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Paper

Intrinsic localized mode as in-plane vibration in two-dimensional Fermi-Pasta-Ulam lattices

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Abstract: Intrinsic localized modes (ILMs) in two dimensional Fermi-Pasta-Ulam lattices are investigated. We consider in-plane vibrations of particles which have two degrees of freedom. We find two types of ILMs, quasi-one dimensional ILM and two dimensional ILM. Effect of interaction of second nearest neighbor lattices on stricture is also discussed.

Key Words: intrinsic localized mode, discrete breather, FPU lattice

1. Introduction

Intrinsic localized modes (ILMs) or discrete breathers (DBs) have been studied extensively since the first report by Sievers and Takeno [1, 2]. ILM is a time-periodic and space-localized structure which appears in nonlinear lattices. Both theoretical and numerical studies on ILM have been performed [3]. Existence of stationary ILMs has been proved in widely range of nonlinear lattices based on the anti–continuous approach. Moreover numerical results have been shown that excitation of ILMs in various systems including systems which have not been proved the existence of ILM. Recently, some experimental results for observing the excitation of ILMs have been also reported [4, 5].

So far, many studies about ILM have been performed in one–dimensional lattices such as Fermi–Pasta-Ulam (FPU) lattices, nonlinear Klein-Gordon lattices and discrete nonlinear Schrödinger lattices. Even in the one dimensional systems, structure of ILMs is quite different from other structures which are observed in continuum nonlinear systems due to discreteness of the systems. For example, two types of ILM are possible in the one dimensional lattice in terms of symmetry of structures: odd mode or Sievers–Takeno (ST) mode [2] and even mode or Page (P) mode [6]. In odd modes, center of localization is one particle. In even modes, on the other hand, center of localization is a midpoint of two neighboring particles.

Excitation of ILM does not depend on dimension of systems. Flach has investigated energy thresholds of ILM in high dimensional systems [7]. Marín has reported excitation of moving ILM from initial perturbations in two dimensional lattices with Lenard-Jones potential [8]. Butt has studied about ILM in two dimensional square FPU lattice with one degree of freedom [9]. Recently excitation of

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ILMs in high dimensional systems with realistic interaction potential has been also studied [10].

In the present study, we investigate structure of ILM in two dimensional FPU which contains interaction between nearest neighbor and second nearest neighbor particles. Ikeda has been studied chaotic breathers in two dimensional FPU systems [11]. In this study, two types of localized structure are excited in the system. However detailed structure in this system has not been clarified yet. Structure of ILM deeply depends on the structure of lattice. We focus on effects of second nearest neighbor interaction on the shape of ILMs.

The remainder of the paper is organized as follows. In section 2, we present the two dimensional FPU lattices. Brief explanation of numerical method is given in section 3. In section 4, numerical results for ILM in two dimensional FPU lattice and discussions are given. In section 5, we present the conclusions of the present study.

2. Models

We consider two dimensional Fermi-Pasta-Ulam systems. Models are shown in Fig. 1. Particles are positioned repeatedly in both x and y direction. Motion of particles is limited in (x, y) plane. A particle interacts with its nearest neighbor particles and second nearest neighbor particles. Hamiltonian of the system is given by

$$H = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}m\dot{\mathbf{X}}_{i,j}^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}[V(\mathbf{X}_{i+1,j}-\mathbf{X}_{i,j}, d) + V(\mathbf{X}_{i,j+1}-\mathbf{X}_{i,j}, d)]$$
$$+k\sum_{i=1}^{N}\sum_{j=1}^{N}\left[V(\mathbf{X}_{i+1,j+1}-\mathbf{X}_{i,j}, \bar{d}) + V(\mathbf{X}_{i-1,j+1}-\mathbf{X}_{i,j}, \bar{d})\right]. \quad (1)$$

Where $\mathbf{X}_{i,j} = (X_{i,j}, Y_{i,j})$, m, d and $\bar{d}$ represent the position of (i, j)-th particle, a mass of particle, the equilibrium distance for the paring potential linking nearest neighbor particles and the equilibrium distance for the paring potential linking second nearest neighbor particles, respectively. We set $m = 1$ and $d = 1$ in our simulations. The inter-site interaction potential V is FPU-β type as follow:

$$V(\mathbf{r}, d) = \frac{1}{2}(|\mathbf{r}| - d)^2 + \frac{1}{4}\beta(|\mathbf{r}| - d)^4, \quad (2)$$

where β is the nonlinear parameter of the system. We set $\beta = 1$ in our numerical simulations. Interaction between the second nearest neighbor particles is $k(< 1)$ times smaller than interaction between the nearest neighbor particles. Periodic boundary conditions are considered for both the x and y directions.

$$\mathbf{X}_{0,j} = \mathbf{X}_{N,j}, \mathbf{X}_{N+1,j} = \mathbf{X}_{1,j}, \quad (j = 1, 2, \ldots, N) \quad (3)$$
$$\mathbf{X}_{i,0} = \mathbf{X}_{i,N}, \mathbf{X}_{i,N+1} = \mathbf{X}_{i,1}, \quad (j = 0, 1, \ldots, N + 1) \quad (4)$$

The equations of motion are given as

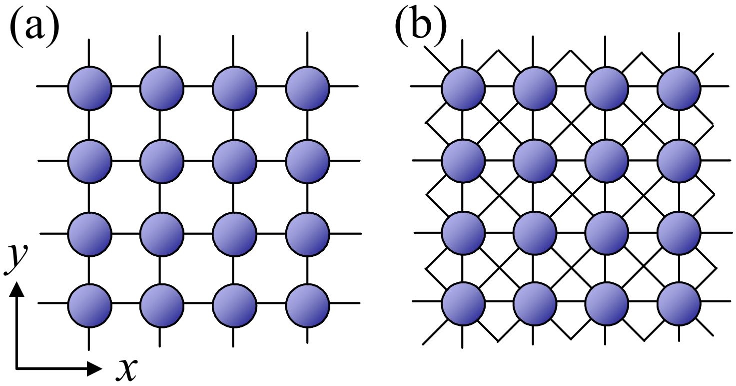


Fig. 1. Model with (a) nearest neighbor interactions and (b) nearest and second nearest neighbor interactions.

$$
\begin{aligned}
m\ddot{\mathbf{X}}_{i,j} = \quad & V'(\mathbf{X}_{i+1,j} - \mathbf{X}_{i,j}, d) + V'(\mathbf{X}_{i-1,j} - \mathbf{X}_{i,j}, d) \\
& + V'(\mathbf{X}_{i,j+1} - \mathbf{X}_{i,j}, d) + V'(\mathbf{X}_{i,j-1} - \mathbf{X}_{i,j}, d) \\
& + kV'(\mathbf{X}_{i+1,j+1} - \mathbf{X}_{i,j}, \bar{d}) + kV'(\mathbf{X}_{i-1,j-1} - \mathbf{X}_{i,j}, \bar{d}) \\
& + kV'(\mathbf{X}_{i-1,j+1} - \mathbf{X}_{i,j}, \bar{d}) + kV'(\mathbf{X}_{i+1,j-1} - \mathbf{X}_{i,j}, \bar{d}). \qquad (5)
\end{aligned}
$$

Where prime indicates $\partial/\partial \mathbf{X}_{i,j}$, which is explicitly written by

$$V'(\mathbf{x}, d) = f(|\mathbf{x}| - d)\frac{\mathbf{x}}{|\mathbf{x}|}, \qquad (6)$$

where $f(r)$ represents strength of force

$$f(r) = r + \beta r^3. \qquad (7)$$

For following discussions, we introduce new variables $\mathbf{x}_{i,j} = (x_{i,j}, y_{i,j})$ or displacements from the equilibrium positions:

$$\mathbf{x}_{i,j} = \mathbf{X}_{i,j} - \mathbf{X}_{0i,j}, \qquad (8)$$

where $\mathbf{X}_{0i,j} = (id, jd)$ is the equilibrium position of (i,j)-th particle. The equations of motions can be rewritten in terms of $\mathbf{x}_{i,j}$

$$
\begin{aligned}
m\ddot{\mathbf{x}}_{i,j} = \quad & V'(\mathbf{x}_{i+1,j} - \mathbf{x}_{i,j} + d\mathbf{e}_1, d) + V'(\mathbf{x}_{i-1,j} - \mathbf{x}_{i,j} - d\mathbf{e}_1, d) \\
& + V'(\mathbf{x}_{i,j+1} - \mathbf{x}_{i,j} + d\mathbf{e}_2, d) + V'(\mathbf{x}_{i,j-1} - \mathbf{x}_{i,j} - d\mathbf{e}_2, d) \\
& + kV'(\mathbf{x}_{i+1,j+1} - \mathbf{x}_{i,j} + d\mathbf{e}_1 + d\mathbf{e}_2, \bar{d}) + kV'(\mathbf{x}_{i-1,j-1} - \mathbf{x}_{i,j} - d\mathbf{e}_1 - d\mathbf{e}_2, \bar{d}) \\
& + kV'(\mathbf{x}_{i-1,j+1} - \mathbf{x}_{i,j} - d\mathbf{e}_1 + d\mathbf{e}_2, \bar{d}) + kV'(\mathbf{x}_{i+1,j-1} - \mathbf{x}_{i,j} + d\mathbf{e}_1 - d\mathbf{e}_2, \bar{d}), \qquad (9)
\end{aligned}
$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vector for x and y direction, respectively.

Substituting a plane wave solutions $\mathbf{x}_{i,j} = \mathbf{a}_{i,j} \exp(\mathbf{q} \cdot \mathbf{X}_{0i,j} - \omega t)$ into linearized equation of (9), we obtain the linear dispersion relation as

$$
\begin{aligned}
\omega_+^2 &= -4A\cos q_1 \cos q_2 + 2(1+2A) - (\cos q_1 + \cos q_2) + \sqrt{(\cos q_1 - \cos q_2)^2 + 16B^2 \sin^2 q_1 \sin^2 q_2}, \\
\omega_-^2 &= -4A\cos q_1 \cos q_2 + 2(1+2A) - (\cos q_1 + \cos q_2) - \sqrt{(\cos q_1 - \cos q_2)^2 + 16B^2 \sin^2 q_1 \sin^2 q_2}.
\end{aligned}
\qquad (10)
$$

where $\mathbf{q} = (q_1, q_2)$ is wave vector, A and B are constants and defined as,

$$A = \frac{k}{4}(4 - \sqrt{2}\bar{d} + 16\beta - 18\sqrt{2}\bar{d}\beta + 12\bar{d}^2\beta - \sqrt{2}\bar{d}^3\beta), \qquad (11)$$

$$B = \frac{k}{4}(\sqrt{2}\bar{d} + 8\beta - 6\sqrt{2}\bar{d}\beta + \sqrt{2}\bar{d}^3\beta). \qquad (12)$$

Therefore the maximum angular frequency is given by

$$\omega_{\max}^2 = 4(1+2A). \qquad (13)$$

Note that A and B is simplified when $\bar{d} = \sqrt{2}$ and $B = 0$ as $A = B = k/2$.

3. Numerical method

A Stationary ILM which has a internal frequency ω_{ILM} corresponds to a periodic orbit in the phase space with a period $T = 2\pi/\omega_{\mathrm{ILM}}$. We consider a $4N^2$ dimensional phase space $\{\mathbf{x}_i, \dot{\mathbf{x}}_i\}$. We define a map F from a point $\mathbf{P}(t_0)$ from a point $\mathbf{P}(t_0 + \Delta t)$ which is a temporal evolution of $\mathbf{P}(t_0)$ during Δt by the equation of motion (9),

$$\mathbf{P}(t_0 + \Delta t) = F(\mathbf{P}(t_0); \Delta t). \qquad (14)$$

If a point $P(t_0)$ is on a periodic orbit which corresponds to ILM with an internal frequency ω_{ILM}, we obtain a relation

$$\mathbf{P}(t_0) = \mathbf{P}(t_0 + 2\pi/\omega_{\mathrm{ILM}}) = F(\mathbf{P}(t_0); 2\pi/\omega_{\mathrm{ILM}}). \tag{15}$$

Therefore we can search the numerical solutions of ILM by solving a equations for $\mathbf{P}(t_0)$ as follows:

$$F(\mathbf{P}(t_0); 2\pi/\omega_{\mathrm{ILM}}) - \mathbf{P}(t_0) = \mathbf{0}. \tag{16}$$

We solve Eq. (16) by Newton-Raphson method. The map F cannot be given explicitly. We calculate F as the temporal evolution of (9) of the initial guess $\bar{\mathbf{X}}(t_0)$. For the calculation of the temporal evolution, we use 6th order symplectic integration method.

Selection of the initial guess $\bar{\mathbf{X}}(t_0)$ is important for good progress of iterations. In quasi one-dimensional ILM case, we make the initial guess from the ILM solution of one-dimensional lattice like:

$$X_{0,0} = a_0 \tag{17}$$
$$X_{\pm 1,0} = -b_0, \tag{18}$$
$$X_{i,j} = 0 \quad \text{(otherwise)}, \tag{19}$$
$$Y_{i,j} = 0, \tag{20}$$

where a_0 and $b_0 (< a_0)$ are positive constants. In two dimensional ILM case, we make a simple initial guess like:

$$X_{0,0} = a_0, Y_{0,0} = a_0, \tag{21}$$
$$X_{1,0} = -a_0, Y_{1,0} = a_0, \tag{22}$$
$$X_{1,1} = -a_0, Y_{1,1} = -a_0, \tag{23}$$
$$X_{0,1} = a_0, Y_{0,1} = -a_0, \tag{24}$$
$$X_{i,j} = 0, Y_{i,j} = 0 \quad \text{(otherwise)}. \tag{25}$$

Once we obtain the numerical solutions from the initial guess in above, we can continue the solution of the parameters of small difference by using the new initial guess as the numerical solutions. Therefore our numerical solutions is a family from a specific initial guess. More details on the continuation of ILMs in systems with Euclidean invariance can be found in [12, 13].

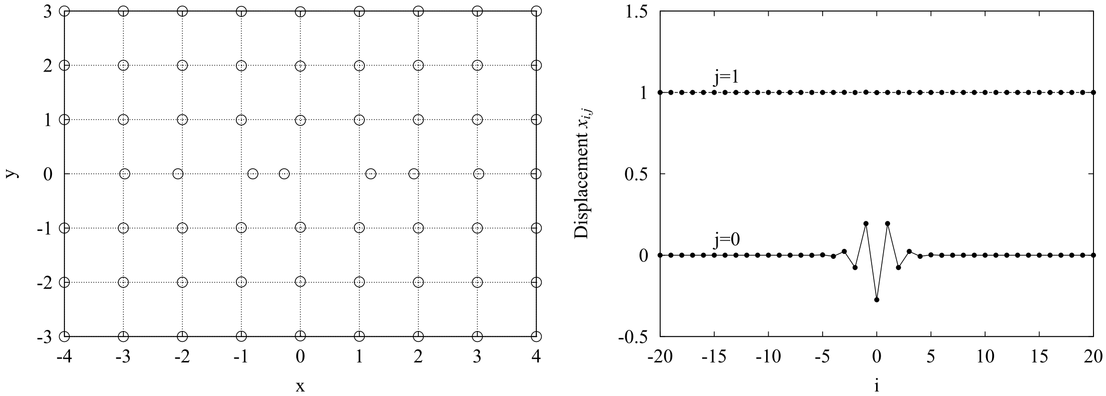
4. Results and discussion

We find two types of ILMs: quasi-one dimensional ILMs and two dimensional ILMs. Structure of quasi-one dimensional ILMs is similar to ILM in one dimensional lattices. Two dimensional ILM has large displacement in both x and y directions. We discuss structure of two types of ILM in following subsections in detail.

4.1 Quasi-1D case

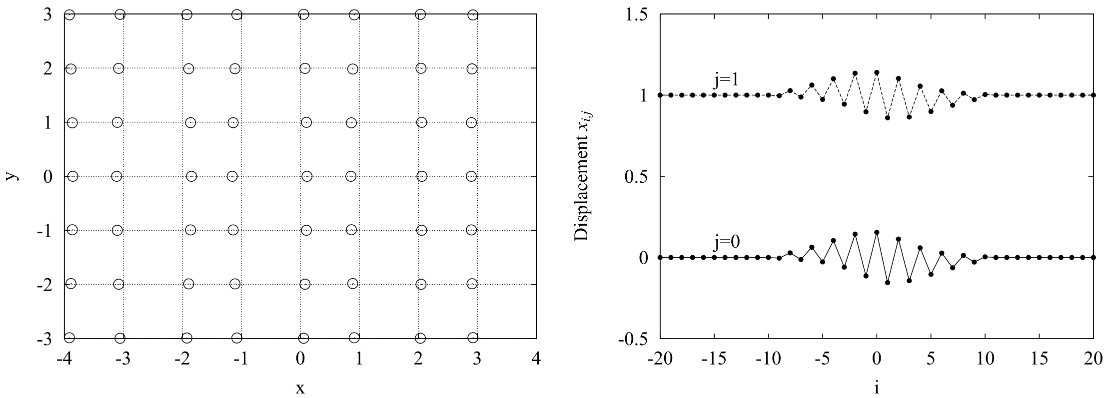
Quasi-one dimensional ILM is excited on one lattice line in the system. Figure 2 shows structure of quasi-one dimensional ILM. We can see that particles on a lattice line $j = 0$ are displaced. Displacement of particles is parallel to the lattice line $j = 0$, that is, particles in ILM vibrate longitudinally. Displacement of the particle on the lattice line $j = \pm 1$ shows fast decays due to the effect of large nonlinear interactions (very small order but nonzero). Therefore, structure of quasi-one dimensional ILM is similar to ILM in one-dimensional lattices. We can obtain both even-type and odd-type ILMs. Note that moving quasi-one dimensional ILM has been reported by Marín [8]. In our simulations, static quasi-one dimensional ILM in the two dimensional FPU lattice also becomes moving one by small perturbations.

Interaction of second-nearest neighbor particles affects structure of quasi-one dimensional ILM. Figure 3 shows the structure of ILMs in the system with second nearest neighbor lattices. It is

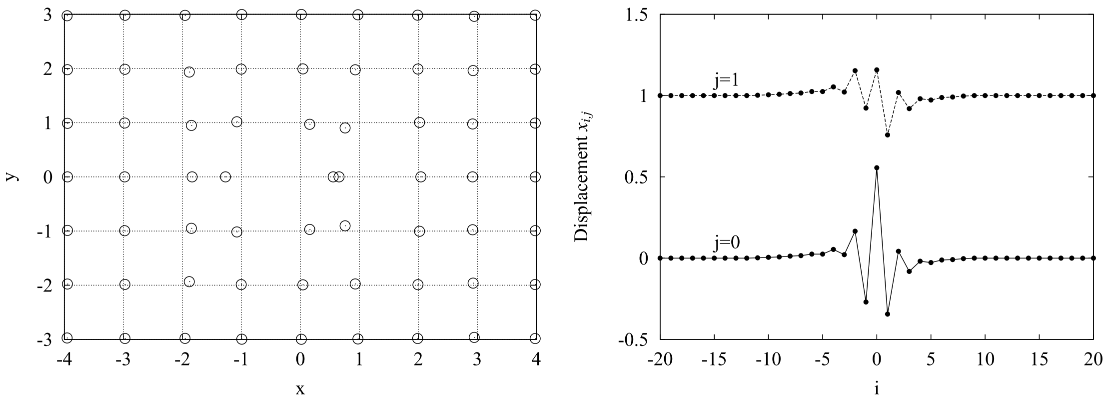


(a) Structure in plane (b) Displacement in x-direction

Fig. 2. Structure of quasi-one dimensional ILM in the system with $k = 0$. Internal frequency is $\omega = 2.4$.



(a) Structure in plane (b) Displacement in x-direction



(a) Structure in plane (b) Displacement in x-direction

Fig. 3. Structure of quasi-one dimensional ILM in the system with $k = 1$ and $\bar{d} = 1$. Internal frequency is $\omega = 4.2$ (top) and $\omega = 4.6$ (bottom).

found that particles on $j \neq 0$ is also displaced. Displacement of particles on $j \neq 0$ are in phase with displacement of particles on $j = 0$. In high localization cases, however, displacement which is perpendicular to ILM's displacement also excited shown in bottom plane in Fig. 3. This is because the displacement along the perpendicular direction decay slowly due to the linear effect between interactions when $k \neq 0$. It is note that, in call cases, shape of ILM is symmetry at the lattice line $j = 0$.

Most significant effect of second nearest interaction is deformation of shape of ILMs. In Fig. 3, we find that ILMs lost its symmetry at the center of localization. ILM no longer has even or odd symmetry.

In what follows we derive the approximated equations of motion in x-direction of particles on the lattice line ($j = 0$) in which ILM is excited. We make following assumptions from the numerical results:

1. Displacement of x-direction on the lattice line $j = 0$ is larger than others.
2. Displacement of y-direction of the lattice line $j = 0$ is zero.
3. Displacement of y-direction is smaller than x-direction.

$$\mathbf{x}_{i,j} = \begin{cases} (x_{i,0}, 0) & \text{for j=0} \\ (\varepsilon x_{i,j}, \varepsilon^2 y_{i,j}) & \text{for others} \end{cases} \tag{26}$$

4. Displacement of particles takes mirror symmetry at the lattice line $j = 0$.

$$\begin{cases} x_{i,j} = x_{i,-j} \\ y_{i,j} = -y_{i,-j} \end{cases} \tag{27}$$

Substituting (26) and (27) into (9) and after some calculations, we obtain the approximated equation of motion for particles on $j = 0$:

$$m\ddot{x}_{i,0} = f_x + f_y + kf_{xy}, \tag{28}$$

where f_x, f_y and f_{xy} are forces which act between the nearest-neighbor particles in x-direction, those in y-direction, and second nearest neighbor particles, respectively:

$$f_x = (x_{i-1,0} + x_{i+1,0} - 2x_{i,0}) + \beta[(x_{i-1,0} - x_{i,0})^3 + (x_{i+1,0} - x_{i,0})^3], \tag{29}$$

$$f_y = \frac{2}{R(x_{i,0})}\left\{-(x_{i,0} - \varepsilon\frac{x_{i,1}d^2}{R(x_{i,0})^2})f(R(x_{i,0}) - d) + \varepsilon\frac{x_{i,1}x_{i,0}^2}{R(x_{i,0})}g(R(x_{i,0}) - d)\right\} \tag{30}$$

$$\begin{aligned} f_{xy} \quad = \quad & \frac{2(d - x_{i,0})}{R_1}f(R_1 - \bar{d}) - \frac{2(d + x_{i,0})}{R_2}f(R_2 - \bar{d}) \\ & +\varepsilon\frac{2x_{i+1,1}}{R_1^2}\left\{\frac{d^2}{R_1}f(R_1 - \bar{d}) + (d - x_{i,0})^2 g(R_1 - \bar{d})\right\} \\ & +\varepsilon\frac{2x_{i-1,1}}{R_2^2}\left\{\frac{d^2}{R_2}f(R_2 - \bar{d}) + (d + x_{i,0})^2 g(R_2 - \bar{d})\right\}. \end{aligned} \tag{31}$$

where $R_1 = R(d - x_{i,0})$, $R_2 = R(d + x_{i,0})$, f is given by (7), g and R are given by

$$g(r) = f'(r) = 1 + 3\beta r^2 \tag{32}$$

$$R(x) = \sqrt{x^2 + d^2}. \tag{33}$$

The first term f_x represents that interaction between the particles on the lattice lines of ILM ($j = 0$). This is similar to interaction of one dimensional FPU-β lattices.

The second term f_y represents interaction between a particle in ILM and its nearest neighbor particles along to y-direction. In numerical results, we find that displacement of particles on adjacent lattice lines is synchronized with displacement of ILMs, that is,

$$x_{i,\pm 1} = \alpha_i x_{i,0}, \tag{34}$$

where $\alpha_i < 1$ are constants. Substituting (34) into (30), we obtain

$$f_y(x_{i,0}) = -\frac{2}{R(x_{i,0})}\left\{x_{i,0}(1 - \varepsilon\frac{\alpha_i d^2}{R(x_{i,0})^2})f(R(x_{i,0}) - d) - \varepsilon x_{i,0}^3\frac{\alpha_i}{R(x_{i,0})}g(R(x_{i,0}) - d)\right\}. \tag{35}$$

In the case of quasi-one dimensional ILM, a function f_y depends only on $x_{i,0}$. Considering $R(x_{i,0}) = R(-x_{i,0})$, it is found that the function $f_y(x_{i,0})$ is a odd function in terms of $x_{i,0}$. Therefore f_y can be regarded as a effective force due to a nonlinear on-site potential in which each particle on quasi-one dimensional ILM is trapped. Moreover this on-site potential is symmetric at $x_{i,0} = 0$.

The third term f_{xy} represents interaction between a particles and the second nearest neighbor particles. f_{xy} can be written in terms of $x_{i,0}$ as follows:

$$f_{xy}(x_{i,0}) = f_{xy}^{(1)}(x_{i,0}) + f_{xy}^{(2)}(x_{i,0}), \tag{36}$$

where

$$\begin{aligned} f_{xy}^{(1)}(x_{i,0}) &= \frac{2(d-x_{i,0})}{R_1} f(R_1-\bar{d}) - \frac{2(d+x_{i,0})}{R_2} f(R_2-\bar{d}) + \varepsilon(x_{i+1,1}-x_{i-1,1}) \\ &\times \left[\frac{1}{R_1^2} \left\{ \frac{d^2}{R_1} f(R_1-\bar{d}) + (d-x_{i,0})^2 g(R_1-\bar{d}) \right\} \right. \\ &\left. + \frac{1}{R_2^2} \left\{ \frac{d^2}{R_2} f(R_2-\bar{d}) + (d+x_{i,0})^2 g(R_2-\bar{d}) \right\} \right]. \end{aligned} \tag{37}$$

$$\begin{aligned} f_{xy}^{(2)}(x_{i,0}) &= \varepsilon(x_{i+1,1}+x_{i-1,1}) \\ &\times \left[\frac{1}{R_1^2} \left\{ \frac{d^2}{R_1} f(R_1-\bar{d}) + (d-x_{i,0})^2 g(R_1-\bar{d}) \right\} \right. \\ &\left. + \frac{1}{R_2^2} \left\{ \frac{d^2}{R_2} f(R_2-\bar{d}) + (d+x_{i,0})^2 g(R_2-\bar{d}) \right\} \right]. \end{aligned} \tag{38}$$

The part of $f_{xy}^{(2)}$ does not vanish generally, since the condition $x_{i+1,1} + x_{i-1,1} = 0$ is not generally satisfied, Therefore f_{xy} can be regarded as a effective force due to a nonlinear symmetric on-site potential.

In the both cases of the $k = 0$ and $k \neq 0$, it is found that the particles on quasi-one dimensional ILM act as FPU-β with symmetric on-site potential. However, in the presence of the interaction between second nearest interactions, deformation of ILM is observed. This is quite different from the mechanism for DC-effect in the one dimensional lattice with asymmetric interaction potential [14]. One possibility of mechanism for the deformation is that the bifurcation of the structure of ILM could occurs at the some values of $k \neq 0$. Detailed analysis of this problem is left for the future investigations.

Figure 4 shows the relation between angular frequency and amplitude of quasi-one dimensional ILM which has even symmetry. As frequency of ILM becomes larger, amplitude grows. Angular frequency becomes larger as the equilibrium length $\bar{d}$ becomes shorter. It should be also noted that no bifurcations are observed in this relation.

4.2 2D case

We also find two dimensional ILM. This type of ILM is excited only in the system with interaction between the second nearest neighbor particles. Figure 5(a) shows structure of two dimensional ILM. Four particles on one lattice square vibrate along diagonal direction. Therefore two dimensional ILM is the mixture of a longitudinal vibration in same phase and a transverse vibration in opposite phase. Longitudinal displacement of two dimensional ILM takes even modes in one dimension lattice. Therefore a center of ILM is just a center of a square lattice.

Figure 6 shows the relation between angular frequency and amplitude of two dimensional ILM. ILM appears above the maximum frequency of the linear phonon band. Angular frequency becomes larger as amplitude becomes larger. This is as same as the case of quasi-one dimensional ILM.

We find some bifurcations of structure of two dimensional ILMs. In Fig. 6, bifurcation points are found at $\omega \approx 3.326$ and $\omega \approx 4.354$. Figures 5(b) and (c) shows bifurcated structures of two dimensional ILMs. Mode 1 and mode 2, and mode 3 which correspond to the curves labeled in Fig. 6, respectively.

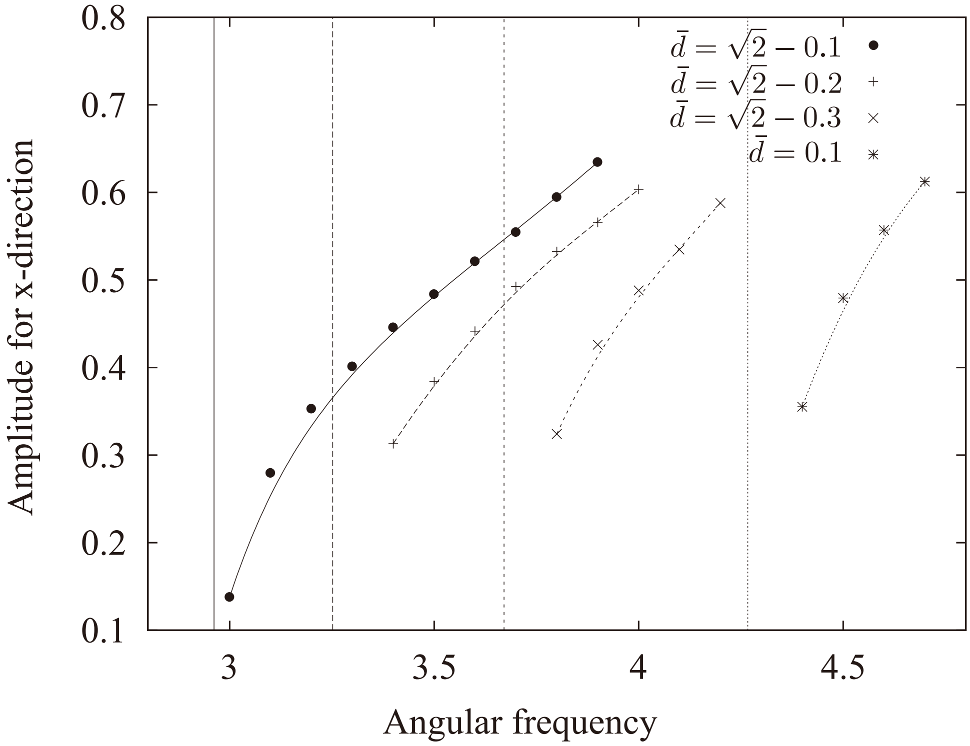


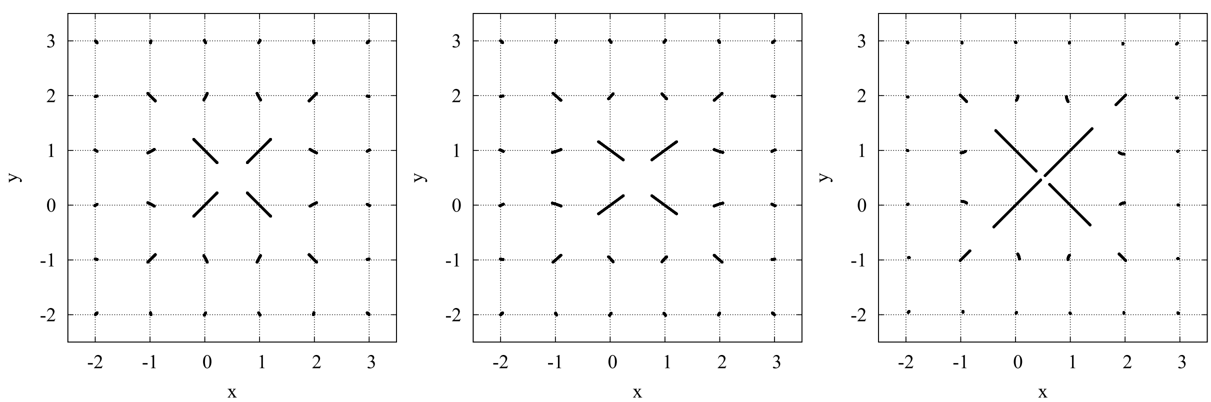
Fig. 4. Relation between angular frequency ω and the maximum amplitude in x-direction of quasi-one dimensional ILM. The parameter $k = 1$.

In the case of mode 1, size of displacement of four particles in ILM is same:

$$
\begin{aligned}
&(x_{m,n}, y_{m,n}) = (-d, -d),\\
&(x_{m+1,n}, y_{m+1,n}) = (d, -d),\\
&(x_{m+1,n+1}, y_{m+1,n+1}) = (d, d),\\
&(x_{m,n+1}, y_{m,n+1}) = (-d, d),
\end{aligned}
\tag{39}
$$

where d is a constant. A lattice square vibrates in time with preserving square shape shown in Fig. 5(a).

In the case of mode 2, displacement in x-direction is different from that in y-direction. Shape of ILM deforms into rectangular shape shown in Fig. 5(b). Displacement of four particles is given by



(a) mode 1 (b) mode 2 (c) mode 3

Fig. 5. Displacement pattern of two dimensional ILM for $k = 1$ and $\bar{d} = \sqrt{2} - 0.15$. Angular frequency is (a) $\omega = 3.4$, (b) $\omega = 3.34$ and (c) $\omega = 4.4$.

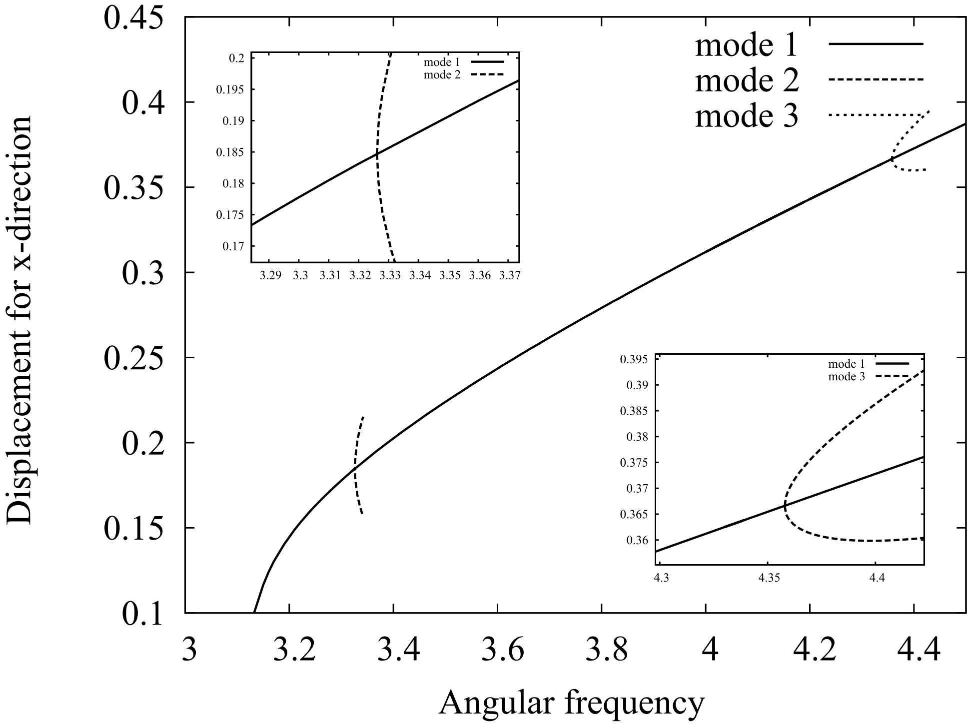


Fig. 6. Relation between angular frequency and the amplitude in x-direction of ILMs. Two inlets show detailed structure near the bifurcation points, $\omega \approx 3.326$ and $\omega \approx 4.357$.

$$
\begin{aligned}
(x_{m,n}, y_{m,n}) &= (-d-a, -d+a),\\
(x_{m+1,n}, y_{m+1,n}) &= (d+a, -d+a),\\
(x_{m+1,n+1}, y_{m+1,n+1}) &= (d+a, d-a),\\
(x_{m,n+1}, y_{m,n+1}) &= (-d-a, d-a),
\end{aligned} \tag{40}
$$

where a is a constant. Two branches of mode 2 are represented by the sign of a. The upper branch corresponds to the case that $a > 0$ and the lower branch corresponds to the case that $a < 0$.

In the case of mode 3, shape of ILM deforms into rhombus shape shown in Fig. 5(c). Displacement of one pair of particles on diagonal becomes shorter than that of the other pair. Displacement is given by

$$
\begin{aligned}
(x_{m,n}, y_{m,n}) &= (-d-b, -d-b),\\
(x_{m+1,n}, y_{m+1,n}) &= (d-b, -d+b),\\
(x_{m+1,n+1}, y_{m+1,n+1}) &= (d+b, d+b),\\
(x_{m,n+1}, y_{m,n+1}) &= (-d+b, d-b).
\end{aligned} \tag{41}
$$

where b is a constant. When $b > 0$, structure of ILM corresponds to the upper branch of mode 3. When $b < 0$, on the other hand, ILM becomes corresponds to the lower branch. Two branches has the same shape. However, the direction of the pair of the particle which has longer displacement is different.

It is found that bifurcation of two dimensional ILMs breaks the symmetry of ILMs. In bifurcation from mode 1 to mode 2, square shape of displacement of four particle becomes rectangular shape. In bifurcation mode 1 to mode 3, square shape becomes rhombus shape. ILMs of two branches which bifurcated from same point have same shapes but different orientation. We obtain one branches of shape by rotating other mode by $\pi/2$ around the center of the lattice square.

5. Conclusions

We investigate structure of ILM in two dimensional FPU-β lattices with the second nearest neighbor interactions. We find two types of ILM as the in-place vibrations: quasi-one dimensional ILM and two dimensional ones. In both types of ILMs, presence of the second nearest neighbor interactions breaks symmetry of shape of ILMs.

1. In quasi-one dimensional ILMs, deformation of ILM is obserbed in the presence of the second nearest neighbor interaction. However the mechanism of these deformation is different from the DC-effect in the one dimensional lattice with asymmetric interaction potential.

2. In two dimensional ILMs, square shape of ILM changes its shape with lower symmetry due to bifurcations. Shape change is described one parameter perturbations against to the square shapes.

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