

Title	Generalized Chebyshev maps of $C^2$ and their perturbations
Author(s)	Uchimura, Keisuke
Citation	Osaka Journal of Mathematics. 46(4) P.995-P.1017
Issue Date	2009-12
Text Version	publisher
URL	<a href="https://doi.org/10.18910/8477">https://doi.org/10.18910/8477</a>
DOI	10.18910/8477
rights	
Note	

***Osaka University Knowledge Archive : OUKA***

<https://ir.library.osaka-u.ac.jp/repo/ouka/all/>

## GENERALIZED CHEBYSHEV MAPS OF $\mathbf{C}^2$ AND THEIR PERTURBATIONS

KEISUKE UCHIMURA

(Received December 14, 2007, revised July 29, 2008)

### Abstract

The Chebyshev map is a typical chaotic map. We consider generalized Chebyshev maps  $T_k$  of  $\mathbf{C}^2$ . The support of the maximal entropy measure of  $T_k$  is connected. We perturb  $T_k$  in a certain direction. Then we can show the support of the maximal entropy measure of this map is a Cantor set.

### 1. Introduction

The Chebyshev map is a typical chaotic map. Compared with the dynamics of the Chebyshev map in one variable very few things are known about the dynamics of generalized Chebyshev maps in higher dimensions. Generalized Chebyshev maps were studied by several researchers, Koornwinder [9], Lidl [10], Veselov [15], Hoffman and Withers [8]. Their constructions are based on the theory of complex Lie algebras.

Veselov [15] defined generalized Chebyshev maps as follows.

Let  $G$  be a simple complex Lie algebra of rank  $n$ ,  $H$  be its Cartan subalgebra,  $H^*$  be its dual space,  $\mathcal{L}$  be a lattice of weights in  $H^*$  generated by the fundamental weights  $\omega_1, \dots, \omega_n$  and  $L$  be the dual lattice in  $H$  (see [3]). One defines the mapping  $\phi_G: H/L \rightarrow \mathbf{C}^n$ ,  $\phi_G = (\varphi_1, \dots, \varphi_n)$ ,  $\varphi_k = \sum_{w \in W} \exp[2\pi i w(\omega_k)]$ , where  $W$  is the Weyl group, acting on the space  $H^*$ . Chevalley asserts that  $\varphi_1, \dots, \varphi_n$  generate the algebra of exponential invariants freely.

With each simple complex Lie algebra  $G$  of rank  $n$  is associated an infinite series of integrable polynomial mappings  $P_G^k$  from  $\mathbf{C}^n$  to  $\mathbf{C}^n$ ,  $k = 2, 3, \dots$ , determined by the condition:

$$\phi_G(kx) = P_G^k(\phi_G(x)).$$

For  $n = 1$  there is a unique simple algebra  $A_1$ . Here  $\phi_{A_1} = 2 \cos(2\pi x)$  and the  $P_{A_1}^k$  are, within a linear substitution, Chebyshev polynomials. Here  $A_n$  is the Lie algebra of  $SL(n+1, \mathbf{C})$ .

In this paper we will study the mappings  $P_G^k$  in the case  $G = A_2$ . When  $G = B_2 \simeq C_2$ , from [11] we know that the extended polynomial maps  $P_G^k$  from  $\mathbf{P}^2$  to  $\mathbf{P}^2$

are represented as symmetric products of two maps of  $\mathbf{P}^1$ . So we study  $P_G^k$  in the case  $G = A_2$ . We denote  $P_{A_2}^k$  by  $T_k$ .

The generalized Chebyshev maps  $T_k$  from  $\mathbf{C}^2$  to  $\mathbf{C}^2$  ( $k \in \mathbf{Z}$ ), are given by  $T_k(x, y) = (g^{(k)}(x, y), g^{(k)}(y, x))$ . Here  $g^{(k)}(x, y)$  is a generalized Chebyshev polynomial defined by Lidl [10]. Let  $x = t_1 + t_2 + t_3$ ,  $y = t_1t_2 + t_1t_3 + t_2t_3$ ,  $1 = t_1t_2t_3$ . Then we set  $g^{(k)}(x, y) := t_1^k + t_2^k + t_3^k$ . So  $g^{(k)}(y, x) = (1/t_1)^k + (1/t_2)^k + (1/t_3)^k = g^{(-k)}(x, y)$ . For instance,  $T_2(x, y) = (x^2 - 2y, y^2 - 2x)$ ,  $T_3(x, y) = (x^3 - 3xy + 3, y^3 - 3xy + 3)$ ,  $T_4(x, y) = (x^4 - 4x^2y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y)$ . Recurrence relations for these polynomials are given by (see [10])

$$(1.1) \quad g^{(k)}(x, y) = xg^{(k-1)}(x, y) - yg^{(k-2)}(x, y) + g^{(k-3)}(x, y).$$

In this paper we consider the dynamics of generalized Chebyshev maps  $T_k$  of  $\mathbf{C}^2$ . We show they have similar properties to those of Chebyshev maps of  $\mathbf{C}$ . The support of the invariant probability measure  $\mu$  of maximal entropy of  $T_k$  is connected. We will study external rays for the support of  $\mu$  and foliations of the Julia set  $J_1$ . We also show the external rays have relations with the affine Weyl group of  $A_2$ .

Next we consider perturbations of the generalized Chebyshev maps. We perturb the generalized Chebyshev maps of  $\mathbf{C}^2$  in a certain direction. Then we will show that the support of  $\mu$  of the perturbed map is a Cantor set. In one variable case, this is parallel to the following fact. For typical quadratic maps  $f_c(z) = z^2 + c$ , when  $c = -2$ ,  $f_{-2}(z)$  is a Chebyshev map. Our result corresponds to the well known fact that when  $c < -2$ , the Julia set of  $f_c$  is a Cantor set.

The generalized Chebyshev maps  $T_k$  have relations with the classical Lie algebra  $A_2$  and so the maps are very symmetric by nature. In this paper, we will show that these symmetric objects collapse under certain perturbations.

In Section 2, we will study the properties of generalized Chebyshev map of  $\mathbf{C}^2$  and their dynamics. In Lemma 2.1, we show an exact formula for the critical set of  $T_k(x, y)$ . The set  $K(T_k(x, y))$  of points with bounded orbits is a closed domain on the real plane  $\{x = \bar{y}\}$ .

In Proposition 2.3, we show an exact form of the invariant measure  $\mu$  of maximal entropy for  $T_k(x, y)$  and its support is equal to  $K(T_k(x, y))$ . We apply the definitions of external rays of polynomial endomorphism given by Bedford and Jonsson [1] to  $T_k(x, y)$  and we give exact forms of external rays of  $T_k(x, y)$  and show they have similar properties to those of one-dimensional Chebyshev maps  $T_k(x)$  (see Proposition 2.4, Fig. 2.4 and Fig. 2.5). We also show the affine Weyl group of  $A_2$  acts on a set of rays.

In Section 3, we will consider perturbations of  $T_k(x, y)$  in a certain direction and find Cantor sets. We define  $c$ -Chebyshev maps  $f_c(x, y)$  as follows: Let  $T_k(x, y) = (u(x, y), u(y, x))$ , where  $u(x, y)$  is a polynomial of degree  $k$  in variables  $x$  and  $y$ . We introduce a new parameter  $c$  and make a homogeneous polynomial  $u(x, y, c)$  of degree  $k$  with  $u(x, y, 1) = u(x, y)$ . Then we define  $c$ -Chebyshev maps by  $f_c(x, y) :=$

$(u(x, y, c), u(y, x, c))$  (see Definition 3.1). The main result of this paper is that if  $c > 1$ , then the support of the maximal entropy measure  $\mu$  of a  $c$ -Chebyshev map  $f_c(x, y)$  is a Cantor set which lies on the plane  $\{x = \bar{y}\}$  (see Theorem 3.1 and Proposition 3.6). The key observation of the proof of this result is the following: if  $c > 1$ , then the orbit  $\{f_c^n(C)\}$  of the critical set  $C$  of  $f_c(x, y)$  approaches the line at infinity uniformly (see Proposition 3.1). To prove this observation we use a topological argument principle, dynamics on the invariant plane  $\{x = \bar{y}\}$  and the external rays of  $T_k(x, y)$ . When  $c = 1$ , the support of the maximal entropy measure  $\mu$  of  $f_c(x, y)$  is connected. But when  $c > 1$ , the support of  $\mu$  is not connected. Then a bifurcation occurs at  $c = 1$ . The symmetric objects of generalized Chebyshev maps  $T_k$  collapse under the perturbations.

**2. Generalized Chebyshev maps of  $\mathbf{C}^2$**

In this section we study some properties of the generalized Chebyshev maps  $T_k(x, y)$  and their dynamics. From the definition of the generalized Chebyshev maps we can find a branched covering map. The following diagram is commutative,

$$\begin{CD} (\mathbf{C} - \{0\})^2 @>g_k>> (\mathbf{C} - \{0\})^2 \\ @VV\Psi V @VV\Psi V \\ \mathbf{C}^2 @>T_k>> \mathbf{C}^2 \end{CD}$$

where  $g_k(t_1, t_2) = (t_1^k, t_2^k)$ , and

$$(2.1) \quad (x, y) = \Psi(t_1, t_2) = \left( t_1 + t_2 + \frac{1}{t_1 t_2}, \frac{1}{t_1} + \frac{1}{t_2} + t_1 t_2 \right).$$

The covering map

$$\Psi: \mathbf{C}^2 \setminus \Psi^{-1}(D) \rightarrow \mathbf{C}^2 \setminus D$$

is a 6-sheeted covering map. The branch locus  $D$  of  $\Psi$  is written as

$$x^2 y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0.$$

$T_k(x, y)$  admits an invariant plane  $\{x = \bar{y}\}$ .  $T_k(x, y)$  restricted to the real plane  $\{x = \bar{y}\}$  may be regarded as a Chebyshev polynomial defined by Koornwinder [9]

$$P_{k,0}^{-1/2}(z, \bar{z}) = e^{ik\sigma} + e^{-ik\tau} + e^{i(k\tau - k\sigma)}, \quad \sigma, \tau \in \mathbf{R}.$$

Set  $z(\sigma, \tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau - \sigma)} = u + iv$ . Based on [9], we review some known facts. Let  $R$  be a closed domain bounded by the triangle with vertices  $O = (0, 0)$ ,  $A = (\pi/\sqrt{2}, -\pi/\sqrt{6})$  and  $B = (\pi/\sqrt{2}, \pi/\sqrt{6})$  in the  $(s, t)$  plane. Here we use the coordinate  $(s, t)$  which is related to the coordinate  $(\sigma, \tau)$  by a coordinate transformation:

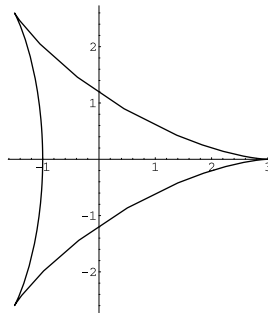


Fig. 2.1. The domain  $S$ .

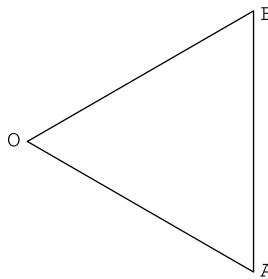


Fig. 2.2. The domain  $R$ .

$s = (\sigma + \tau)/(2\sqrt{2})$ ,  $t = \sqrt{3}(\sigma - \tau)/(2\sqrt{2})$ . Let  $J_1$ ,  $J_2$  and  $J_3$  denote the reflections in the edges  $OA$ ,  $OB$  and  $AB$  in the  $(s, t)$  plane, respectively. Then  $R$  is a fundamental domain for the group generated by  $J_1$ ,  $J_2$  and  $J_3$ .

In other words,  $R$  is similar to the closure of an alcove of the Lie algebra  $A_2$ . The set of simple root vectors is written as  $\{\alpha_1 = (\sqrt{2}, 0), \alpha_2 = (-1/\sqrt{2}, \sqrt{3}/\sqrt{2})\}$  and the highest root vector is  $\tilde{\alpha} = \alpha_1 + \alpha_2$ . (See [3].)

We denote the images of  $R$  under the inverse of the coordinate transformation by  $R_1$ . Let  $S$  be a closed domain bounded by Steiner's hypocycloid

$$(u^2 + v^2 + 9)^2 + 8(-u^3 + 3uv^2) - 108 = 0$$

in the  $(u, v)$  plane. Then the mapping  $z: (\sigma, \tau) \rightarrow (u, v)$  is a diffeomorphism from the interior  $R_1^\circ$  of  $R_1$  to  $S^\circ$  and the boundary  $\partial R_1$  is mapped one to one and onto the boundary  $\partial S$ . Then  $S = \{e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)}: 0 \leq \sigma, \tau \leq 2\pi\}$ .

Combining the inverse of the map  $z(\sigma, \tau)$  with the coordinate transformation, we get a continuous map  $\varphi$  from  $S$  to  $R$  such that  $\varphi$  is a diffeomorphism from  $S^\circ$  onto  $R^\circ$  and  $\partial S$  is mapped onto  $\partial R$ . See Figs. 2.1 and 2.2.

In [13], we show that for any segment  $l$  in  $R$  parallel to one of the three root vectors  $\alpha_1, \alpha_2$  and  $\tilde{\alpha}$ ,  $\varphi^{-1}(l)$  is also a segment in  $S$ . Then such a segment  $\varphi^{-1}(l)$  may

be viewed as a “geodesic”. (See Fig. 2.4.)

In the first place we consider the critical set of  $T_k(x, y)$  defined by

$$C(T_k) := \{(x, y) \in \mathbf{C}^2 : \det(DT_k) = 0\}.$$

**Lemma 2.1.** *Let  $k \in \mathbf{Z}$ . Assume that*

$$x = t_1 + t_2 + t_3, \quad y = t_1t_2 + t_1t_3 + t_2t_3, \quad t_1t_2t_3 = 1.$$

Then

$$\det(DT_k) = k^2 \frac{t_1^k - t_2^k}{t_1 - t_2} \cdot \frac{t_1^k - t_3^k}{t_1 - t_3} \cdot \frac{t_2^k - t_3^k}{t_2 - t_3}.$$

Proof. We note that

$$\det(DT_k) = \frac{\det(D(T_k \circ \Psi))}{\det(D\Psi)}.$$

By direct computations we have

$$\det(DT_k \circ \Psi) = \frac{k^2}{t_1t_2} (t_1^k - t_2^k)(t_1^k - t_3^k)(t_2^k - t_3^k),$$

and

$$\det(D\Psi) = \frac{1}{t_1t_2} (t_1 - t_2)(t_1 - t_3)(t_2 - t_3). \quad \square$$

Dinh [4] shows that generalized Chebyshev maps are critically finite. From Lemma 2.1 we see that  $C(T_k)$  can be parameterized as

$$(2.2) \quad x = t + \varepsilon t + \frac{1}{\varepsilon t^2}, \quad y = \frac{1}{t} + \frac{1}{\varepsilon t} + \varepsilon t^2 \quad (\varepsilon = e^{2j\pi\sqrt{-1}/k}, \quad j \in \mathbf{N}).$$

We will prove in Proposition 2.3 below that  $S$  is equal to the support of the maximal entropy measure for  $T_k$ .

Let  $f$  be a map from a complex manifold  $X$  to  $X$ . We define  $K(f) := \{x \in X : \text{the orbit } \{f^n(x)\} \text{ is bounded}\}$ . Then  $K(T_k)$  is described in the following form.

**Proposition 2.1** ([15]).

$$K(T_k) = \Psi(\{|t_1| = |t_2| = 1\}) = S \subset \{x = \bar{y}\}.$$

Proof. From the following commutative diagram we can prove this proposition.

$$\begin{array}{ccc}
 (t_1, t_2) & \xrightarrow{g^k} & (t_1^k, t_2^k) \\
 \downarrow \Psi & & \downarrow \Psi \\
 (x, y) & \xrightarrow{T_k} & (g^{(k)}, g^{(-k)}).
 \end{array}
 \quad \square$$

Next we study the properties of periodic points of  $T_k$ .

**Lemma 2.2.** *Assume that  $k \geq 2$ . All the periodic points of  $T_k$  lie in  $S$  on the plane  $\{x = \bar{y}\}$  and any periodic point in the interior  $S^\circ$  is repelling.*

Proof. Clearly, any periodic point of  $T_k$  lies in the set  $K(T_k)$ . By the semi-conjugacy (2.1), we see that any periodic point in  $S^\circ$  is repelling.  $\square$

We consider the function  $T_k$  restricted to  $\{x = \bar{y}\}$ , which is denoted by  $S_k$ .

$$S_k := T_k|_{\{x=\bar{y}\}} : \mathbf{R}^2 \rightarrow \mathbf{R}^2,$$

e.g.

$$S_2(z) = z^2 - 2\bar{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v).$$

We use the bijection  $\varphi$  from  $S$  in the  $(u, v)$  space to  $R$  in the  $(s, t)$  space (see Figs. 2.1 and 2.2). We divide the closed triangular region  $R$  into  $k^{2n}$  congruent closed triangular regions  $\Delta$ .

**Proposition 2.2.** *Each region  $\Delta$  has a periodic point of  $\varphi \circ S_k \circ \varphi^{-1}$  of period  $n$ .*

Proof. We set  $\kappa := \varphi \circ S_k \circ \varphi^{-1}$ . Then  $\kappa(s, t) = (ks, kt)$ . We prove this lemma when  $k = 2$ . The proof in the general case is similar.

$R$  is the closed domain bounded by an equilateral triangle  $\triangle OAB$ . See Fig. 2.3. We divide the triangle  $\triangle OAB$  into four congruent equilateral triangles. Let the closed domain bounded by  $\triangle DEF, \triangle OEF, \triangle ADF, \triangle BED$  denote  $\Delta(0), \Delta(1), \Delta(2), \Delta(3)$ , respectively. Then the image of  $\Delta(2)$  under the map  $\kappa$  is the closed domain bounded by  $\triangle AA^*D^*$ . This closed domain is equivalent to the fundamental domain  $R$  by reflections. Then we can define a homeomorphism  $k_2$  from  $\Delta(2)$  onto  $R$ . Let  $h_2$  be the inverse of  $k_2$ . Hence  $h_2$  is a continuous map from  $R$  to  $R$ . Then, by the fixed point theorem of a closed disk, we have a fixed point  $p_2^{(1)}$  of  $h_2$ . Hence  $p_2^{(1)}$  is a fixed point of  $k_2$  in  $\Delta(2)$ . By the same arguments, we have a fixed point of  $\kappa$  on each  $\Delta(j)$ , ( $j = 0, 1, 2, 3$ ). Further, we divide each  $\Delta(j)$  into four smaller congruent equilateral triangular domains  $\Delta(jl)$ . In the same way, we can prove that there is a periodic point

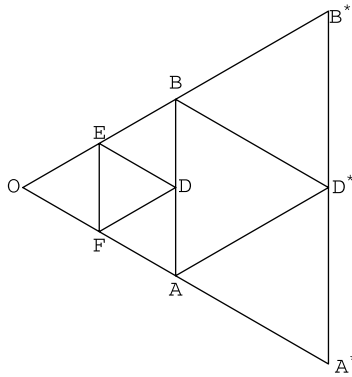


Fig. 2.3. Divisions of a regular triangle and their extensions.

of period 2 of  $\kappa$  on each  $\Delta(jl)$ . Repeating this procedure, we have a periodic point of period  $n$  on each  $\Delta(j_1, j_2, \dots, j_n)$ . We can show these are distinct  $k^{2n}$  periodic point of period  $n$ . Indeed. Note that  $\kappa(\partial R) \subset \partial R$  and  $\kappa^n(\partial \Delta(j_1, j_2, \dots, j_n)) = \partial R$ . The point  $O$  is a fixed point and the points  $A$  and  $B$  are periodic points of period 2.  $\square$

**Corollary 2.1.** *The map  $\kappa|_{\partial R}$  from  $\partial R$  onto  $\partial R$  is a  $k$ -sheeted covering map.*

Proof. Set  $a = OB$ ,  $b = AB$  and  $c = OA$ . Then  $\kappa(\partial R)$  is represented as  $(abc)^k$  in a counterclockwise orientation.  $\square$

Now we consider the invariant measure  $\mu$  of maximal entropy for  $T_k$ .

**Proposition 2.3.** (1)  $\text{supp } \mu = S$ .

(2)

$$\mu = \frac{3}{\pi^2} \frac{dx_1 dx_2}{\sqrt{-x^2 \bar{x}^2 + 4x^3 + 4\bar{x}^3 - 18x\bar{x} + 27}}, \quad x = x_1 + ix_2.$$

Proof. A theorem of [2] reads as follows. Let  $\mu_n$  be the measure  $\mu_n := (1/k^{2n}) \times \sum_{f^n(y)=y, y \text{ repelling}} \delta_y$ . Then the sequence  $\{\mu_n\}$  converges weakly to the invariant measure  $\mu$ .

By Lemma 2.2, we see that all the periodic points in  $S^\circ$  are repelling. We use the notations in the proof of Proposition 2.2. A small triangle  $\Delta(j_1, j_2, \dots, j_m)$  in  $R^\circ$  has exactly one repelling periodic point of  $\kappa$  of period  $m$ . Set  $\tilde{\mu}_n := \varphi_* \mu_n$ . Then,  $\tilde{\mu}_m(\Delta(j_1, j_2, \dots, j_m)) = 1/k^{2m}$ . Dividing  $\Delta(j_1, j_2, \dots, j_m)$  into smaller equilateral triangles, we see that if  $n \geq m$ ,  $\tilde{\mu}_n(\Delta(j_1, j_2, \dots, j_m)) = 1/k^{2m}$ . Thus, we deduce that the sequence  $\{\tilde{\mu}_n\}$  converges weakly to  $\sqrt{3}/\pi^2 \tilde{\mu}$ , where  $\tilde{\mu}$  is the Lebesgue measure in the



$(s, t)$  plane. Then  $\mu = \varphi^*(2\sqrt{3}/\pi^2 \tilde{\mu})$ . Theorem 3.5 in [9] states that

$$ds dt = \frac{\sqrt{3} dx_1 dx_2}{2\sqrt{-x^2 \bar{x}^2 + 4x^3 + 4\bar{x}^3 - 18x\bar{x} + 27}}.$$

Hence this proposition follows. □

Next we study external rays of  $T_k(x, y)$ . We use the definitions of external rays given by Bedford and Jonsson [1] in our situation. We extend the map  $T_k(x, y): \mathbf{C}^2 \rightarrow \mathbf{C}^2$  to a holomorphic map from  $\mathbf{P}^2$  to  $\mathbf{P}^2$ . Let  $W^s(J_\Pi, T_k)$  be the stable set of the Julia set  $J_\Pi$  on the line at infinity. Bedford and Jonsson [1] show that there exists a homeomorphism  $\Psi$  (an inverse Böttcher coordinate) such that

$$\Psi: W^s(J_\Pi, f_k) \rightarrow W^s(J_\Pi, T_k)$$

conjugating  $f_k$  to  $T_k$ , where  $f_k(t_1, t_2) = (t_1^k, t_2^k)$ . They define a local stable manifold  $W_{loc}^s(a)$ , ( $a \in J_\Pi$ ) and then a stable disk  $W_a \supset W_{loc}^s(a)$  and external rays  $R(a, \theta)$ .

Nakane [11] observed the following results on  $T_2(x, y)$ :  
 Nakane defined

$$(2.3) \quad \tilde{\Psi}(t_1, t_2) = \left( t_1 + \frac{1}{t_2} + \frac{t_2}{t_1}, \frac{1}{t_1} + t_2 + \frac{t_1}{t_2} \right).$$

Two maps  $\Psi$  in (2.1) and  $\tilde{\Psi}$  in (2.3) are essentially the same and  $\tilde{\Psi}$  is an inverse Böttcher coordinate conjugating  $f_k$  to  $T_k$ . The stable disk  $W_a$  is the image of  $\{(t, at): |t| > 1\}$  under the map  $\tilde{\Psi}$ . Then the stable disk  $W_a$  can be written as the set of points  $R(r, \sigma, \tau)$  in the form

$$(2.4) \quad \begin{aligned} x &= r e^{-2\pi i \tau} + \frac{1}{r} e^{2\pi i(\tau - \sigma)} + e^{2\pi i \sigma}, & y &= r e^{2\pi i(\sigma - \tau)} + \frac{1}{r} e^{2\pi i \tau} + e^{-2\pi i \sigma}, \\ a &= e^{2\pi i \sigma}, & r &> 1. \end{aligned}$$

Any external ray is written as  $R(\sigma, \tau) := \{R(r, \sigma, \tau): r > 1\}$ . Each point  $z$  in  $S$  is a landing point of exactly 1, 3, or 6 external rays if  $z$  is a cusp point on  $\partial S$ ,  $z$  is a non-cusp point on  $\partial S$  or  $z \in S^\circ$ , respectively.

We can show that Nakane’s results are also true for any  $T_k(x, y)$ ,  $k \neq 0$ . Further, we give a structure of foliations  $W_a$  of  $W^s(J_\Pi, T_k)$  and show their relations to a Lie algebra.

**Proposition 2.4.** *For any point  $z \in S^\circ$ , there exist three stable disks  $W_a$  such that boundaries of these three disks lie on  $S$  and intersect at  $z$ . At that point, two external rays on each  $W_a$  land from opposite directions.*

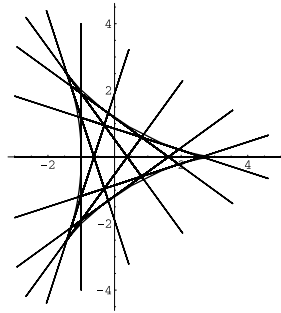


Fig. 2.4. Geodesics and external rays.

Metaphorically speaking, it is like three mouths (stable disks) biting a sandwich ( $\text{supp } \mu = S$ ). When  $r = 1$ , the point  $R(1, \sigma, \tau)$  lies on  $S \subset \{x = \bar{y}\}$ . The boundary of  $W_a$  is written as  $\{R(1, \sigma, \tau) : 0 \leq \tau \leq 2\pi\}$  and covers a segment on  $S$  twice. The segment is a geodesic and is inscribed in the hypocycloid  $\partial S$ . See Fig. 2.4. We can extend the segment across the hypocycloid in both directions. The two half-lines are external rays of  $T_k$  in  $\{x = \bar{y}\}$ . We consider the affine Weyl group  $W$  of the Lie algebra  $A_2$ . The affine Weyl group  $W$  of  $A_2$  is expressed by the following six transformations (see [9], p.360):

$$J_1(\sigma, \tau) = (-\sigma + \tau, \tau), \quad J_2(\sigma, \tau) = (\sigma, \sigma - \tau), \quad J_3(\sigma, \tau) = (-\tau, -\sigma),$$

$$J_4(\sigma, \tau) = (-\tau, \sigma - \tau), \quad J_5(\sigma, \tau) = (\tau - \sigma, -\sigma), \quad J_0(\sigma, \tau) = (\sigma, \tau).$$

When  $r = 1$  in (2.4), then a point  $(x, y) = R(1, \sigma, \tau)$  lies in  $S \subset \{x = \bar{y}\}$  and the point  $(x, y)$  is fixed under any transformation  $J_j$ .

When  $r > 1$ , any element  $J_j$  of affine Weyl group  $W$  at  $z$  in  $S^\circ$  acts on a set of external rays. Two external rays  $R(\sigma, \tau)$  and  $R(\sigma, \sigma - \tau)$  corresponding to  $J_0$  and  $J_2$  lie on a stable disk  $W_a$  ( $a = e^{2\pi i\sigma}$ ) and land at the same point  $z$ . Any two points  $R(r, \sigma, \tau)$  and  $R(r, \sigma, \sigma - \tau)$  are symmetrical about  $\{x = \bar{y}\}$  in the following sense.

- (1) The midpoint of the segment  $R(r, \sigma, \tau)R(r, \sigma, \sigma - \tau)$  lies on the plane  $\{x = \bar{y}\}$ .
- (2) The segment is perpendicular to the plane  $\{x = \bar{y}\}$ .

The same properties hold for external rays  $R(-\tau, -\sigma)$  and  $R(-\tau, \sigma - \tau)$  corresponding to  $J_3$  and  $J_4$  on a stable disk  $W_a$  ( $a = e^{-2\pi i\tau}$ ) and also for external rays  $R(\tau - \sigma, \tau)$  and  $R(\tau - \sigma, -\sigma)$  corresponding to  $J_1$  and  $J_5$  on a stable disk  $W_a$  ( $a = e^{2\pi i(\tau - \sigma)}$ ). These six external rays land on the same point  $z$ . □

We compare the external rays of  $T_k(x, y)$  with those of a Chebyshev map  $P_{A_1}^k(z) = T_k(z)$  on  $\mathbb{C}$ . Any external ray of  $T_k(z)$  is written as

$$R(r, \phi) = r e^{2\pi i\phi} + \frac{1}{r} e^{2\pi i(-\phi)}, \quad r > 1.$$

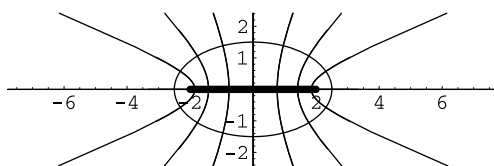


Fig. 2.5. External rays of  $T_k(z)$ .

Clearly,  $R(r, -\phi) = re^{2\pi i(-\phi)} + (1/r)e^{2\pi i\phi}$ . Then  $R(r, \phi) = \overline{R(r, -\phi)}$ . Hence  $R(r, \phi)$  and  $R(r, -\phi)$  are symmetrical about the real axis. See Fig. 2.5.

Note that the affine Weyl group of  $A_1$  acts on a set of external rays of  $T_k(z)$ . On the other hand, the affine Weyl group  $W$  of  $A_2$  acts on a set of external rays of  $T_k(x, y)$ .

### 3. Perturbations of generalized Chebyshev maps of $\mathbf{C}^2$ and Cantor sets

In this section we perturb generalized Chebyshev maps  $T_k(x, y)$  in a certain direction. Recall that  $T_k(x, y) = (g^{(k)}(x, y), g^{(k)}(y, x))$ , where  $g^{(k)}(x, y)$  is a polynomial of degree  $|k|$ . We introduce a new parameter  $c$  in  $T_k(x, y)$  as follows. We make a homogeneous polynomial  $g^{(k)}(x, y, c)$  of degree  $|k|$  by adding a new variable  $c$  such that  $g^{(k)}(x, y, 1) = g^{(k)}(x, y)$ . Then we define maps  $f_c^{(k)}(x, y)$  from  $\mathbf{C}^2$  to  $\mathbf{C}^2$  by

$$f_c^{(k)}(x, y) = (g^{(k)}(x, y, c), g^{(k)}(y, x, c)).$$

DEFINITION 3.1.  $f_c^{(k)}(x, y)$  is called a  $c$ -Chebyshev map of degree  $|k|$ .

When  $c = 1$ ,  $f_1^{(k)}(x, y) = T_k(x, y)$ . From (1.1) we see that if  $k \geq 1$ ,  $g^{(k)}(x, y) = x^k + \pi_{k-1}(x, y)$ , where  $\pi_{k-1}(x, y)$  denotes a polynomial in  $x$  and  $y$  of degree  $\leq k - 1$ . Then the map  $f_c^{(k)}$  extends holomorphically to  $\mathbf{P}^2$ . We state the main result in this section.

**Theorem 3.1.** *Assume that  $c > 1$ . Then the support of the maximal entropy measure  $\mu$  of the  $c$ -Chebyshev map  $f_c^{(k)}(x, y)$  is a Cantor set for any  $k \in \mathbf{Z} \setminus \{0, 1, -1\}$ .*

The map  $f_c^{(2)}(x, y)$  restricted to the line  $\{x = y\}$  is the map  $q_c^{(2)}(x) = x^2 - 2cx$  which is conjugate to the map  $p_\lambda(x) = x^2 + \lambda$ . The interval  $1 \leq c < \infty$  corresponds to the interval  $-\infty < \lambda \leq -2$  which is a half-line beginning at the top of antenna in the Mandelbrot set. By Proposition 2.3, we know that when  $c = 1$ , the support of the maximal entropy measure of  $f_1^{(k)}(x, y)$  ( $= T_k(x, y)$ ) is the connected set  $S$  on the plane  $\{x = \bar{y}\}$ . However if  $c > 1$ , the support of the maximal entropy measure of  $f_c^{(k)}(x, y)$  is not connected. This shows that a bifurcation occurs at  $c = 1$ . This theorem is parallel to a well-known result that if  $\lambda < -2$ , then the Julia set of  $p_\lambda$  is a Cantor set.

When  $k = 2$ , the theorem is proved in [14]. This is a generalization of the result in [14].

We fix the value  $k$  and use an abbreviation  $f_c(x, y)$  for  $f_c^{(k)}(x, y)$ . One of the reasons why we define a  $c$ -Chebyshev map in such a form is shown in the next lemma.

**Lemma 3.1.** *Let  $x = c(t_1 + t_2 + t_3)$ ,  $y = c(1/t_1 + 1/t_2 + 1/t_3)$  and  $t_1 t_2 t_3 = 1$ . Then*

$$f_c(x, y) = \left( c^{|k|}(t_1^k + t_2^k + t_3^k), c^{|k|}\left(\frac{1}{t_1^k} + \frac{1}{t_2^k} + \frac{1}{t_3^k}\right) \right).$$

*Proof.* Set  $x' = t_1 + t_2 + t_3$  and  $y' = 1/t_1 + 1/t_2 + 1/t_3$ . By definition,  $f_c(x, y) = (g^{(k)}(x, y, c), g^{(k)}(y, x, c))$ .

Clearly,  $g^{(k)}(cx', cy', c) = c^{|k|}g^{(k)}(x', y')$ . Then

$$\begin{aligned} f_c(x, y) &= c^{|k|}(g^{(k)}(x', y'), g^{(k)}(y', x')) = c^{|k|}T_k(x', y') \\ &= c^{|k|}\left(t_1^k + t_2^k + t_3^k, \frac{1}{t_1^k} + \frac{1}{t_2^k} + \frac{1}{t_3^k}\right). \end{aligned} \quad \square$$

**Lemma 3.2.** *The critical set  $C(f_c)$  and the critical value set  $f_c(C)$  are written as follows:*

$$\begin{aligned} C(f_c): x &= c\left((1 + \varepsilon)t + \frac{1}{\varepsilon t^2}\right), \quad y = c\left(\frac{1}{t} + \frac{1}{\varepsilon t} + \varepsilon t^2\right), \\ f_c(C): x &= c^{|k|}\left(2t^k + \frac{1}{t^{2k}}\right), \quad y = c^{|k|}\left(\frac{2}{t^k} + t^{2k}\right), \end{aligned}$$

where  $\varepsilon = e^{2j\pi\sqrt{-1}/k}$  and  $t \in \mathbf{C} \setminus \{0\}$ .

*Proof.* It can be easily observed that

$$\det(Df_c(x, y)) = c^{2(|k|-1)} \det\left(DT_k\left(\frac{x}{c}, \frac{y}{c}\right)\right).$$

Then by (2.2) we have the parameterization of  $C(f_c)$ . The parameterization of  $f_c(C)$  is obtained from Lemma 3.1. □

The key observation in the proof of Theorem 3.1 is the following property.

**Proposition 3.1.** *If  $c > 1$ ,  $K(f_c) \cap C(f_c) = \emptyset$ .*

This is equivalent to the statement if  $c > 1$ , then for any  $(x, y) \in C(f_c)$ ,  $\|f_c^n(f_c(x, y))\| \rightarrow \infty$  as  $n \rightarrow \infty$  with respect to the Euclidean norm. By Lemma 3.2, we know that the critical value set  $f_c(C)$  is parameterized by  $t$  in  $\mathbf{C} \setminus \{0\}$ . We will shrink the domain of definition  $\mathbf{C} \setminus \{0\}$ .

In the first place we assume that  $k \geq 2$ . Set

$$(u_n(t), v_n(t)) := f_c^n(u_0(t), u_0(1/t)), \quad \text{where } u_0(t) = c^k(2t^k + 1/t^{2k}).$$

Since  $u_0(t)$  is the first component of the critical value set  $f_c(C)$  (see Lemma 3.2),  $(u_n(t), v_n(t))$  represents an element of  $f_c^n(f_c(C))$ . We consider the maps  $f_c^{(-k)}(x, y)$  with  $k \geq 2$ . In Section 1, we show that  $g^{(-k)}(x, y) = g^{(k)}(y, x)$ . Then  $f_c^{(-k)}(x, y) = f_c^{(k)}(y, x)$ . Hence we have  $(f_c^{(-k)})^2(x, y) = (f_c^{(k)})^2(x, y)$ . By Lemma 3.2, we know that the critical value of  $f_c^{(-k)}(x, y)$  is parameterized as  $x = c^k(2/t^k + t^{2k})$ ,  $y = c^k(2t^k + 1/t^{2k})$ . Note that  $c^k(2t^k + 1/t^{2k})$  is the first component of the critical value of  $f_c^{(k)}(x, y)$ . Then

$$(f_c^{(-k)})^2(C(f_c^{(-k)})) = (f_c^{(k)})^2(C(f_c^{(k)})).$$

Hence

$$\text{if } K(f_c^{(k)}) \cap C(f_c^{(k)}) = \emptyset, \quad \text{then } C(f_c^{(-k)}) \cap K(f_c^{(-k)}) = \emptyset.$$

Thus it suffices to prove Proposition 3.1 when  $k \geq 2$ .

**Lemma 3.3.** (1) *We assume that  $k \geq 2$ . Then*

$$\text{if } K(f_c^{(k)}) \cap C(f_c^{(k)}) = \emptyset, \quad \text{then } C(f_c^{(-k)}) \cap K(f_c^{(-k)}) = \emptyset.$$

(2) *For any  $n \in \mathbf{N}$ ,  $v_n(t) = u_n(1/t)$ .*

*Proof.* We can prove (2) by induction on  $n$ . □

By Lemma 3.3 (2) we see that proving Proposition 3.1 requires only proving the following proposition. Indeed. For  $t \in \mathbf{C} \setminus \{0\}$ , we consider two cases (1)  $|t| \leq 1$  and (2)  $|t| > 1$ . In case (1), Proposition 3.2 implies Proposition 3.1. In case (2), we note that  $u_n(t) = v_n(1/t)$ .

**Proposition 3.2.** *For any  $t \in \bar{\mathbf{D}} \setminus \{0\}$ ,  $|v_n(t)| \rightarrow \infty$  as  $n \rightarrow \infty$  where  $\mathbf{D}$  denotes the unit disk.*

To prove this proposition we need two steps.

**Proposition 3.3.** *If  $c > 1$ , then  $|v_n(t)|$  has its minimum value on a boundary  $\partial\mathbf{D}$  of  $\bar{\mathbf{D}} \setminus \{0\}$  for any  $n \in \mathbf{N}$ .*

**Proposition 3.4.** *If  $c > 1$ , then  $|v_n(e^{i\theta})| \rightarrow \infty$  as  $n \rightarrow \infty$  for any  $\theta$  in  $[0, 2\pi)$ .*

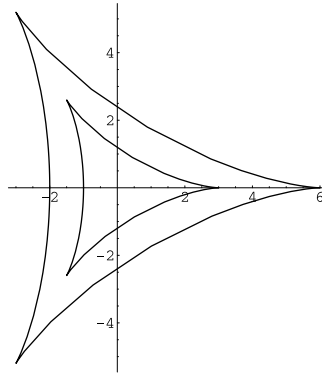


Fig. 3.1.  $cS$  and  $c^k S$ .

To prove these two propositions, we need to study the dynamics of  $f_c$  on an invariant plane. When  $c$  is real,  $f_c(x, y)$  admits an invariant plane  $\{x = \bar{y}\}$ . Then we consider the map  $g_c(z)$  on the plane  $\{x = \bar{y}\}$ . That is,  $g_c(z) := f_c(z, \bar{z})$ . The map  $g_c(z)$  may be viewed as a map from  $\mathbf{R}^2$  to  $\mathbf{R}^2$ .

**Lemma 3.4.** *Assume that  $c > 1$ . The critical set of  $g_c$  is equal to the set*

$$\left\{ c \left( e^{i\theta} + \varepsilon e^{i\theta} + \frac{1}{\varepsilon e^{2\theta i}} \right) : 0 \leq \theta < 2\pi \right\}.$$

The critical value set of  $g_c$  is equal to the set

$$\{c^k (2e^{k\theta i} + e^{-2k\theta i}) : 0 \leq \theta < 2\pi\}.$$

*Proof.* Let  $z = z_1 + iz_2$ , ( $z_j \in \mathbf{R}$ ) and this be an element on the plane  $\{x = \bar{y}\}$ . By Proposition A.1 in Appendix A, we have  $|Df_c(z, \bar{z})| = |Dg_c(z_1, z_2)|$ . Hence the critical set of  $g_c$  is equal to  $C(f_c) \cap \{x = \bar{y}\}$ . The critical set  $C(f_c)$  is described in Lemma 3.2. Clearly,  $((1 + \varepsilon)t + 1/(\varepsilon t^2), 1/t + 1/(\varepsilon t) + \varepsilon t^2)$  belongs to the plane  $\{x = \bar{y}\}$  if and only if  $|t| = 1$ . □

We consider the map  $g_c(z)$  restricted to the closed domain  $cS = \{cz : z \in S\}$ , where  $S$  is the closed domain defined in Section 2. For any point  $cz$  in  $cS$  with  $z = e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)} \in S$ , we have

$$T_k(z, \bar{z}) = (e^{ik\sigma} + e^{-ikt\tau} + e^{ik(\tau-\sigma)}, e^{-ik\sigma} + e^{ikt\tau} + e^{ik(\sigma-\tau)})$$

and so

$$f_c(cz, c\bar{z}) = c^k (e^{ik\sigma} + e^{-ikt\tau} + e^{ik(\tau-\sigma)}, e^{-ik\sigma} + e^{ikt\tau} + e^{ik(\sigma-\tau)}).$$

Then the map  $g_c(z)$  from  $cS$  onto  $c^k S$  is similar to the map  $T_k(x, y)$ .

**Lemma 3.5.** (1) For any point  $z$  in the interior  $c^k S^\circ$ ,  $g_c^{-1}(z) \subset cS^\circ$  and  $g_c^{-1}(z)$  consists of  $k^2$  distinct points.

(2)  $g_c|_{\mathbf{C} \setminus cS}: \mathbf{C} \setminus cS \rightarrow \mathbf{C} \setminus c^k S$  is a  $k$ -sheeted unbranched covering map and it is sense preserving.

(3)  $g_c|_{\partial cS}: \partial cS \rightarrow \partial c^k S$  is a  $k$ -to-one map.

*Proof.* (1) From the above remarks, we can consider  $g_1(z)$  in place of  $g_c(z)$ . Clearly,  $g_1(z) = S_k(z)$ . (Note that  $S_k = T_k|_{\{x=\bar{y}\}}$ .) Then by similar arguments used in the proof of Proposition 2.2, we can prove this assertion. The map  $\varphi \circ g_1(z) \circ \varphi^{-1}$  enlarges the fundamental domain  $R$  by  $k$  times. Then  $k^2$  small subdivisions of  $R$  are mapped onto  $R$  under  $\varphi \circ g_1(z) \circ \varphi^{-1}$ .

(2) By Lemma 3.4, we know that the critical set of  $g_c$  is contained in the set  $cS$  since  $S = \{e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)}: 0 \leq \sigma, \tau \leq 2\pi\}$ . Then the map  $g_c|_{\mathbf{C} \setminus cS}$  does not have any critical points. Hence  $g_c|_{\mathbf{C} \setminus cS}$  is an unbranched covering map. From the recurrence equation (1.1) of generalized Chebyshev polynomials, we know that  $g_c(z) = z^k + \pi_{k-1}(z, \bar{z})$ , where  $\pi_n$  denotes a polynomial in  $z$  and  $\bar{z}$  of degree  $\leq n$ . We will show  $\det(Dg_c(z)) > 0$ , for any  $z \in \mathbf{C} \setminus cS$ . Indeed,  $\det(Dg_c(z)) = |\partial g_c / \partial z|^2 - |\partial g_c / \partial \bar{z}|^2$ . When  $|z|$  is large,  $\det(Dg_c(z)) > 0$ . Therefore from the fact the critical set of  $g_c$  is contained in  $cS$  we deduce that  $\det(Dg_c(z)) > 0$ , for any  $z \in \mathbf{C} \setminus cS$ . Then  $g_c|_{\mathbf{C} \setminus cS}$  is sense preserving.

Consider a circle  $\gamma$  of center at the origin and with radius  $R_0 \gg 1$ . Then the image of  $\gamma$  under  $g_c$  lies outside  $\gamma$  and the winding number of  $g_c(\gamma)$  around  $\gamma$  is  $k$ . We use a topological argument principle (see [12], p.350); Let  $h(z)$  be a continuous mapping such that only a finite number of its  $p$ -points lie inside a simple loop  $\Gamma$ . Then the total number of  $p$ -points inside  $\Gamma$  (counted with their topological multiplicities) is equal to the winding number of  $h(\Gamma)$  around  $p$ . We apply this to our mapping  $g_c$ . Instead of the annulus  $\mathbf{C} \setminus cS$  we consider a topological disk  $\hat{\mathbf{C}} \setminus cS$  and use the usual substitution  $z = \phi(\zeta) = 1/\zeta$  where  $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ . Let  $G_c := \phi^{-1} \circ g_c \circ \phi$  be the function from  $\hat{\mathbf{C}} \setminus \phi^{-1}(cS)$  to  $\hat{\mathbf{C}} \setminus \phi^{-1}(c^k S)$ . Since  $\det(D\phi(\zeta)) > 0$  for any  $\zeta \in \hat{\mathbf{C}} \setminus cS$ , we have  $\det(DG_c(\zeta)) > 0$ .

We will select a point  $p$  inside  $G_c(\phi^{-1}(\gamma))$  and near  $\zeta = 0$  in the following manner. Let  $g_c(z) = z^k + \pi_{k-1}(z, \bar{z}) = \sum a_{ij} z^i \bar{z}^j$ . We set  $h_c(z) := \sum |a_{ij}| |z|^{i+j}$ . Then  $|g_c(z)| \leq h_c(z)$ . Let  $h_c(z) = |z|^k (1 + Q(|z|))$ . There exists a large positive number  $R_1$  satisfying  $1 + Q(R_1) < 2$ . Then we set  $\phi(p) = 2R_1^k$ . Thus we see that if  $|z| \leq R_1$  then  $|g_c(z)| < \phi(p)$ . Hence if  $u$  is any element of  $g_c^{-1}(\phi(p))$  then  $|u| > R_1$ . We set  $R_0 = R_1$ . Then all the points of  $g_c^{-1}(\phi(p))$  lie outside  $\gamma$  and  $\phi(p)$  lies outside  $g_c(\gamma)$ .

Since  $G_c$  is sense preserving, the topological multiplicity of any  $p$ -point is 1. Since the winding number of  $G_c(\phi^{-1}(\gamma))$  around  $p$  is  $k$ , the total number of  $p$ -points inside  $\phi^{-1}(\gamma)$  is  $k$ . Hence the number of points of  $G_c^{-1}(p)$  is  $k$ . Therefore that of  $g_c^{-1}(p)$  is  $k$ . Since  $\mathbf{C} \setminus c^k S$  is connected, all fibers of the covering  $g_c|_{\mathbf{C} \setminus cS}$  have the same cardinality. Hence  $g_c|_{\mathbf{C} \setminus cS}$  is a  $k$ -sheeted covering.

(3) To prove this, it suffices to consider the case when  $c = 1$ . In this case, the proof is the same as that used in Corollary 2.1.  $\square$

To prove Proposition 3.3, we will use the topological argument principle again. Let  $W(\gamma, p)$  be the winding number of a closed curve  $\gamma$  around a point  $p$ . Let  $S^1$  denote the unit circle  $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$  with a counterclockwise orientation. We assume that a hypocycloid  $c^k \partial S$  has a counterclockwise orientation.

Calculating the winding number  $W(u_n(S^1), 0)$  is not easy because the relation from  $u_n(t)$  to  $u_{n+1}(t)$  is not given by a simple mapping. But when  $t = e^{i\theta}$ ,  $(u_n(t), v_n(t))$  lies on the plane  $\{x = \bar{y}\}$ . Then we can use the map  $g_c$  when  $|t| = 1$ .

**Lemma 3.6.** *Let  $n$  be any positive integer and  $c > 1$ .*

- (1)  $u_n(e^{i\theta}) = g_c^n(c^k(2e^{k\theta i} + e^{-2k\theta i}))$ .
- (2)  $W(u_n(S^1), 0) = k \times W(g_c^n(c^k \partial S), 0)$ .

*Proof.* (1) Induction on  $n$ .

(2) A closed path  $\gamma : [0, 2\pi] \rightarrow \mathbf{C}$  given by  $\gamma(\theta) = c^k(2e^{k\theta i} + e^{-2k\theta i})$  follows the hypocycloid  $c^k \partial S$   $k$  times. Then the assertion follows.  $\square$

In the proof of Lemma 3.5 (2), by analyzing a winding number near  $\infty$  we see that the number of fibers of  $g_c|_{\mathbf{C} \setminus cS}$  is  $k$ . Conversely from the number of fibers we can calculate the winding number  $W(g_c^n(c^k \partial S), 0)$ .

**Lemma 3.7.** *If  $c > 1$ , then  $W(g_c^n(c^k \partial S), 0) = k^n$ .*

*Proof.* Let  $\Gamma$  be any simple loop in the topological disk  $\hat{\mathbf{C}} \setminus \phi^{-1}(cS)$  not passing through  $\zeta = 0$  oriented in a counterclockwise orientation. By Lemma 3.5 (2) we can select a point  $p$  very near  $\zeta = 0$  such that  $G_c^{-n}(p)$  consists of  $k^n$  distinct points and all such  $p$ -points lie in the interior of the loop  $\Gamma$ . The total number of  $p$ -points of  $G_c^n$  inside  $\Gamma$  counted with topological multiplicities is equal to  $W(G_c^n(\Gamma), p)$ . Since  $G_c$  is sense preserving, the topological multiplicity of any  $p$ -point is equal to 1. Hence it follows that  $W(G_c^n(\Gamma), p) = k^n$ . Then we can easily see that  $W(G_c^n(\Gamma), 0) = k^n$ . From this we can easily obtain that  $W(g_c^n(\phi(\Gamma)), 0) = k^n$ .  $\square$

From this lemma and Lemma 3.3, we can deduce the following.

**Corollary 3.1.** *If  $c > 1$ , then  $-W(v_n(S^1), 0) = W(u_n(S^1), 0) = k^{n+1}$ .*

From the paragraph below Proposition 3.1, we see that  $v_n(t)$  is a rational function in the variable  $t$  that has only a pole at  $t = 0$ .

**Lemma 3.8.** *The multiplicity of the pole at  $t = 0$  of  $v_n(t)$  is at most  $k^{n+1}$ .*



Proof. Since  $v_n(t) = u_n(1/t)$ , it suffices to show that

$$u_n(t) = a(k^{n+1})t^{k^{n+1}} + \dots + a(0) + a(-1)t^{-1} + \dots + a(-2k^{n+1})t^{-2k^{n+1}},$$

where  $a(j) \in \mathbf{R}[c]$ ,  $-2k^{n+1} \leq j \leq k^{n+1}$ .

We may view  $c$  as a variable. Then we will prove the following:

- (1) the maximum degree of  $t$  of  $u_n(t)$  is at most  $k^{n+1}$ ,
- (2) the minimum degree of  $t$  of  $u_n(t)$  is equal to  $-2k^{n+1}$ .

Since  $g^{(k)}(x, y) = x^k + \pi_{k-1}(x, y)$ , we can easily prove (2) by induction on  $n$ . Next we will calculate the maximum degree of  $t$  in  $u_{n+1}(t)$ . We consider the weighted degree of  $g^{(k)}(x, y, c)$ . We define the weighted degree of a monomial  $p(c)x^\alpha y^\beta$  to be  $\alpha + 2\beta$ . From the recurrence relation (1.1) for  $\{g^{(m)}(x, y)\}$ , we see that the maximum weighted degree of  $g^{(k)}(x, y)$  is  $k$  and so that of  $g^{(k)}(x, y, c)$  is  $k$ . Hence, when  $k = \alpha + 2\beta$ , the maximum degree of  $t$  of  $u_n(t)^\alpha v_n(t)^\beta$  is at most  $(\alpha + 2\beta)k^{n+1} = k^{n+2}$ .  $\square$

Proof of Proposition 3.3. We apply the argument principle to the rational function  $v_n(t)$ . Thus  $W(v_n(S^1), 0) = N - M$  where  $N$  is the number of zeros in the unit disk  $\mathbf{D}$  and  $M$  is the number of poles in  $\mathbf{D}$ . From the proof of Lemma 3.7, we see that  $u_n(S^1)$  and so  $v_n(S^1)$  does not pass through the origin. Combining Corollary 3.1 and Lemma 3.8, we see that  $v_n(t)$  does not have any zeros in  $\mathbf{D}$  and it is holomorphic in  $\mathbf{D} \setminus \{0\}$ . If  $|t| \ll 1$ ,  $|v_n(t)|$  is large. Then  $v_n(t)$  has its minimum-modulus on the boundary  $\partial\mathbf{D}$ .  $\square$

We begin with the proof of Proposition 3.4. Since  $v_n(e^{i\theta}) = u_n(e^{-i\theta})$ , to prove Proposition 3.4, it suffices to prove that if  $c > 1$ , then  $|u_n(e^{i\theta})| \rightarrow \infty$  ( $n \rightarrow \infty$ ) for any  $\theta$ . To prove this we will define a function  $\|z\|$  with  $\|z\| > 1$ , for any  $z \in \mathbf{C} \setminus cS$  and we will show that if  $c > 1$ ,  $\|g_c(z)\| > \|z\|^k$  and  $|g_c^n(z)| \rightarrow \infty$  ( $n \rightarrow \infty$ ). If this is true, then from Lemma 3.6, Proposition 3.4 follows.

We restrict the map  $\tilde{\Psi}$  in (2.3) to the set  $\{t_1 = t, t_2 = \bar{t}\}$ . We denote the map by  $\psi$ . Since  $\tilde{\Psi}(t, \bar{t}) = (t + 1/\bar{t} + \bar{t}/t, 1/t + \bar{t} + t/\bar{t})$ ,  $\psi(t) = t + 1/\bar{t} + \bar{t}/t$  and  $\tilde{\Psi}(t, \bar{t})$  lies on the plane  $\{x = \bar{y}\}$ . Since  $\psi(re^{i\theta}) = (r + 1/r)e^{i\theta} + e^{-2i\theta}$ , ( $r > 1$ ), the map  $\psi$  from  $\mathbf{C} \setminus \bar{\mathbf{D}}$  to  $\mathbf{C} \setminus S$  is a homeomorphism.

The image of a radial line  $\{re^{i\theta} : r > 1\}$  under the map  $\psi$  is also a half-line. Let  $h_\lambda$  be a function from  $\mathbf{C} \setminus S$  to  $\mathbf{C} \setminus \lambda S$  defined by  $h_\lambda(z) = \lambda z$  with  $\lambda \geq 1$ . The composition  $h_\lambda \circ \psi$  is a map from  $\mathbf{C} \setminus \bar{\mathbf{D}}$  onto  $\mathbf{C} \setminus \lambda S$ . Then the image of a radial line under the map  $h_\lambda \circ \psi$  is a half-line which is called a  $\lambda$ -external ray. We define  $\|z\| := |(h_c \circ \psi)^{-1}(z)|$  for  $z \in \mathbf{C} \setminus cS$ .

**Proposition 3.5.** *We assume that  $c > 1$ . For any point  $z$  in  $\mathbf{C} \setminus cS$ ,  $\|g_c(z)\| > \|z\|^k$  and  $|g_c^n(z)| \rightarrow \infty$  ( $n \rightarrow \infty$ ).*

Set  $\|z\| = r_0$  and  $\|g_c(z)\| = r_1$ . To prove Proposition 3.5, we consider  $c$ -external rays and  $c^k$ -external rays. We first note the symmetry of  $c$ -external rays. Let  $\omega$  be a

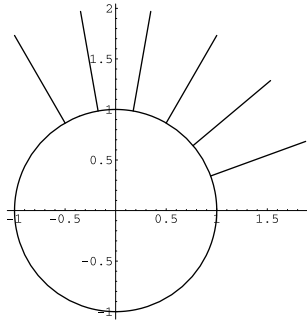


Fig. 3.2. Radial lines.

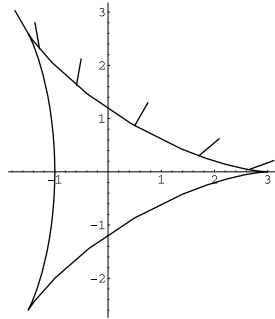


Fig. 3.3. External rays.

cubic root of unity. Then  $\psi(\omega t) = \omega\psi(t)$  and  $\psi(\omega^2 t) = \omega^2\psi(t)$ . Then it suffices to consider only  $c$ -external rays  $\{c\psi(re^{i\theta}): 0 \leq \theta \leq 2\pi/3, 1 < r\}$ . Further for  $\alpha$  with  $0 \leq \alpha \leq \pi/3$ , two  $c$ -external rays  $\{c\psi(re^{(\pi/3-\alpha)i}): r > 1\}$  and  $\{c\psi(re^{(\pi/3+\alpha)i}): r > 1\}$  are symmetric with respect to a  $c$ -external ray  $\{c\psi(re^{\pi i/3}): r > 1\}$ . See Fig. 3.4. Hence we consider only  $c$ -external rays  $\{c\psi(re^{i\theta}): 0 \leq \theta \leq \pi/3, 1 < r\}$ .

For a point  $z \in \mathbb{C} \setminus cS$ , we denote its image  $g_c(z)$  by  $P$ . Let the landing point of the  $c$ -external ray through  $P$  be  $Q_1$ . Let  $Q_2$  be the point of intersection of the segment  $PQ_1$  and the curve  $\partial c^k S$ . Let  $Q_3$  be the landing point of the  $c^k$ -external ray through  $P$ . See Fig. 3.5. Let  $|AB|$  denote the Euclidian length of a segment  $AB$ . We will evaluate the length  $|PQ_1| = |PQ_2| + |Q_2Q_1|$ . Then, to prove Proposition 3.5, it suffices to prove the third assertion of the following lemma.

**Lemma 3.9.** (1) *The slope of  $PQ_3$  is greater than that of  $PQ_2$ .*

- (2)  $|PQ_2| \geq |PQ_3|$ .
- (3)  $|PQ_1| = c(r_1 + 1/r_1 - 2) > |PQ_3| = c^k(r_0^k + 1/r_0^k - 2) > c(r_0^k + 1/r_0^k - 2)$ .

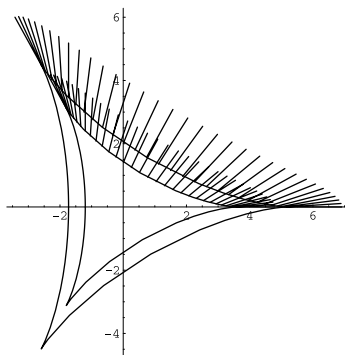


Fig. 3.4.  $c$ - and  $c^k$ -external rays.

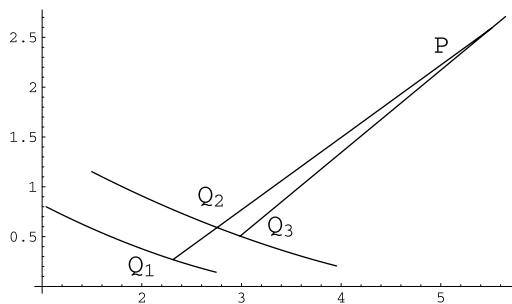


Fig. 3.5. Two external rays through  $P$ .

Proof. (1) Let  $Q_4$  be the point of intersection of the segment  $Q_3O$  and  $\partial cS$  where  $O$  denotes the origin. Let  $\{c((r + 1/r)e^{i\sigma} + e^{-2\sigma i}): r > 1\}$  be the  $c$ -external ray through  $Q_4$ . Let  $l$  denote this  $c$ -external ray. The  $c^k$ -external ray through  $Q_3$  is parallel to this half line  $l$ . The slope of  $l$  is  $\tan \sigma$ . Since  $Q_4 = c(2e^{i\sigma} + e^{-2\sigma i})$ , the slope of the segment  $OQ_4$  is  $(2 \sin \sigma - \sin 2\sigma)/(2 \cos \sigma + \cos 2\sigma)$ . We can easily verify that if  $0 \leq \sigma \leq \pi/3$ , then  $\tan \sigma \geq (2 \sin \sigma - \sin 2\sigma)/(2 \cos \sigma + \cos 2\sigma)$  by identities of trigonometric functions. Then the slope of  $OQ_4$ , which is equal to that of  $OQ_3$ , is less than or equal to the slope of  $l$ . Hence  $l$  lies above the  $c^k$ -external ray through  $Q_3$ . If the  $c$ -external ray  $l$  moves along the curve  $\partial cS$  downward, then it will touch the point  $P$ . Then we get the assertion (1).

(2) Let  $l_1$  be the line through  $Q_3$  that is perpendicular to  $PQ_3$ . To prove the above inequality, we see from (1) that it suffices to prove that the point  $Q_2$  lies below the line  $l_1$ . The hypocycloid  $\partial c^kS$  is written as  $\{c^k(2e^{i\tau} + e^{-2\tau i}): 0 \leq \tau < 2\pi\}$ . When  $0 \leq \tau \leq \pi/3$ , it is convex. Hence it suffices to prove the above fact in the case when  $Q_2$  is equal to the point  $Q_5 := c^k(2e^{i\pi/3} + e^{-2\pi i/3})$ . Hence, we will prove that the slope

of  $Q_3Q_5$  is greater than that of the line  $l_1$ . This requires only showing that

$$-\cot \tau \leq \frac{\sqrt{3}/2 - 2 \sin \tau + \sin 2\tau}{1/2 - 2 \cos \tau - \cos 2\tau}, \quad \text{when } 0 \leq \tau < \pi/3.$$

We denote the right hand side of the above inequality by  $m(\tau)$ . Then

$$\frac{dm}{d\tau} = \frac{-32 \sin(\pi/6 - \tau/2) \cos(\pi/6 + \tau) \sin(3\tau/2)}{(-1 + 4 \cos \tau + 2 \cos 2\tau)^2}.$$

Then  $m(\tau)$  is monotone decreasing in the range  $[0, \pi/3)$  and  $m(\tau)$  approaches  $-1/\sqrt{3}$  as  $\tau \rightarrow \pi/3$ . On the other hand,  $-\cot \tau$  is monotone increasing and  $-\cot(\pi/3) = -1/\sqrt{3}$ .

(3) Since  $\|g_c(z)\| = r_1$ ,  $g_c(z) = c((r_1 + 1/r_1)e^{i\phi} + e^{-2i\phi})$  and  $Q_1 = c(2e^{i\phi} + e^{-2i\phi})$ . Then  $|PQ_1| = c(r_1 + 1/r_1 - 2)$ . Since  $\|z\| = r_0$ ,  $z = c((r_0 + 1/r_0)e^{i\theta} + e^{-2i\theta})$ . Then from Lemma 3.1, we know that  $P = g_c(z) = c^k((r_0^k + 1/r_0^k)e^{k\theta i} + e^{-2k\theta i})$ . Since  $Q_3 = c^k(2e^{k\theta i} + e^{-2k\theta i})$ ,  $|PQ_3| = c^k(r_0^k + 1/r_0^k - 2)$ . □

From Lemma 3.4 we know that the critical value set of  $g_c$  is a compact set  $\partial c^k S$  included in  $\mathbf{C} \setminus cS$ . From Proposition 3.5, we see that the sequence  $\{g_c^n(C(f_c) \cap \{x = \bar{y}\})\}$  converges uniformly to  $\infty$ . This completes the proof of Proposition 3.4. Thus we have proved that  $\{f_c^n(C(f_c))\}$  converges uniformly to the line at infinity. This completes the proof of Proposition 3.1. □

Next we will prove that if  $c > 1$ , then  $K(f_c)$  is a Cantor set. To prove this we need some preparations. In the first place we assume that  $k \geq 2$ .

**Lemma 3.10.** *If  $c > 1$ , the number of periodic points of  $g_c(z)$  of period  $n$  is  $k^{2n}$ .*

Proof. The proof of this lemma is almost the same as that of Proposition 2.2. □

From Corollary 3.2 in [5], we know that the number of periodic points of period  $n$  of  $f_c(x, y)$  is  $k^{2n}$ . Then we have the following.

**Corollary 3.2.** *If  $c > 1$ , any periodic point of  $f_c(x, y)$  lies on the plane  $\{x = \bar{y}\}$  and belongs to the set  $K(g_c)$  in the plane  $\{x = \bar{y}\}$ .*

Now we return to the proof of Theorem 3.1.

Note that the map  $f_c^{(k)}$  is a regular endomorphism of  $\mathbf{C}^2$ . We use Theorem 3.8 in [6]. Then combining Theorem 3.8 in [6] and Proposition 3.1 yields  $K(f_c^{(k)}) = \text{supp } \mu$ , for any  $k \in \mathbf{Z} \setminus \{0, 1, -1\}$ .

Before starting a proof of Theorem 3.1, we will state a precise version of Theorem 3.1 and prove it when  $k \geq 2$ .

**Proposition 3.6.** *Assume that  $c > 1$  and  $k \geq 2$ . Then:*

- (1)  $K(f_c) = \text{supp } \mu = K(g_c) \subset \{x = \bar{y}\}$ .
- (2)  $K(g_c)$  is a Cantor set.

Proof. (1) From Theorem 3.8 in [6] and Corollary 3.2, we see that

$$\begin{aligned} K(f_c) &\subset \overline{\{\text{repelling periodic points of } f_c\}} \subset \overline{\{\text{periodic points of } f_c\}} \\ &\subset K(g_c) \subset K(f_c). \end{aligned} \quad \square$$

The proof of (2) is essentially the same as that used in Theorem 5.1 in [13]. Recall that  $g_c(z) = z^k + \pi_{k-1}(z, \bar{z})$ , where  $\pi_{k-1}$  denotes a polynomial of degree  $\leq k - 1$ . Then there is a constant  $R \gg 1$  such that if  $|z| > R$  then  $g_c^n(z) \rightarrow \infty$  ( $n \rightarrow \infty$ ). Then we set

$$\mathbf{D}_R := \{z \in \mathbf{C} : |z| < R\}.$$

**Lemma 3.11.** *Assume that  $c > 1$ . Then there exists a nonnegative integer  $n$  such that  $g_c^{-n}(\bar{\mathbf{D}}_R) \subset c^k S^\circ$ .*

Proof. This follows from Proposition 3.5. □

**Lemma 3.12.** *Let  $c > 1$ . Assume that  $g_c^{-n}(\bar{\mathbf{D}}_R)$  is not contained in  $c^k S^\circ$  and  $g_c^{-n}(\bar{\mathbf{D}}_R)$  is arcwise connected. Then  $g_c^{-n-1}(\bar{\mathbf{D}}_R)$  is arcwise connected.*

Proof. Let  $P$  be any point of  $g_c^{-n}(\bar{\mathbf{D}}_R) \setminus c^k S^\circ$ . Then there is a path  $\gamma$  in  $g_c^{-n}(\bar{\mathbf{D}}_R)$  connecting  $P$  and a fixed point  $Q$  of  $g_c$ . Let a point of intersection of  $\gamma$  and  $\partial c^k S$  be  $M$ . Let  $P_{-1}$  be any point of  $g_c^{-1}(P)$ . We will construct a path in  $g_c^{-n-1}(\bar{\mathbf{D}}_R)$  connecting  $P_{-1}$  and  $Q$ . Recall that the map  $g_c(z)$  is the map  $f_c^{(k)}(x, y)$  of degree  $k$  restricted to the plane  $\{x = \bar{y}\}$ .

From the recurrence relation (1.1) for Chebyshev polynomials, we have the following claim.

**Claim 3.1.** *Let  $\omega$  be a cubic root of unity.*

- (1) If  $k \equiv 0 \pmod 3$ ,  $g_c(z) = g_c(\omega z) = g_c(\omega^2 z)$ .
- (2) If  $k \equiv 1 \pmod 3$ ,  $g_c(\omega z) = \omega g_c(z)$ ,  $g_c(\omega^2 z) = \omega^2 g_c(z)$ .
- (3) If  $k \equiv 2 \pmod 3$ ,  $g_c(\omega z) = \omega^2 g_c(z)$ ,  $g_c(\omega^2 z) = \omega g_c(z)$ .

Since  $M \in \partial c^k S \cap g_c^{-n}(\bar{\mathbf{D}}_R)$ , it follows that  $\omega^j M \in \partial c^k S \cap g_c^{-n}(\bar{\mathbf{D}}_R)$ . Then there is a path connecting  $\omega^j M$  and  $Q$ . Next we consider the set  $g_c^{-1}(g_c^{-n}(\bar{\mathbf{D}}_R))$ . We may regard the closed domain  $cS$  bounded by a hypocycloid as the triangular region  $R$ . We prove this lemma when  $k = 2$ . (In other cases, the similar proofs hold.) We consider the closed domain bounded by  $\triangle OAB$  in Fig. 2.3. The components of  $g_c^{-1}(c^k S)$  may be regarded as small triangular regions  $\triangle OEF$ ,  $\triangle ADF$ ,  $\triangle BDE$ ,  $\triangle DEF$ . Any element  $M_{jl}$

of  $g_c^{-1}(\omega^j M)$  lies on an edge of a small triangle. Let  $g_c^{-1}(Q) = \{Q = Q_0, Q_1, Q_2, Q_3\}$ . Each  $Q_i$  lies in an interior of a small triangle. Two points  $Q_i$  and  $Q_h$  are connected by a path in  $g_c^{-n-1}(\bar{\mathbf{D}}_R)$  through some points  $M_{jl}$ . From Lemma 3.5, we know that there is a path in  $g_c^{-n-1}(\bar{\mathbf{D}}_R)$  connecting  $P_{-1}$  and some point  $M_{jl}$ . Then we have a path in  $g_c^{-n-1}(\bar{\mathbf{D}}_R)$  connecting  $P_{-1}$  and  $Q$ .

In the same way we can prove this lemma for any  $P \in g_c^{-n}(\bar{\mathbf{D}}_R) \cap c^k S^\circ$ . □

Proof of Proposition 3.6 (2). When  $k = 2$ , this lemma has already been proved in Theorem 5.1 in [13]. In the same way we can prove this lemma when  $k \geq 3$ . So we show only an outline of the proof. From Lemma 3.11 and Lemma 3.12, we see that there is a nonnegative integer  $N$  such that  $g_c^{-N}(\bar{\mathbf{D}}_R)$  is contained in the interior of  $c^k S$  and  $g_c^{-N}(\bar{\mathbf{D}}_R)$  is connected. So,  $g_c$  does not have any critical values in  $g_c^{-N}(\bar{\mathbf{D}}_R)$ . We use inverse branches  $\phi_j, j = 1, \dots, k^2$ , of  $g_c$ . Set

$$K(i_1, \dots, i_n) := \phi_{i_1} \circ \dots \circ \phi_{i_n}(g_c^{-N}(\bar{\mathbf{D}}_R)).$$

Then in the same way used in the proof of Theorem 5.1 in [13], we can prove that for any given sequence  $(j_n)$ ,  $\text{diameter}[K(j_1, \dots, j_n)] \rightarrow 0$  as  $n \rightarrow \infty$ . Then

$$K(g_c) = \bigcap_{n=0}^{\infty} \left( \bigcup_{j_1, \dots, j_n=1}^{k^2} K(j_1, \dots, j_n) \right)$$

is a Cantor set. □

This completes the proof of Proposition 3.6. □

Proof of Theorem 3.1. When  $k \geq 2$ , Theorem 3.1 follows from Proposition 3.6. We will prove Theorem 3.1 for the maps  $f_c^{(-k)}(x, y)$  with  $k \geq 2$ . In the proof of Lemma 3.3 (1) we prove that  $(f_c^{(-k)})^2 = (f_c^{(k)})^2$ . Then  $K(f_c^{(-k)}) = K((f_c^{(-k)})^2) = K((f_c^{(k)})^2) = K(f_c^{(k)})$ . Combining Proposition 3.1 and Theorem 3.8 in [6] yields  $K(f_c^{(-k)}) = \text{supp } \mu$ . Then by Proposition 3.6, we conclude that  $\text{supp } \mu$  is a Cantor set. □

**Appendix A.**

We show a relation between complex Jacobian matrices and real Jacobian matrices on  $\{x = \bar{y}\}$  for symmetric polynomial endomorphisms.

Let  $h(z_1, z_2) \in \mathbf{R}[z_1, z_2]$ . We define a map  $f$  from  $\mathbf{C}^2$  to  $\mathbf{C}^2$  by

$$f(z_1, z_2) := (h(z_1, z_2), h(z_2, z_1)).$$

Then the holomorphic map  $f$  admits an invariant plane  $\{z_1 = \bar{z}_2\}$ . Let  $g(z) := h(z, \bar{z})$ . Then  $g$  may be viewed as a map from  $\mathbf{R}^2$  to  $\mathbf{R}^2$ .

**Proposition A.1.** *Let  $Df(z, \bar{z})$  and  $Dg(z)$  be the complex Jacobian matrix of  $f$  at  $(z, \bar{z})$  and the real Jacobian matrix of  $g$  at  $z$ , respectively. Then*

$$U^{-1}Df(z, \bar{z})U = Dg(z),$$

where  $U$  is a unitary matrix given by

$$U = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

*Proof.* Let  $z_k = x_k + iy_k$  ( $k = 1, 2$ ). Let

$$(A.1) \quad f(z_1, z_2) = (p_1 + iq_1, p_2 + iq_2),$$

where  $p_k(x_1, x_2, y_1, y_2)$  and  $q_k(x_1, x_2, y_1, y_2)$  are real valued functions. It is well known (e.g. [7]) that if we define  $2 \times 2$  matrices  $A$  and  $B$  by

$$(A.2) \quad A = \begin{pmatrix} \frac{\partial p_j}{\partial x_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial q_j}{\partial y_k} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{\partial q_j}{\partial x_k} \end{pmatrix} = -\begin{pmatrix} \frac{\partial p_j}{\partial y_k} \end{pmatrix},$$

then  $Df = A + iB$ .

By (A.1), we know that any term in  $p_k$  (resp.  $q_k$ ) is represented as  $a(x_1, x_2)y_1^m y_2^m$  where  $m+n$  is nonnegative and even (resp. odd). On the plane  $\{z_1 = \bar{z}_2\}$ ,  $x_1 = x_2$  and  $y_1 = -y_2$ . Then we have the followings:

- (1)  $\partial p_1/\partial x_1 = \partial p_2/\partial x_2$  and  $\partial p_1/\partial x_2 = \partial p_2/\partial x_1$ ,
- (2)  $\partial q_1/\partial x_1 = -\partial q_2/\partial x_2$  and  $\partial q_1/\partial x_2 = -\partial q_2/\partial x_1$ ,

at any point  $(z, \bar{z})$  in the plane  $\{z_1 = \bar{z}_2\}$ . Hence by (A.2) we see that

$$Df(z, \bar{z}) = \begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} \\ \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_1} \end{pmatrix} \Big|_{(z, \bar{z})} + i \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} \\ -\frac{\partial q_1}{\partial x_2} & -\frac{\partial q_1}{\partial x_1} \end{pmatrix} \Big|_{(z, \bar{z})}.$$

Therefore

$$U^{-1}Df(z, \bar{z})U^{-1} = \begin{pmatrix} \frac{\partial p_1}{\partial x_1} + \frac{\partial p_1}{\partial x_2} & -\frac{\partial q_1}{\partial x_1} + \frac{\partial q_1}{\partial x_2} \\ \frac{\partial q_1}{\partial x_1} + \frac{\partial q_1}{\partial x_2} & \frac{\partial p_1}{\partial x_1} - \frac{\partial p_1}{\partial x_2} \end{pmatrix} \Big|_{(z, \bar{z})}.$$

Set  $z := u + iv$ ,  $p(u, v) := p_1(u, u, v, -v)$  and  $q(u, v) := q_1(u, u, v, -v)$ . Then

$$\begin{aligned} \frac{\partial p}{\partial u} &= \frac{\partial p_1}{\partial x_1} + \frac{\partial p_1}{\partial x_2}, & \frac{\partial p}{\partial v} &= \frac{\partial p_1}{\partial y_1} - \frac{\partial p_1}{\partial y_2} = -\frac{\partial q_1}{\partial x_1} + \frac{\partial q_1}{\partial x_2}, \\ \frac{\partial q}{\partial u} &= \frac{\partial q_1}{\partial x_1} + \frac{\partial q_1}{\partial x_2}, & \frac{\partial q}{\partial v} &= \frac{\partial q_1}{\partial y_1} - \frac{\partial q_1}{\partial y_2} = \frac{\partial p_1}{\partial x_1} - \frac{\partial p_1}{\partial x_2}. \end{aligned}$$

Since  $g(u, v) = (p(u, v), q(u, v))$ , it follows that

$$U^{-1}Df(u + iv, u - iv)U = Dg(u, v). \quad \square$$

ACKNOWLEDGEMENTS. The author thanks the referee for helping to improve the paper. The author also thanks Mr. Timmy Jensen for correcting the manuscript.

---

### References

- [1] E. Bedford and M. Jonsson: *Dynamics of regular polynomial endomorphisms of  $\mathbb{C}^k$* , Amer. J. Math. **122** (2000), 153–212.
- [2] J.-Y. Briend and J. Duval: *Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de  $\mathbb{C}P^k$* , Acta Math. **182** (1999), 143–157.
- [3] N. Bourbaki: *Groups et Algebras de Lie*, chapitres IV, V, VI, Hermann, Paris, 1968.
- [4] T.-C. Dinh: *Sur les endomorphismes polynomiaux permutables de  $\mathbb{C}^2$* , Ann. Inst. Fourier (Grenoble) **51** (2001), 431–459.
- [5] J.E. Fornæss and N. Sibony: *Complex dynamics in higher dimension*, I, Astérisque **222** (1994), 201–231.
- [6] J.E. Fornæss and N. Sibony: *Dynamics of  $\mathbb{P}^2$  (examples)*; in *Laminations and Foliations in Dynamics, Geometry and Topology* (Stony Brook, NY, 1998), Contemp. Math. **269**, Amer. Math. Soc., Providence, RI., 2001, 47–85.
- [7] K. Fritzsche and H. Grauert: *From Holomorphic Functions to Complex Manifolds*, Springer, New York, 2002.
- [8] M.E. Hoffman and W.D. Withers: *Generalized Chebyshev polynomials associated with affine Weyl groups*, Trans. Amer. Math. Soc. **308** (1988), 91–104.
- [9] T.H. Koornwinder: *Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators*, III, IV, Indag. Math. **36** (1974), 357–369, 370–381.
- [10] R. Lidl: *Tschebyscheffpolynome in mehreren Variablen*, J. Reine Angew. Math. **273** (1975), 178–198.
- [11] S. Nakane: *External rays for polynomial maps of two variables associated with Chebyshev maps*, J. Math. Anal. Appl. **338** (2008), 552–562.
- [12] T. Needham: *Visual Complex Analysis*, Oxford Univ. Press, New York, 1997.
- [13] K. Uchimura: *The sets of points with bounded orbits for generalized Chebyshev mappings*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **11** (2001), 91–107.
- [14] K. Uchimura: *Dynamics of symmetric polynomial endomorphisms of  $\mathbb{C}^2$* , Michigan Math. J. **55** (2007), 483–511.
- [15] A.P. Veselov: *Integrable mappings and Lie algebras*, Soviet Math. Dokl. **35** (1987), 211–213.

Department of Mathematics  
 Tokai University  
 Hiratsuka, 259–1292  
 Japan  
 e-mail: uchimura@tokai-u.jp