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TIME CHANGE, JUMPING MEASURE AND FELLER MEASURE

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Abstract

In this paper, we shall investigate some potential theory for time change of Markov processes. Under weak duality, it is proved that the jumping measure and Feller measure are actually independent of time change, and the jumping measure of a time changed process induced by a PCAF supported on V coincides with the sum of the Feller measure on V and the trace of the original jumping measure on V.

1. Introduction

In this paper we shall mainly discuss some properties concerning to time change of Markov processes under weak duality setting. Roughly in §2 we first give a formula which describes how energy functionals of the process and time changed process are related to each other. We then prove that jumping measure is independent of time change induced by (strictly increasing) CAF's. In §3 Feller measure on a set is introduced and it is proved that Feller measure is also independent of time change induced by (strictly increasing) CAF's. Finally in §4 using the invariance of jumping measure and Feller measure, we give an expression of the jumping measure of a time changed process. This generalizes a result in [10].

To explain the motivation behind this work, let us first present the classical Douglas integral ([5]):

$$\frac{1}{2}\int_D |\nabla Hf(x)|^2\,dx = \frac{1}{2}\int_{\partial D\times\partial D\setminus d}(f(\xi)-f(\eta))^2 N(\xi,\eta)\,d\xi\,d\eta, \tag{1.1}$$

where Hf denotes the harmonic function on the planar unit disk D with boundary value f and $N(\xi,\eta)=1/(4\pi(1-\cos(\xi-\eta)))$. In 1962, J.L. Doob [4] extended formula (1.1) to the case where D is a general Green space and ∂D is its Martin boundary by adopting as U the Naim kernel, which was identified with the Feller kernel soon after by Fukushima in [8]. The Feller kernel had been introduced by W. Feller [6] for the minimal Markov process on a countable state space for the purpose of describing all possible boundary conditions on some ideal boundaries. In the recent work [10], the Douglas integral has been generalized to the case of symmetric diffusions using the Feller measure introduced there.

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It should be mentioned that in symmetric case, the identification has been done in [1]. A recent work of Chen-Fukushima-Ying [2] proved the identification in fairly general setting. However our approach is very different from theirs.

2. Invariance of jumping measure under time change

Let X and $\hat{X}$ be two Borel right Markov processes on the state space $(E, \mathscr{E})$ with transition semigroup (P_t) and $(\hat{P}_t)$, in weak duality, namely, for any non-negative measurable function f, g on E, it holds that

$$(P_t f, g)_m = (f, \hat{P}_t g)_m,$$

where $(\cdot, \cdot)_m$ denotes the inner product in $L^2(E, m)$. Without loss of generality, we assume that they are both realized on the same probability space, and only differ by their probability laws $\{\mathbb{P}^x\}$ and $\{\hat{\mathbb{P}}^x\}$. A consequence of this duality is that $t \mapsto X_t$ has left limits on $(0, \infty)$ for $\mathbb{P}^m$-almost all sample paths (see, e.g., [17]). We shall always use $\hat{}$ or prefix 'co-' to denote the dual objects. Since their roles are symmetric, the conclusion holds for X should also hold accordingly for the dual $\hat{X}$. The measure m is usually called the duality measure. For the terminologies and notations such as excessive measures, excessive functions, additive functional, Revuz measures, energy functionals, appeared in the sequel, we refer readers to [13]. For general theory of Markov processes and time change, please refer to [16] and §65 in it.

Let A be a positive continuous additive functional (PCAF in short) of X with fine support V, which is finely perfect, and τ the right continuous inverse of A. Let Y the time change of X by τ or A, namely

$$Y = (\Omega, \mathscr{F}, \mathscr{F}_{\tau_t}, X_{\tau_t}, \theta_{\tau_t}, \mathbb{P}^x),$$

which is a right process on V. Let $u := \xi_A^m$ be the Revuz measure of A with respect to m, which is supported on V. This PCAF A has a natural unique duality $\hat{A}$ (refer to [12]), a PCAF of $\hat{X}$, which has μ as its Revuz measure with respect to m. Hence the process Y has a natural duality $\hat{Y}$, which is the time change of $\hat{X}$ by $\hat{A}$, with μ as the duality measure.

Assume that A is strictly increasing or finely supported on E. In this case the inverse of A is also continuous and time change is invertible, namely X is a time change of Y. Hence X and Y are actually time change of each other. By the Blumenthal-Getoor-McKean theorem (Theorem 5.1 in [3]), if both Y and Z are time change of X by strictly increasing PCAF's, then Y is a time change of Z. Hence time change by strictly increasing PCAF is an equivalence relation in the space of all Borel right processes on E.

It is easy to verify by the identity $\tau_t \circ \theta_{\tau_s} + \tau_s$ that if H is a PCAF of X, then $H_t : t \mapsto H_{\tau_t}$ is an additive functional of Y, but not necessarily continuous. If both A and B are PCAF's with fine support V, respective inverse τ and σ and respective time change Y and Z, then B_τ is an AF of Y. However B_τ is actually continuous and strictly increasing, since B has the same fine support as A and is constant on interval (τ_{t-}, τ_t) for any $t > 0$. Hence Z is a time change of Y by a strictly increasing PCAF.

The energy functional of a right process is important in probabilistic potential theory. For the definition and properties, refer to [13]. Let L^X and L^Y denote the energy functional of X and Y, respectively. It is easy to check that μ is excessive for Y and if A is strictly increasing, m is the Revuz measure of τ, a PCAF of Y, relative to μ. If $\eta_n U \uparrow m$, then $\eta_n U_A \uparrow \mu$ (refer to [13]), where U_A is the potential operator of A and is nothing but the potential operator of Y, since a change of variable gives an identity

$$\mathbb{E}^x \int_0^\infty f(X_{\tau_t})\,dt = \mathbb{E}^x \int_0^\infty f(X_t)\,dA_t.$$

We shall now state a lemma. Note that part (2) was actually proved in [7].

Lemma 2.1. *Assume that A is a strictly increasing PCAF.*
(1) *If X and Y are transient, then X and Y have the same class of excessive functions, and the same class of excessive measures. Furthermore, their energy functionals satisfy that for any excessive function h,*

$$L^X(m, h) = L^Y(u, h).$$

(2) *If H is an AF of X, then the Revuz measure $\xi_H^{X,m}$ of H computed against X and its excessive measure m coincides the Revuz measure $\xi_{H_\tau}^{Y,\mu}$ of H_τ computed against Y and its excessive measure μ.*

Proof. (1) Denote by U^Y the potential operator of Y. It is known that $U^Y = U_A$. Hence any potential of Y is excessive for X, and it follows that an excessive function of Y is excessive for X due to the transience. The converse is true since X is also a time change of Y. By the transience again, there exists a sequence $\{\eta_n\}$ of measures such that $\eta_n U \uparrow m$. Then $\eta_n U^Y \uparrow \mu$ and

$$L^X(m, h) = \lim_n \eta_n(h) = \lim_n L^Y(\eta_n U^Y, h) = L^Y(u, h).$$

(2) We first assume that X is transient. For any non-negative measurable function f on E, (refer to [13]),

$$\xi_H^{X,m}(f) = L(m, U_H f).$$

However a change of variable proves

$$U_H f(x) = \mathbb{E}^x \int_0^\infty f(X_t)\,dH_t$$
$$= \mathbb{E}^x \int_0^\infty f(X_{\tau_t})\,dH_{\tau_t} = U_{H_\tau}^Y f(x).$$

It then follows from (1) that

$$\xi_H^{X,m}(f) = L(m, U_H f) = L^Y\left(\mu, U_{H_\tau}^Y f\right) = \xi_{H_\tau}^{Y,\mu}(f).$$

In general, let X^1 be 1-subprocess of X. Then the potential of X^1 is U^1, which is proper, i.e., X^1 is transient. The Revuz measure of H computed against X^1 and m is the same as $\xi_H^{X,m}$. Let $\{\mathbb{P}_1^x\}$ be the 1-subprocess measure on $(\Omega, \mathscr{F})$. Since $\{\mathbb{P}_1^x\}$ is equivalent to $\{\mathbb{P}_1^x\}$, A is a PCAF for 1-subprocess. Let us compute the potential of time change Y' of subprocess by A. For $f \in \mathscr{E}_+$ and $x \in E$, we have

$$\begin{aligned}\mathbb{E}_1^x \int_0^\infty e^{-qt} f(X_{\tau_t})\, dt &= \mathbb{E}_1^x \int_0^\infty e^{-qA_t} f(X_t)\, dA_t \\ &= \mathbb{E}^x \int_0^\infty e^{-qA_t} e^{-t} f(X_t)\, dA_t \\ &= \mathbb{E}^x \int_0^x e^{-qt} e^{-\tau_t} f(Y_t)\, dt.\end{aligned}$$

It follows that Y' is a subprocess of Y killed by a continuous decreasing multiplicative functional $(e^{-\tau_t})$ of Y. Then the Revuz measure of H_τ computed against Y' and μ is the same as one against Y and μ. That completes the proof. □

We may simply write $\xi_H^{X,m}$ as ξ_H^m, and $\xi_{H_\tau}^{Y,\mu}$ as $\xi_{H_\tau}^\mu$. We now turn to Lévy system. It is known as in §73 of [16], there exists a Lévy system of X which characterizes how X jumps. A kernel n on E and a PCAF H of X, (n, H), is called a Lévy system of X if for any $(\mathscr{F}_t)$-predictable process $Z = (Z_t)$, it holds that for any function F on $E \times E$ vanishing on the diagonal, and $x \in E$

$$\mathbb{E}^x \sum_{t<\infty} Z_t F(X_{t-}, X_t) = \mathbb{E}^x \int_0^\infty Z_t nF(X)\, dH_t, \tag{2.1}$$

where $nF(x) := \int_E F(x, y)\, n(x, dy)$. The jumping measure of X, a measure on $E \times E$ not charging on the diagonal, is defined as

$$J(F) := \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}^m \sum_{0<s\le t} F(X_{s-}, X_s).$$

From (2.1), it follows that $J(dx, dy) = n(x, dy)\xi_H^m(dx)$, where ξ_H^m is the Revuz measure of H relative to m. It is easy to see that (n, H_τ) is a Lévy system of Y. The next result shows that the jumping measure is independent of time change.

Theorem 2.1. *Assume that A and B are two PCAF with the same fine support V and Y, Z are their respective time changed processes on V. Then the jumping measures, J^Y and J^Z, of Y and Z with respect to their duality measure respectively are identical.*

Proof. Since Y and Z are time change of each other by strictly increasing PCAF's, we may assume that A is strictly increasing and prove that Y has the same jumping measure as X. By the theorem above,

$$J^Y(dx, dy) = n(x, dx)\,\xi^{\mu}_{H_\tau}(dx) = n(x, dy)\,\xi^m_H(dx) = J(dx, dy).$$

That completes the proof. □

The easy consequence is that jumping measure (relative to natural excessive measure) is invariant under time change induced by strictly increasing CAF.

3. Invariance of Feller measure under time change

Fix a finely open set D and denote $V = D^c$ and $T := T_V$ the hitting time of V. Assume that $\mathbb{P}^x(T < +\infty) = 1$ for any $x \in E$ and V is finely perfect, i.e., any point of V is regular for V. Let

$$Q_t(x, A) := \mathbb{P}^x(X_t \in A,\ t < T), \quad x \in D,\ A \subset D$$

the transition semigroup of X^D (the restriction of X on D). Clearly X^D and $\hat{X}^D$ are also in weak duality with respect to the measure $m_D := 1_D \cdot m$. Let μ be an excessive measure of X^D. Then for any $f \in \mathscr{E}_+(\mathrm{V})$,

$$t \mapsto \frac{1}{t}\mathbb{P}^{\mu}(T \le t,\ f(X_T))$$

is decreasing and there exists a unique measure, denoted by ξ^{μ}_T, on V such that for any $f \in \mathscr{E}_+(C)$,

$$\xi^{\mu}_T(f) = \lim_{t\downarrow 0} \frac{1}{t}\mathbb{P}^{\mu}(T \le t,\ f(X_T)).$$

The measure ξ^{μ}_T may be called the Revuz measure of T with respect to μ.

Let P^q_V denote the q-balayage operator on E, i.e.,

$$P^q_V(x, A) := \mathbb{P}^x(e^{-qT}1_A(X_T)), \quad x \in D,\ A \in \mathscr{E}.$$

Then $P^q_V(x,\ \cdot\ )$ is carried by V, since V is finely closed. It is easy to verify that $P^q_V f$ is a q-excessive function for X^D. It follows then that $P_V f \cdot m_D$ is an excessive measure for $\hat{X}^D$ (or co-excessive measure for X^D) and similarly $\hat{P}_V f \cdot m_D$ is an excessive measure for X^D. Hence there exists a unique measure N on $V \times V$ such that, for $f, g \in \mathscr{E}_+(V)$,

$$N(g \otimes f) = \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^{\hat{P}_V g\cdot m_D}(f(X_T),\ T \le t).$$

The right hand side is actually the co-Revuz measure of T with respect to the X^D-excessive measure $\hat{P}_V g \cdot m_D$. The measure N may be called the Feller measure on V with respect to m, since its definition is similar to the well-known Feller kernel.

Let L^D be the energy functional of X^D which is a function on excessive measures and functions of X^D. For any excessive function $\hat{u}$ of $\hat{X}^D$, $\hat{u} \cdot m$ is an excessive measure of X^D and hence we define, for any excessive functions v of X^D

$$L^D(\hat{u}, u) := L^D(\hat{u} \cdot m_D, v),$$

which is a function on co-excessive and excessive functions of X^D. It follows that

$$N(g \otimes f) = L^D(\hat{P}_V g, P_V f) = \hat{L}^D(P_V f, \hat{P}_V g)$$

the second equality follows from duality. The argument in this part about Feller measure is similar to that in §2 of [10], though the process considered there is symmetric.

We shall now prove that Feller measure is also independent of time change. Let A be a strictly increasing CAF and Y a time change of X as in §2. It is known that A has a dual $\hat{A}$, which is also a strictly increasing CAF corresponding to μ, i.e., $\hat{\xi}^m_{\hat{A}} = \mu$. Then $\hat{Y}$, the time change of $\hat{X}$ by the inverse $\hat{\tau}$ of $\hat{A}$, and Y are in weak duality with respect to μ.

Theorem 3.1. *The Feller measure is independent of time change. Precisely if N^Y is the Feller measure of Y on V, then $N^Y = N$.*

Proof. First of all, since the process and its time change have identical hitting distributions, $P^Y_V = P_V$ and $\hat{P}^{\hat{Y}}_V = \hat{P}_V$. It is known from [14] that $\xi^{\hat{h}\cdot m}_H = \hat{h} \cdot \xi^m_H$ if $\hat{h}$ is an excessive function for $\hat{X}$. It is known from [7] or [18] that $\xi^{\hat{P}_V g \cdot m_D}_A = \hat{P}_V g \cdot \mu_D$. We then need to check that time change and killing upon leaving D commute. Let T' be the hitting time of Y to V. Then

$$\begin{aligned} T' &= \inf\{t > 0 \colon Y_t \in V\} = \inf\{t > 0 \colon X(\tau_t) \in V\} \\ &= \inf\{A_t \colon X_t \in V\} = A_T. \end{aligned}$$

Let Z be time change of the killed process X^D by A. But $X^D = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \mathbb{Q}^x)$ and $\mathbb{Q}^x H = \mathbb{P}^x(H \circ k_T)$ where $H \in \mathscr{F}$ and (k_t) are the killing operators (refer to §61 of [16]). Let us compute the potential of Z. For any $f \in \mathscr{E}_+$ and $q > 0$, we have

$$\begin{aligned} \mathbb{Q}^x \int_0^\infty e^{-qt} f(X_{\tau_t})\, dt &= \mathbb{Q}^x \int_0^\infty e^{-qA_t} f(X_t)\, dA_t \\ &= \mathbb{P}^x \int_0^T e^{-qA_t} f(X_t)\, dA_t = \mathbb{P}^x \int_0^{A_T} e^{-qt} f(X_{\tau_t})\, dt. \end{aligned}$$

Therefore the potential operator of Z is the same as the one of Y killed at $A_T = T'$, i.e., time change and killing commute.

Denote now by $(L^Y)^D$ the energy functional of Y killed upon leaving D, and $(L^D)^Y$ the other way. By Theorem 2.1 (1), we have

$$\begin{aligned} N^Y(g \otimes f) &= (L^Y)^D\big(\hat{P}_V^{\hat{Y}} g \cdot \mu_D, P_V^Y f\big) \\ &= (L^Y)^D(\hat{P}_V g \cdot \mu_D, P_V f) \\ &= (L^D)^Y(\hat{P}_V g \cdot \mu_D, P_V f) \\ &= L^D\big(\xi_A^{\hat{P}_V g \cdot m_D}, P_V f\big) \\ &= L^D(\hat{P}_V g \cdot m_D, P_V f) = N(g \otimes f). \end{aligned}$$

That completes the proof. □

4. Jumping measure of time changed process

We now assume that both X and its dual $\hat{X}$ are conservative. In this case, m is actually invariant for both. For any AF A, $\mathbb{P}^m \int_0^t f(X_s)\, dA_s$ is linear in t. Hence the Revuz measure can written as $\xi_A^m(f) = \mathbb{E}^m \int_0^1 f(X_s)\, dA_s$. For any $\omega \in \Omega$, we define

$$M(\omega) = \overline{\{t \in [0, \infty) \colon X_t(\omega) \in V\}}.$$

Clearly the relatively open set $M(\omega)^c$ in $[0, \infty)$ consists of all of excursion intervals away from F of the sample path ω. We denote by I the set of left endpoints of excursion intervals in M^c. M is homogeneous, i.e., $M \circ \theta_s + s = M$ if $M \subset [s, \infty)$. I is also homogeneous.

For $t > T$, we define

$$L(t) := \sup[0, t] \cap M$$

and

$$R(t) := \inf(t, \infty) \cap M = \inf\{s > t \colon X_s \in V\}$$

with the convention that $\inf \emptyset = \infty$. When $t > T$, we call $(L(t), R(t))$ the excursion straddling on t. Clearly $t \mapsto R(t)$ is right continuous and increasing and it is easy to verify that $R(t) = T \circ \theta_t + t$, and that for any $s, t \geq 0$, $R(t) \circ \theta_s + s = R(t + s)$. Due to the right continuity, $X_{R(t)} \in V$ on $\{R(t) < \infty\}$. We can also see that, for $t > T$, $R(t-) < R(t)$ if and only if $t \in I$ and in this case $t = R(t-) = L(t)$. We shall further verify that P_x-a.s. $X_{R(t-)-} \in V$ for every $t > T$ with $R(t) < \infty$ for m-a.e. $x \in E$.

Define the inverse operator at t,

$$\gamma_t \omega(s) := \omega((t - s)-),$$

or $X_s \circ \gamma_t = X_{(t-s)-}$, $s \in [0,t)$. Then $L(t)\circ\gamma_t = t - T$ and

$$X_{L(t)-} \circ \gamma_t = X_{t-L(t)\circ\gamma_t} = X_T.$$

Since X and $\hat{X}$ are dual with respect to m, the image of $\mathbb{P}^m$ on $\mathscr{F}_t$ under the inverse operator γ_t is precisely $\hat{\mathbb{P}}^m$. We state it as a lemma, which is well-known and may be proved by an argument similar to the proof of Lemma 4.1.2 in [11].

Lemma 4.1. *For any $t > 0$ and any non-negative $\mathscr{F}_t$-measurable random variable Y,*

$$\mathbb{E}^m(Y\circ\gamma_t) = \hat{\mathbb{E}}^m Y, \quad \mathbb{E}^m Y = \hat{\mathbb{E}}^m Y = \hat{\mathbb{E}}^m(Y\circ\gamma_t).$$

Lemma 4.2. *Let $t = t_1 < t_2 < \cdots < t_n$, and $S \subset (0,t_1)$, $U \subset (t_n,1)$, $A, B \in \mathscr{B}(V)$, $C_1,\dots,C_n \in \mathscr{B}(D)$. Then*

$$\begin{aligned}
&\mathbb{P}^m(L(t)\in S,\ X_{L(t)-}\in A,\ X_{t_1}\in C_1,\dots,X_{t_n}\in C_n,\ X_{R(t)}\in B,\ R(t)\in U)\\
&= \int_{x\in C_1, x_2\in C_2,\dots,x_n\in C_n} \hat{\mathbb{P}}^x(t-T\in S,\ X_T\in A)\\
&\qquad\qquad \cdot P^D_{t_2-t_1}(x,dx_2)\cdots P^D_{t_n-t_{n-1}}(x_{n-1},dx_n)\,\mathbb{P}^{x_n}(X_T\in B,\ T\in U-t)\,m(dx).
\end{aligned}$$

Proof. Clearly $\{L(t)\in S,\ X_{L(t)-}\in A\}\in\mathscr{F}_t$. By the Markov property and Lemma 4.1, we have

$$\begin{aligned}
&\mathbb{P}^m(L(t)\in S,\ X_{L(t)-}\in A,\ X_{t_1}\in C_1,\dots,X_{t_n}\in C_n,\ X_{R(t)}\in B,\ R(t)\in U)\\
&= \mathbb{E}^m(L(t)\in S,\ X_{L(t)-}\in A,\ \mathbb{P}^{X_t}(X_0\in C_1,\dots,X_{t_n-t}\in C_n,\ X_T\in B,\ T+t\in U))\\
&= \hat{\mathbb{E}}^m[(1_{\{L(t)\in S,\ X_{L(t)-}\in A\}}\phi(X_t))\circ\gamma_t]\\
&= \hat{\mathbb{E}}^m[\phi(X_0);\ X_T\in S,\ T\in t-S]
\end{aligned}$$

where

$$\begin{aligned}
\phi(x) &= \mathbb{P}^x(X_0\in C_1,\dots,X_{t_n-t}\in C_n,\ X_T\in B,\ T+t\in U)\\
&= 1_{C_1}(x)P^D_{t_2-t_1}(x,dx_2)\cdots P^D_{t_n-t_n-1}(x_{n-1},x_n)\,\mathbb{P}^{x_n}(X_T\in B,\ T\in U-t).
\end{aligned}$$

Combining these together, the conclusion follows. □

Corollary 4.1. *We have the following two identities.*

$$\begin{aligned}
&\mathbb{P}^m(L(t)\in S,\ X_{L(t)-}\in A,\ X_t\in D,\ X_{R(t)}\in B,\ R(t)\in U)\\
&= \int_D \hat{\mathbb{P}}^x(T\in t-S,\ X_T\in A)\,\mathbb{P}^x(T\in U-t,\ X_T\in B)\,m(dx).
\end{aligned}$$

$$\mathbb{P}^m(L(t) < t,\ X_{L(t)-} \in A,\ X_{R(t)} \in B)$$
$$= \int_D \hat{\mathbb{P}}^x(T \le t,\ X_T \in A)\, P_V(x, B)\, m(dx).$$

We consider

$$A_t := \sum_{s\in I:\ 0<s\le t} f(X_{L(s)-}, X_{R(s)}),$$

where f is a non-negative continuous function on $V \times V$ vanishing on the diagonal: $f(a,a) = 0$ for any $a \in V$. Clearly $R(s)$ is a right continuous additive functional of X, and $s \in I$ if and only if $R(s-) < R(s)$ and $R(s-) = L(s)$. Thus

$$A_t = \sum_{0<s\le t:\ R(s-)<R(s)} f(X_{R(s-)-}, X_{R(s)}).$$

Thus A is a raw additive functional of X. A raw AF means an increasing right continuous real process which is additive. An adapted raw AF is an AF. Refer to (35.5) in [16]. Hence there exists a measure M on $V \times V \setminus d$ such that

$$M(f) = \lim_{t\to 0} \frac{1}{t}\mathbb{E}^m \sum_{0<s\le t:\ R(s-)<R(s)} f(X_{L(s)-}, X_{R(s)}).$$

Intuitively M may be called a measure induced by excursions away from V. From the argument of Theorem 2.1, it is seen that the part (2) is also true when H is only a raw AF. Hence it follows that the measure M is also independent of time change.

Theorem 4.1. *If X is conservative, then $M = N$.*

Proof. For $n >= 1$, let $D_n := \{t_{n,k} = k/2^n : k \ge 0\}$ and $I_{n,k} = [t_{n,k-1}, t_{n,k})$. If $L(t) < t < R(t)$ for some $t \in D_n$, then we have $L(t) = L(t_{n,k}) \in I_{n,k}$ for one and only one k. On the other hand, for any $t > 0$, the excursion interval $(L(t), R(t))$ will have a binary point in D_n for n large enough. Thus any excursion interval will be counted finally and at most once in this way. Then by Corollary 4.1, we have for continuous functions f, g on V with non-intersected supports,

$$\begin{aligned} M(g \times f) &= \lim_{t\to 0} \frac{1}{t}\mathbb{E}^m \sum_{0<s\le t:\ R(s-)<R(s)} g(X_{L(s)-})\, f(X_{R(s)}) \\ &= \lim_{t\to 0} \frac{1}{t} \lim_n \mathbb{E}^m \sum_{k:\ t_{n,k}\le t} g(X_{L(t_{n,k})-})\, f(X_{R(t_{n,k})}) 1_{\{L(t_{n,k})\in I_{n,k}\}} \\ &= \lim_{t\to 0} \frac{1}{t} \lim_n \sum_{k:\ t_{n,k}\le t} \mathbb{E}^m[g(X_{L(t_{n,k})})\, f(X_{R(t_{n,k})}) 1_{\{L(t_{n,k})\in I_{n,k}\}}] \end{aligned}$$

$$= \lim_{t\to 0} \frac{1}{t} \lim_n \sum_{k:\, t_{n,k}\le t} \int_D \hat{\mathbb{E}}^x(T \in (0, 2^{-n}], g(X_T)) P_V f(x)\, m_D(dx)$$

$$= \lim_{t\to 0} \frac{1}{t} \lim_n [2^n t] \hat{\mathbb{E}}^{P_V f\cdot m_D}(g(X_T); T \le 2^{-n})$$

$$= N(g \otimes f),$$

where $[2^n t]$ is the biggest integer dominated by $2^n t$. □

We shall compute the jumping measure of time changed process. Let X be a conservative Borel right process on E and A be a PCAF of X with V as its fine support, and μ its Revuz measure with respect to m. Let $\tau = (\tau_t)$ be the right continuous inverse of A. Set $Y_t := X_{\tau_t}$, the time change of X. Then Y has its weak duality with respect to measure μ. Hence

$$Y_{t-} = X_{\tau_{t-}-}$$

exists in V a.e. $\mathbb{P}^\mu$. The jumping measure of Y relative to the duality measure μ is defined as

$$J^Y(f) := \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}^\mu \sum_{0<s\le t} f(X_{s-}, X_s) = \lim_{q\to\infty} q\mathbb{E}^\mu \sum_{0<t<\infty} e^{-qt} f(Y_{t-}, Y_t),$$

where f is any non-negative measurable function on $V \times V$ which vanishes on its diagonal. The main result of this section gives an expression of J^Y.

Theorem 4.2. *If X is conservative, then it holds that $J^Y = N + J_{V\times V}$, $J_{V\times V}$ should be understood as the jumping measure of X restricted on $V \times V$.*

Proof. Firstly it is known from §59 and §64 of [16] that $\tau_0 = \inf\{t: A_t > 0\} = T$. By continuity of A, $A_{\tau_t} = t$ provided $\tau_t < \infty$. Then it follows that

$$\tau_{A_t} = \inf\{s: A_s > A_t\} = \inf\{s: A_{s-t} \circ \theta_t > t\} = T \circ \theta_t + t = R(t).$$

Thus for any $t > 0$, $\tau_{A_t} = R(t) > t$ a.s. Let f be a non-negative measurable function on $V \times V$ which vanishes on the diagonal. When $m(E) < \infty$, it has been shown in the proof of Theorem 5.1 of [10] that

$$J^Y(f) = \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}^m \sum_{0<u\le t} f(X_{R(u-)-}, X_{R(u)}).$$

Obviously R is continuous at u if and only if $u = R(u)$ and $X_u \in V$. Hence by Theorem 4.1, we have

$$J^Y(f) = \lim_{t\to 0} \frac{1}{t} \mathbb{E}^m \sum_{0<u\le t} f(X_{R(u-)-}, X_{R(u)})$$

$$= \lim_{t\to 0}\frac{1}{t}\mathbb{E}^m \sum_{0<u\le t,\ R(u-)<R(u)} f(X_{L(u)-}, X_{R(u)})$$
$$+ \lim_{t\to 0}\frac{1}{t}\mathbb{E}^m \sum_{0<u\le t,\ R(u-)=R(u)} f(X_{u-}, X_u)$$
$$= N(f) + J_{V\times V}(f).$$

The conclusion follows.

If m is only σ-finite, take a bounded, strictly positive and integrable function ϕ on E, and let $B_t := \int_0^t \phi(X_s)\,ds$ and denote its inverse by σ. Then B is strictly increasing, and consider the time change X' of X by B. The duality measure $m' := \phi \cdot m$ of X' is finite. It is easily seen that Y is identical to the time change of X' by its PCAF $A' := A_\sigma$, which has the same fine support V as A does, since σ is strictly increasing and continuous. Since $m'(E) = m(\phi) < \infty$, it follows from the result above that $J^Y = N' + J'_{V\times V}$, where N' and J' are the Feller measure on V and jumping measure of X' (relative to m'). Then the invariance plays a role. By Theorem 2.1, $J' = J$, and by Theorem 3.1, $N' = N$. Hence we have $J^Y = N + J_{V\times V}$. □

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References

[1] Z.-Q. Chen, M. Fukushima and J. Ying: *Traces of symmetric Markov processes and their characterization*, Ann. Probab. **34** (2006), 1052–1102.

[2] Z. Chen, M. Fukushima and J. Ying: *Entrance law, exit system and Lévy system of time changed processes*, to appear on Illinois J. Math.

[3] R.M. Blumenthal and R.K. Getoor: Markov Processes and Potential Theory, Academic Press, New York, 1968.

[4] J.L. Doob: *Boundary properties of functions with finite Dirichlet integrals*, Ann. Inst. Fourier (Grenoble) **12** (1962), 573–621.

[5] J. Douglas: *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), 263–321.

[6] W. Feller: *On boundaries and lateral conditions for the Kolmogorov differential equations*, Ann. of Math. (2) **65** (1957), 527–570.

[7] P.J. Fitzsimmons and R.K. Getoor: *Revuz measures and time changes*, Math. Z. **199** (1988), 233–256.

[8] M. Fukushima: *On Feller's kernel and the Dirichlet norm*, Nagoya Math. J. **24** (1964), 167–175.

[9] M. Fukushima: *On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities*, J. Math. Soc. Japan **21** (1969), 58–93.

[10] M. Fukushima, P. He and J. Ying: *Time changes of symmetric diffusions and Feller measures*, Ann. Probab. **32** (2004), 3138–3166.

[11] M. Fukushima, Y. Ōshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes. Walter de Gruyter, Berlin, 1994.

[12] R.K. Getoor: *Multiplicative functionals of dual processes*, Ann. Inst. Fourier (Grenoble) **21** (1971), 43–83.
[13] R.K. Getoor: Excessive Measures, Birkhäuser, Boston, MA, 1990.
[14] R.K. Getoor and M.J. Sharpe: *Naturality, standardness, and weak duality for Markov processes*, Z. Wahrsch. Verw. Gebiete **67** (1984), 1–62.
[15] P. Hsu: *On excursions of reflecting Brownian motion*, Trans. Amer. Math. Soc. **296** (1986), 239–264.
[16] M. Sharpe: General Theory of Markov Processes, Academic Press, Boston, MA, 1998.
[17] J.B. Walsh: *Markov processes and their functionals in duality*, Z. Wahrsch. Verw. Gebiete **24** (1972), 229–246.
[18] J. Ying: *Revuz measures and the related formulas on energy functional and capacity*, Potential Anal. **8** (1998), 1–19.

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