is completed. This proof does not work when Γ is a multiple section.

2. Proof of Theorem 2

Let X be a log Fano variety of dimension d ($d \geq 2$) satisfying the hypothesis in Theorem 2. Write $-K_X \sim_q rH$ where r is a positive rational number such that $r > d - 2$ and H is an ample Cartier divisor. This is possible by the hypothesis in Theorem 2. We need the following :

Theorem 2.1 (cf. Theorem 0.5 in [1]). *Let X be a log Fano variety. With the above notations and assumptions, then $|H|$ is non-empty and base component free, and a general member X_{d-1} in $|H|$ is a normal projective variety with only log terminal singularities.*

By the adjunction formula, we have $-K_{X_{d-1}} \sim_q (r-1)H|_{X_{d-1}}$. Hence when $d \geq 3$, X_{d-1} is a log Fano variety by the definition and its Fano index $r(X_{d-1}) \geq r-1 > \dim X_{d-1} - 2$.

Applying Alexeev's Theorem $(d-1)$ -times, we get a *ladder* below such that the assertions in the following lemma hold true :

$$(X_d = X, H_d = H), (X_{d-1}, H_{d-1}), \dots, (X_1, H_1).$$

Lemma 2.2. (1) We have $X_i \in |H_{i+1}|$, $H_i = H_{i+1|X_i}$ and $-K_{X_i} \sim_{\mathbb{Q}} (r+i-d)H_i$ for all $i \geq 1$. Moreover, X_i 's ($i \geq 2$) are log Fano varieties of Fano index $r(X_i) > \dim X_i - 2$.

(2) If $r = d - 1$, then X_1 is a nonsingular elliptic curve.
 (3) If $r > d - 1$, then X_1 is a nonsingular rational curve.

The following lemma is, though easy, crucial in order to reduce to the dimension two case.

Lemma 2.3. Let X be a log Fano variety of dimension d ($d \geq 2$). With the notations and assumptions at the beginning of the section, we have :

(1) $|H|$ has at most isolated base points and the base locus $\text{Bs}|H|$ is contained in the smooth part X^0 of X .
 (2) The singular locus $\text{Sing } X_{d-1}$ of X_{d-1} contains $X_{d-1} \cap \text{Sing } X$.
 (3) If X has only canonical singularity then so does X_{d-1} .

Proof. Note that (2) follows from the condition that X_{d-1} is a Cartier divisor on X . The assertion (3) follows from (1) and the definition of canonical singularity.

Now we shall prove (1). Consider the following exact sequence :

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(H) \rightarrow \mathcal{O}_{X_{d-1}}(H) \rightarrow 0.$$

By Kawamata's vanishing theorem (cf. [6, Theorem 1-2-5]), we have $H^1(X, \mathcal{O}_X) = 0$ and hence we have a surjection :

$$(*) \quad H^0(X, \mathcal{O}_X(H)) \rightarrow H^0(X_{d-1}, \mathcal{O}_{X_{d-1}}(H)).$$

By the result (*), we have $\text{Bs}|H| = \text{Bs}|H_{d-1}|$ where $H_{d-1} := H|_{X_{d-1}}$. So if $|H_{d-1}|$ has at most isolated base points then the same is true for $|H|$. If a point P in $\text{Bs}|H_{d-1}|$ is a smooth point on X_{d-1} then P is also a smooth point on X because X_{d-1} is a Cartier divisor on X . Thus, we are reduced to prove a statement similar to (1) for X_{d-1} (cf. Lemma 2.2). By the same argument, we can reduce to prove (1) for X_2 . So to prove (1), we may assume that $\dim X = 2$.

By Alexeev's Theorem, we may assume that H is normal. Hence H is nonsingular because $\dim H = 1$. This, together with the condition that H is a Cartier divisor on X , implies that H is contained in the smooth part of X . Hence follows the second part of (1). Thus (1) follows because $|H|$ is base component free by Alexeev's Theorem.

Now we shall apply the following :

Theorem 2.4 (cf. [11, Cor. 2.3]). Let \tilde{X} be a nonsingular projective variety. Let \tilde{H} be a divisor on \tilde{X} such that the Kodaira D-dimension $\kappa(\tilde{X}, \tilde{H}) \geq 2$. Let Δ be a Zariski-closed proper subset. Let U be any open tubular neighborhood of

\tilde{H} in \tilde{X} . Then the following natural homomorphism is a surjection :

$$\pi_1(U - \Delta \cap U) \rightarrow \pi_1(\tilde{X} - \Delta).$$

Let $f : \tilde{X} \rightarrow X$ be a resolution of singularities such that $f^*X_{d-1} + \Delta$ is a normal crossing divisor. Here Δ is the exceptional divisor of f . By Lemma 2.3(2), $X_{d-1}^o := X_{d-1} - \text{Sing } X_{d-1}$ is a Zariski-open subset of $X_{d-1} - \text{Sing } X = f^*X_{d-1} - \Delta$. So one has a surjective homomorphism

$$\pi_1(X_{d-1}^o) \rightarrow \pi_1(f^*X_{d-1} - \Delta).$$

Applying Nori's Theorem to $\tilde{H} := f^*X_{d-1}$, one obtains a surjective homomorphism :

$$\pi_1(f^*X_{d-1} - \Delta) \rightarrow \pi_1(X^o).$$

Combining these two surjections, we have proved the following Lemma 2.5 when $i = d-1$. Applying the same arguments several times, one can prove Lemma 2.5 for all $i \geq 1$.

Lemma 2.5. *The natural homomorphism $\pi_1(X_i^o) \rightarrow \pi_1(X_{i+1}^o)$ is surjective for all $i \geq 1$.*

Now Theorem 2(1) follows from Lemma 2.5 because it is true in the dimension two case by [3, 4]. Theorem 2(2) follows from Lemma 2.5 and Lemma 2.3(3) since it is true in the dimension two case by [8, 9].

Theorem 2(4) follows from Lemma 2.5 and Lemma 2.2(3) because now $X_1^o = X_1 \cong \mathbb{P}^1$ and $\pi_1(X_1^o) = \{1\}$.

In view of Lemma 2.5 and Theorem 2(4), it suffices to prove Theorem 2(3) in the case where $d = \dim X = 2$ and $r(X) = d-1 = 1$. It is easy to see that $H_1(X^o, \mathbb{Z})$ is finite (cf. e.g. [18, Lemma 1.3]). By Lemma 2.5, $\pi_1(X^o)$ is a surjective image of $\pi_1(X_1^o) = \mathbb{Z} \times \mathbb{Z}$ because $X_1^o = X_1$ is a nonsingular elliptic curve by Lemma 2.2(2). Theorem 2(3) is proved via [9] for X_2 is Gorenstein now by Th. 2.1.

We have proved Theorem 2 stated in the Introduction. Actually, the proofs for (3) and (4) of Theorem 2 are self-contained. In other words, we obtained a simpler proof for the result in [3, 4] when $r(X) \geq \dim X - 1 = 1$.

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