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Author(s)	Kato, Kazuya; Nakayama, Chikara; Usui, Sampei
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Classifying spaces of degenerating mixed Hodge structures, V: Extended period domains and algebraic groups

Kazuya Kato, Chikara Nakayama, Sampei Usui

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Abstract

For a linear algebraic group G over \mathbf{Q} , we consider the period domains D classifying G -mixed Hodge structures, and construct the extended period domains D_{BS} , $D_{\text{SL}(2)}$, and $\Gamma \backslash D_\Sigma$. In particular, we give toroidal partial compactifications of mixed Mumford–Tate domains.

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0 Introduction

In this paper, we generalize the theories of Mumford–Tate domains (see Green–Griffiths–Kerr’s book [16]) and their toroidal partial compactifications by Kerr–Pearlstein ([25]) to mixed Hodge theory. We also construct the corresponding Borel–Serre spaces and the spaces of $\mathrm{SL}(2)$ -orbits. In [22], we described this generalization briefly with an application to construct the toroidal partial compactifications of higher Albanese manifolds. In this paper, we give the details.

In Section 1, for a linear algebraic group G , we define the period domain D as a space of G -mixed Hodge structures. Here a G -mixed Hodge structure means an exact \otimes -functor from the category of linear representations of G endowed with weight filtrations to the

category of mixed Hodge structures. In the case where G is reductive, our period domain D is essentially the Mumford–Tate domain studied in [16].

In Sections 2, 3, and 4, we construct the extended period domains D_{BS} , $D_{\text{SL}(2)}$, and $\Gamma \setminus D_\Sigma$, the space of Borel–Serre orbits, the space of $\text{SL}(2)$ -orbits, and the space of nilpotent orbits, respectively. In the case where G is reductive, $\Gamma \setminus D_\Sigma$ is essentially the space of Kerr–Pearlstein ([25]).

Our method to construct D_{BS} , $D_{\text{SL}(2)}$, and $\Gamma \setminus D_\Sigma$ is similar to that for the usual period domain developed in [24] and in the preceding parts of this series of papers [21]. Actually, the present paper is regarded as a generalization of our previous works in Parts I–IV of [21]. We prove similar results (the Hausdorffness etc.) for these spaces to what we have proved so far for the corresponding spaces in the case of usual period domain. We obtain a fundamental diagram

$$\begin{array}{ccccccc}
 & & & D_{\text{SL}(2),\text{val}}^* & \xrightarrow{\eta^*} & D_{\text{BS},\text{val}} & \\
 & & & \downarrow & & \downarrow & \\
 D_{\Sigma,[\text{val}]}^\sharp & \xrightarrow{\psi} & D_{\text{SL}(2),\text{val}}^I & \rightarrow & D_{\text{SL}(2),\text{val}}^{II} & \xrightarrow{\eta^*} & D_{\text{BS},\text{val}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Gamma \setminus D_{\Sigma,\text{val}} & \leftarrow & D_{\Sigma,:}^\sharp & \xrightarrow{\psi} & D_{\text{SL}(2)}^I & \rightarrow & D_{\text{SL}(2)}^{II} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Gamma \setminus D_\Sigma & \leftarrow & D_\Sigma^\sharp & & & & D_{\text{SL}(2)}^* \\
 & & & & & & \downarrow \\
 & & & & D_{\text{SL}(2)}^{\star,+} & \xrightarrow{\eta^*} & D_{\text{SL}(2)}^{\star,W} \\
 & & & & \downarrow & & \downarrow \\
 & & & & D_{\text{SL}(2)}^* & \xrightarrow{\eta^*} & D_{\text{SL}(2)}^* \\
 & & & & \downarrow & & \downarrow \\
 & & & & D_{\text{SL}(2)}^* & \xrightarrow{\eta^*} & D_{\text{SL}(2)}^* \\
 & & & & \downarrow & & \downarrow \\
 & & & & D_{\text{SL}(2)}^* & \xrightarrow{\eta^*} & D_{\text{SL}(2)}^*
 \end{array}$$

of the same style as the fundamental diagram in Part IV of [21], which is commutative and in which the maps respect the structures of the spaces. The good properties of these spaces are proved starting from the spaces on the right-hand-side and then moving to the left-hand-side, similarly as in [24] and [21].

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1 The period domain D

Let G be a linear algebraic group over \mathbf{Q} . Let G_u be the unipotent radical of G , and let $G_{\text{red}} = G/G_u$ be the reductive quotient of G . Let $\text{Rep}(G)$ be the category of finite-dimensional linear representations of G over \mathbf{Q} .

1.1 Definition of the period domain D

1.1.1. In this Section 1.1, we give the definition of our period domain D considered in this paper. As we will see in Section 1.2, this is the period domain D in our previous work [22]. In this Section 1.1, we introduce D from the following point of view: We want to have a period domain endowed with an action of G . More precisely, we want to have a

complex analytic period domain D for \mathbf{Q} -mixed Hodge structures which is endowed with a real analytic action of the subgroup $G(\mathbf{R})G_u(\mathbf{C})$ of $G(\mathbf{C})$. Then we naturally come to the definition of our D .

1.1.2. For a subfield E of \mathbf{R} , let $(EMHS)$ be the category of E -mixed Hodge structures.

By a G -mixed Hodge structure (G -MHS for short), we mean an exact \otimes -functor ([14] 2.7) $H : \text{Rep}(G) \rightarrow (\mathbf{QMHS})$; $V \mapsto (H(V)_{\mathbf{Q}}, W, F)$. (This definition of G -MHS is slightly different from the definition in our previous work [22].)

1.1.3. Let $D_{\text{all}}(G)$ be the set of all isomorphism classes of G -MHS $H : \text{Rep}(G) \rightarrow (\mathbf{QMHS})$ which preserves the underlying \mathbf{Q} -vector spaces (that is, for any $V \in \text{Rep}(G)$, $H(V)_{\mathbf{Q}} = V$ as a \mathbf{Q} -vector space) and which satisfy the following conditions (i) and (ii).

(i) For every $V \in \text{Rep}(G)$, the weight filtration W of $H(V)$ is stabilized by the action of G on V .

(ii) For every $V \in \text{Rep}(G)$, G_u acts trivially on $\text{gr}^W V$.

Then the subgroup $G(\mathbf{R})G_u(\mathbf{C})$ of $G(\mathbf{C})$ acts on $D_{\text{all}}(G)$: For $H \in D_{\text{all}}(G)$ and $g \in G(\mathbf{R})G_u(\mathbf{C})$, $gH \in D_{\text{all}}(G)$ is defined as follows. For $V \in \text{Rep}(G)$, $(gH(V))_{\mathbf{Q}} = V$, the weight filtration W of $gH(V)$ is that of $H(V)$, and the Hodge filtration F of $gH(V)$ is given by $F^p(gH(V)) := gF^p H(V) \subset V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{Q}} V$, where $F^p H(V)$ is the Hodge filtration of $H(V)$.

1.1.4. By a *period domain of G -MHS*, we mean a $G(\mathbf{R})G_u(\mathbf{C})$ -orbit in $D_{\text{all}}(G)$.

This is the definition of the period domain of this paper, which we denote by D .

1.1.5. As will be shown in 1.2.15, for a reasonable topology on $D_{\text{all}}(G)$, each period domain is open and closed in $D_{\text{all}}(G)$ and hence as a topological space, $D_{\text{all}}(G)$ is a disjoint union of all period domains in $D_{\text{all}}(G)$.

Remark 1.1.6. We think this definition of D is a natural way to have a period domain for \mathbf{Q} -MHS with an action of $G(\mathbf{R})G_u(\mathbf{C})$.

To consider a G -MHS is a natural way to have an action of G on the period domain because each $V \in \text{Rep}(G)$ has an action of G .

The group $G(\mathbf{R})$ acts on $D_{\text{all}}(G)$ by the condition (i) in 1.1.3. We remark how this condition (i) is important. Note that if (V, W, F) is a \mathbf{Q} -MHS with $V \in \text{Rep}(G)$, W the weight filtration, and F the Hodge filtration, then for $g \in G(\mathbf{R})$, we have an \mathbf{R} -MHS $(V_{\mathbf{R}}, gW_{\mathbf{R}}, gF)$. But if $gW_{\mathbf{R}}$ is rational for all $g \in G(\mathbf{R})$ (we need this to have the action of $G(\mathbf{R})$ on the set of \mathbf{Q} -MHS), W must be stabilized by the connected component of $G(\mathbf{R})$ containing 1. (That is, rational weight filtrations can not move continuously.) Hence it is natural to put the condition that G stabilizes W .

We have the action of $G_u(\mathbf{C})$ on $D_{\text{all}}(G)$ by the condition (ii) in 1.1.3.

Remark 1.1.7. The condition G stabilizes W may be natural also for the following reason. In the definition of a period domain of mixed Hodge structures, it is natural to fix the weight filtration and move the Hodge filtration. This is because in a variation of mixed Hodge structure, the weight filtration is put on the local system and the Hodge filtration is put on the vector bundle, and hence the weight filtration is constant locally and Hodge filtration varies continuously.

1.1.8. For a subfield E of \mathbf{R} , by an E -HS, we will mean an E -MHS which is a direct sum of pure E -Hodge structures. (They are also called *split* E -mixed Hodge structure.) Let $(E\text{-HS})$ be the category of E -HS. By $G\text{-HS}$, we mean a G -MHS which has values in $(\mathbf{Q}\text{-HS}) \subset (\mathbf{Q}\text{-MHS})$.

Proposition 1.1.9. *Assume that G is reductive and let $H \in D_{\text{all}}(G)$. Then H is a G -HS. For $V \in \text{Rep}(G)$ and for $w \in \mathbf{Z}$, if V_w denotes the underlying \mathbf{Q} -vector space of the part of $H(V)$ of weight w , V_w is G -stable in V .*

Proof. Since any representation of a reductive group is semisimple, the weight filtration on V has a splitting which is compatible with the action of G . Hence there is a decomposition $V = \bigoplus_w V_w$ such that V_w is G -stable and $W_w V_w = V_w$, $W_{w-1} V_w = 0$. We have $H(V) = \bigoplus_w H(V_w)$ and $H(V_w)$ is a pure Hodge structure of weight w . \square

1.2 Relation to [11] Section 5 and [12] 1.5–1.8 of Deligne

1.2.1. Let $S_{\mathbf{C}/\mathbf{R}}$ be the Weil restriction of \mathbf{G}_m from \mathbf{C} to \mathbf{R} . It represents the functor $A \mapsto (\mathbf{C} \otimes_{\mathbf{R}} A)^\times$ for commutative rings A over \mathbf{R} . We have $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^\times$, which is understood as \mathbf{C}^\times regarded as an algebraic group over \mathbf{R} .

Let $w : \mathbf{G}_{m,\mathbf{R}} \rightarrow S_{\mathbf{C}/\mathbf{R}}$ be the homomorphism induced from the natural maps $A^\times \rightarrow (\mathbf{C} \otimes_{\mathbf{R}} A)^\times$ for commutative rings A over \mathbf{R} .

In [11] and [12], Deligne related $S_{\mathbf{C}/\mathbf{R}}$ to the theory of Hodge structures as follows.

1.2.2. The following (1) and (2) are identified.

- (1) An \mathbf{R} -HS.
- (2) A (finite-dimensional) linear representation of $S_{\mathbf{C}/\mathbf{R}}$ over \mathbf{R} .

In fact, a linear representation of $S_{\mathbf{C}/\mathbf{R}}$ over \mathbf{R} is equivalent to a finite-dimensional \mathbf{R} -vector space V endowed with a decomposition

$$V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} V = \bigoplus_{p,q \in \mathbf{Z}} V_{\mathbf{C}}^{p,q}$$

such that for any p, q , $V_{\mathbf{C}}^{q,p}$ coincides with the complex conjugate of $V_{\mathbf{C}}^{p,q}$ (that is, the image of $V_{\mathbf{C}}^{p,q}$ under $\mathbf{C} \otimes_{\mathbf{R}} V \rightarrow \mathbf{C} \otimes_{\mathbf{R}} V$; $z \otimes v \mapsto \bar{z} \otimes v$). For a linear representation V of $S_{\mathbf{C}/\mathbf{R}}$, the corresponding decomposition is defined by

$$V_{\mathbf{C}}^{p,q} = \{v \in V_{\mathbf{C}} \mid [z]v = z^p \bar{z}^q v \text{ for } z \in \mathbf{C}^\times\}.$$

Here $[z]$ denotes z regarded as an element of $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$. By taking $V_{\mathbf{C}}^{p,q}$ as the (p, q) -Hodge component of $V_{\mathbf{C}}$, this is understood as an \mathbf{R} -HS with underlying \mathbf{R} -vector space V .

The weight m -part of this \mathbf{R} -HS is the part on which $z \in \mathbf{R}^\times = \mathbf{G}_{m,\mathbf{R}}(\mathbf{R})$ acts as multiplication by z^m via $w : \mathbf{G}_{m,\mathbf{R}} \rightarrow S_{\mathbf{C}/\mathbf{R}}$.

1.2.3. (1) The following (1.1) and (1.2) are identified.

- (1.1) An exact \otimes -functor $H : \text{Rep}(G) \rightarrow (\mathbf{R}\text{-HS})$ with the underlying \mathbf{R} -vector space $H(V)_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} V$ ($V \in \text{Rep}(G)$).
- (1.2) A homomorphism $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$.

(2) The following (2.1) and (2.2) are identified.

(2.1) A G -HS $H : \text{Rep}(G) \rightarrow (\text{QHS})$ (1.1.8) preserving the underlying \mathbf{Q} -vector spaces.

(2.2) A homomorphism $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ such that $h \circ w : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}}$ is defined over \mathbf{Q} .

We explain (1). If we are given a homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ in (1.2), for any $V \in \text{Rep}(G)$, $H(V)_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} V$ has an action of $S_{\mathbf{C}/\mathbf{R}}$ and is regarded as an \mathbf{R} -HS. Conversely, if we have a functor $H : \text{Rep}(G) \rightarrow (\mathbf{RHS})$ having the property in (1.1), then by 1.2.2 and by the theory of Tannakian categories, we have a homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$. This shows (1).

Taking account of \mathbf{Q} -structures, we get (2) from (1).

1.2.4. Let $\hat{D}_{\text{all}}(G)$ be the set of all isomorphism classes of exact \otimes -functors $H : \text{Rep}(G) \rightarrow (\mathbf{RMHS})$ with the underlying \mathbf{R} -vector space $H(V)_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} V$ ($V \in \text{Rep}(G)$) such that for every $V \in \text{Rep}(G)$, the action of G on V stabilizes the weight filtration W on $V_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} V$ and the action of G_u on $\text{gr}^W V_{\mathbf{R}}$ is trivial. (Note that $D_{\text{all}}(G)$ in 1.1.3 is identified with the set of those functors whose weight filtrations are \mathbf{Q} -rational.)

Let $\hat{D}'_{\text{all}}(G)$ be the set of all pairs (h, δ) , where h is a homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ and δ is an element of $\text{Lie}(G)_{\mathbf{R}}$ satisfying the following condition (i). Consider the adjoint action of G on $\text{Lie}(G)$ and the induced action of $S_{\mathbf{C}/\mathbf{R}}$ on $\text{Lie}(G)_{\mathbf{R}}$ via h . By this, $\text{Lie}(G)_{\mathbf{R}}$ becomes an \mathbf{R} -HS. For $m, n \in \mathbf{Z}$, let $H^{m,n}$ be the Hodge (m, n) -component of the weight $(m+n)$ -part of $\text{Lie}(G)_{\mathbf{C}}$.

(i) $\delta \in \bigoplus_{m < 0, n < 0} H^{m,n}$.

Proposition 1.2.5. *We have a bijection*

$$\hat{D}'_{\text{all}}(G) \rightarrow \hat{D}_{\text{all}}(G) ; (h, \delta) \mapsto e^{i\delta} H,$$

where H is the functor $\text{Rep}(G) \rightarrow (\mathbf{RHS})$ corresponding to h in 1.2.3 (1).

Proof. This follows from [10] Proposition (2.20). \square

Lemma 1.2.6. *Let $H \in \hat{D}_{\text{all}}(G)$ (1.2.4). Consider the adjoint action of G on $\text{Lie}(G)$ and consider the weight filtration of $\text{Lie}(G)_{\mathbf{R}}$ and the Hodge filtration of $\text{Lie}(G)_{\mathbf{C}}$ defined by H .*

(1) *Let $\ell \in \text{Lie}(G)_{\mathbf{R}}$ and $w \in \mathbf{Z}$. Then $\ell \in W_w \text{Lie}(G)_{\mathbf{R}}$ if and only if, for any $V \in \text{Rep}(G)$ and any $k \in \mathbf{Z}$, the Lie action $\text{Lie}(G)_{\mathbf{R}} \times V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ satisfies $\ell W_k V_{\mathbf{R}} \subset W_{k+w} V_{\mathbf{R}}$.*

(2) *Let $\ell \in \text{Lie}(G)_{\mathbf{C}}$ and $p \in \mathbf{Z}$. Then $\ell \in F^p \text{Lie}(G)_{\mathbf{C}}$ if and only if, for any $V \in \text{Rep}(G)$ and any $r \in \mathbf{Z}$, the Lie action $\text{Lie}(G)_{\mathbf{C}} \times V_{\mathbf{C}} \rightarrow V_{\mathbf{C}}$ satisfies $\ell F^r V_{\mathbf{C}} \subset F^{r+p} V_{\mathbf{C}}$.*

Proof. The only if parts are clear.

Take a $V \in \text{Rep}(G)$ such that the map $\text{Lie}(G) \rightarrow \text{End}(V)$ induced by the Lie action is injective. Then since this is a homomorphism of \mathbf{R} -MHS, we have $W_w \text{Lie}(G)_{\mathbf{R}} = \text{Lie}(G)_{\mathbf{R}} \cap W_w \text{End}(V_{\mathbf{R}}) = \{h \in \text{Lie}(G)_{\mathbf{R}} \mid h W_k V_{\mathbf{R}} \subset W_{k+w} V_{\mathbf{R}} \text{ for any } k \in \mathbf{Z}\}$, and a similar relation of Hodge filtrations. This proves the if parts. \square

1.2.7. (1) Let $\Psi_W(G)$ be the set of all homomorphisms $k : \mathbf{G}_m \rightarrow G_{\text{red}}$ satisfying the following conditions (i) and (ii).

(i) The image of k is contained in the center of G_{red} .

(ii) For some homomorphism (and hence for any homomorphism) $\tilde{k} : \mathbf{G}_m \rightarrow G$ which lifts k , the adjoint action of \mathbf{G}_m on $\text{Lie}(G_u)$ via \tilde{k} is of weight ≤ -1 .

Note that in the case where G is reductive, the condition (ii) is automatically satisfied.

(2) Let $\Psi_H(G)$ be the set of all homomorphisms $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$ satisfying the following condition: There exists $k \in \Psi_W(G)$ such that $h \circ w : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$ comes from k (see 1.2.1 for w).

We have a canonical map $\Psi_H(G) \rightarrow \Psi_W(G) ; h \mapsto k$.

1.2.8. Let $k \in \Psi_W(G)$.

Then, for any $V \in \text{Rep}(G)$, the action of \mathbf{G}_m on V via a lifting \tilde{k} of k defines a rational increasing filtration W on V called the *weight filtration*, which is independent of the lifting. For any $V \in \text{Rep}(G)$, $W_w V$ is G -stable for any $w \in \mathbf{Z}$ and the action of G_u in $\text{gr}^W V$ is trivial.

Proposition 1.2.9. (1) Let $H \in \hat{D}_{\text{all}}(G)$ (1.2.4). Then the following conditions (i) and (ii) are equivalent.

(i) $H \in D_{\text{all}}(G)$ (1.1.3).

(ii) The restriction of H to $\text{Rep}(G_{\text{red}})$ is a G -HS (1.1.8), and the corresponding homomorphism $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$ (1.2.3 (2)) belongs to $\Psi_H(G)$ (1.2.7 (2)).

(2) Let $H \in \hat{D}_{\text{all}}(G)$ (1.2.4). If the equivalent conditions in (1) are satisfied, the weight filtrations of H are given by the image k of h in $\Psi_W(G)$ as in 1.2.8.

(3) If G is reductive, the map $D_{\text{all}}(G) \rightarrow \Psi_H(G) ; H \mapsto h$ is bijective.

Proof. In the case where G is reductive, (1) follows from Proposition 1.1.9.

(3) follows from the case of (1) where G is reductive.

We prove (1) and (2).

Assume (i). Then, by Proposition 1.1.9, the restriction is a G -HS. Further, the induced homomorphism $h \circ w : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$ is defined over \mathbf{Q} and its image is in the center of $G_{\text{red},\mathbf{R}}$. We prove that $\text{Lie}(G_u)$ is of weight ≤ -1 , which implies (ii). Since G_u acts on $\text{gr}^W V$ trivially for any $V \in \text{Rep}(G)$, the Lie action of $\text{Lie}(G_u)$ on V induces the zero action on $\text{gr}^W V$. By Lemma 1.2.6, this proves that $\text{Lie}(G_u)$ has weights ≤ -1 .

Assume (ii). Since H comes from $\hat{D}'_{\text{all}}(G)$ (Proposition 1.2.5), the weight filtration W of H is given by the image k of h in $\Psi_W(G)$ as in 1.2.8. Hence the weight filtration of H is rational, that is, $H \in D_{\text{all}}(G)$. Thus we have proved (1) and (2). \square

By Proposition 1.2.9 (1), we have a map

$$D_{\text{all}}(G) \rightarrow \Psi_H(G) ; H \mapsto h.$$

Corollary 1.2.10. If $D_{\text{all}}(G)$ is not empty, G_u is contained in the commutator subgroup of G .

Proof. If $D_{\text{all}}(G)$ is not empty, via the maps $D_{\text{all}}(G) \rightarrow \Psi_H(G) \rightarrow \Psi_W(G)$, we see that $\Psi_W(G)$ is not empty. For a lifting $\tilde{k} : \mathbf{G}_m \rightarrow G$ of an element $k : \mathbf{G}_m \rightarrow G_{\text{red}}$ of $\Psi_W(G)$, since $\text{Lie}(G_u)$ is of weight ≤ -1 for \tilde{k} , we have $G_u = [\tilde{k}(\mathbf{G}_m), G_u]$. \square

Proposition 1.2.11. *Consider the maps $D_{\text{all}}(G) \rightarrow \Psi_H(G) \rightarrow \Psi_W(G)$.*

- (1) *Let $H \in D_{\text{all}}(G)$ and let h be the image of H in $\Psi_H(G)$. Then for $g_1 \in G(\mathbf{R})$ and $g_2 \in G_u(\mathbf{C})$, the image of $g_1g_2H \in D_{\text{all}}(G)$ in $\Psi_H(G)$ is $g_{1,\text{red}}hg_{1,\text{red}}^{-1} : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$. Here $g_{1,\text{red}}$ is the image of g_1 in $G_{\text{red}}(\mathbf{R})$.*
- (2) *Let $h \in \Psi_H(G)$ and let k be the image of h in $\Psi_W(G)$. Then for $g \in G_{\text{red}}(\mathbf{R})$, the image of ghg^{-1} in $\Psi_W(G)$ is k .*
- (3) *The map $D_{\text{all}}(G) \rightarrow \Psi_H(G)$ is surjective. The action of $G_u(\mathbf{C})$ on each fiber of this map is transitive.*

Proof. (1) and (2) are straightforwards.

We prove (3). Let $h \in \Psi_H(G)$. By Proposition 1.2.5, the fiber of h in $D_{\text{all}}(G)$ consists of the images $e^{i\delta}H$ of $(\tilde{h}, \delta) \in \hat{D}'_{\text{all}}(G)$ in $\hat{D}_{\text{all}}(G)$, where $\tilde{h} : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ are liftings of h and H denotes the functor $\text{Rep}(G) \rightarrow (\text{RHS})$ corresponding to \tilde{h} . The fiber is not empty because a lifting \tilde{h} of h exists. Let $(\tilde{h}, \delta), (\tilde{h}', \delta') \in \hat{D}'_{\text{all}}(G)$, where \tilde{h}, \tilde{h}' are liftings of h and let H and H' be the functors corresponding to \tilde{h} and \tilde{h}' , respectively. For $V \in \text{Rep}(G)$, the weight filtration W on V given by $e^{i\delta}H$ and that given by $e^{i\delta'}H'$ coincide because they are given by the image of h in $\Psi_W(G)$ as in 1.2.8. Let $s_V : \text{gr}^W V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ and $s'_V : \text{gr}^W V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ be the splittings of W over \mathbf{R} defined by $\tilde{h} \circ w$ and $\tilde{h}' \circ w$, respectively. Then $(s'_V \circ s_V^{-1} : V \rightarrow V)_V$ comes from an element u of $G_u(\mathbf{R})$ and $\tilde{h}' = u\tilde{h}u^{-1}$. We have $H' = uH$ and hence $e^{i\delta'}H' = ge^{i\delta}H$, where $g = e^{i\delta'}ue^{-i\delta} \in G_u(\mathbf{C})$. \square

We will prove a more precise result on the fibers in Theorem 1.4.6 below.

1.2.12. Fix $k \in \Psi_W(G)$. Then the inverse image of k in $D_{\text{all}}(G)$ under $D_{\text{all}}(G) \rightarrow \Psi_W(G)$ is identified with the set of all G -MHS which preserve the underlying \mathbf{Q} -vector spaces and whose weight filtrations are given by k as in 1.2.8.

1.2.13. Fix a homomorphism

$$h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$$

which belongs to $\Psi_H(G)$ (1.2.7). Then, by Proposition 1.2.11, there is a unique $G(\mathbf{R})G_u(\mathbf{C})$ -orbit $D(G, h_0)$ in $D_{\text{all}}(G)$ whose image in $\Psi_H(G)$ is the set of all $G(\mathbf{R})$ -conjugates of h_0 . We call this period domain of G -MHS (1.1.4) the *period domain associated to h_0* . By 1.2.12, $D(G, h_0)$ coincides with the period domain associated to h_0 defined in our previous paper [22].

In the case where G is reductive, $D(G, h_0)$ is identified with the set of all $G(\mathbf{R})$ -conjugates of h_0 (Proposition 1.2.9 (3)). In this case, the definition of the period domain $D(G, h_0)$ as the set of $G(\mathbf{R})$ -conjugates of h_0 appears in Section 5 of [11] and 1.5 of [12] of Deligne. We borrow the notation h_0 from [12].

1.2.14. If G is reductive and h_0 in 1.2.13 satisfies the Shimura data in 1.5 of [12], then as in 1.8 of ibid.,

- (1) $G(\mathbf{Q}) \backslash (D(G, h_0) \times G(\mathbf{A}_{\mathbf{Q}}^{\infty})) / K$, where $\mathbf{A}_{\mathbf{Q}}^{\infty}$ is the adele ring of \mathbf{Q} without ∞ -component and K is an open compact subgroup of $G(\mathbf{A}_{\mathbf{Q}}^{\infty})$,

is a *Shimura variety over \mathbf{C} associated to h_0* .

We expect that the set (1) in general (it is a complex analytic space as in Section 1.4 below) is also important in number theory. [19] is a trial of the study in this direction.

We show that each period domain is isolated in $D_{\text{all}}(G)$.

Proposition 1.2.15. (1) *If G is reductive, each $G(\mathbf{R})$ -orbit is open in $D_{\text{all}}(G)$, where we endow $D_{\text{all}}(G)$ with the topology induced by the compact-open topology of $\text{Hom}_{\text{cont}}(\mathbf{C}^{\times}, G(\mathbf{R}))$ via the injection $D_{\text{all}}(G) \simeq \Psi_H(G) \rightarrow \text{Hom}_{\text{cont}}(\mathbf{C}^{\times}, G(\mathbf{R}))$.*

(2) *For a general G , for any topology of $D_{\text{all}}(G)$ such that the map $D_{\text{all}}(G) \rightarrow D_{\text{all}}(G_{\text{red}})$ is continuous for the above topology of $D_{\text{all}}(G_{\text{red}})$, each $G(\mathbf{R})G_u(\mathbf{C})$ -orbit is open in $D_{\text{all}}(G)$. That is, as a topological space, $D_{\text{all}}(G)$ is the disjoint union of period domains.*

Proof. (1) The set of homomorphisms from $\mathbf{G}_{m,\mathbf{R}}$ to the center of $G_{\mathbf{R}}$ is discrete for the topology induced by the compact-open topology of $\text{Hom}_{\text{cont}}(\mathbf{R}^{\times}, G(\mathbf{R}))$. Hence we are reduced to proving that for $S_{\mathbf{C}/\mathbf{R}}^{(1)} := \text{Ker}(\text{norm} : S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathbf{G}_{m,\mathbf{R}})$, every $G(\mathbf{R})$ -conjugacy class in $\text{Hom}(S_{\mathbf{C}/\mathbf{R}}^{(1)}, G_{\mathbf{R}})$ is open for the topology induced by the compact-open topology of $\text{Hom}_{\text{cont}}(S_{\mathbf{C}/\mathbf{R}}^{(1)}(\mathbf{R}), G(\mathbf{R}))$. But this follows from the case $K = S_{\mathbf{C}/\mathbf{R}}^{(1)}(\mathbf{R}) = \{z \in \mathbf{C}^{\times} \mid |z| = 1\}$ and $L = G(\mathbf{R})$ of the result of Lee and Wu [26] that for a compact group K and for a locally compact group L , each L -conjugacy class in $\text{Hom}_{\text{cont}}(K, L)$ is open for the compact-open topology.

(2) follows from (1) because D is the inverse image of its image D_{red} in $D_{\text{all}}(G_{\text{red}})$ and D_{red} is open in $D_{\text{all}}(G_{\text{red}})$ by (1). \square

1.2.16. We compare the above period domain with the Griffiths period domain [17] and its generalization [35] to MHS. (A more precise comparison is given in Section 1.6.)

The period domain in [35] classifies MHS with a fixed weight filtration and fixed Hodge numbers of each gr_w^W , and there the Hodge filtrations move.

In the definition of the period domain of G -MHS in the present paper, fixing W in ibid. corresponds to fixing $k_0 \in \Psi_W(G)$ as in 1.2.8, and fixing Hodge numbers of each gr_w^W in ibid. corresponds to the fact that we fix the $G_{\text{red}}(\mathbf{C})$ -conjugacy class of the composition $\mathbf{G}_{m,\mathbf{C}} \rightarrow \mathbf{G}_{m,\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}} = S_{\mathbf{C}/\mathbf{R},\mathbf{C}} \xrightarrow{h_0} G_{\text{red},\mathbf{C}}$, where the first arrow is $z \mapsto (z, 1)$. Moving the Hodge filtration in ibid. corresponds to moving $H \in D$ by $G(\mathbf{R})G_u(\mathbf{C})$.

1.3 The real analytic structure of D

Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$ be as in 1.2.13 and let $D = D(G, h_0)$ be the associated period domain (1.2.13). We consider the real analytic structure of D .

1.3.1. D is regarded as $(G(\mathbf{R})G_u(\mathbf{C}))/I_x$ for the isotropy subgroup I_x of $G(\mathbf{R})G_u(\mathbf{C})$ at $x \in D$. Furthermore, I_x is a real algebraic subgroup of $G(\mathbf{R})G_u(\mathbf{C})$. This gives a real analytic structure on D , and it is independent of the choice of $x \in D$.

1.3.2. The image of the composite map $D \rightarrow \Psi_H(G) \rightarrow \Psi_W(G)$ is a one point $k_0 \in \Psi_W(G)$. Hence the weight filtration W on $V \in \text{Rep}(G)$ given by $x \in D$ is independent of x and it is defined by k_0 as in 1.2.8.

Let $\text{spl}(W)$ be the set of all isomorphisms of \otimes -functors from $\text{Rep}(G)$ to the category of \mathbf{R} -vector spaces

$$(V \mapsto \text{gr}^W V_{\mathbf{R}}) \xrightarrow{\sim} (V \mapsto V_{\mathbf{R}}) \quad \text{preserving the weight filtrations.}$$

(The notation $\text{spl}(W_{\mathbf{R}})$ may be better because we are considering splittings over \mathbf{R} , but we use the notation $\text{spl}(W)$ for simplicity and for the compatibility with our notation in [21] Parts I–IV.) Then $\text{spl}(W)$ is a $G_u(\mathbf{R})$ -torsor. Hence it is regarded as a real analytic manifold.

1.3.3. Let $D_{\text{red}} = D(G_{\text{red}}, h_0)$. This set is identified with the set of all $G(\mathbf{R})$ -conjugates of h_0 in $\text{Hom}(S_{\mathbf{C}/\mathbf{R}}, G_{\text{red}, \mathbf{R}})$. We have a canonical surjective projection $D \rightarrow D_{\text{red}}$. For $x \in D$, let x_{red} be the image of x in D_{red} .

1.3.4. Let $\mathcal{L} = \bigoplus_{w \leq -2} \text{gr}_w^W \text{Lie}(G)_{\mathbf{R}}$.

Consider the adjoint action of G on $\text{Lie}(G)$. Since the weight filtration is stable under the action of G and since G_u acts trivially on $\text{gr}^W \text{Lie}(G)$ (1.2.8), G_{red} acts on $\text{gr}^W \text{Lie}(G)$. For $p \in D_{\text{red}}$, define $\mathcal{L}(p) = \text{gr}^W \text{Lie}(G)_{\mathbf{R}} \cap (\bigoplus_{m < 0, n < 0} H^{m,n}) \subset \mathcal{L}$, where $H^{m,n}$ denotes the Hodge (m, n) -component of $\text{gr}_{m+n}^W \text{Lie}(G)_{\mathbf{C}}$ with respect to p .

By Proposition 1.2.5, any element x of D is written uniquely as

$$x = s(e^{i\delta} p) : V \mapsto s(e^{i\delta} p(\text{gr}^W(V))),$$

where $p \in D_{\text{red}}$, $s \in \text{spl}(W)$, $\delta \in \mathcal{L}(p)$. In fact, this is the understanding of x as the image of $(\tilde{h}, s\delta s^{-1}) \in \hat{D}'_{\text{all}}(G)$ in $\hat{D}_{\text{all}}(G)$, where $\tilde{h} : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ is the lifting of the homomorphism $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$, corresponding to p , defined as $\tilde{h}(z) = sh(z)s^{-1}$ ($z \in S_{\mathbf{C}/\mathbf{R}}$).

We denote this $\delta \in \mathcal{L}(p)$ by $\delta(x)$.

1.3.5. We have a canonical real analytic map

$$\text{spl}_W : D \rightarrow \text{spl}(W) ; x \mapsto \text{spl}_W(x)$$

which is a modification $\text{spl}_W(x) = s \circ \exp(\zeta)$ of the real analytic map $D \rightarrow \text{spl}(W) ; x \mapsto s$ in 1.3.4 by an element $\zeta \in \mathcal{L}(p)$ explained below. This splitting $\text{spl}_W(x)$ is called the *canonical splitting of W at x* .

We explain $\zeta \in \mathcal{L}(p)$. For each $V \in \text{Rep}(G)$, $\zeta_V \in \mathcal{L}_{p(V)}$ is defined as a universal Lie polynomial of the Hodge (j, k) -components $\delta_{V,j,k}$ of δ_V ([10] (3.28), (6.60); see also [20] 1.4, Appendix, [21] Part II 1.2). We can show $\zeta_{V \otimes V'} = \zeta_V \otimes \text{id}_{V'} + \text{id}_V \otimes \zeta_{V'}$, and we have $\zeta \in \mathcal{L}(p)$.

Proposition 1.3.6. *We have a canonical isomorphism of real analytic manifolds*

$$D \xrightarrow{\sim} \{(p, s, \delta) \in D_{\text{red}} \times \text{spl}(W) \times \mathcal{L} \mid \delta \in \mathcal{L}(p)\}$$

given by $x \mapsto (x_{\text{red}}, \text{spl}_W(x), \delta(x))$.

We have an isomorphism of the same form even if we replace the map $\text{spl}_W : D \rightarrow \text{spl}(W)$ by the map $D \rightarrow \text{spl}(W) ; x \mapsto s$ of 1.3.4, but the isomorphism in Proposition 1.3.6 behaves better in degeneration (see Remark 1.3.10 below).

1.3.7. If $p, p' \in D_{\text{red}}$, $p' = gp$ for some $g \in G_{\text{red}}(\mathbf{R})$, and we have $\mathcal{L}(p') = \text{Ad}(g)\mathcal{L}(p)$. Hence all $\mathcal{L}(p)$ ($p \in D_{\text{red}}$) are isomorphic as graded \mathbf{R} -vector spaces.

Let $L = \mathcal{L}(p)$ for some $p \in D_{\text{red}}$.

By Proposition 1.3.6, we have

Corollary 1.3.8. *D is an L-bundle over $D_{\text{red}} \times \text{spl}(W)$.*

1.3.9. Let D_{spl} be the part of D consisting of exact \otimes -functors $\text{Rep}(G) \rightarrow (\mathbf{QMHS})$ such that the image of the composition $\text{Rep}(G) \rightarrow (\mathbf{QMHS}) \rightarrow (\mathbf{RMHS})$ is contained in (\mathbf{RMHS}) (that is, such that the images are \mathbf{R} -split mixed Hodge structures). Then

$$D_{\text{spl}} = \{s(p) \mid s \in \text{spl}(W), p \in D_{\text{red}}\} = \{x \in D \mid \delta(x) = 0\}$$

and D_{spl} is a closed real analytic submanifold of D . Here $s(p) : \text{Rep}(G) \rightarrow (\mathbf{QMHS})$ sends a $V \in \text{Rep}(G)$ to the \mathbf{Q} -MHS on the underlying \mathbf{Q} -vector space of V which is induced by $p(\text{gr}^W V)$ and $s(V) : \text{gr}^W V_{\mathbf{R}} \simeq V_{\mathbf{R}}$. Let $D_{\text{nspl}} = D \setminus D_{\text{spl}}$.

In the rest of this Section 1.3, we explain how $\text{spl}_W : D \rightarrow \text{spl}(W)$ is important and why we prefer this map to the map $D \rightarrow \text{spl}(W)$; $x \mapsto s$ in 1.3.4.

Remark 1.3.10. When we consider degeneration along a nilpotent orbit, the canonical splitting $\text{spl}_W : D \rightarrow \text{spl}(W)$ behaves better than the splitting $D \rightarrow \text{spl}(W)$ in 1.3.4. We explain this. Assume that (N_1, \dots, N_n, F) generates a nilpotent orbit as in 4.1.3 below. For $y = (y_j)_{1 \leq j \leq n} \in \mathbf{R}^n$ with y_j are sufficiently large for all j , let $\text{spl}_W(y)$ and $s(y)$ be the above splittings associated to the mixed Hodge structure $(W, \exp(\sum_{j=1}^n iy_j N_j)F)$. Then, $\text{spl}_W(y)$ converges in $\text{spl}(W)$ when $y_j/y_{j+1} \rightarrow \infty$ ($1 \leq j \leq n$, y_{n+1} denotes 1) ([20] Theorem 0.5 (1), [21] Part II 2.4.2 (i)), whereas $s(y)$ can diverge. For this convergence, the term ζ in the canonical splitting $\text{spl}_W(y)$ plays a crucial role ([20] Example 13.3).

The canonical splitting of W of MHS has a special importance and a characterization related to the theory of $\text{SL}(2)$ -orbits as in Remark 1.3.11 and Remark 1.3.12 below ([20] 8.7).

Remark 1.3.11. Assume that we are given a nilpotent orbit $(H_{\mathbf{R}}, \langle \cdot, \cdot \rangle, N, F)$ of weight w as in [24] 5.4.1. Let W' be the $-w$ twist of the monodromy filtration M of N . (That is, W' is the twist of M such that the central graded quotient of W' is of weight w whereas the central graded quotient of M is of weight 0.) Then (W', F) is a MHS. The canonical splitting of W' of this MHS is explained as follows.

For $y \gg 0$, $(H_{\mathbf{R}}, \langle \cdot, \cdot \rangle, \exp(iyN)F)$ is a polarized Hodge structure. Let $s_{\text{BS}}(y)$ be the unique splitting of W' such that $s_{\text{BS}}(y)(\text{gr}_w^{W'})$ and $s_{\text{BS}}(y)(\text{gr}_{w'}^{W'})$ with $w \neq w'$ are orthogonal with respect to the Hodge metric of $(H_{\mathbf{R}}, \langle \cdot, \cdot \rangle, \exp(iyN)F)$. (This splitting of W' is treated in the theory of Borel–Serre lifting. See 2.2.2 below.) As is proved in [20], when $y \rightarrow \infty$, $s_{\text{BS}}(y)$ converges to the canonical splitting $\text{spl}_{W'}(F)$ of W' associated to the MHS (W', F) .

In the theory of $\text{SL}(2)$ -orbits, we have a homomorphism of algebraic groups $\rho : \text{SL}(2)_{\mathbf{R}} \rightarrow \text{Aut}_{\mathbf{R}}(H_{\mathbf{R}})$ over \mathbf{R} associated to (N, F) . It is the unique homomorphism such that $\rho \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix}$ acts on the part of $H_{\mathbf{R}}$ of weight j for $\text{spl}_{W'}(F)$ as t^{j-w} and such that the Lie algebra homomorphism $\text{Lie}(\rho) : \mathfrak{sl}(2)_{\mathbf{R}} \rightarrow \text{End}_{\mathbf{R}}(H_{\mathbf{R}})$ sends the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to N . That is, the action of the diagonal part of $\text{SL}(2)$ in the theory of $\text{SL}(2)$ -orbits gives the canonical splitting $\text{spl}_{W'}(F)$. This $\text{spl}_{W'}(F)$ is denoted as $\text{spl}_{W'}^{\text{BS}}(F)$ in 3.3.8.

Remark 1.3.12. The canonical splitting of the weight filtration is functorial for MHS. Any MHS F' is embedded in a MHS F which appears in Remark 1.3.11 ([20]). Hence Remark 1.3.11 characterizes the canonical splitting.

1.4 The complex analytic structure of D

We consider the complex analytic structure of D .

Fix a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ as in 1.2.13.

1.4.1. Let Y be the set of all isomorphism classes of exact \otimes -functors from $\text{Rep}(G)$ to the following category \mathcal{C} preserving the underlying vector spaces and the weight filtrations.

\mathcal{C} is the category of triples (V, W, F) , where V is a finite-dimensional \mathbf{Q} -vector space, W is an increasing filtration on V (called the weight filtration), and F is a decreasing filtration on $V_{\mathbf{C}}$ (called the Hodge filtration).

Then $G(\mathbf{C})$ acts on Y by changing the Hodge filtration F . We have $D \subset Y$ and D is stable in Y under the action of $G(\mathbf{R})G_u(\mathbf{C})$.

Let

$$\check{D} := G(\mathbf{C})D \subset Y.$$

Since the action of $G(\mathbf{C})$ on \check{D} is transitive and the isotropy group of each point of \check{D} is an algebraic subgroup of $G(\mathbf{C})$, \check{D} has a natural structure of a complex analytic manifold as a $G(\mathbf{C})$ -homogeneous space.

Proposition 1.4.2. *For $x \in \check{D}$, the tangent space of \check{D} at x is canonically isomorphic to $\text{Lie}(G)_{\mathbf{C}}/F(x)^0\text{Lie}(G)_{\mathbf{C}}$, where $F(x)$ denotes the Hodge filtration of x on $\text{Lie}(G)_{\mathbf{C}}$ defined by the adjoint action of G on $\text{Lie}(G)$.*

The tangent bundle of \check{D} is canonically isomorphic to $\text{Lie}(G)_{\mathcal{O}}/F^0\text{Lie}(G)_{\mathcal{O}}$, where $\text{Lie}(G)_{\mathcal{O}} := \mathcal{O} \otimes \text{Lie}(G)$ with $\mathcal{O} = \mathcal{O}_{\check{D}}$ the sheaf of holomorphic functions on \check{D} .

Proof. By definition of $F(x)$, $F(x)^0\text{Lie}(G)_{\mathbf{C}}$ is the Lie algebra of the isotropy subgroup of $G(\mathbf{C})$ at x under the action of $G(\mathbf{C})$ on \check{D} in 1.4.1. The assertions of this proposition follow. \square

Proposition 1.4.3. *D is open in \check{D} .*

Proof. Let $x \in D$. Since the Hodge filtration $F(x)^{\bullet}\text{Lie}(G_{\text{red}})_{\mathbf{C}}$ is pure of weight 0, the map $\text{Lie}(G_{\text{red}})_{\mathbf{R}} \rightarrow \text{Lie}(G_{\text{red}})_{\mathbf{C}}/F(x)^0\text{Lie}(G_{\text{red}})_{\mathbf{C}}$ is surjective. Hence the map $\text{Lie}(G)_{\mathbf{R}} + \text{Lie}(G_u)_{\mathbf{C}} \rightarrow \text{Lie}(G)_{\mathbf{C}}/F(x)^0\text{Lie}(G)_{\mathbf{C}}$ is surjective. Since $\text{Lie}(G)_{\mathbf{C}}/F(x)^0\text{Lie}(G)_{\mathbf{C}}$ is the tangent space of \check{D} at x (Proposition 1.4.2), the last surjectivity shows that $G(\mathbf{R})G_u(\mathbf{C})x$ is a neighborhood of x in \check{D} . \square

Corollary 1.4.4. *D is a complex analytic manifold.*

Remark 1.4.5. This Proposition 1.4.3 is Proposition 3.2.7 of [22]. The proof of it given there is wrong.

The real analytic structure of D given in 1.3.1 coincides with the one induced by this complex analytic structure.

Theorem 1.4.6. *The map $D \rightarrow D_{\text{red}}$ is smooth and surjective. For $x \in D$, the fiber over $x_{\text{red}} \in D_{\text{red}}$ (1.3.3) in D is isomorphic to $G_u(\mathbf{C})/F(x)^0 G_u(\mathbf{C})$. Here $F(x)^0 G_u(\mathbf{C})$ is the \mathbf{C} -valued points of the algebraic subgroup of $G_{u,\mathbf{C}}$ whose Lie algebra is $F(x)^0 \text{Lie}(G_u)_{\mathbf{C}}$.*

Proof. The smoothness follows from the smoothness of D and D_{red} and the surjectivity of the map $T_x D \rightarrow T_{x_{\text{red}}}(D_{\text{red}})$ of tangent spaces which follows, by Proposition 1.4.2, from the surjectivity of $\text{Lie}(G) \rightarrow \text{Lie}(G_{\text{red}})$. Consider the action of $G_u(\mathbf{C})$ on the fiber. The action is transitive by Proposition 1.2.11 (3). We prove that the isotropy subgroup of $G_u(\mathbf{C})$ at x is $F(x)^0 G_u(\mathbf{C})$. Let $g \in G_u(\mathbf{C})$. Then $g \in F(x)^0 G_u(\mathbf{C})$ if and only if $\log(g) \in F(x)^0 \text{Lie}(G_u)_{\mathbf{C}}$, that is, if and only if $\log(g) \in F(x)^0 \text{Lie}(G)_{\mathbf{C}}$. By Lemma 1.2.6 (2), the last condition is equivalent to the condition that $\log(g) F(x)^p V_{\mathbf{C}} \subset F(x)^p V_{\mathbf{C}}$ for any $V \in \text{Rep}(G)$ for the Lie action. This condition is equivalent to the condition that $g F(x)^p V_{\mathbf{C}} = F(x)^p V_{\mathbf{C}}$ for any $V \in \text{Rep}(G)$, that is, g fixes x . \square

1.4.7. In this paper, we will often use the following fact (see [27] Theorem 4.14): If \mathcal{G} is a linear algebraic group over a field E and if V_1 is a finite-dimensional faithful representation of \mathcal{G} , V_1 generates the \otimes -category $\text{Rep}_k(\mathcal{G})$ of all finite-dimensional representations of \mathcal{G} over E . That is, all $V \in \text{Rep}_k(\mathcal{G})$ can be constructed from V_1 by taking \otimes , \oplus , the dual, and subquotients.

Lemma 1.4.8. *For $V \in \text{Rep}(G)$, define $D(V)$ (resp. $\check{D}(V)$) as the set $\{FH(V) \mid H \in D\}$ (resp. $\{FH(V) \mid H \in \check{D}\}$) of decreasing filtrations on $V_{\mathbf{C}}$.*

Assume that $V \in \text{Rep}(G)$ is faithful. Then the map $D \rightarrow D(V)$ (resp. $\check{D} \rightarrow \check{D}(V)$); $H \mapsto FH(V)$ is a bijection. If $H \in D$ (resp. \check{D}), $D(V)$ (resp. $\check{D}(V)$) coincides with the $G(\mathbf{R})G_u(\mathbf{C})$ -orbit (resp. $G(\mathbf{C})$ -orbit) in the set of decreasing filtrations on $V_{\mathbf{C}}$ containing $FH(V)$.

Proof. Since V is faithful, the map $D \rightarrow D(V)$ (resp. $\check{D} \rightarrow \check{D}(V)$); $H \mapsto FH(V)$ is injective by 1.4.7, and hence bijective by definition of $D(V)$ (resp. $\check{D}(V)$). The action of $G(\mathbf{R})G_u(\mathbf{C})$ (resp. $G(\mathbf{C})$) on D (resp. \check{D}) is transitive. The second assertion follows. \square

1.5 Polarizability

For a linear algebraic group G , let $G' := [G, G]$ be the commutator algebraic subgroup. Note that $G_u \subset G'$ if $D_{\text{all}}(G)$ is non-empty (1.2.10).

1.5.1. There are two formulations of polarization of a Hodge structure: the “classical formulation” ([17] I 2) and the formulation by Deligne ([12]). We adopted the former in Part I–Part IV of this series of papers [21].

Let H be a \mathbf{Q} -Hodge structure of weight w . Then a polarization in the “classical sense” is a \mathbf{Q} -bilinear form $\langle \cdot, \cdot \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \rightarrow \mathbf{Q}$ which is symmetric if w is even and anti-symmetric if w is odd, satisfying $\langle F^p, F^{w+1-p} \rangle = 0$ for any p , where F is the Hodge filtration, and the condition that the Hermitian form $(\cdot, \cdot) : H_{\mathbf{C}} \times H_{\mathbf{C}} \rightarrow \mathbf{C}$ defined by $(x, y) = \langle x, i^{p-q} \bar{y} \rangle$ for $x \in H_{\mathbf{C}}$ and $y \in H_{\mathbf{C}}^{p,q}$ is positive definite. (This positive definite Hermitian form (\cdot, \cdot) is called the *Hodge metric* of the polarization. Note that the restriction of the Hodge metric (\cdot, \cdot) of the polarization to $H_{\mathbf{R}} \times H_{\mathbf{R}}$ is a positive definite

symmetric bilinear form $H_{\mathbf{R}} \times H_{\mathbf{R}} \rightarrow \mathbf{R}$ and it is written as $(x, y) \mapsto \langle x, h(i)y \rangle$ for the action h of $S_{\mathbf{C}/\mathbf{R}}$ on $H_{\mathbf{R}}$ (1.2.3).)

On the other hand, a polarization in the sense of Deligne is a homomorphism $p : H_{\mathbf{Q}} \otimes H_{\mathbf{Q}} \rightarrow \mathbf{Q}(-w) = \mathbf{Q} \cdot (2\pi i)^{-w}$ of Hodge structures of weight $2w$ such that the induced \mathbf{Q} -bilinear form $H_{\mathbf{Q}} \times H_{\mathbf{Q}} \rightarrow \mathbf{Q}$; $(x, y) \mapsto (2\pi i)^w p(x \otimes y)$ is a polarization in the above classical sense.

To keep the consistency with Parts I–IV of this series of papers, we adopt in this Part V the formulation of polarization in the classical sense. When we use the formulation of Deligne, we will say that it is a polarization in the sense of Deligne.

We formulate polarizations of \mathbf{R} -Hodge structures in the same way.

1.5.2. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ be as in 1.2.13. Let $C := h_0(i)$ (Weil operator) be the image of $i \in \mathbf{C}^\times = S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$ by h_0 in $G_{\text{red}}(\mathbf{R})$.

We say that h_0 is \mathbf{R} -polarizable if $\{a \in (G_{\text{red}})'(\mathbf{R}) \mid Ca = aC\}$ is a maximal compact subgroup of $(G_{\text{red}})'(\mathbf{R})$.

That is, h_0 is \mathbf{R} -polarizable if and only if $\text{Ad}(C)$ on $\text{Lie}(G_{\text{red}}')_{\mathbf{R}}$ is a Cartan involution.

In the following lemma, which is a variant of [13] Section 2, we compare several polarizabilities (the above \mathbf{R} -polarizability is put as the condition (4.0)).

Lemma 1.5.3. *Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ be as in 1.2.13. Then, for $a = 1, 2, 3$, the following conditions (a.1) and (a.2) are equivalent. Furthermore, the conditions (4.0), (4.1), and (4.2) are equivalent. For the conditions (a) := (a.1) \Leftrightarrow (a.2) ($a = 1, 2, 3$), and the condition (4) := (4.0) \Leftrightarrow (4.1) \Leftrightarrow (4.2), we have the implications (1) \Rightarrow (2) \Rightarrow (4) and (1) \Rightarrow (3) \Rightarrow (4).*

(1.1) (resp. (1.2)). *There is a homomorphism $t : G_{\text{red}} \rightarrow \mathbf{G}_m$ such that $t(h_0(w(x))) = x^{-2}$ ($x \in \mathbf{G}_m$). Furthermore, if we consider the action of G on $\mathbf{Q} \cdot (2\pi i)^r$ ($r \in \mathbf{Z}$) via $t^r : G_{\text{red}} \rightarrow \mathbf{G}_m$ and identify $H(\mathbf{Q} \cdot (2\pi i)^r)$ for $H \in D = D(G, h_0)$ with the Hodge structure $\mathbf{Q}(r)$, then, for every $H \in D$ and every $V \in \text{Rep}(G)$ (resp. for some $H \in D$ and some faithful representation $V \in \text{Rep}(G)$) and for each $w \in \mathbf{Z}$, there exists a homomorphism $\text{gr}_w^W(V) \otimes \text{gr}_w^W(V) \rightarrow \mathbf{Q} \cdot (2\pi i)^{-w}$ in $\text{Rep}(G)$ which polarizes the Hodge structure $\text{gr}_w^W H(V)$ of weight w in the sense of Deligne.*

(2.1) (resp. (2.2)). *There is a homomorphism $t : G_{\text{red}, \mathbf{R}} \rightarrow \mathbf{G}_{m, \mathbf{R}}$ such that $t(h_0(w(x))) = x^{-2}$ ($x \in \mathbf{G}_{m, \mathbf{R}}$) and such that, for every $H \in D$ and every $V \in \text{Rep}(G)$ (resp. for some $H \in D$ and some faithful representation $V \in \text{Rep}(G)$) and for each $w \in \mathbf{Z}$, there exists an \mathbf{R} -bilinear form $\langle \cdot, \cdot \rangle : \text{gr}_w^W(V)_{\mathbf{R}} \times \text{gr}_w^W(V)_{\mathbf{R}} \rightarrow \mathbf{R}$ satisfying $\langle gx, gy \rangle = t(g)^{-w} \langle x, y \rangle$ ($g \in G_{\text{red}, \mathbf{R}}$) which polarizes the \mathbf{R} -Hodge structure $\text{gr}_w^W H(V)_{\mathbf{R}}$ of weight w .*

(3.1) (resp. (3.2)). *For every $H \in D$ and every $V \in \text{Rep}(G)$ (resp. for some $H \in D$ and some faithful representation $V \in \text{Rep}(G)$) and for each $w \in \mathbf{Z}$, there exists a \mathbf{Q} -bilinear form $\langle \cdot, \cdot \rangle : \text{gr}_w^W(V) \times \text{gr}_w^W(V) \rightarrow \mathbf{Q}$ satisfying $\langle gx, gy \rangle = \langle x, y \rangle$ for $g \in G_{\text{red}}$ which polarizes the Hodge structure $\text{gr}_w^W H(V)$ of weight w .*

(4.0) *The homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ is \mathbf{R} -polarizable in the sense of 1.5.2.*

(4.1) (resp. (4.2)). *For every $H \in D$ and every $V \in \text{Rep}(G)$ (resp. for some $H \in D$ and some faithful representation $V \in \text{Rep}(G)$) and for each $w \in \mathbf{Z}$, there exists an \mathbf{R} -bilinear form $\langle \cdot, \cdot \rangle : \text{gr}_w^W(V)_{\mathbf{R}} \times \text{gr}_w^W(V)_{\mathbf{R}} \rightarrow \mathbf{R}$ satisfying $\langle gx, gy \rangle = \langle x, y \rangle$ for $g \in G_{\text{red}, \mathbf{R}}'$ which polarizes the \mathbf{R} -Hodge structure $\text{gr}_w^W H(V)_{\mathbf{R}}$ of weight w .*

Proof. For $a = 1, 2, 3, 4$, the implication $(a.1) \Rightarrow (a.2)$ is clear and the implication $(a.2) \Rightarrow (a.1)$ follows from 1.4.7. For $b = 1, 2$, the implications $(1.b) \Rightarrow (2.b) \Rightarrow (4.b)$ and the implications $(1.b) \Rightarrow (3.b) \Rightarrow (4.b)$ are clear.

We prove the equivalence of (4.0) and (4.2).

To see the equivalence, by taking gr^W , we may assume that G is reductive. Then the equivalence is an analogue of [13] lemme 2.8, and proved as follows. We assume that G is reductive.

Assume (4.0). We prove that (4.2) is satisfied.

Let $K := \{a \in G'(\mathbf{R}) \mid Ca = aC\}$. Let $S_{\mathbf{C}/\mathbf{R}}^{(1)}$ be the kernel of the norm map $S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathbf{G}_{m,\mathbf{R}}$ and let $K_1 \subset G(\mathbf{R})$ be the image of $S_{\mathbf{C}/\mathbf{R}}^{(1)}(\mathbf{R}) = \{z \in \mathbf{C}^\times \mid |z| = 1\}$ under h_0 . We first show that there is a compact subgroup K_2 of $G(\mathbf{R})$ which contains both K and K_1 . Let $J = \{a \in G_{\mathbf{R}} \mid Ca = aC\}$ and let K_2 be a maximal compact subgroup of $J(\mathbf{R})$ which contains K_1 . Then, since K_2 contains some conjugate of K in $J(\mathbf{R})$ and since K is normal in $J(\mathbf{R})$, we have $K \subset K_2$.

By [28], there is a finite-dimensional faithful representation V of $G_{\mathbf{R}}$ and a positive definite symmetric \mathbf{R} -bilinear form $(\cdot, \cdot) : V \times V \rightarrow \mathbf{R}$ which is fixed by K_2 such that $G_{\mathbf{R}}$ is stable in $\text{Aut}_{\mathbf{R}}(V)$ under the transpose $g \mapsto {}^t g$ with respect to (\cdot, \cdot) . Note that the last condition implies that $G'_{\mathbf{R}}$ is also stable under the transpose.

Claim 1. The Cartan involution $\theta_K : G'_{\mathbf{R}} \rightarrow G'_{\mathbf{R}}$ associated to K is $g \mapsto {}^t g^{-1}$.

Note that this claim is also used in [7].

Proof of Claim 1. This is an algebraic homomorphism and its set of fixed points is compact and contains K . Since K is a maximal compact subgroup of $G'(\mathbf{R})$ by the assumption, K coincides with the set of fixed points of θ_K . This proves Claim 1.

On the other hand, we know that $g \mapsto C^{-1}gC$ is the Cartan involution of $G'_{\mathbf{R}}$ associated to K . Hence, by Claim 1, we have $C^{-1}gC = {}^t g^{-1}$.

Put $\langle x, y \rangle := (x, C^{-1}y)$. We show that it is $G'_{\mathbf{R}}$ -invariant. Let g be in $G'_{\mathbf{R}}$. Then we have $\langle gx, gy \rangle = (gx, C^{-1}gy) = (gx, {}^t g^{-1}C^{-1}y) = (g^{-1}gx, C^{-1}y) = (x, C^{-1}y) = \langle x, y \rangle$.

Let V_w for $w \in \mathbf{Z}$ be the part of V of weight w with respect to h_0 . Let $c \in \mathbf{R}^\times \subset \mathbf{C}^\times = S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$. We prove that ${}^t h_0(c)v = c^w v$ for $v \in V_w$. Since ${}^t h_0(c)$ belongs to $G(\mathbf{R})$ in $\text{Aut}_{\mathbf{R}}(V)$, we have ${}^t h_0(c)V_w = V_w$. For every $v' \in V_w$, we have $({}^t h_0(c)v, v') = (v, h_0(c)v') = (v, c^w v') = (c^w v, v')$. Since $(\cdot, \cdot) : V_w \times V_w \rightarrow \mathbf{R}$ is non-degenerate, we have ${}^t h_0(c)v = c^w v$. We prove $\langle V_w, V_{w'} \rangle = 0$ unless $w = w'$. For $v \in V_w$ and $v' \in V_{w'}$, $c^w(v, v') = (h_0(c)v, v') = (v, {}^t h_0(c)v') = (v, c^{w'}v') = c^{w'}(v, v')$. Hence if $w \neq w'$, then $\langle V_w, V_{w'} \rangle = 0$ and hence $\langle V_w, V_{w'} \rangle = 0$.

Let $\langle \cdot, \cdot \rangle_w : V_{w,\mathbf{R}} \times V_{w,\mathbf{R}} \rightarrow \mathbf{R}$ be the pairing induced by $\langle \cdot, \cdot \rangle$. We prove that $\langle \cdot, \cdot \rangle_w$ is a polarization on V_w . Let $\langle \cdot, \cdot \rangle_w : V_{w,\mathbf{C}} \times V_{w,\mathbf{C}} \rightarrow \mathbf{C}$ be the positive definite Hermitian form induced by $\langle \cdot, \cdot \rangle$. Let $H_w^{p,q}$ ($p + q = w$) be the (p, q) -Hodge component of $V_{w,\mathbf{C}}$ with respect to h_0 . It is sufficient to show that $(H_w^{p,q}, H_w^{p',q'}) = 0$ unless $p = p'$. Let $u \in K_1 = S_{\mathbf{C}/\mathbf{R}}^{(1)}(\mathbf{R})$. Then, since $\langle \cdot, \cdot \rangle_w$ is K_1 -invariant, we have, for $v \in H_w^{p,q}$ and $v' \in H_w^{p',q'}$, $\langle v, v' \rangle = ([u]v, [u]v') = u^{p-q}u^{q'-p'}(v, v') = u^{2(p-p')}(v, v')$. Hence $p = p'$. This proves that the condition (4.2) is satisfied.

Assume (4.2). We prove that (4.0) is satisfied. Take a faithful representation $V \in \text{Rep}(G)$. By our assumption, there is a $G'_{\mathbf{R}}$ -invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V_{\mathbf{R}}$ such that $(x, y) := \langle x, Cy \rangle$ is positive definite and symmetric.

Claim 2. For $g \in G'_{\mathbf{R}}$, we have $C^{-1}gC = {}^tg^{-1}$, where the transpose is with respect to (\cdot, \cdot) .

Proof. This follows from the fact that $\langle \cdot, \cdot \rangle$ is $G'_{\mathbf{R}}$ -stable.

The following Claim 3 is well-known.

Claim 3. Let U be a finite-dimensional \mathbf{R} -vector space endowed with a positive definite symmetric bilinear form (\cdot, \cdot) . Let \mathcal{G} be an algebraic subgroup of GL_U which is stable under $g \mapsto {}^tg$, where the transpose is with respect to (\cdot, \cdot) . Then $\{g \in \mathcal{G}(\mathbf{R}) \mid {}^tg = g^{-1}\}$ is a maximal compact subgroup of $\mathcal{G}(\mathbf{R})$.

By Claim 2 and Claim 3, $\{g \in G'(\mathbf{R}) \mid C^{-1}gC = g\}$ is a maximal compact subgroup of $G'(\mathbf{R})$. Thus (4.2) implies (4.0). \square

1.5.4. The conditions (1)–(4) in Lemma 1.5.3 are different from each other as the following examples show.

Example 1. Let E be a cubic extension field of \mathbf{Q} having one real place and one complex place and let $G = E^\times$ regarded as a torus over \mathbf{Q} of dimension 3. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ be the homomorphism such that the induced map $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^\times \rightarrow G(\mathbf{R}) = (E \otimes_{\mathbf{Q}} \mathbf{R})^\times = \mathbf{R}^\times \times \mathbf{C}^\times$ sends $z \in \mathbf{C}^\times$ to $(z\bar{z}, z^2)$. This example satisfies the conditions (2) and (4), but does not satisfy the conditions (1), (3). In fact, (2) is satisfied because we have $t : G_{\mathbf{R}} \rightarrow \mathbf{G}_{m,\mathbf{R}}$ which sends $(r, z) \in G(\mathbf{R}) = \mathbf{R}^\times \times \mathbf{C}^\times$ to $r(z\bar{z})^{-1}$, and for $V = E \in \mathrm{Rep}(G)$, we have an \mathbf{R} -bilinear form on $E_{\mathbf{R}} = \mathbf{R} \times \mathbf{C}$ with values in $\mathbf{R} \cdot (2\pi i)^{-2} = \mathbf{R}$ given by $((x_1, y_1), (x_2, y_2)) \mapsto x_1x_2 - y_1\bar{y}_2 - \bar{y}_1y_2$ ($x_j \in \mathbf{R}$, $y_j \in \mathbf{C}$) which polarizes the Hodge structure of E of weight 2 associated to h_0 . But the condition (3) is not satisfied because there is no bilinear form on the \mathbf{Q} -vector space E which polarizes the Hodge structure associated to h_0 .

Example 2. Let E be an imaginary quadratic field over \mathbf{Q} and let $G = E^\times$ regarded as a torus over \mathbf{Q} of dimension 2. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ be the homomorphism such that the induced map $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^\times \rightarrow G(\mathbf{R}) = (E \otimes_{\mathbf{Q}} \mathbf{R})^\times = \mathbf{C}^\times$ sends $z \in \mathbf{C}^\times$ to z/\bar{z} . This example satisfies the conditions (3) and (4), but does not satisfy the conditions (1), (2). In fact, there is no homomorphism $t : G_{\mathbf{R}} \rightarrow \mathbf{G}_{m,\mathbf{R}}$ such that $t(h_0(w(x))) = x^{-2}$. The condition (3) is satisfied because $E \times E \rightarrow \mathbf{Q}$; $(x_1, x_2) \mapsto -x_1\bar{x}_2 - \bar{x}_1x_2$ ($x_j \in E$) is the polarization of the Hodge structure of E of weight 0 associated to h_0 .

We thank Teruhisa Koshikawa for his advice on \mathbf{R} -polarizability and for showing the above Example 1 to us.

1.5.5. Let \mathcal{G} be a semisimple algebraic group over \mathbf{R} .

Let \mathfrak{X} be the set of all maximal compact subgroups of $\mathcal{G}(\mathbf{R})$. Then \mathfrak{X} is not empty. The group $\mathcal{G}(\mathbf{R})$ acts on \mathfrak{X} by conjugation, and this action is transitive. See [28] Theorem 3.1, [4] Proposition 1.12. For $K \in \mathfrak{X}$, we have a bijection $\mathcal{G}(\mathbf{R})/K \rightarrow \mathfrak{X}$; $g \mapsto gKg^{-1}$. Via this bijection, we regard \mathfrak{X} as a real analytic manifold. This real analytic structure is independent of the choice of K . This \mathfrak{X} is called the *symmetric space associated to \mathcal{G}* .

1.5.6. Consider the commutator subgroup $G'_{\mathrm{red}} = [G_{\mathrm{red}}, G_{\mathrm{red}}]$ of $G_{\mathrm{red}} = G/G_u$. This is a semisimple algebraic group. Let \mathfrak{X} be the symmetric space associated to $G'_{\mathrm{red},\mathbf{R}}$.

Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathrm{red},\mathbf{R}}$ be \mathbf{R} -polarizable as in 1.5.2. From $D_{\mathrm{red}} = D(G_{\mathrm{red}}, h_0)$ (1.3.3), we have a canonical map

$$D_{\mathrm{red}} \rightarrow \mathfrak{X}$$

which sends $h \in D_{\text{red}} \subset \text{Hom}(S_{\mathbf{C}/\mathbf{R}}, G_{\text{red}, \mathbf{R}})$ to a maximal compact subgroup $K = \{g \in G'_{\text{red}}(\mathbf{R}) \mid gh(i) = h(i)g\}$ of $G'_{\text{red}}(\mathbf{R})$ associated to the Weil operator $h(i)$.

1.5.7. Let Γ be a subgroup of $G(\mathbf{Q})$. We call Γ an *arithmetic subgroup* (resp. *semi-arithmetic subgroup*) of $G(\mathbf{Q})$ if the following condition (A) (resp. (SA)) is satisfied.

(A) There are $n \geq 1$ and an injective homomorphism $\rho : G \rightarrow \text{GL}(n)$ such that Γ is a subgroup of $\{g \in G(\mathbf{Q}) \mid \rho(g) \in \text{GL}(n, \mathbf{Z})\}$ of finite index.

(SA) There are $n \geq 1$ and an injective homomorphism $\rho : G \rightarrow \text{GL}(n)$ such that $\rho(\Gamma) \subset \text{GL}(n, \mathbf{Z})$. That is, there is a faithful representation $V \in \text{Rep}(G)$ and a \mathbf{Z} -lattice L in V such that L is stable under the action of Γ .

The terminology arithmetic subgroup is used by many people. We hope the terminology semi-arithmetic group is acceptable.

The next two lemmas are straightforward.

Lemma 1.5.8. (1) *The condition (A) is equivalent to the following condition ('A).*

('A) *For every $n \geq 1$ and every homomorphism $\rho : G \rightarrow \text{GL}(n)$, $\rho(\Gamma) \cap \text{GL}(n, \mathbf{Z})$ is of finite index in $\rho(\Gamma)$ and in $\rho(G(\mathbf{Q})) \cap \text{GL}(n, \mathbf{Z})$.*

(2) *The condition (SA) is equivalent to the following condition ('SA).*

('SA) *For every $n \geq 1$ and every homomorphism $\rho : G \rightarrow \text{GL}(n)$, $\rho(\Gamma) \cap \text{GL}(n, \mathbf{Z})$ is of finite index in $\rho(\Gamma)$.*

Lemma 1.5.9. *Let $f : G_1 \rightarrow G_2$ be a homomorphism of linear algebraic groups over \mathbf{Q} , let Γ_1 be a subgroup of $G_1(\mathbf{Q})$ and let Γ_2 be the image of Γ_1 in $G_2(\mathbf{Q})$.*

(1) *If Γ_1 is a semi-arithmetic subgroup of $G_1(\mathbf{Q})$, Γ_2 is a semi-arithmetic subgroup of $G_2(\mathbf{Q})$.*

(2) *If Γ_1 is an arithmetic subgroup of $G_1(\mathbf{Q})$ and if f is surjective, Γ_2 is an arithmetic subgroup of $G_2(\mathbf{Q})$.*

Remark 1.5.10. If Γ is a semi-arithmetic subgroup of $G(\mathbf{Q})$, Γ is discrete in $G(\mathbf{R})$. (The converse is not valid: A subgroup of $G(\mathbf{Q})$ which is discrete in $G(\mathbf{R})$ need not be semi-arithmetic. For example, let $G = \text{SL}(2)$, and let Γ be the subgroup of $G(\mathbf{Q})$ consisting of diagonal matrices with diagonal entries $(2^n, 2^{-n})$ ($n \in \mathbf{Z}$). Then Γ is discrete in $G(\mathbf{R}) = \text{SL}(2, \mathbf{R})$ but Γ is not semi-arithmetic. Another example is $G = \mathbf{G}_m$ and $\Gamma = \{2^n \mid n \in \mathbf{Z}\}.$)

Proposition 1.5.11. *Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ be as in 1.2.13. Assume that h_0 is \mathbf{R} -polarizable (1.5.2). Let Γ be a semi-arithmetic subgroup of $G'(\mathbf{Q})$ (1.5.7). Then the following holds.*

(1) *The action of Γ on D is proper and the quotient space $\Gamma \backslash D$ is Hausdorff.*

(2) *If Γ is torsion-free, the action of Γ on D is free (that is, if $\gamma \in \Gamma$ and if $\gamma p = p$ for some $p \in D$, then $\gamma = 1$), and the projection $D \rightarrow \Gamma \backslash D$ is a local homeomorphism.*

This Proposition 1.5.11 follows from its stronger version Theorem 2.6.1. (We will not use this 1.5.11 before we prove 2.6.1.)

Remark 1.5.12. Proposition 1.5.11 for a semi-arithmetic subgroup of $G(\mathbf{Q})$ need not be true as is shown in the following examples.

Example 1. Let E be a number field (a finite extension of \mathbf{Q}), and let G be the algebraic group E^\times over \mathbf{Q} (that is, $G(R) = (E \otimes_{\mathbf{Q}} R)^\times$ for any commutative ring R over \mathbf{Q}). Then the unit group O_E^\times of O_E is an arithmetic subgroup of $G(\mathbf{Q})$. Take any homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ (for example, the trivial homomorphism). Then h_0 is \mathbf{R} -polarizable and D consists of one point. The action of $\Gamma := O_E^\times$ on the one-point set D is proper if and only if Γ is finite. But O_E^\times need not be finite, e.g., for a real quadratic extension field E over \mathbf{Q} .

Example 2. Let E be a real quadratic field, and let G be the algebraic group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in E^\times, b \in E \right\}$$

over \mathbf{Q} (that is, $G(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in (E \otimes_{\mathbf{Q}} R)^\times, b \in E \otimes_{\mathbf{Q}} R \right\}$ for any commutative ring R over \mathbf{Q}). Then the group $\Gamma := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in O_E^\times, b \in O_E \right\}$ is an arithmetic subgroup of $G(\mathbf{Q})$. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ be the composite $S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathbf{G}_{m, \mathbf{R}} \rightarrow E_{\mathbf{R}}^\times = G_{\text{red}, \mathbf{R}}$, where the first homomorphism is the norm inverse and the second is the natural inclusion, that is, the one such that the induced map $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^\times \rightarrow (E \otimes \mathbf{R})^\times = G_{\text{red}}(\mathbf{R})$ sends $z \in \mathbf{C}^\times$ to $|z|^{-2}$. Then h_0 is \mathbf{R} -polarizable (simply because $(G_{\text{red}})'$ is trivial). Let $H \in D$ be the element corresponding to the composition $S_{\mathbf{C}/\mathbf{R}} \xrightarrow{h_0} E_{\mathbf{R}}^\times \subset G_{\mathbf{R}}$. By Theorem 1.4.6, or by 1.4.8 applied to $V = E^2$, we have $D = G_u(\mathbf{C}) = E \otimes_{\mathbf{Q}} \mathbf{C}$, where $g \in G_u(\mathbf{C})$ is identified with $gH \in D$. Via this identification, the action of Γ on D comes from the adjoint action of G on G_u , and described as

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \cdot (x \otimes w) = (ax + b) \otimes w \quad (a \in O_E^\times, b \in O_E, x \in E, w \in \mathbf{C}).$$

The subspace $\Gamma \backslash (E \otimes_{\mathbf{Q}} \mathbf{R})$ of the quotient $\Gamma \backslash D$ is not Hausdorff because it is homeomorphic to the quotient of the real torus $O_E \backslash (E \otimes_{\mathbf{Q}} \mathbf{R})$ by the action of O_E^\times , and the last action has a dense orbit ([29]). Hence, $\Gamma \backslash D$ is also not Hausdorff.

Proposition 1.5.13. *Assume that the condition (1) of Lemma 1.5.3 is satisfied. Then for a semi-arithmetic subgroup Γ of $G(\mathbf{Q})$, $\Gamma \cap G'(\mathbf{Q})$ is of finite index in Γ .*

Proof. By $G_u \subset G'$ (1.2.10), we have $G/G' \xrightarrow{\sim} G_{\text{red}}/G'_{\text{red}}$. Hence by replacing Γ by the image of Γ in $G_{\text{red}}(\mathbf{Q})$, we are reduced to the case G is reductive.

Assume that G is reductive. Let $\Gamma_0 = \Gamma \cap Z(G)(\mathbf{Q})$ and $\Gamma_1 = \Gamma \cap G'(\mathbf{Q})$. Since $Z(G) \times G' \rightarrow G$ is an isogeny, the image of $\Gamma_0 \times \Gamma_1 \rightarrow \Gamma$ is of finite index. Hence it is sufficient to prove that Γ_0 is finite. We prove this.

The image of Γ under $t : G(\mathbf{Q}) \rightarrow \mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^\times$ is contained in $\{\pm 1\}$. Hence in the faithful representation V of G in (1.2) in Lemma 1.5.3, for $H \in D(G, h_0)$, the action of some subgroup of Γ_0 of finite index preserves the Hodge filtration and the polarization of $H(\text{gr}_w^W V)$ and hence preserves the Hodge metric for every w . The elements of $G(\mathbf{R})$ which preserve these Hodge metrics for all w form a compact subgroup and Γ_0 is discrete, and hence Γ_0 is finite. \square

1.6 Relations with usual period domains and Mumford–Tate domains

In this section, in 1.6.1–1.6.5, we explain that the classical Griffiths domains [17] and their mixed Hodge generalization in [35] are essentially regarded as special cases of the period domains in this paper. In this case, our partial compactifications essentially coincide with those in [21] Part III. Next in 1.6.6, we explain that the Mumford–Tate domains studied in [16] are regarded as important cases of our period domains for reductive G . For Mumford–Tate domains, our toroidal partial compactifications essentially coincide with those in [25].

1.6.1. Let $\Lambda = (H_{0,\mathbf{Q}}, W, (\langle \cdot, \cdot \rangle_w)_w, (h^{p,q})_{p,q})$ be as in [21] Part III 2.1.1. That is, $H_{0,\mathbf{Q}}$ is a finite-dimensional \mathbf{Q} -vector space, W is a finite increasing filtration on $H_{0,\mathbf{Q}}$, $\langle \cdot, \cdot \rangle_w$ for each $w \in \mathbf{Z}$ is a non-degenerate \mathbf{Q} -bilinear form $\text{gr}_w^W \times \text{gr}_w^W \rightarrow \mathbf{Q}$ which is symmetric if w is even and anti-symmetric if w is odd, $h^{p,q}$ are non-negative integers given for each $(p, q) \in \mathbf{Z}^2$ such that $\sum_{p,q} h^{p,q} = \dim_{\mathbf{Q}} H_{0,\mathbf{Q}}$, $\sum_{p+q=w} h^{p,q} = \dim_{\mathbf{Q}} \text{gr}_w^W$ for every $w \in \mathbf{Z}$, and $h^{p,q} = h^{q,p}$ for all (p, q) .

Let G be the subgroup of $\text{Aut}(H_{0,\mathbf{Q}}, W) \times \mathbf{G}_m$ consisting of all elements (g, t) such that $\langle gx, gy \rangle_w = t^w \langle x, y \rangle_w$ for all w and for all $x, y \in \text{gr}_w^W H_{0,\mathbf{Q}}$.

Let $D(\Lambda)$ be the period domain of [35]. As a set, it is the set of all decreasing filtrations F on $H_{0,\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Q}} H_{0,\mathbf{Q}}$ such that $\dim_{\mathbf{C}}(\text{gr}_F^p(\text{gr}_w^W H_{0,\mathbf{C}})) = h^{p,w-p}$ for all $w, p \in \mathbf{Z}$ and such that $(\text{gr}_w^W, F(\text{gr}_w^W), \langle \cdot, \cdot \rangle_w)$ is a polarized Hodge structure for any $w \in \mathbf{Z}$.

Let $D^\pm(\Lambda)$ be the set of all decreasing filtrations on $H_{0,\mathbf{C}}$ such that $\dim_{\mathbf{C}}(\text{gr}_F^p(\text{gr}_w^W)) = h^{p,w-p}$ for all $w, p \in \mathbf{Z}$ and such that either $(\text{gr}_w^W, F(\text{gr}_w^W), \langle \cdot, \cdot \rangle_w)$ is a polarized Hodge structure for any $w \in \mathbf{Z}$ or $(\text{gr}_w^W, F(\text{gr}_w^W), (-1)^w \langle \cdot, \cdot \rangle_w)$ is a polarized Hodge structure for any $w \in \mathbf{Z}$. Then $D^\pm(\Lambda) = D(\Lambda)$ if and only if $\text{gr}_w^W H_0 = 0$ for all odd w . If $\text{gr}_w^W \neq 0$ for some odd w , there is a $(g, t) \in G(\mathbf{Q})$ such that $t < 0$ and that $D^\pm(\Lambda) = D(\Lambda) \coprod gD(\Lambda)$.

1.6.2. Assume that $D(\Lambda)$ is not empty and fix an $\mathbf{r} \in D(\Lambda)$. Then the Hodge decomposition of $\mathbf{r}(\text{gr}^W)$ induces a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$. We have $\langle h_0(z)x, h_0(z)y \rangle_w = |z|^{2w} \langle x, y \rangle_w$ for $z \in \mathbf{C}^\times = S_{\mathbf{C}/\mathbf{R}}(\mathbf{R})$ and $x, y \in \text{gr}_w^W$. The condition (1) of Lemma 1.5.3 is satisfied.

Proposition 1.6.3. *We have an isomorphism*

$$D \xrightarrow{\sim} D^\pm(\Lambda) ; H \mapsto H(H_{0,\mathbf{Q}}),$$

where $D = D(G, h_0)$ is the period domain of the present paper.

Proof. This is seen by Lemma 1.4.8. \square

1.6.4. In the above situation, the extended period domains D_{BS} , $D_{\text{SL}(2)}$, and $\Gamma \backslash D_\Sigma$ in this paper generalize those in [21] Part I–Part IV.

1.6.5. (Classical) Example. Take Λ in 1.6.1 as follows. $H_{0,\mathbf{Q}} = \mathbf{Q}^2 = \mathbf{Q}e_1 + \mathbf{Q}e_2$, $W_1 = H_{0,\mathbf{Q}}$, $W_0 = 0$, $\langle \cdot, \cdot \rangle_1$ is the anti-symmetric form characterized by $\langle e_2, e_1 \rangle_1 = 1$, $h^{1,0} = h^{0,1} = 1$, and other $h^{p,q}$ are 0. Identify $\text{GL}(2)$ with the subgroup G in 1.6.1 by

$g \mapsto (g, \det g)$. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ be the homomorphism whose homomorphism of \mathbf{R} -valued points is $\mathbf{C}^\times \rightarrow \mathrm{GL}(2, \mathbf{R})$; $z \mapsto \langle z \rangle$, where

$$\langle z \rangle = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{for } z = a + bi \ (a, b \in \mathbf{R}, (a, b) \neq (0, 0)).$$

Then we have unique isomorphisms

$$\mathfrak{H}^\pm \simeq D := D(G, h_0), \quad \mathbf{P}^1(\mathbf{C}) \simeq \check{D},$$

which send $i \in \mathfrak{H} \subset \mathfrak{H}^\pm$ to h_0 such that the former is $\mathrm{GL}(2, \mathbf{R})$ -equivariant and the latter is $\mathrm{GL}(2, \mathbf{C})$ -equivariant. Here \mathfrak{H}^\pm denotes the disjoint union of the upper half plain and the lower half plain, which are interchanged by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In fact, consider the natural action of $G = \mathrm{GL}(2)$ on $V = \mathbf{Q}^2$. The eigenvalues of $h_0(z)$ are z, \bar{z} , and the eigenspace decomposition is $V_{\mathbf{C}} = \mathbf{C} \begin{pmatrix} i \\ 1 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} -i \\ 1 \end{pmatrix}$, which yields the Hodge decomposition corresponding to $i \in \mathfrak{H}$.

1.6.6. Let H be a polarized \mathbf{Q} -Hodge structure.

Let \mathcal{C} be the Tannakian category of \mathbf{Q} -Hodge structures generated by H and $\mathbf{Q}(1)$. Let M be the Tannakian group of \mathcal{C} . This means that \mathcal{C} is identified with the Tannakian category $\mathrm{Rep}(M)$. This M is reductive. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow M_{\mathbf{R}}$ be the associated homomorphism. Then our period domain $D(M, h_0)$ coincides with the Mumford–Tate domain in [16] associated to H .

Take $\Lambda = (H_{0,\mathbf{Q}}, W, (\langle \cdot, \cdot \rangle_w)_{w \in \mathbf{Z}}, (h^{p,q})_{p,q})$ in 1.6.1 as follows. $H_{0,\mathbf{Q}}$ is the \mathbf{Q} -structure $H_{\mathbf{Q}}$ of H . W is the weight filtration on $H_{\mathbf{Q}}$ (that is, if w_0 denotes the weight of $H_{\mathbf{Q}}$, $W_w = H_{\mathbf{Q}}$ if $w \geq w_0$ and $W_w = 0$ if $w < w_0$). $\langle \cdot, \cdot \rangle_w$ is the polarization of H (times $(2\pi i)^{w_0}$). $h^{p,q}$ is the dimension of the (p, q) -Hodge component of H . Let L be the algebraic group G in 1.6.1. Then M is identified with the smallest algebraic closed subgroup M of L defined over \mathbf{Q} such that $M_{\mathbf{R}}$ contains the image of the homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow L_{\mathbf{R}}$ associated to H . The Mumford–Tate domain $D(M, h_0)$ is identified with the $M(\mathbf{R})$ -orbit in $D(\Lambda)$ containing the class of H .

This h_0 satisfies the condition (1) in Lemma 1.5.3.

The period domain $D(M/Z, \bar{h}_0)$, where Z is the center of M and \bar{h}_0 denotes the composition $S_{\mathbf{C}/\mathbf{R}} \xrightarrow{h_0} M_{\mathbf{R}} \rightarrow (M/Z)_{\mathbf{R}}$, for the semisimple algebraic group M/Z is also considered as a Mumford–Tate domain, for example, as in [25]. The period domain $D(M, h_0)$ is identified with an open and closed subspace of the period domain $D(M/Z, \bar{h}_0)$ (Proposition 4.10.12).

2 The space of Borel–Serre orbits

We define and study the space $D_{\mathrm{BS}} \supset D$ of Borel–Serre orbits. This is the G -MHS version of the space $D(\Lambda)_{\mathrm{BS}} \supset D(\Lambda)$ for the classical period domain $D(\Lambda)$ (1.6) defined and studied in [23] (in the pure case) [21] Part I (in the mixed case).

For an algebraic group G , G° denotes the connected component of G as an algebraic group which contains $1 \in G$. A parabolic subgroup P of G is a closed algebraic subgroup $P \subset G^\circ$ such that G/P is a projective variety (see e.g. [3] IV (11.2)).

The organization of this Section 2 is as follows. After preliminaries in Sections 2.1 and 2.2 and a review of Borel–Serre space \mathfrak{X}_{BS} in Section 2.3, we define our space D_{BS} as a set in Section 2.4, which we endow with a real analytic structure (precisely speaking, a structure of a real analytic manifold with corners) in Section 2.5. In Section 2.6, we prove the nice properties of D_{BS} (e.g. Hausdorffness of the quotient by a semi-arithmetic subgroup of $G'(\mathbb{Q})$).

2.1 Real analytic manifolds with corners

As will be explained in Section 2.5, our space D_{BS} is a real analytic manifold with corners.

In this Section 2.1, we review this notion real analytic manifold with corners ([7] Appendix by A. Douady and L. Héault) and consider spherical compactifications as examples of real analytic manifolds with corners.

2.1.1. Let $m, n \geq 0$ and consider the topological space $S = \mathbf{R}^m \times \mathbf{R}_{\geq 0}^n$ which is endowed with the inverse image \mathcal{O}_S of the sheaf of (\mathbf{R} -valued) real analytic functions on \mathbf{R}^{m+n} . That is, \mathcal{O}_S is the sheaf of functions which are locally extendable to real analytic functions on an open subset of $\mathbf{R}^m \times \mathbf{R}^n$.

A *real analytic manifold with corners* is a locally ringed space over \mathbf{R} which has an open covering whose each member is isomorphic to an open set of the above $S = \mathbf{R}^m \times \mathbf{R}_{\geq 0}^n$ with the restriction of \mathcal{O}_S for some $m, n \geq 0$.

2.1.2. Consider a finite-dimensional graded real vector space $V = \bigoplus_{w \leq -1} V_w$. In the rest of this Section 2.1, we review the compactification \overline{V} of V defined as a real analytic manifold with corners in [21] Part I Section 7. We call it the *spherical compactification* of V because as a topological space, it coincides with the spherical compactification of V in [1] Definition 2.1. We will use this \overline{V} in Section 2.4.

As in [24] and [21], the property compact includes Hausdorff in our terminology.

2.1.3. Consider the action of the multiplicative group $\mathbf{R}_{>0}$ on V given by $t \circ (\sum_w v_w) := \sum_w t^w v_w$ ($t \in \mathbf{R}_{>0}$, $v_w \in V_w$).

Let

$$\overline{V} := V \times^{\mathbf{R}_{>0}} \mathbf{R}_{\geq 0} \setminus \{(0, 0)\},$$

which we endow with the natural topology.

Recall that for a group H , for a set X on which H acts from the right, and for a set Y on which H acts from the left, $X \times^H Y$ denotes the quotient of $X \times Y$ by the equivalence relation $(xh, y) \sim (x, hy)$ ($x \in X$, $y \in Y$, $h \in H$). When we use the notation $X \times^H Y$ in this paper, H is a commutative group (as above) and hence right or left in the action doesn't matter.

Embed V into \overline{V} by $v \mapsto \text{class}(v, 1)$ ($v \in V$). We have $\overline{V} \setminus V = \{\text{class}(v, 0) \mid v \in V \setminus \{0\}\}$, and for $v \in V \setminus \{0\}$, when $t \in \mathbf{R}_{>0}$ converges to 0, $t \circ v = \text{class}(tv, 1) = \text{class}(v, t)$ converges to $\text{class}(v, 0)$. This space \overline{V} is covered by the two open sets, V and the complement $\overline{V} \setminus \{0\}$ of $0 \in V \subset \overline{V}$.

2.1.4. We regard \overline{V} as a real analytic manifold with corners as follows.

There is a real analytic map $f : V \setminus \{0\} \rightarrow \mathbf{R}_{>0}$ such that

$$(1) \quad f(t \circ v) = tf(v) \text{ for any } t \in \mathbf{R}_{>0} \text{ and } v \in V \setminus \{0\}.$$

For example, taking a base $(e_{w,j})_j$ of V_w for each w and taking an integer $m < 0$ satisfying $m/w \in 2\mathbf{Z}$ for any w such that $V_w \neq 0$, let $f(\sum_{w,j} x_{w,j} e_{w,j}) := (\sum_{w,j} x_{w,j}^{m/w})^{1/m}$. Then, this f satisfies the condition (1).

Let $f : V \setminus \{0\} \rightarrow \mathbf{R}_{>0}$ be a real analytic map satisfying (1). Let $V^{(1)} = f^{-1}(1)$. Then $V^{(1)} \times \mathbf{R}_{>0} \xrightarrow{\sim} V \setminus \{0\}$; $(v, t) \mapsto t \circ v$. The inverse map is $v \mapsto (f(v)^{-1} \circ v, f(v))$. We have a canonical homeomorphism $V^{(1)} \times \mathbf{R}_{\geq 0} \rightarrow \overline{V} \setminus \{0\}$. We endow $\overline{V} \setminus \{0\}$ with the structure of a real analytic manifold with corners via this homeomorphism. This structure is independent of the choice of f . This is because if $f' : V \setminus \{0\} \rightarrow \mathbf{R}_{>0}$ is also a real analytic map satisfying (1) and if $V^{(1)'} := (f')^{-1}(1)$, the isomorphism $V^{(1)} \times \mathbf{R}_{\geq 0} \xrightarrow{\sim} V^{(1)'} \times \mathbf{R}_{\geq 0}$; $(v, t) \mapsto (f'(v)^{-1} \circ v, tf'(v))$ of real analytic manifolds with corners is compatible with the homeomorphisms to $\overline{V} \setminus \{0\}$.

Furthermore, the restriction of this structure to $V \setminus \{0\}$ coincides with the natural structure of it as a real analytic manifold. Hence there is a unique structure on \overline{V} of a real analytic manifold with corners whose restriction to $\overline{V} \setminus \{0\}$ is the structure which we just defined and whose restriction to V is the natural structure of V as a real analytic manifold.

2.1.5. The map $V^{(1)} \times \mathbf{R}_{\geq 0} \rightarrow \overline{V} \setminus \{0\}$ in 2.1.4 extends to a surjective continuous map $V^{(1)} \times [0, \infty] \rightarrow \overline{V}$ which sends all points (v, ∞) ($v \in V^{(1)}$) to $0 \in \overline{V}$. Via this map, \overline{V} is homeomorphic to the quotient of the compact space $V^{(1)} \times [0, \infty]$ obtained by identifying all (v, ∞) ($v \in V^{(1)}$). Hence \overline{V} is compact.

2.2 Borel–Serre liftings

2.2.1. *Borel–Serre lifting.* Let \mathcal{G} be a semisimple algebraic group over \mathbf{R} . Let \mathcal{P} be a parabolic subgroup of \mathcal{G} , and let $\mathcal{S}_{\mathcal{P}}$ be the largest \mathbf{R} -split torus in $\mathcal{P}_{\text{red}} = \mathcal{P}/\mathcal{P}_u$, where \mathcal{P}_u is the unipotent radical of \mathcal{P} . Let K be a maximal compact subgroup of $\mathcal{G}(\mathbf{R})$.

Then we have a unique homomorphism $\mathcal{S}_{\mathcal{P}} \rightarrow \mathcal{P}$; $a \mapsto a_K$ which lifts the inclusion map $\mathcal{S}_{\mathcal{P}} \hookrightarrow \mathcal{P}_{\text{red}}$ and which satisfies $\theta_K(a_K) = a_K^{-1}$ for any $a \in \mathcal{S}_{\mathcal{P}}(\mathbf{R})$, where θ_K is the Cartan involution of $\mathcal{G}(\mathbf{R})$ associated to K ([7] Proposition 1.6).

We call this a_K the *Borel–Serre lifting* of $a \in \mathcal{S}_{\mathcal{P}}$ at K .

Remark 2.2.2. We remark that the Borel–Serre lifting is understood as the splitting of a filtration by taking the orthogonal complements.

By [28] (see also [4] Section 1), there is a finite-dimensional faithful representation V of \mathcal{G} and a positive definite symmetric \mathbf{R} -bilinear form $(\cdot, \cdot) : V \times V \rightarrow \mathbf{R}$ which is fixed by K such that \mathcal{G} is stable under the transpose $g \mapsto {}^t g$ for (\cdot, \cdot) . Furthermore, there are \mathbf{R} -subspaces of V such that $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$ which are stable under \mathcal{P} satisfying $(g - 1)V_j \subset V_{j-1}$ for all $g \in \mathcal{P}_u$ and $1 \leq j \leq m$. For example, we can take $V_j = \{v \in V \mid (g_1 - 1) \cdots (g_j - 1)v = 0 \text{ for all } g_1, \dots, g_j \in \mathcal{P}_u\}$ (then V_j are \mathcal{P} -stable and $V_m = V$ for some m).

We have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{P}}(\mathbf{R}) & \subset & \mathcal{P}_{\text{red}} \rightarrow \prod_{j=1}^m \text{Aut}_{\mathbf{R}}(V_j/V_{j-1}) \\ \downarrow & & \downarrow \\ \mathcal{P}(\mathbf{R}) & \xrightarrow{\subset} & \text{Aut}_{\mathbf{R}}(V), \end{array}$$

where the left vertical arrow is the Borel–Serre lifting at K and the right vertical arrow is given by the orthogonal decomposition of the filtration $(V_j)_j$. That is, we have a unique decomposition $V = \bigoplus_{j=1}^m V_{[j]}$ such that $V_j = \bigoplus_{k \leq j} V_{[k]}$ for all j and such that $V_{[j]}$ and $V_{[k]}$ are orthogonal for (\cdot, \cdot) if $j \neq k$. The right vertical arrow is given by $V_j/V_{j-1} \simeq V_{[j]}$.

We prove that the above diagram is commutative, that is, a_K for $a \in \mathcal{S}_{\mathcal{P}}(\mathbf{R})$ preserves $V_{[j]}$. Note that the Cartan involution of $\mathcal{G}(\mathbf{R})$ associated to K is given by $\theta_K(g) = ({}^t g)^{-1}$ for $g \in \mathcal{G}(\mathbf{R})$. Hence ${}^t(a_K) = \theta_K(a_K)^{-1} = a_K$. Since $V_{[j]} = (\bigoplus_{k \geq j} V_{[k]}) \cap V_j$ and a_K preserves V_j , it is sufficient to prove that a_K preserves $\bigoplus_{k \geq j} V_{[k]}$. Note that $\bigoplus_{k \geq j} V_{[k]}$ is the annihilator of V_{j-1} for (\cdot, \cdot) . For $x \in \bigoplus_{k \geq j} V_{[k]}$ and $y \in V_{j-1}$, we have $(a_K x, y) = (x, {}^t(a_K) y) = (x, a_K y) = 0$. Hence $a_K x \in \bigoplus_{k \geq j} V_{[k]}$.

2.3 Review of Borel–Serre theory

Let G be a semisimple algebraic group over \mathbf{Q} . Let \mathfrak{X} be the associated symmetric space as in 1.5.5.

In this Section 2.3, we review how the Borel–Serre space $\mathfrak{X}_{\text{BS}} \supset \mathfrak{X}$ is constructed in the paper of Borel–Serre [7].

2.3.1. Let P be a parabolic subgroup of G . Let S_P be the largest \mathbf{Q} -split torus in the center of $P_{\text{red}} := P/P_u$. Let $X(S_P)$ be the character group of S_P . Let $X(S_P)^+$ be the submonoid of $X(S_P)$ generated by roots. Here a *root* means an element $\chi \in X(S_P)$ such that for some (equivalently, for any) lifting $s : S_P \rightarrow P$ of the embedding $S_P \hookrightarrow P_{\text{red}}$, there is a non-zero element v of $\text{Lie}(P_u)$ such that $\text{Ad}(s(t))v = \chi(t)^{-1}v$ for $t \in S_P$. Then $X(S_P)^+$ is a free monoid, that is, $X(S_P)^+ \simeq \mathbf{N}^n$ for some $n \geq 0$. The basis $\Delta(P)$ of the monoid $X(S_P)^+$ is called the set of *fundamental roots* (or, of *simple roots*).

Let

$$A_P := \text{Hom}(X(S_P)^+, \mathbf{R}_{>0}^{\text{mult}}) = \mathbf{R}_{>0}^{\Delta(P)} \subset \bar{A}_P := \text{Hom}(X(S_P)^+, \mathbf{R}_{\geq 0}^{\text{mult}}) = \mathbf{R}_{\geq 0}^{\Delta(P)}.$$

Since $X(S_P)^+$ generates a subgroup of $X(S_P)$ of finite index, the identification $S_P(\mathbf{R}) = \text{Hom}(X(S_P), \mathbf{R}^\times)$ induces an isomorphism $S_P(\mathbf{R})^\circ \xrightarrow{\sim} A_P$, where $S_P(\mathbf{R})^\circ$ denotes the connected component of the topological group $S_P(\mathbf{R})$ containing 1.

2.3.2. *Borel–Serre action* (*geodesic action*, in the terminology of [7] 3.3). Let P be as in 2.3.1. We have the action \circ of the group A_P on \mathfrak{X} , which we call the Borel–Serre action, defined as follows. For $K \in \mathfrak{X}$ and $a \in A_P$, let $a_K \in P(\mathbf{R})$ be the Borel–Serre lifting of a at K obtained by applying 2.2.1 to $\mathcal{G} = G_{\mathbf{R}}$, $\mathcal{P} = P_{\mathbf{R}}$ and $\mathcal{S}_{\mathcal{P}} \supset S_{P,\mathbf{R}}$. Define

$$a \circ K = a_K K a_K^{-1}.$$

2.3.3. As a set, the Borel–Serre space \mathfrak{X}_{BS} is defined by $\mathfrak{X}_{\text{BS}} := \{(P, Z)\}$, where P runs over parabolic subgroups of G and Z is an A_P -orbit for the Borel–Serre action.

2.3.4. For a parabolic subgroup P of G , let

$$\mathfrak{X}_{\text{BS}}(P) := \{(Q, Z) \in \mathfrak{X}_{\text{BS}} \mid Q \supset P\}.$$

Then we have a bijection

$$\mathfrak{X}_{\text{BS}}(P) \simeq \mathfrak{X} \times^{A_P} \bar{A}_P, \quad (Q, Z) \leftrightarrow (K, a),$$

defined as follows.

Let Q be a parabolic subgroup of G such that $Q \supset P$. Then, $\Delta(Q)$ is regarded as a subset of $\Delta(P)$, A_Q is regarded as a subgroup of A_P , and $Q \mapsto \Delta(Q)$ is a bijection from the set of all parabolic subgroups of G such that $Q \supset P$ to the set of all subsets of $\Delta(P)$.

For $(Q, Z) \in \mathfrak{X}_{\text{BS}}(P)$, K is any element of Z and $a \in \bar{A}_P$ is defined by $a(\chi) = 0$ if $\chi \in \Delta(Q)$ and $a(\chi) = 1$ if $\chi \notin \Delta(Q)$. For $(K, a) \in \mathfrak{X} \times \bar{A}_P$, Q is the parabolic subgroup of G containing P such that $\Delta(Q) = \{\chi \in \Delta(P) \mid a(\chi) = 0\}$ and $Z := \{a' \circ K \mid a' \in A_P, \chi(a') = a(\chi) \text{ for any } \chi \in \Delta(P) - \Delta(Q)\}$. Note that, when $Q \supset P$, at a common K , the Borel–Serre action of A_Q for Q coincides with its action for P regarding A_Q as a subgroup of A_P .

2.3.5. The set \mathfrak{X}_{BS} has a structure of a real analytic manifold with corners defined as follows.

For a parabolic subgroup P of G , there exists a real analytic map $f : \mathfrak{X} \rightarrow A_P$ satisfying

$$(1) \quad f(a \circ K) = af(K) \text{ for all } a \in A_P \text{ and all } K \in \mathfrak{X}.$$

We sketch the proof of this assertion. Let ${}^{\circ}P := \bigcap_{\chi} \text{Ker}(\chi^2 : P \rightarrow \mathbf{G}_m)$, where χ runs over all homomorphisms $P \rightarrow \mathbf{R}^{\times}$ of algebraic groups over \mathbf{Q} . Then, $P_u \subset {}^{\circ}P$ and the composition $A_P \rightarrow P(\mathbf{R})/P_u(\mathbf{R}) \rightarrow P(\mathbf{R})/{}^{\circ}P(\mathbf{R})$ is an isomorphism by [7] 1.2, which we use by taking P_{red} as G there. Let $\pi : P(\mathbf{R}) \rightarrow P(\mathbf{R})/{}^{\circ}P(\mathbf{R}) \simeq A_P$. Fix $K \in \mathfrak{X}$. Since $G(\mathbf{R}) = P(\mathbf{R})K$ (see [2] §11) and since π kills the compact subgroup $P(\mathbf{R}) \cap K$, there exists a unique map $G(\mathbf{R}) \rightarrow A_P$ sending pk to $\pi(p)$ ($p \in P(\mathbf{R})$, $k \in K$), which factors through $f : \mathfrak{X} \simeq G(\mathbf{R})/K \rightarrow A_P$. Since the action of $a \in A_P$ on $G(\mathbf{R})$ is $a \circ pk = pa_Kk$, f satisfies the property $f(a \circ K) = af(K)$.

The set $\mathfrak{X}_{\text{BS}}(P)$ is regarded as a real analytic manifold with corners as follows.

Let $\mathfrak{X}^{(1)} := f^{-1}(1)$. Then we have

$$(2) \quad \mathfrak{X}^{(1)} \times A_P \xrightarrow{\sim} \mathfrak{X}; \quad (K, a) \mapsto a \circ K, \text{ an isomorphism of real analytic manifolds,}$$

$$(3) \quad \mathfrak{X}^{(1)} \times \bar{A}_P \rightarrow \mathfrak{X} \times^{A_P} \bar{A}_P = \mathfrak{X}_{\text{BS}}(P), \text{ a bijection of sets.}$$

We regard $\mathfrak{X}_{\text{BS}}(P)$ as a real analytic manifold with corners via the bijection (3). This structure of $\mathfrak{X}_{\text{BS}}(P)$ does not depend on the choice of f because if $f' : \mathfrak{X} \rightarrow A_P$ satisfies the same condition as f and if $Y := \mathfrak{X}^{(1)} = f^{-1}(1)$ and $Y' := (f')^{-1}(1)$, the map (3) and the map $Y' \times \bar{A}_P \rightarrow \mathfrak{X}_{\text{BS}}(P)$ are compatible with the isomorphism $Y \times \bar{A}_P \xrightarrow{\sim} Y' \times \bar{A}_P$; $(K, a) \mapsto (f'(K)^{-1} \circ K, f'(K)a)$ whose inverse is $(K, a) \mapsto (f(K)^{-1}K, f(K)a)$.

\mathfrak{X}_{BS} has a unique structure of a real analytic manifold with corners such that for any parabolic subgroup P of G , $\mathfrak{X}_{\text{BS}}(P)$ is open in \mathfrak{X}_{BS} and the structure of $\mathfrak{X}_{\text{BS}}(P)$ as a real

analytic manifold with corners defined above coincides with the restriction of that of \mathfrak{X}_{BS} . This follows from the following (4) and (5).

(4) Let Q be a parabolic subgroup of G such that $Q \supset P$. Then $\mathfrak{X}_{\text{BS}}(Q)$ is an open subset of $\mathfrak{X}_{\text{BS}}(P)$ and the structure of $\mathfrak{X}_{\text{BS}}(Q)$ as a real analytic manifold with corners coincide with the restriction of that of $\mathfrak{X}_{\text{BS}}(P)$.

(4) is shown as follows. Let $f : \mathfrak{X} \rightarrow A_P$ be a real analytic map satisfying (1) and let $Y = f^{-1}(1)$. Let p be the projection $A_P \rightarrow A_Q$. Then $f' := h \circ f : \mathfrak{X} \rightarrow A_Q$ satisfies the condition on f with Q replacing P . Let $Y' = (f')^{-1}(1)$. Let T be the kernel of p . Then the map (3) and the map $Y' \times \bar{A}_Q \rightarrow \mathfrak{X}_{\text{BS}}(Q)$ are compatible with the open immersion $Y' \times \bar{A}_Q \simeq Y \times T \times A_Q \hookrightarrow Y \times \bar{A}_P$, where the first isomorphism is induced by $Y \times T \xrightarrow{\sim} Y'$; $(K, a) \mapsto a \circ K$ and the second map is induced by $T \times \bar{A}_Q \rightarrow \bar{A}_P$; $(t, a) \mapsto ta$.

(5) For parabolic subgroups P and Q of G , $\mathfrak{X}_{\text{BS}}(P) \cap \mathfrak{X}_{\text{BS}}(Q) = \mathfrak{X}_{\text{BS}}(P * Q)$. Here $P * Q$ is the algebraic subgroup of G generated by P and Q , which is a parabolic subgroup of G .

2.3.6. We will often use the following basic things about proper actions. Let H be a Hausdorff topological group.

(1) If H acts properly on a topological space X , the quotient space $H \setminus X$ is Hausdorff.

(2) If H is discrete and if H acts properly on a Hausdorff space X , and if this action is free (that is, $hx = x$ with $h \in H$ and $x \in X$ implies $h = 1$), the projection $X \rightarrow H \setminus X$ is a local homeomorphism.

(3) Assume that H acts on topological spaces X and Y , and let $f : X \rightarrow Y$ be an equivariant continuous map.

(3.1) If H acts properly on Y and if X is Hausdorff, then H acts properly on X .

(3.2) If H acts on X properly and if the map f is proper and surjective, then H acts properly on Y .

Here in (3.2) and throughout this paper, as in [24] and [21], for a continuous map $f : X \rightarrow Y$ of topological spaces, f is proper means that it satisfies the following (a) and (b). (a) For any topological space Z , the induced map $X \times_Y Z \rightarrow Z$ is a closed map. (b) The map $X \rightarrow X \times_Y X$ is a closed map.

(4) Let H_1 be a closed normal subgroup of H and let $H_2 := H/H_1$. Assume that H (resp. H_2) acts on a topological space X_1 (resp. X_2) continuously. Assume that for $j = 1, 2$, the action of H_j on X_j is proper and free. Assume that there are a neighborhood U of 1 in H_2 and a continuous map $U \rightarrow H$ which lifts the inclusion map $U \rightarrow H_2$. Then the diagonal action of H on $X_1 \times X_2$ is proper and free.

(5) Let H_1 be a closed subgroup of H of finite index. Assume that H acts on a topological space X continuously. If the action of H_1 on X is proper, then the action of H on X is proper.

For proofs of (1), (2), (3), see [9] Ch.3 §4 no.2 Proposition 3, ibid. Ch.3 §4 no.4 Corollary, ibid. Ch.3 §4 no.2 Proposition 5, respectively. The proof of (4) is given in [21] Part III Definition 4.2.4. (5) is proved in an elementary manner.

2.3.7. Let Γ be a semi-arithmetic subgroup of $G(\mathbf{Q})(1.5.7)$. Then:

(1) The action of Γ on \mathfrak{X}_{BS} is proper, and the quotient space $\Gamma \setminus \mathfrak{X}_{\text{BS}}$ is Hausdorff.

(2) If Γ is torsion-free, the action of Γ on \mathfrak{X}_{BS} is free, and the map $\mathfrak{X}_{\text{BS}} \rightarrow \Gamma \backslash \mathfrak{X}_{\text{BS}}$ is a local homeomorphism.

(3) If Γ is an arithmetic subgroup of $G(\mathbf{Q})$, $\Gamma \backslash \mathfrak{X}_{\text{BS}}$ is compact.

These are given in [7] 9.3 Theorem and 9.5 for arithmetic subgroups Γ of $G(\mathbf{Q})$. Since a semi-arithmetic subgroup of $G(\mathbf{Q})$ is a subgroup of an arithmetic subgroup of $G(\mathbf{Q})$, the properness in (1) is reduced to the case of an arithmetic subgroup. The Hausdorffness in (1) follows from the properness in (1) (2.3.6 (1)). We prove (2). We may assume that Γ has a subgroup Γ_1 of finite index which is a subgroup of a torsion-free arithmetic subgroup. Assume that $\gamma \in \Gamma$ fixes some point of \mathfrak{X}_{BS} . Take $n \geq 1$ such that $\gamma^n \in \Gamma_1$. By the result for an arithmetic subgroup, we have $\gamma^n = 1$. Since Γ is torsion-free, we have $\gamma = 1$. The rest of (2) follows from the properness in (1) and from this free property (2.3.6 (2)).

2.4 The set D_{BS}

Let G be a linear algebraic group over \mathbf{Q} and let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}}$ be a homomorphism as in 1.2.13 which is \mathbf{R} -polarizable.

2.4.1. Let $G'_{\text{red}} = [G_{\text{red}}, G_{\text{red}}]$ be the commutator subgroup of $G_{\text{red}} = G/G_u$. This is a semisimple algebraic group.

We have bijections between the sets

$$(\text{parabolic subgroups of } G) \leftrightarrow (\text{parabolic subgroups of } G_{\text{red}}) \leftrightarrow (\text{parabolic subgroups of } G'_{\text{red}})$$

given as follows. The bijection from the second set to the first set is to take the inverse image in G . The bijection from the second set to the third set is to take the intersection with G'_{red} .

2.4.2. Let P be a parabolic subgroup of G_{red} .

Let $P' := P \cap G'_{\text{red}}$. Then P' is a parabolic subgroup of G'_{red} . We will denote $A_{P'}$ by A_P . This A_P is also described as follows.

Let S_P be the largest \mathbf{Q} -split torus in the center of P_{red} . Let $X(S_P)^+$ be the submonoid of the character group $X(S_P)$ generated by the inverses of characters which appear in the adjoint action on $\text{Lie}(P_u)$ of a lifting of S_P in P (see 2.3.1). Then the canonical map $X(S_P) \rightarrow X(S_{P'})$ induces an isomorphism $X(S_P)^+ \xrightarrow{\sim} X(S_{P'})^+$ and a bijection $\Delta(P) \rightarrow \Delta(P')$ between the bases. Hence we can write

$$A_P = \text{Hom}(X(S_P)^+, \mathbf{R}_{>0}^{\text{mult}}) = \mathbf{R}_{>0}^{\Delta(P)} \subset \bar{A}_P = \text{Hom}(X(S_P)^+, \mathbf{R}_{\geq 0}^{\text{mult}}) = \mathbf{R}_{\geq 0}^{\Delta(P)}.$$

2.4.3. We define the *Borel–Serre action of A_P on D* as follows. Let $a \in A_P$ and $x \in D$. Let $(p, s, \delta) \in D_{\text{red}} \times \text{spl}(W) \times \mathcal{L}$, with $\delta \in \mathcal{L}(p)$, be the element corresponding to x (Proposition 1.3.6). Let $K(p) \in \mathfrak{X}$ be the image of p under $D_{\text{red}} \rightarrow \mathfrak{X}$ (1.5.6). We have the Borel–Serre lifting $A_P \rightarrow P'(\mathbf{R})$; $a \mapsto a_{K(p)}$ associated to $K(p)$. Then $a \circ x$ is defined to be the element of D whose image in $D_{\text{red}} \times \text{spl}(W) \times \mathcal{L}$ is $(a_{K(p)}p, s, \text{Ad}(a_{K(p)})\delta)$.

2.4.4. Let $B_P = \mathbf{R}_{>0} \times A_P$. Then we define the action of B_P on D_{nspl} (see 1.3.9) as follows. Let $b = (t, a) \in \mathbf{R}_{>0} \times A_P$. Then, for $x \in D_{\text{nspl}}$, $b \circ x := a \circ (t \circ x)$, where $t \circ x \in D$ corresponds to the element $(p, s, t \circ \delta)$ of $D_{\text{red}} \times \text{spl}(W) \times \mathcal{L}$ (see 2.1.3). Here (p, s, t) is the element corresponding to x by Proposition 1.3.6.

2.4.5. Let D_{BS} be the set of all pairs (P, Z) , where P is a parabolic subgroup of G_{red} and Z is either an A_P -orbit in D or a B_P -orbit in D_{nspl} for the Borel–Serre action.

We denote by $D_{\text{BS}}^{\text{mild}}$ the part of D_{BS} consisting of A_P -orbits and by $D_{\text{nspl}, \text{BS}}$ the subset of D_{BS} consisting of all elements of the form (P, Z) such that $Z \subset D_{\text{nspl}}$. ($D_{\text{BS}}^{\text{mild}}$ and $D_{\text{nspl}, \text{BS}}$ correspond to $D_{\text{BS}}^{(A)}$ and $D_{\text{BS}}^{(B)}$ in the notation of [21] Part I 8.1, respectively.)

2.5 The real analytic structure of D_{BS}

We are in the same setting as in 2.4. We endow D_{BS} with a structure of a real analytic manifold with corners.

Lemma 2.5.1. *Let P be a parabolic subgroup of G . Then there is a real analytic map $f : D \rightarrow A_P$ such that $f(a \circ x) = af(x)$ for any $a \in A_P$ and $x \in D$.*

Proof. Take a real analytic map $f_{\mathfrak{X}} : \mathfrak{X} \rightarrow A_P$ such that $f_{\mathfrak{X}}(a \circ K) = af_{\mathfrak{X}}(K)$ for any $a \in A_P$ and $K \in \mathfrak{X}$ (2.3.5), and define f to be the composite map $D \rightarrow D_{\text{red}} \rightarrow \mathfrak{X} \xrightarrow{f_{\mathfrak{X}}} A_P$, where the second arrow is as in 1.5.6. \square

Lemma 2.5.2. *Let P be a parabolic subgroup of G . Then there is a real analytic map $f : D_{\text{nspl}} \rightarrow B_P$ such that $f(b \circ x) = bf(x)$ for any $b \in B_P$ and $x \in D_{\text{nspl}}$.*

Proof. By Lemma 2.5.1, there is a real analytic map $f_D : D_{\text{red}} \rightarrow A_P$ such that $f_D(a \circ p) = af_D(p)$ for any $a \in A_P$ and any $p \in D_{\text{red}}$.

Let $\mathcal{L} = W_{-2}\text{gr}^W \text{Lie}(G)_{\mathbf{R}}$ be as in 1.3.4. It is a graded \mathbf{R} -vector space with weights ≤ -2 . By 2.1.4, there is a real analytic map $f_{\mathcal{L}} : \mathcal{L} \setminus \{0\} \rightarrow \mathbf{R}_{>0}$ such that $f_{\mathcal{L}}(t \circ \delta) = tf_{\mathcal{L}}(\delta)$ for any $t \in \mathbf{R}_{>0}$ and $\delta \in \mathcal{L} \setminus \{0\}$. Define $f : D_{\text{nspl}} \subset D_{\text{red}} \times \text{spl}(W) \times (\mathcal{L} \setminus \{0\}) \rightarrow B_P = A_P \times \mathbf{R}_{>0}$ by $f(p, s, \delta) = (f_D(p), f_{\mathcal{L}}(\text{Ad}(f_D(p)_p)^{-1}\delta))$ ($p \in D_{\text{red}}$, $s \in \text{spl}(W)$, $\delta \in \mathcal{L}(p)$), where $f_D(p)_p$ is the Borel–Serre lifting of $f_D(p) \in A_P$ at p . This f satisfies $f(b \circ x) = bf(x)$. \square

2.5.3. Let P be a parabolic subgroup of G . Let $D_{\text{BS}}(P) := \{(Q, Z) \in D_{\text{BS}} \mid Q \supset P\}$.

Let $D_{\text{BS}}^{\text{mild}}(P) := D_{\text{BS}}(P) \cap D_{\text{BS}}^{\text{mild}}$ and $D_{\text{nspl}, \text{BS}}(P) := D_{\text{BS}}(P) \cap D_{\text{nspl}, \text{BS}}$. Then, in the same way as in the case of $\mathfrak{X}_{\text{BS}}(P)$, we have

$$D_{\text{BS}}^{\text{mild}}(P) = D \times^{A_P} \bar{A}_P, \quad D_{\text{nspl}, \text{BS}}(P) = D \times^{B_P} \bar{B}_P.$$

We endow $D_{\text{BS}}(P)$ with the structure of a real analytic manifold in the following way.

First, the set $U_1 := D_{\text{BS}}^{\text{mild}}(P)$ is regarded as a real analytic manifold with corners as follows. By Lemma 2.5.1, there is a real analytic map $f_A : D \rightarrow A_P$ satisfying

$$(1) \quad f_A(a \circ x) = af_A(x) \text{ for all } a \in A_P \text{ and all } x \in D.$$

Let $D_A^{(1)} = f_A^{-1}(1)$. Then

$$(2) \quad D_A^{(1)} \times A_P \xrightarrow{\sim} D; (x, a) \mapsto a \circ x.$$

This map (2) induces a bijection

$$(3) \quad D_A^{(1)} \times \bar{A}_P \rightarrow D \times^{A_P} \bar{A}_P = D_{\text{BS}}^{\text{mild}}(P).$$

We regard $U_1 = D_{\text{BS}}^{\text{mild}}(P)$ as a real analytic manifold with corners via the bijection (3). This structure of $D_{\text{BS}}^{\text{mild}}(P)$ does not depend on the choice of f_A .

Next, the set $U_2 := D_{\text{nspl}, \text{BS}}(P)$ is regarded as a real analytic manifold with corners as follows. By Lemma 2.5.2, there is a real analytic map $f_B : D_{\text{nspl}} \rightarrow B_P$ satisfying

$$(4) \quad f_B(b \circ x) = bf_B(x) \text{ for all } b \in B_P \text{ and all } x \in D_{\text{nspl}}.$$

Let $D_B^{(1)} = f_B^{-1}(1)$. Then

$$(5) \quad D_B^{(1)} \times B_P \xrightarrow{\sim} D_{\text{nspl}} ; (x, b) \mapsto b \circ x.$$

This map (5) induces a bijection

$$(6) \quad D_B^{(1)} \times \bar{B}_P \rightarrow D \times^{B_P} \bar{B}_P = D_{\text{nspl}, \text{BS}}(P).$$

We regard $U_2 := D_{\text{nspl}, \text{BS}}(P)$ as a real analytic manifold with corners via the bijection (6). This structure of $D_{\text{nspl}, \text{BS}}(P)$ does not depend on the choice of f_B .

It is easy to prove that for $j = 1, 2$ and for the structure of a real analytic manifold with corners on U_j , the intersection $U_1 \cap U_2$ is an open set of U_j and that the restriction of the structure of the real analytic manifold with corners of U_1 to $U_1 \cap U_2$ coincides with that of U_2 . Hence there is a unique structure of a real analytic manifold with corners on $U_1 \cup U_2 = D_{\text{BS}}(P)$ for which U_1 and U_2 are open sets and whose restriction to U_j coincides with that of U_j for $j = 1, 2$.

Proposition 2.5.4. *The set D_{BS} has a unique structure of a real analytic manifold with corners for which $D_{\text{BS}}(P)$ is an open set and whose restriction to $D_{\text{BS}}(P)$ coincides with that of $D_{\text{BS}}(P)$ defined in 2.5.3 for any parabolic subgroup P of G .*

Proof. It is easy to see that for parabolic subgroups P and Q of G such that $Q \supset P$, $D_{\text{BS}}(Q)$ is an open set of $D_{\text{BS}}(P)$ and the restriction of the structure of $D_{\text{BS}}(P)$ as a real analytic manifold with corners to $D_{\text{BS}}(Q)$ coincides with that of $D_{\text{BS}}(Q)$. Furthermore, $D_{\text{BS}}(P) \cap D_{\text{BS}}(Q) = D_{\text{BS}}(P * Q)$. This proves Proposition 2.5.4. \square

Remark 2.5.5. When G is semisimple, the Borel–Serre partial compactification $\overline{G(\mathbf{R})}_{\text{BS}}$ in [6] Proposition 6.3 coincides with our $D(G, h_0)_{\text{BS}}$ in Proposition 2.5.4 as real analytic manifolds with corners.

2.5.6. We have a morphism

$$D_{\text{BS}} \rightarrow \mathfrak{X}_{\text{BS}}$$

of real analytic manifolds with corners induced by $D_{\text{red}, \text{BS}} \rightarrow \mathfrak{X}_{\text{BS}} ; (P, Z) \mapsto (P, Z')$, where Z' is the image of $Z \subset D_{\text{red}}$ under $D_{\text{red}} \rightarrow \mathfrak{X}$ (1.5.6).

Proposition 2.5.7. *If G is reductive, the map $D_{\text{BS}} \rightarrow \mathfrak{X}_{\text{BS}}$ is proper.*

Proof. Since G is reductive, we have $D_{\text{BS}}(P) = D_{\text{BS}}^{\text{mild}}(P)$. Hence it is sufficient to prove that $D_{\text{BS}}(P) \rightarrow \mathfrak{X}_{\text{BS}}(P)$ is proper. Since $D_{\text{BS}}(P) = D_A^{(1)} \times \bar{A}_P$ (2.5.3 (3)) and $\mathfrak{X}_{\text{BS}}(P) = \mathfrak{X}^{(1)} \times \bar{A}_P$ (2.3.5 (3)), the properness of $D_{\text{BS}}(P) \rightarrow \mathfrak{X}_{\text{BS}}(P)$ is reduced to that of $D_A^{(1)} \rightarrow \mathfrak{X}^{(1)}$. The latter is reduced to the properness of $D \rightarrow \mathfrak{X}$ by 2.5.3 (2) and 2.3.5 (2). Take a point $p \in D$, let K' be the isotropic subgroup of $G(\mathbf{R})$ at p , and let K be the isotropic subgroup of $G(\mathbf{R})$ at the image of p in \mathcal{X} . Then K is a maximal compact subgroup of $G(\mathbf{R})$ and $K' \subset K$ is a compact subgroup. Hence $D = G(\mathbf{R})/K' \rightarrow \mathfrak{X} = G(\mathbf{R})/K$ is proper. \square

2.5.8. Let P be a parabolic subgroup of G . Fix $f : D_{\text{red}} \rightarrow A_P$, which is induced from Lemma 2.5.1, let $Y = f^{-1}(1) \subset D_{\text{red}}$ and let b_f be the composition $D_{\text{red,BS}}(P) = Y \times \bar{A}_P \rightarrow Y \subset D_{\text{red}}$ (see 2.4.1). Consider the isomorphism of real analytic manifolds

$$D \xrightarrow{\sim} \{(p, s, \delta) \in D_{\text{red}} \times \text{spl}(W) \times \mathcal{L} \mid \delta \in \mathcal{L}(b_f(p))\} ; x \mapsto (p, \text{spl}_W(x), \text{Ad}(f(p)_p)^{-1}\delta(x)),$$

where $p = x_{\text{red}}$ and $f(p)_p$ is the Borel–Serre lifting of $f(p)$ to P at p .

Proposition 2.5.9. *This isomorphism extends uniquely to an isomorphism of real analytic manifolds with corners*

$$D_{\text{BS}}(P) \xrightarrow{\sim} \{(p, s, \delta) \in D_{\text{red,BS}}(P) \times \text{spl}(W) \times \bar{\mathcal{L}} \mid \delta \in \bar{\mathcal{L}}(b_f(p))\}.$$

This map sends $x = (Q, Z)$ to (p, s, δ) defined as follows. $p = (Q, Z_{\text{red}})$. $s = \text{spl}_W(y)$ for any $y \in Z$, which is independent of the choice of y . $\delta = \text{Ad}(f(z)_z)^{-1}(\delta(y))$, where $y \in Z$ and $z = y_{\text{red}}$, which is independent of the choice of y .

By this map, the image of $x \in D_{\text{BS}}(P)$ in $\bar{\mathcal{L}}$ belongs to \mathcal{L} if and only if x is an A_P -orbit.

Proposition 2.5.10. *The map $D_{\text{BS}} \rightarrow D_{\text{red,BS}} \times \text{spl}(W)$ is proper. It is an \bar{L} -bundle. Note that \bar{L} here is $\bar{\mathcal{L}}(p)$ for any p .*

We hope that our notation $D_{\text{red,BS}}$ is not confusing with the reductive Borel–Serre space in [37].

2.6 Global properties of D_{BS}

We are still in the setting in Section 2.4.

Theorem 2.6.1. *Let Γ be a semi-arithmetic subgroup of $G'(\mathbf{Q})$ (1.5.7).*

- (1) *The action of Γ on D_{BS} is proper and the quotient space $\Gamma \backslash D_{\text{BS}}$ is Hausdorff.*
- (2) *If Γ is torsion-free, the action of Γ on D_{BS} is free and the map $D_{\text{BS}} \rightarrow \Gamma \backslash D_{\text{BS}}$ is a local homeomorphism.*
- (3) *If Γ is an arithmetic subgroup of $G'(\mathbf{Q})$, $\Gamma \backslash D_{\text{BS}}$ is compact.*

2.6.2. We first prove that D_{BS} is Hausdorff, that is, the case $\Gamma = \{1\}$ of 2.6.1 (1).

The map $D_{\text{BS}} \rightarrow \mathfrak{X} \times \text{spl}(W)$ is proper by Propositions 2.5.7 and 2.5.10. Since \mathfrak{X} is Hausdorff (2.3.7) as well as $\text{spl}(W)$ which is isomorphic to $G_u(\mathbf{R})$, we have that D_{BS} is Hausdorff.

2.6.3. We say that a subgroup Γ of $G(\mathbf{Q})$ is *neat* if it satisfies the following condition (1).

- (1) There is a faithful representation $V \in \text{Rep}(G)$ such that for every element $\gamma \in \Gamma$, the subgroup of \mathbf{C}^\times generated by all the eigenvalues of the action of γ on $V_{\mathbf{C}}$ is torsion-free.

By 1.4.7, the condition (1) is equivalent to the following condition (1').

- (1') For every $\gamma \in \Gamma$, every $n \geq 1$, and every homomorphism $\rho : G \rightarrow \text{GL}(n)$, the subgroup of \mathbf{C}^\times generated by all the eigenvalues of $\rho(\gamma)$ is torsion-free.

A neat subgroup of $G(\mathbf{Q})$ is torsion-free.

Every semi-arithmetic subgroup Γ (1.5.7) has a neat subgroup of finite index ([2] 17.4).

2.6.4. We prove Theorem 2.6.1. Let Γ be a semi-arithmetic subgroup of $G'(\mathbf{Q})$ (1.5.7).

The proof is similar to that of [21] Part I Theorem 9.1.

By 2.3.6 (5), the properness in (1) is reduced to the case where Γ is neat. We prove the properness in (1) and the free property in (2) assuming Γ is neat. We apply 2.3.6 (4) to $H = \Gamma$, $H_1 = \Gamma_u := \Gamma \cap G_u(\mathbf{Q})$, $X_1 = \text{spl}(W)$, $X_2 = \mathfrak{X}$. The action of Γ/Γ_u on \mathfrak{X} is proper and free by (1) and (2) of 2.3.7, and the action of Γ_u on $\text{spl}(W)$ is proper and free because $\text{spl}(W) \simeq G_u(\mathbf{R})$ on which Γ_u acts through the inclusion $\Gamma_u \rightarrow G_u(\mathbf{R})$. Hence the action of Γ on $\text{spl}(W) \times \mathfrak{X}$ is proper and free. By using the canonical continuous equivariant map $D_{\text{BS}} \rightarrow \text{spl}(W) \times \mathfrak{X}$, we have that the action of Γ on D_{BS} is proper and free by 2.6.2 and 2.3.6 (3.1).

The Hausdorffness in (1) follows from the properness in (1) (2.3.6 (1)) and the local homeomorphism property in (2) follows from it and from the properness in (1) (2.3.6 (2)).

(3) follows from the compactness of $(\Gamma/\Gamma_u) \setminus \mathfrak{X}_{\text{BS}}$, where $\Gamma_u = \Gamma \cap G_u(\mathbf{Q})$, the compactness of $\Gamma_u \setminus G_u(\mathbf{R})$, and Proposition 2.5.10.

In Theorem 2.6.1, we can use an arithmetic subgroup and a semi-arithmetic subgroup of $G(\mathbf{Q})$ (not of $G'(\mathbf{Q})$) in the following situations in Remark 2.6.5 and Proposition 2.6.6.

Remark 2.6.5. If either G is semisimple or the condition (1) in Lemma 1.5.3 is satisfied, 2.6.1 holds for a semi-arithmetic subgroup Γ of $G(\mathbf{Q})$. In fact, $\Gamma \cap G'(\mathbf{Q})$ is of finite index (cf. Proposition 1.5.13 for the latter case). Hence by 2.3.6 (5), we can replace Γ by the semi-arithmetic subgroup $\Gamma \cap G'(\mathbf{Q})$ of $G'(\mathbf{Q})$.

Proposition 2.6.6. *Assume that G is reductive. Let Γ be a subgroup of $G(\mathbf{Q})$. Then:*

- (1) *If Γ is a semi-arithmetic subgroup of $G(\mathbf{Q})$, $\Gamma \setminus D_{\text{BS}}$ is Hausdorff.*
- (2) *Let Z be the center of G . If Γ is a semi-arithmetic subgroup of $G(\mathbf{Q})$ and the image of Γ in $(G/Z)(\mathbf{Q})$ is torsion-free, then the map $D_{\text{BS}} \rightarrow \Gamma \setminus D_{\text{BS}}$ is a local homeomorphism.*
- (3) *If Γ is an arithmetic subgroup of $G(\mathbf{Q})$, $\Gamma \setminus D_{\text{BS}}$ is compact.*

See 4.10.18 for the proof.

3 The space of $\text{SL}(2)$ -orbits

Let G be a linear algebraic group defined over \mathbf{Q} . We define and study the extended period domains $D_{\text{SL}(2)}$ and $D_{\text{SL}(2)}^*$ of $\text{SL}(2)$ -orbits which contain D . This $D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$) is the G -MHS version of the space $D(\Lambda)_{\text{SL}(2)}$ (resp. $D(\Lambda)_{\text{SL}(2)}^*$) for the classical period domain $D(\Lambda)$ (1.6) defined and studied in [23] (in the pure case) and [21] Part II (in the mixed case) (resp. in [21] Part IV).

Assume that we are given an element $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$ of $\Psi_H(G)$ (1.2.7). We assume the \mathbf{R} -polarizability (1.5.2). We denote by $k_0 \in \Psi_W(G)$ the homomorphism $\mathbf{G}_m \rightarrow G_{\text{red}}$ (defined over \mathbf{Q}) which induces $h_0 \circ w : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\text{red},\mathbf{R}}$.

3.1 The set $D_{\text{SL}(2)}$ when G is reductive

In this Section 3.1, we assume that G is reductive.

3.1.1. Let $n \geq 0$. By an $\mathrm{SL}(2)$ -orbit in n variables, we mean a pair (ρ, φ)

$$\rho : \mathrm{SL}(2)_{\mathbf{R}}^n \rightarrow G_{\mathbf{R}}, \quad \varphi : \mathbf{P}^1(\mathbf{C})^n \rightarrow \check{D},$$

where ρ is a homomorphism of algebraic groups over \mathbf{R} and φ is a holomorphic map satisfying the following conditions (i)–(iii).

- (i) $\varphi(gz) = \rho(g)\varphi(z)$ ($g \in \mathrm{SL}_2(\mathbf{C})^n$, $z \in \mathbf{P}^1(\mathbf{C})^n$).
- (ii) $\varphi(\mathfrak{H}^n) \subset D$.

(iii) For $z \in \mathfrak{H}^n$, the homomorphism $\mathrm{Lie}(\rho) : \mathfrak{sl}(2, \mathbf{R})^n \rightarrow \mathrm{Lie}(G_{\mathbf{R}}) = \mathrm{Lie}(G)$ of Lie algebras induced by ρ is a homomorphism of \mathbf{R} -Hodge structures if we endow $\mathfrak{sl}(2, \mathbf{Q})^n$ with the \mathbf{Q} -Hodge structure of weight 0 associated to z (see below) and endow $\mathrm{Lie}(G)$ with the \mathbf{Q} -Hodge structure of weight 0 associated to $\varphi(z) \in D$ and the adjoint action of G on $\mathrm{Lie}(G)$.

Here the \mathbf{Q} -Hodge structure of $\mathfrak{sl}(2, \mathbf{Q})^n$ associated to $z = (z_j)_{1 \leq j \leq n}$ is the direct sum of the \mathbf{Q} -Hodge structure of the j -th $\mathfrak{sl}(2, \mathbf{Q})$ induced by the \mathbf{Q} -Hodge structure of $V_{\mathbf{C}} = \mathbf{C}^2$ corresponding to z_j (e.g. [24] 1.2.3) and the adjoint action of $\mathrm{GL}(2)_{\mathbf{Q}}$ on $\mathfrak{sl}(2, \mathbf{Q})$.

3.1.2. (Classical) *Example.* Assume $G = \mathrm{GL}(2)$ and let h_0 be the homomorphism $z \mapsto \langle z \rangle$ in 1.6.5. Let $\rho : \mathrm{SL}(2)_{\mathbf{R}} \rightarrow G_{\mathbf{R}}$ be the inclusion map and let $\varphi : \mathbf{P}^1(\mathbf{C}) \rightarrow \check{D}$ be the identity map in 1.6.5. Then (ρ, φ) is an $\mathrm{SL}(2)$ -orbit in one variable.

Lemma 3.1.3. Let $n \geq 0$. Then the following sets (i), (ii), (iii), (iii)', and (iv) can be identified.

- (i) The set of all $\mathrm{SL}(2)$ -orbits (ρ, φ) in n variables.
- (ii) The set of all pairs (ρ, φ) , where ρ is a homomorphism $\mathrm{SL}(2)_{\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$ of algebraic groups over \mathbf{R} and φ is a holomorphic map $\mathfrak{H}^n \rightarrow D$ satisfying the condition (i) in 3.1.1 for $g \in \mathrm{SL}_2(\mathbf{R})^n$ and $z \in \mathfrak{H}^n$ and satisfying the condition (iii) in 3.1.1.
- (iii) The set of all pairs (ρ, \mathbf{r}) , where ρ is a homomorphism $\mathrm{SL}(2)_{\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$ of algebraic groups over \mathbf{R} and $\mathbf{r} \in D$ satisfying the following condition.

The homomorphism $\mathrm{Lie}(\rho) : \mathfrak{sl}(2, \mathbf{R})^n \rightarrow \mathrm{Lie}(G_{\mathbf{R}})$ is a homomorphism of \mathbf{R} -Hodge structures with respect to the Hodge structure of $\mathfrak{sl}(2, \mathbf{R})^n$ associated to \mathbf{i} and the Hodge structure of $\mathrm{Lie}(G_{\mathbf{R}})$ associated to \mathbf{r} .

(iii)' The set of all pairs (ρ, \mathbf{r}) , where ρ is a homomorphism $\mathrm{SL}(2)_{\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$ of algebraic groups over \mathbf{R} and $\mathbf{r} \in D$ satisfying the following condition.

Let $S_{\mathbf{C}/\mathbf{R}}^{(1)}$ be the kernel of norm $S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathbf{G}_{m, \mathbf{R}}$; $z \mapsto z\bar{z}$. Let $\xi_1 : S_{\mathbf{C}/\mathbf{R}}^{(1)} \rightarrow G_{\mathbf{R}}$ be the homomorphism defined by

$$\xi_1(u) := \mathbf{r}(u)\rho(\langle u \rangle, \dots, \langle u \rangle)^{-1},$$

where we regard \mathbf{r} as a homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ and $\langle u \rangle \in \mathrm{SO}(2)_{\mathbf{R}}$ is as in 1.6.5. Then $\xi_1(u)\rho(g) = \rho(g)\xi_1(u)$ for any $u \in S_{\mathbf{C}/\mathbf{R}}^{(1)}$ and $g \in \mathrm{SL}(2)_{\mathbf{R}}^n$.

(iv) The set of all homomorphisms $\mathrm{SL}(2)_{\mathbf{R}}^n \times S_{\mathbf{C}/\mathbf{R}}^{(1)} \rightarrow G_{\mathbf{R}}$ satisfying the following conditions (iv.1) and (iv.2). Let ρ (resp. ξ_1) be the restriction of this homomorphism to $\mathrm{SL}(2)_{\mathbf{R}}^n$ (resp. $S_{\mathbf{C}/\mathbf{R}}^{(1)}$).

(iv.1) $k_0(-1) = \xi_1(-1)\rho(-1, \dots, -1)$. Here -1 inside $\rho(-)$ denotes the scalar matrix -1 in $\mathrm{SL}(2)_{\mathbf{R}}$.

(iv.2) Define the homomorphism $\mathbf{r} : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ by $\mathbf{r}(tu) := k_0(t)\xi_1(u)\rho(\langle u \rangle, \dots, \langle u \rangle)$, where $t \in \mathbf{G}_{m,\mathbf{R}} \subset S_{\mathbf{C}/\mathbf{R}}$ and $u \in S_{\mathbf{C}/\mathbf{R}}^{(1)}$. Then $\mathbf{r} \in D$.

Proof. We will prove that the first three sets are identified. In fact, the bijectivities of the evident maps (i) \rightarrow (ii) \rightarrow (iii) are well-known (for their 1-variable cases, see [32], [36] Remark (2.2), and [31]), but we give a proof here. To prove that (i) \rightarrow (ii) \rightarrow (iii) are bijective, since they are injective, it is sufficient to prove that (i) \rightarrow (iii) is surjective. Let (ρ, \mathbf{r}) be an element of the set (iii). We prove that if $g \in H := \{g \in \mathrm{SL}(2, \mathbf{C})^n \mid g\mathbf{i} = \mathbf{i}\}$, then $\rho(g)\mathbf{r} = \mathbf{r}$. This will imply that we have a map $\varphi : \mathbf{P}^1(\mathbf{C})^n \rightarrow \check{D} ; g\mathbf{i} \mapsto \rho(g)\mathbf{r}$ having the desired properties. Since H is connected, it is sufficient to prove that for $X \in \mathrm{Lie}(H) = F_{\mathbf{i}}^0 \mathfrak{sl}(2, \mathbf{C})^n$, $\mathrm{Lie}(\rho)(X)$ respects the Hodge filtration of \mathbf{r} . But this follows from $\mathrm{Lie}(\rho)(X) \in F_{\mathbf{r}}^0 \mathrm{Lie}(G)_{\mathbf{C}}$.

We prove (iii) $=$ (iii)'. Since the Hodge structures on the Lie algebras in the condition in the definition of (iii) are of weight 0, the condition in (iii) is equivalent to the condition that for $u \in S_{\mathbf{C}/\mathbf{R}}^{(1)}$, the action $\mathrm{Ad}(\langle u \rangle, \dots, \langle u \rangle)$ on $\mathfrak{sl}(2, \mathbf{R})^n$ and the adjoint action of $\mathbf{r}(u)$ on $\mathrm{Lie}(G_{\mathbf{R}})$ are compatible via $\mathrm{Lie}(\rho)$. This is equivalent to the condition that the action $\mathrm{Ad}(\xi_1(u))$ on $\mathrm{Lie}(\rho)(\mathfrak{sl}(2, \mathbf{R})^n)$ is trivial for every u and hence to the condition that $\xi_1(u)$ and $\rho(g)$ commute for every u and $g \in \mathrm{SL}(2)_{\mathbf{R}}^n$.

The evident map (iii)' \rightarrow (iv) is bijective because the inverse map is given as in (iv.2). Thus we can identify the sets (i), (ii), (iii), (iii)', and (iv). \square

Proposition 3.1.4. *Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit in n variables and let $\mathbf{r} = \varphi(\mathbf{i}) : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ and $\xi_1 : S_{\mathbf{C}/\mathbf{R}}^{(1)} \rightarrow G_{\mathbf{R}}$ be as in Lemma 3.1.3. Define a homomorphism $\xi : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ of algebraic groups over \mathbf{R} by*

$$\xi(t) := k_0(t)\rho(\mathrm{diag}(1/t, t), \dots, \mathrm{diag}(1/t, t)) \quad \text{for } t \in \mathbf{G}_{m,\mathbf{R}} \subset S_{\mathbf{C}/\mathbf{R}},$$

$$\xi(u) := \xi_1(u) \quad \text{for } u \in S_{\mathbf{C}/\mathbf{R}}^{(1)},$$

where ξ_1 is as in Lemma 3.1.3 (iv). Then the Hodge filtration of ξ is $\varphi(\mathbf{0})$, where $\mathbf{0} = (0, \dots, 0) \in \mathbf{P}^1(\mathbf{C})^n$.

Proof. To relate ξ and \mathbf{r} , we use the Cayley element

$$\mathbf{c} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C}).$$

We use the following properties (i) and (ii) of \mathbf{c} .

(i) By the action of $\mathrm{SL}_2(\mathbf{C})$ on $\mathbf{P}^1(\mathbf{C})$, \mathbf{c} sends $0 \in \mathbf{P}^1(\mathbf{C})$ to $i \in \mathbf{P}^1(\mathbf{C})$.

(ii) $\langle u \rangle = \mathbf{c} \mathrm{diag}(u^{-1}, u) \mathbf{c}^{-1}$ for $u \in S_{\mathbf{C}/\mathbf{R}}^{(1)}$.

Let $V \in \mathrm{Rep}(G)$, let $V_{\mathbf{C}, \xi}$ (resp. $V_{\mathbf{C}, \mathbf{r}}$) be $V_{\mathbf{C}}$ endowed with the action of $S_{\mathbf{C}/\mathbf{R}}$ via ξ (resp. \mathbf{r}), and let $V_{\mathbf{C}, \xi}^{p,q}$ (resp. $V_{\mathbf{C}, \mathbf{r}}^{p,q}$) be the Hodge (p, q) -component of $V_{\mathbf{C}, \xi}$ (resp. $V_{\mathbf{C}, \mathbf{r}}$).

Claim. $\rho(\mathbf{c}, \dots, \mathbf{c})(\bigoplus_q V_{\mathbf{C}, \xi}^{p,q}) = \bigoplus_q V_{\mathbf{C}, \mathbf{r}}^{p,q}$ in $V_{\mathbf{C}}$.

We prove Claim. For $(a, b, c) \in \mathbf{Z}^3$, let $V_{\mathbf{C}}^{(a,b,c)}$ be the part of $V_{\mathbf{C}}$ on which $k_0(t)$ for $t \in \mathbf{C}^\times$ acts as t^a , $\xi_1(u)$ for $u \in S_{\mathbf{C}/\mathbf{R}}^{(1)}$ acts as u^b , and $\rho(\mathrm{diag}(1/t, t), \dots, \mathrm{diag}(1/t, t))$ for $t \in \mathbf{C}^\times$ acts as t^c . Then $V_{\mathbf{C}}$ is the direct sum of these $V_{\mathbf{C}}^{(a,b,c)}$. On $V_{\mathbf{C}}^{(a,b,c)}$, $\xi(tu)$ ($t \in \mathbf{G}_{m,\mathbf{R}} \subset S_{\mathbf{C}/\mathbf{R}}$,

$u \in S_{\mathbf{C}/\mathbf{R}}^{(1)}$ acts as $t^{a+c}u^b$ and hence $V_{\mathbf{C}}^{(a,b,c)}$ has the Hodge type $((a+b+c)/2, (a-b+c)/2)$. On the other hand, by the above property (ii) of $\mathbf{c}, \mathbf{r}(tu)$ acts on $\rho(\mathbf{c}, \dots, \mathbf{c})V_{\mathbf{C}}^{(a,b,c)}$ as $t^a u^{b+c}$ and hence $\rho(\mathbf{c}, \dots, \mathbf{c})V_{\mathbf{C}}^{(a,b,c)}$ has the Hodge type $((a+b+c)/2, (a-b-c)/2)$. This proves Claim.

Since $\varphi(\mathbf{0}) = \rho(\mathbf{c}, \dots, \mathbf{c})^{-1}\varphi(\mathbf{i})$ by the property (i) of \mathbf{c} , Claim shows that ξ gives $\varphi(\mathbf{0})$. \square

The fact that the Cayley element relates $\varphi(\mathbf{0})$ and $\varphi(\mathbf{i})$ was used also in [36].

3.1.5. Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit in n variables. Then the *rank* of (ρ, φ) is defined to be the number of j ($1 \leq j \leq n$) such that the j -th component $\rho_j : \mathrm{SL}(2)_{\mathbf{R}} \rightarrow G_{\mathbf{R}}$ of ρ is a non-trivial homomorphism.

Let r be the rank of (ρ, φ) and let (ρ', φ') be the $\mathrm{SL}(2)$ -orbit in r variables defined as follows. Let $\{s(1), \dots, s(r)\}$ ($s(1) < \dots < s(r)$) be the set of all j ($1 \leq j \leq n$) such that ρ_j is non-trivial. Define $\rho' : \mathrm{SL}(2)_{\mathbf{R}}^r \rightarrow G_{\mathbf{R}}$ to be the unique homomorphism satisfying $\rho(g_1, \dots, g_n) = \rho'(g_{s(1)}, \dots, g_{s(r)})$ and let φ' be the unique map $\mathbf{P}^1(\mathbf{C})^r \rightarrow \check{D}$ such that $\varphi(z_1, \dots, z_n) = \varphi'(z_{s(1)}, \dots, z_{s(r)})$.

We call (ρ', φ') the $\mathrm{SL}(2)$ -orbit in r variables of rank r *associated to* (ρ, φ) .

3.1.6. Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit in n variables of rank n . For $1 \leq j \leq n$, define homomorphisms $\tau_j^*, \tau_j : \mathbf{G}_{m, \mathbf{R}} \rightarrow G_{\mathbf{R}}$ as follows.

$$\tau_j^*(t) = \rho(g_1, \dots, g_n) \quad (t \in \mathbf{R}^\times), \quad \text{where}$$

$$g_k = \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix} \text{ if } 1 \leq k \leq j, \quad g_k = 1 \text{ if } j < k \leq n.$$

$$\tau_j(t) := \tau_j^*(t)k_0(t) \quad (t \in \mathbf{R}^\times),$$

where $k_0 \in \Psi_W(G)$ is as in the beginning of Section 3.

3.1.7. We define the equivalence relation between $\mathrm{SL}(2)$ -orbits.

An $\mathrm{SL}(2)$ -orbit in n variables and an $\mathrm{SL}(2)$ -orbit in n' -variables are equivalent if and only if they have the same rank r and their associated $\mathrm{SL}(2)$ -orbits of rank r are equivalent in the following sense.

For $\mathrm{SL}(2)$ -orbits in n variables of rank n , the equivalence relation is given as follows: $(\rho, \varphi) \sim (\rho', \varphi')$ if and only if there is a $t = (t_j)_{1 \leq j \leq n} \in \mathbf{R}_{>0}^n$ such that

$$\rho'(g) = \tau(t)\rho(g)\tau(t)^{-1}, \quad \varphi'(z) = \tau(t)\varphi(z)$$

for any $g \in \mathrm{SL}(2)_{\mathbf{R}}^n$ and $z \in \mathfrak{H}^n$, where $\tau(t)$ denotes $\prod_{j=1}^n \tau_j(t_j)$ associated to (ρ, φ) .

In terms of Lemma 3.1.3 (iv), $(\rho, \xi_1) \sim (\rho', \xi'_1)$ if and only if there is a $t = (t_j)_{1 \leq j \leq n} \in \mathbf{R}_{>0}^n$ such that $\rho'(g) = \tau(t)\rho(g)\tau(t)^{-1}$ for every $g \in \mathrm{SL}(2)_{\mathbf{R}}^n$ and $\xi_1 = \xi'_1$.

Remark 3.1.8. We have the same equivalence relation \sim even if we replace τ_j by τ_j^* . This is because $\tau(t)\rho(g)\tau(t)^{-1} = \tau^*(t)\rho(g)\tau^*(t)^{-1}$ and the actions of $\tau(t)$ and $\tau^*(t)$ on D are the same. Here $\tau^*(t) = \prod_j \tau_j^*(t_j)$ which is associated to (ρ, φ) .

3.1.9. Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit in n variables.

Let $1 \leq j \leq n$. For $V \in \mathrm{Rep}(G)$, let $W_{\bullet}^{(j)}V_{\mathbf{R}}$ be the increasing filtration on $V_{\mathbf{R}}$ defined by the action of $\mathbf{G}_{m,\mathbf{R}}$ via τ_j . That is, for $w \in \mathbf{Z}$, $W_w^{(j)}V_{\mathbf{R}}$ is the part of $V_{\mathbf{R}}$ on which the action of $\mathbf{G}_{m,\mathbf{R}}$ via τ_j is of weights $\leq w$.

For $1 \leq j \leq n$, define $N_j \in \mathrm{Lie}(G_{\mathbf{R}})$ by

$$N_j = \mathrm{Lie}(\rho)(g_1, \dots, g_n) \quad \text{with } g_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g_k = 0 \text{ for } k \neq j,$$

where $\mathrm{Lie}(\rho)$ is the homomorphism $\mathfrak{sl}(2, \mathbf{R})^n \rightarrow \mathrm{Lie}(G_{\mathbf{R}})$ induced by ρ . Let $W^{(0)} = W$. Then for $0 \leq j \leq k \leq n$ and for any $a_l \in \mathbf{R}_{>0}$ ($j < l \leq k$) and any $V \in \mathrm{Rep}(G)$, the filtration $W_{\bullet}^{(k)}V_{\mathbf{R}}$ is the relative weight filtration of the nilpotent operator $\sum_{j < l \leq k} a_l N_l : V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ with respect to $W_{\bullet}^{(j)}V_{\mathbf{R}}$.

For $1 \leq j \leq n$, the following conditions (i)–(iv) are equivalent. (i) ρ_j is trivial. (ii) τ_j is trivial. (iii) $W^{(j)} = W^{(j-1)}$. (iv) $N_j = 0$.

We say that the weight filtrations of (ρ, φ) are *rational* if the weight filtrations $W^{(j)}$ ($1 \leq j \leq n$) of the associated $\mathrm{SL}(2)$ -orbit in n variables of rank n has the following property: For any $V \in \mathrm{Rep}(G)$, the filtrations $W_{\bullet}^{(j)}V_{\mathbf{R}}$ ($1 \leq j \leq n$) on $V_{\mathbf{R}}$ are \mathbf{Q} -rational. This rationality depends only on the equivalence class of (ρ, φ) .

3.1.10. We define the set $D_{\mathrm{SL}(2)}$ as the set of all equivalence classes of $\mathrm{SL}(2)$ -orbits whose weight filtrations are rational.

Remark 3.1.11. The rationality condition on the weight filtrations will become important to have the good properties (Hausdorff property etc.) of the quotient space $\Gamma \backslash D_{\mathrm{SL}(2)}$ for a semi-arithmetic subgroup Γ of $G'(\mathbf{Q})$ (cf. Section 4.14).

3.1.12. Let $p \in D_{\mathrm{SL}(2)}$. Let (ρ, φ) be the $\mathrm{SL}(2)$ -orbit in n variables of rank n whose class is p .

Let

$$\tau_p, \tau_p^* : \mathbf{G}_{m,\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$$

be the homomorphisms τ and τ^* associated to (ρ, φ) as in 3.1.7, respectively. Let $\tau_{p,j}$ and $\tau_{p,j}^*$ ($1 \leq j \leq n$) be the j -th component $\mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}}$ of τ_p (resp. τ_p^*).

We will call

$$\{\varphi(iy_1, \dots, iy_n) \in D \mid y_j \in \mathbf{R}_{>0} (1 \leq j \leq n)\} = \tau_p(\mathbf{R}_{>0}^n)\varphi(\mathbf{i}) = \tau_p^*(\mathbf{R}_{>0}^n)\varphi(\mathbf{i})$$

the *torus orbit* associated to p .

3.2 The sets $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2)}^*$

When we do not assume that G is reductive, unlike the case where G is reductive, there are several sets of $\mathrm{SL}(2)$ -orbits. Here we define the most important two sets $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2)}^*$. Other sets of $\mathrm{SL}(2)$ -orbits $D_{\mathrm{SL}(2)}^{\star,+}$, $D_{\mathrm{SL}(2)}^{\star,W}$ will be defined later.

3.2.1. Let $D_{\text{red}} = D(G_{\text{red}}, h_0)$ be as in 1.3.3. Let $D_{\text{red}, \text{SL}(2)} = D(G_{\text{red}}, h_0)_{\text{SL}(2)}$ be $D_{\text{SL}(2)}$ for (G_{red}, h_0) in 3.1.10. This $D_{\text{red}, \text{SL}(2)}$ is a G -MHS version of the space $D_{\text{SL}(2)}(\text{gr}^W)^\sim$ in [21] Part II 3.5.1.

For $p \in D_{\text{red}, \text{SL}(2)}$ and let $\tau_p, \tau_p^* : \mathbf{G}_{m, \mathbf{R}}^n \rightarrow G_{\text{red}, \mathbf{R}}$ be the homomorphism in 3.1.12 associated to p , where n is the rank of p .

Let $D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$) be the set of pairs (p, Z) , where p is an element of $D_{\text{red}, \text{SL}(2)}$ and Z is a subset of D whose image in D_{red} coincides with the torus orbit (3.1.12) of p , satisfying one of the following conditions (A) and (B) for every (equivalently, for all) $z \in Z$. Let $s = \text{spl}_W(z)$.

(A) Let $h : \mathbf{G}_{m, \mathbf{R}}^n \rightarrow G_{\mathbf{R}}$ be the homomorphism defined by

$$h(t) := s\tau_p(t)s^{-1} \quad (\text{resp. } s\tau_p^*(t)s^{-1}).$$

Then Z is a $h(\mathbf{R}_{>0}^n)$ -orbit.

(B) Let $h : \mathbf{G}_m \times \mathbf{G}_{m, \mathbf{R}}^n \rightarrow G$ be the homomorphism defined by

$$h(t_0, (t_j)_{1 \leq j \leq n}) := s(k_0(t_0) \prod_{j=1}^n \tau_p(t_j))s^{-1} \quad (\text{resp. } s(k_0(t_0) \prod_{j=1}^n \tau_p^*(t_j))s^{-1}).$$

Then Z is an $h(\mathbf{R}_{>0} \times \mathbf{R}_{>0}^n)$ -orbit contained in D_{nspl} .

3.2.2. Let $x = (p, Z) \in D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$). We call x an A -orbit in the case (A), and a B -orbit in the case (B).

We define the *mild part* of $D_{\text{SL}(2)}^*$ as the part consisting of all A -orbits and denote it by $D_{\text{SL}(2)}^{*, \text{mild}}$.

We denote by $D_{\text{SL}(2), \text{spl}}$ the subset of $D_{\text{SL}(2)}$ consisting of (p, Z) whose Z is contained in D_{spl} , and by $D_{\text{SL}(2), \text{nspl}}$ its complement.

3.2.3. For $x = (p, Z) \in D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$), the homomorphism h in 3.2.1 is independent of the choice of $z \in Z$. In the case $x \in D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$), we denote this homomorphism h by τ_x (resp. τ_x^*).

3.3 Weight filtrations associated to $\text{SL}(2)$ -orbits

In this Section 3.3, we give preliminary definitions which are used in Section 3.4.

3.3.1. Let E be a field of characteristic 0 and let \mathcal{G} be a linear algebraic group over E . Let $\text{Rep}_E(\mathcal{G})$ be the category of finite-dimensional representations of \mathcal{G} over E . Let W' be an increasing filtration on the functor $V \mapsto V$ from $\text{Rep}_E(\mathcal{G})$ to the category of finite-dimensional E -vector spaces. Then, by [30] Chap. IV 2.4, the following two conditions (i) and (ii) are equivalent.

(i) There is a homomorphism $a : \mathbf{G}_{m, E} \rightarrow \mathcal{G}$ which defines W' . That is, for $V \in \text{Rep}_E(\mathcal{G})$, $W'_w V$ is the part of V on which the action of $\mathbf{G}_{m, E}$ via a is of weights $\leq w$.

(ii) W' is an exact \otimes -filtration in the sense of [30] Chapter IV 2.1. That is, $\text{gr}_w^{W'}$ is an exact functor for every $w \in \mathbf{Z}$, and $W'_w(V_1 \otimes V_2) = \sum_{j+k=w} W'_j V_1 \otimes_E W'_k V_2$ for every $V, V_2 \in \text{Rep}_E(\mathcal{G})$ and $w \in \mathbf{Z}$.

We denote by $\mathfrak{W}(\mathcal{G})$ the set of all W' satisfying the above equivalent conditions (i) and (ii).

If $E \subset E'$, via the injection $\mathfrak{W}(\mathcal{G}) \rightarrow \mathfrak{W}(\mathcal{G}_{E'})$ induced by the homomorphism $\text{Hom}(\mathbf{G}_{m,E}, \mathcal{G}) \rightarrow \text{Hom}(\mathbf{G}_{m,E'}, \mathcal{G}_{E'})$, we often regard $\mathfrak{W}(\mathcal{G})$ as a subset of $\mathfrak{W}(\mathcal{G}_{E'})$.

Actually only the cases $E = \mathbf{Q}$ and $E = \mathbf{R}$ are important to us.

3.3.2. Let $x = (p, Z) \in D_{\text{SL}(2)}$ and let n be the rank of $p = x_{\text{red}}$. In the case where x is an A (resp. B)-orbit, we defined in 3.2.3 a homomorphism

$$\tau_x : \mathbf{G}_{m,\mathbf{R}}^n = \mathbf{G}_{m,\mathbf{R}}^{\{1, \dots, n\}} \rightarrow G_{\mathbf{R}} \quad (\text{resp. } \mathbf{G}_{m,\mathbf{R}} \times \mathbf{G}_{m,\mathbf{R}}^n = \mathbf{G}_{m,\mathbf{R}}^{\{0, \dots, n\}} \rightarrow G_{\mathbf{R}}).$$

For $1 \leq j \leq n$ (resp. $0 \leq j \leq n$), let $W_x^{(j)}$ be the increasing filtration on the functor $V \mapsto V_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} V$ given by $\tau_{x,j} : \mathbf{G}_{m,\mathbf{R}} \rightarrow G$. For $1 \leq j \leq n$, we have $W_x^{(j)} = s(W_p^{(j)})$. That is, for $V \in \text{Rep}(G)$, $W_{x,w}^{(j)} V_{\mathbf{R}} = \bigoplus_{k \in \mathbf{Z}} s(W_{p,w}^{(j)} \text{gr}_k^W V_{\mathbf{R}})$. In the case where x is a B -orbit, $W_x^{(0)} = W$. These $W_x^{(j)}$ ($1 \leq j \leq n$) (resp. ($0 \leq j \leq n$)) are called the *weight filtrations associated to x* .

For $x \in D_{\text{SL}(2)}$, let

$$\Phi(x) \subset \mathfrak{W}(G_{\mathbf{R}})$$

be the set of weight filtrations associated to x .

We have $W \in \Phi(x)$ if and only if x is a B -orbit. If G is reductive, $W \notin \Phi(x)$ and we have $\Phi(x) \subset \mathfrak{W}(G)$.

Proposition 3.3.3. $(p, Z) \in D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$) is determined by $(\Phi(p), Z)$.

Proof. When G is reductive, the proof of [23] Lemma 3.10 for \mathbf{Q} -polarization works for \mathbf{R} -polarization. The general non-reductive case follows easily from this. \square

This will be used in the proof of the injectivity in Propositions 3.4.4 (2), 3.4.5 (2), 3.4.9 (2) in Section 3.4.

Proposition 3.3.4. Let $x \in D_{\text{SL}(2)}$ and let n be the rank of $x_{\text{red}} \in D_{\text{red,SL}(2)}$. Let $V \in \text{Rep}(G)$, $V \neq 0$. Let $0 \leq j \leq n$. Let $W_x^{(0)} := W$ also in the case where x is an A -orbit. Define the mean value $\mu_j \in \mathbf{Q}$ and the variance $\sigma_j^2 \in \mathbf{Q}$ of $W_x^{(j)}$ by

$$\mu_j := \sum_{w \in \mathbf{Z}} \dim_{\mathbf{R}}(\text{gr}_w^{W_x^{(j)}} V_{\mathbf{R}}) w / \dim_{\mathbf{R}}(V_{\mathbf{R}}), \quad \sigma_j^2 := \sum_{w \in \mathbf{Z}} \dim_{\mathbf{R}}(\text{gr}_w^{W_x^{(j)}} V_{\mathbf{R}}) (w - \mu_j)^2 / \dim_{\mathbf{R}}(V_{\mathbf{R}}).$$

Then μ_j is independent of j and $\sigma_j^2 \leq \sigma_k^2$ if $j \leq k$. Furthermore, if V is a faithful representation of G , then $\sigma_j^2 < \sigma_k^2$ for $0 \leq j < k \leq n$.

Proof. This follows from [21] Part II Proposition 2.1.12. \square

3.3.5. We denote by $\mathcal{W}(G)$ the set of all subsets Φ of $\mathfrak{W}(G_{\mathbf{R}})$ such that $\Phi = \Phi(x)$ for some $x \in D_{\text{SL}(2)}$.

For $\Phi \in \mathcal{W}(G)$, let $\Phi_{\text{red}} \in \mathcal{W}(G_{\text{red}})$ be $\{\text{gr}^W(W') \mid W' \in \Phi, W' \neq W\}$. If $x = (p, Z) \in D_{\text{SL}(2)}$, then $\Phi(x)_{\text{red}} = \Phi(p)$.

Let $\Phi \in \mathcal{W}(G)$ and let n be the order of Φ_{red} . In the case $W \notin \Phi$ (resp. $W \in \Phi$), we often identify Φ with the set $\{1, \dots, n\}$ (resp. $\{0, 1, \dots, n\}$) in the unique way that if $W', W'' \in \Phi$ correspond to $i, j \in \{1, \dots, n\}$ (resp. $\{0, 1, \dots, n\}$), respectively, then $i \leq j$ if and only if the variance of $W' \leq$ the variance of W'' (3.3.4).

3.3.6. For $\Phi \in \mathcal{W}(G_{\text{red}})$, let $D_{\text{SL}(2)}(\Phi)$ (resp. $D_{\text{SL}(2)}^*(\Phi)$) be the set of all points (p, Z) of $D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$) such that $\Phi(p) \subset \Phi$.

For $\Phi \in \mathcal{W}(G)$, let $D_{\text{SL}(2)}(\Phi) = \{x \in D_{\text{SL}(2)} \mid \Phi(x) \subset \Phi\}$.

Then, in the next Section 3.4,

$$D_{\text{SL}(2)}^* = \bigcup_{\Phi \in \mathcal{W}(G_{\text{red}})} D_{\text{SL}(2)}^*(\Phi)$$

will be an important open covering of $D_{\text{SL}(2)}^*$,

$$D_{\text{SL}(2)} = \bigcup_{\Phi \in \mathcal{W}(G_{\text{red}})} D_{\text{SL}(2)}(\Phi)$$

will be an important open covering of $D_{\text{SL}(2)}^{II}$, and

$$D_{\text{SL}(2)} = \bigcup_{\Phi \in \mathcal{W}(G)} D_{\text{SL}(2)}(\Phi) = \left(\bigcup_{\Phi \in \mathcal{W}(G), W \notin \Phi} D_{\text{SL}(2)}(\Phi) \right) \cup \left(\bigcup_{\Phi \in \mathcal{W}(G), W \in \Phi} D_{\text{SL}(2), \text{nspl}}(\Phi) \right)$$

(here $D_{\text{SL}(2), \text{nspl}}(\Phi) := D_{\text{SL}(2), \text{nspl}} \cap D_{\text{SL}(2)}(\Phi)$) will be an important open covering of $D_{\text{SL}(2)}^I$.

3.3.7. Let the notation be as in 3.3.1. Assume that \mathcal{G} is reductive and let $W' \in \mathfrak{W}(\mathcal{G})$. Then, by [30] Chap. IV 2.2.5, the stabilizer $P = \mathcal{G}_{W'}^\circ \subset \mathcal{G}^\circ$ of W' in \mathcal{G}° is a parabolic subgroup of \mathcal{G} .

We thus have a map

$$\text{Hom}(\mathbf{G}_{m,E}, \mathcal{G}) \rightarrow \{\text{parabolic subgroup of } \mathcal{G}\}.$$

This map is surjective (cf. [27] Theorem 25.1, [34] 15.1.2 (ii)).

Let $W' \in \mathfrak{W}(\mathcal{G})$ and let P be the associated parabolic subgroup of \mathcal{G} .

If a homomorphism $a : \mathbf{G}_{m,E} \rightarrow \mathcal{G}$ defines W' , then the image of a in \mathcal{G} is contained in P . The composition $\mathbf{G}_{m,E} \xrightarrow{a} P \rightarrow P_{\text{red}}$ is independent of the choice of a and has the image in the center of P_{red} . The adjoint action of $\mathbf{G}_{m,E}$ on $\text{Lie}(P_u)$ via a is of weights ≤ -1 .

Let $\text{spl}(W')$ be the set of all isomorphisms of \otimes -functors from $\text{Rep}_E(\mathcal{G})$ to the category of E -vector spaces

$$(V \mapsto \text{gr}^{W'} V) \xrightarrow{\sim} (V \mapsto V) \quad \text{preserving the increasing filtrations defined by } W'.$$

Then $\text{spl}(W')$ is a $P_u(E)$ -torsor. We have a natural bijection from the set of all homomorphisms $\mathbf{G}_{m,E} \rightarrow \mathcal{G}$ which define W' to $\text{spl}(W')$.

3.3.8. Assume that G is reductive.

Let $\mathfrak{W}_0(G_{\mathbf{R}})$ be the set of all $W' \in \mathfrak{W}(G_{\mathbf{R}})$ satisfying the following condition.

(*) For some (equivalently, for every) homomorphism $a : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}}$ which defines W' , the image of the homomorphism $a^* : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}} ; t \mapsto a(t)k_0(t)^{-1}$ belongs to the commutator subgroup G' of G , where k_0 is as in the beginning of this Section 3.

For a point $p \in D_{\text{SL}(2)}$, we have $\Phi(p) \subset \mathfrak{W}(G) \cap \mathfrak{W}_0(G_{\mathbf{R}})$. In fact, the condition (*) is satisfied by $W' \in \Phi(p)$ because $\text{SL}(2)^n$ coincides with its commutator subgroup and

hence for any linear algebraic group H , the image of any homomorphism $\mathrm{SL}(2)^n \rightarrow H$ is contained in the commutator subgroup H' .

Let $W' \in \mathfrak{W}_0(G_{\mathbf{R}})$. We have a canonical real analytic map

$$\mathrm{spl}_{W'}^{\mathrm{BS}} : D \rightarrow \mathrm{spl}(W') := \mathrm{spl}_{\mathbf{R}}(W')$$

defined as follows. Let $\iota : \mathbf{G}_m \rightarrow G_{W',\mathrm{red}}^\circ$ be the canonical homomorphism in 3.3.7. Let $x \in D$. We have a Borel–Serre lifting $a^* : \mathbf{G}_{m,\mathbf{R}} \rightarrow (G')_{W',\mathbf{R}}^\circ$ of $\mathbf{G}_m \rightarrow (G')_{W',\mathrm{red}}^\circ$; $t \mapsto \iota(t)k_0(t)^{-1}$ associated to the image of x under $D \rightarrow \mathfrak{X}$. We define $\mathrm{spl}_{W'}^{\mathrm{BS}}(x)$ to be the \mathbf{R} -splitting of W' associated to $\mathbf{G}_{m,\mathbf{R}} \rightarrow G_{W',\mathrm{red}}^\circ$; $t \mapsto a^*(t)k_0(t)$.

Note that this splitting $\mathrm{spl}_{W'}^{\mathrm{BS}}(x)$ is also defined as follows. Let $V \in \mathrm{Rep}(G)$. Then for $H = x$, $V = H(V) = \bigoplus_w V_w$ where V_w is the part of V of weight w , and we have a polarization $\langle \cdot, \cdot \rangle_w$ on $V_{w,\mathbf{R}} = H(V_w)_{\mathbf{R}}$ in the condition (4.1) in Lemma 1.5.3. Then the splitting $\mathrm{spl}_{W'}^{\mathrm{BS}}(x)$ of $W'_{\bullet}V_{\mathbf{R}}$ is the direct sum for $w \in \mathbf{Z}$ of the orthogonal decomposition of $W'_{\bullet}V_{w,\mathbf{R}}$ with respect to the Hodge metric associated to $\langle \cdot, \cdot \rangle_w$. See 2.2.2.

An important fact is that, for $p \in D_{\mathrm{SL}(2)}$ of rank n , if \mathbf{r} is an element of the torus orbit of p and $1 \leq j \leq n$, the splitting of $W_p^{(j)}$ given by $\tau_{p,j} : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}}$ coincides with $\mathrm{spl}_{W_p^{(j)}}^{\mathrm{BS}}(\mathbf{r})$. See [23] Lemma 3.9. This fact is in the basis of the relation of $\mathrm{SL}(2)$ -orbits and Borel–Serre orbits considered in this paper and in our former papers [24], [21].

3.3.9. Let $\Phi \in \mathcal{W}_{\mathbf{R}}(G)$. Let $G_{\mathbf{R},\Phi} = \bigcap_{W' \in \Phi} G_{\mathbf{R},W'}$, where $G_{\mathbf{R},W'}$ denotes the stabilizer subgroup of W' in $G_{\mathbf{R}}$. By a *splitting* of Φ , we mean a homomorphism

$$\alpha : \mathbf{G}_{m,\mathbf{R}}^\Phi \rightarrow G_{\Phi,\mathbf{R}}$$

such that for each $W' \in \Phi$, the W' -component $\mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}}$ of α is a splitting of W' .

A splitting of Φ exists. In fact, if $\Phi = \Phi(x)$ for $x \in D_{\mathrm{SL}(2)}$, τ_x (3.3.2) is a splitting of Φ .

In the case $W \notin \Phi$ (resp. $W \in \Phi$), by a *distance to Φ -boundary*, we mean a real analytic map

$$\beta : D \rightarrow \mathbf{R}_{>0}^\Phi \quad (\text{resp. } D_{\mathrm{nspl}} \rightarrow \mathbf{R}_{>0}^\Phi)$$

such that $\beta(\alpha(t)x) = t\beta(x)$ for all splittings α of Φ , all $t \in \mathbf{R}_{>0}^\Phi$, and all $x \in D$ (resp. $x \in D_{\mathrm{nspl}}$).

Proposition 3.3.10. *A distance to Φ -boundary exists.*

This is a G -MHS version of [21] Part II 3.2.5. The proof given below is similar to that of loc. cit.

Proof. We first prove the case where G is reductive of 3.3.10. Take a submonoid V of $X(\mathbf{G}_m^\Phi)$ such that $V \cup V^{-1} = X(\mathbf{G}_m^\Phi)$, $V \supset X(\mathbf{G}_m^\Phi)_+$ and such that $V^\times \cap X(\mathbf{G}_m^\Phi)_+ = \{1\}$. Let $P = P_V$ be the parabolic subgroup of G' associated to V . Then $(G')_\Phi^\circ \subset P$ and, for every splitting α of Φ , the map $\mathbf{G}_{m,\mathbf{R}}^\Phi \rightarrow P_{\mathrm{red},\mathbf{R}}$ is injective and independent of the choice of α , and the image of this homomorphism is contained in S_P . Let $\mathbf{R}_{>0}^\Phi \rightarrow A_P$ be the induced injective map. Take a homomorphism of Lie groups $h : A_P \rightarrow \mathbf{R}_{>0}^\Phi$ which splits the last injection. Let $\pi : P(\mathbf{R}) \rightarrow A_P$ be the canonical map defined in 2.3.5, and let

$f : \mathfrak{X} \rightarrow A_P$ be the map there induced by π by fixing $K \in \mathfrak{X}$. Then the composition $D \rightarrow \mathfrak{X} \xrightarrow{f} A_P \xrightarrow{h} \mathbf{R}_{>0}^\Phi$ is a distance to Φ -boundary.

Now we consider the general case. Let n be the order of Φ_{red} . Let $\beta_{\text{red}} : D_{\text{red}} \rightarrow \mathbf{R}_{>0}^n$ be a distance to the Φ_{red} -boundary.

In the case $W \notin \Phi$, the composition $D \rightarrow D_{\text{red}} \xrightarrow{\beta_{\text{red}}} \mathbf{R}_{>0}^n$ is a distance to Φ -boundary.

Next we prove the case $W \in \Phi$ of 3.3.10. Fix $p_0 \in D_{\text{red}}$. Let Z be the center of $G_{\text{red}}(\mathbf{R})$. By the \mathbf{R} -polarizability, $K = \{g \in G_{\text{red}}(\mathbf{R})/Z \mid gp_0 = p_0\}$ is compact. For each integer $w \leq -2$, take a K -invariant positive symmetric \mathbf{R} -bilinear form $(\cdot, \cdot)_w$ on the part L_w of $L := \mathcal{L}(p_0)$. Define a map $f : L \setminus \{0\} \rightarrow \mathbf{R}_{>0}$ by $f(\delta) := (\sum_{w \leq -2} (\delta_w, \delta_w)^{-1/w})^{-1/2}$, where δ_w denotes the component of δ of weight w . For $p \in D_{\text{red}}$, if g is an element of $G_{\text{red}}(\mathbf{R})$ such that $p = gp_0$, then we have an isomorphism $\text{Ad}(g) : L = \mathcal{L}(p_0) \xrightarrow{\sim} \mathcal{L}(p)$. The map $f_p : \mathcal{L}(p) \setminus \{0\} \rightarrow \mathbf{R}_{>0}, \delta \mapsto f(\text{Ad}(g)^{-1}\delta)$ is independent of the choice of g . This is because $(g')^{-1}g \in K$ if $g, g' \in G_{\text{red}}(\mathbf{R})$ and $gp_0 = g'p_0$. Define $\gamma' : D_{\text{nspl}} \rightarrow \mathbf{R}_{>0}$ by sending an element corresponding to (p, s, δ) ($p \in D_{\text{red}}, s \in \text{spl}(W), \delta \in \mathcal{L}(p) \setminus \{0\}$) to $f_p(\delta)$. For $x \in D_{\text{nspl}}$, define $\gamma(x) = \gamma'(x) \prod_{j=1}^n \beta_{\text{red},j}(x_{\text{red}})^{-1}$. Then $(\gamma, \beta_{\text{red}}) : D_{\text{nspl}} \rightarrow \mathbf{R}_{>0} \times \mathbf{R}_{>0}^n$ is a distance to Φ -boundary. \square

3.4 The real analytic structures of $D_{\text{SL}(2)}$ and $D_{\text{SL}(2)}^*$

In this Section 3.4, we endow $D_{\text{SL}(2)}$ and $D_{\text{SL}(2)}^*$ with the sheaves of real analytic functions and with the log structures with sign. In fact, like in Part II of [21], $D_{\text{SL}(2)}$ is endowed with two kinds of such structures, $D_{\text{SL}(2)}^I$ and $D_{\text{SL}(2)}^{II}$.

A basic property of these structures is that in the case where G is reductive, if (ρ, φ) is an $\text{SL}(2)$ -orbit in n variables with class p in $D_{\text{SL}(2)}$, $\varphi(iy_1, \dots, iy_n)$ converges to p when $y_j/y_{j+1} \rightarrow \infty$ for $1 \leq j \leq n$ (y_{n+1} denotes 1).

We first review the category $\mathcal{B}'_{\mathbf{R}}(\log)$ in 3.4.1–3.4.3. The above spaces with the above structures become objects of $\mathcal{B}'_{\mathbf{R}}(\log)$

3.4.1. The categories $\mathcal{B}'_{\mathbf{R}}$ and $\mathcal{B}'_{\mathbf{R}}(\log)$ (see [21] Part IV 1.3).

Let $\mathcal{B}'_{\mathbf{R}}$ be the category of locally ringed spaces S over \mathbf{R} satisfying the following condition (i) locally on S . Endow \mathbf{R}^n with the sheaf $\mathcal{O}_{\mathbf{R}^n}$ of real analytic functions.

(i) There are $n \geq 0$ and a morphism $\iota : S \rightarrow \mathbf{R}^n$ of locally ringed spaces over \mathbf{R} such that ι is injective, the topology of S coincides with the topology induced from that of \mathbf{R}^n , and the map $\iota^{-1}(\mathcal{O}_{\mathbf{R}^n}) \rightarrow \mathcal{O}_S$ is surjective.

For an object S of $\mathcal{B}'_{\mathbf{R}}$, we often call the structural sheaf \mathcal{O}_S the *sheaf of real analytic functions* on S (though S need not be a real analytic space).

For an object S of $\mathcal{B}'_{\mathbf{R}}$, a *log structure with sign* on S means a log structure M on S endowed with a submonoid sheaf $M_{>0}$ of M satisfying the following (i) and (ii).

(i) $M_{>0} \supset \mathcal{O}_{S, >0}^\times$. Here $\mathcal{O}_{S, >0}^\times$ denotes the subgroup sheaf of \mathcal{O}_S^\times consisting of all local sections whose values are > 0 .

(ii) The map $M_{>0} \times \{\pm 1\} \rightarrow M$; $(f, \epsilon) \mapsto \epsilon f$ is an isomorphism of sheaves. Here we regard $\{\pm 1\} \subset \mathcal{O}_S^\times \subset M$.

Let $\mathcal{B}'_{\mathbf{R}}(\log)$ be the category of objects of $\mathcal{B}'_{\mathbf{R}}$ endowed with an fs log structure with sign.

Real analytic manifolds with corners are regarded as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$ ([21] Part IV 1.3.8 (2)).

The category $\mathcal{B}'_{\mathbf{R}}(\log)$ has fiber products (ibid. Proposition 1.3.11 (1)).

3.4.2. For an object X of $\mathcal{B}'_{\mathbf{R}}(\log)$, for an object Y of $\mathcal{B}'_{\mathbf{R}}$ and for a morphism $Y \rightarrow X$ in $\mathcal{B}'_{\mathbf{R}}$, we have an fs log structure with sign $(M_Y, M_{Y,>0})$ on Y , called the *inverse image* of the log structure with sign $(M_X, M_{X,>0})$ of X , defined as follows. The log structure M_Y is the inverse image of the log structure M_X , that is, M_Y is the push-out of $\mathcal{O}_Y^\times \leftarrow f^{-1}(\mathcal{O}_X^\times) \rightarrow f^{-1}(M_X)$ in the category of sheaves of commutative monoids on Y , and $M_{Y,>0}$ is the push-out of $\mathcal{O}_{Y,>0}^\times \leftarrow f^{-1}(\mathcal{O}_{X,>0}^\times) \rightarrow f^{-1}(M_{X,>0})$ in the category of sheaves of commutative monoids on Y .

3.4.3. ([21] Part IV 1.3.16.) Let X be an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ satisfying the following condition (C).

(C) The canonical map from \mathcal{O}_X to the sheaf of \mathbf{R} -valued functions on X is injective.

Let Y be a subset of X . Then Y has a structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ satisfying (C) such that for any object S of $\mathcal{B}'_{\mathbf{R}}(\log)$ satisfying (C), $\text{Mor}_{\mathcal{B}'_{\mathbf{R}}(\log)}(S, Y)$ is identified with $\{f \in \text{Mor}_{\mathcal{B}'_{\mathbf{R}}(\log)}(S, X) \mid f(S) \subset Y\}$. This structure on Y is defined as follows. The topology is the one as a subspace of X . \mathcal{O}_Y is the image of the canonical map from the pullback of \mathcal{O}_X on Y to the sheaf of \mathbf{R} -valued functions on Y . The log structure with sign of Y is the inverse image of that of X (3.4.2).

We will say this is the *structure of Y as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ induced by the injection $Y \rightarrow X$* (or by the embedding of Y in X).

In the rest of this Section 3.4, we state definitions and properties of the structures $D_{\text{SL}(2)}^I$ and $D_{\text{SL}(2)}^{II}$ of $D_{\text{SL}(2)}$ and the structure of $D_{\text{SL}(2)}^*$ as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$. The almost all proofs will be given in Section 3.5.

We first consider the case where G is reductive. In this case, $D_{\text{SL}(2)}^I = D_{\text{SL}(2)}^{II} = D_{\text{SL}(2)}^*$.

Proposition 3.4.4. *Assume that G is reductive. Let $\Phi \in \mathcal{W}(G)$ (3.3.5). Take a splitting α of Φ and a distance β to Φ -boundary (3.3.9).*

(1) *There is a unique map*

$$\nu_{\alpha,\beta} : D_{\text{SL}(2)}(\Phi) \rightarrow \mathbf{R}_{\geq 0}^\Phi \times D \times \prod_{W' \in \Phi} \text{spl}(W')$$

satisfying the following conditions (i) and (ii).

- (i) *For $p \in D$, $\nu_{\alpha,\beta}(p) = (\beta(p), \alpha\beta(p)^{-1}p, (\text{spl}_{W'}^\text{BS}(p))_{W' \in \Phi}) \in \mathbf{R}_{\geq 0}^\Phi \times D \times \prod_{W' \in \Phi} \text{spl}(W')$.*
- (ii) *For $p \in D_{\text{SL}(2)}(\Phi)$, $\nu_{\alpha,\beta}(p)$ is the limit of $\nu_{\alpha,\beta}(\tau_p(t)r)$ for a fixed point r of the torus orbit (3.1.12) of p and for $t \in \mathbf{R}_{>0}^{\Phi(p)}$ which tends to $(0, \dots, 0) \in \mathbf{R}_{\geq 0}^{\Phi(p)}$.*

(2) *This map $\nu_{\alpha,\beta}$ is injective.*

(3) *The structure of $D_{\text{SL}(2)}(\Phi)$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ induced via the injection $\nu_{\alpha,\beta}$ (3.4.3) is independent of the choice of (α, β) .*

We will denote this map $\nu_{\alpha,\beta}$ as $p \mapsto (\beta(p), b_{\alpha,\beta}(p), (\text{spl}_{W'}(p))_{W'})$.

Next we consider $D_{\text{SL}(2)}^{II}$ and $D_{\text{SL}(2)}^*$ for a general linear algebraic group G over \mathbf{Q} .

Proposition 3.4.5. *Let $\Phi \in \mathcal{W}(G_{\text{red}})$. Let $D_{\text{SL}(2)}(\Phi)$ (resp. $D_{\text{SL}(2)}^*(\Phi)$) be the inverse image of $D_{\text{red},\text{SL}(2)}(\Phi)$ in $D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$).*

Let α be a splitting of Φ and let β be a distance to Φ -boundary (3.3.9).

(1) *There is a unique map*

$$\nu_{\alpha,\beta} : D_{\text{SL}(2)}(\Phi) \rightarrow D_{\text{red},\text{SL}(2)}(\Phi) \times \text{spl}(W) \times \overline{\mathcal{L}}$$

$$(\text{resp. } \nu_{\alpha,\beta}^* : D_{\text{SL}(2)}^*(\Phi) \rightarrow D_{\text{red},\text{SL}(2)}^*(\Phi) \times \text{spl}(W) \times \overline{\mathcal{L}})$$

satisfying the following conditions (i) and (ii).

(i) *For $x \in D$, set $p = x_{\text{red}}$. Then,*

$$\nu_{\alpha,\beta}(x) = (p, \text{spl}_W(x), \text{Ad}(\alpha\beta(p))^{-1}\delta(x))$$

$$(\text{resp. } \nu_{\alpha,\beta}^*(x) = (p, \text{spl}_W(x), \text{Ad}(\alpha^*\beta(p))^{-1}\delta(x))).$$

(ii) *Let $x = (p, Z) \in D_{\text{SL}(2)}(\Phi)$ (resp. $D_{\text{SL}(2)}^*(\Phi)$) (3.2.1). Then $I\nu_{\alpha,\beta}(x)$ (resp. $\nu_{\alpha,\beta}^*(x)$) is the limit of $\nu_{\alpha,\beta}(\tau_x(t)r)$ (resp. $\nu_{\alpha,\beta}^*(\tau_x^*(t)r)$) for a fixed point r of Z , where $t \in \mathbf{R}_{>0}^{\Phi(x)}$ and $t \rightarrow (0, \dots, 0) \in \mathbf{R}_{\geq 0}^{\Phi(x)}$.*

(2) *This map $\nu_{\alpha,\beta}$ (resp. $\nu_{\alpha,\beta}^*$) induces the bijection*

$$D_{\text{SL}(2)}(\Phi) \xrightarrow{\sim} \{(p, s, \delta) \in D_{\text{red},\text{SL}(2)}(\Phi) \times \text{spl}(W) \times \overline{\mathcal{L}} \mid \delta \in \overline{\mathcal{L}}(b_{\alpha,\beta}(p))\}$$

$$(\text{resp. } D_{\text{SL}(2)}^*(\Phi) \xrightarrow{\sim} \{(p, s, \delta) \in D_{\text{red},\text{SL}(2)}^*(\Phi) \times \text{spl}(W) \times \overline{\mathcal{L}} \mid \delta \in \overline{\mathcal{L}}(b_{\alpha,\beta}(p))\}).$$

Here $b_{\alpha,\beta}(p)$ is as after Proposition 3.4.4. The ones for $\nu_{\alpha,\beta}$ and for $\nu_{\alpha,\beta}^*$ coincide.

(3) *The structure of $D_{\text{SL}(2)}(\Phi)$ (resp. $D_{\text{SL}(2)}^*(\Phi)$) as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ induced via the bijection $\nu_{\alpha,\beta}$ (resp. $\nu_{\alpha,\beta}^*$) is independent of the choice of (α, β) .*

In the bijection in the above (2), the image of $x \in D_{\text{SL}(2)}(\Phi)$ (resp. $x \in D_{\text{SL}(2)}^*(\Phi)$) belongs to \mathcal{L} if and only if x is an A -orbit.

For $D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$), this Proposition 3.4.5 is a G -MHS version of [21] Part II Proposition 3.2.6 (ii), Proposition 3.2.7 (ii), and the situation (c) of [21] Part IV Proposition 2.3.9 (resp. of the situation (a) of [21] Part IV Proposition 2.3.9) which treated the extended period domain $D_{\text{SL}(2)} \supset D(\Lambda)$ (resp. $D_{\text{SL}(2)}^* \supset D(\Lambda)$) for a classical period domain $D(\Lambda)$ (1.6).

Remark 3.4.6. This is a remark on the description of a related part of [21]. There are fourteen “resp.,”s in [21] Part IV 2.3.8, which are divided into two groups, and correspond only to the ones belonging to the same group. Precisely, the fourth, the fifth, the seventh, and the last “resp.,”s correspond, and the other ten “resp.,”s correspond. But four and ten do not correspond.

Proposition 3.4.7. *$D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$) has a unique structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ such that for every $\Phi \in \mathcal{W}(G_{\text{red}})$, $D_{\text{SL}(2)}(\Phi)$ (resp. $D_{\text{SL}(2)}^*(\Phi)$) is open in $D_{\text{SL}(2)}$ and the induced structure on it coincides with the structure in Proposition 3.4.5 (3).*

The $D_{\mathrm{SL}(2)}$ as an object $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$ with the above structure is denoted by $D_{\mathrm{SL}(2)}^{II}$.

For $D_{\mathrm{SL}(2)}$ (resp. $D_{\mathrm{SL}(2)}^*$), this is the G -MHS version of the situation (c) (resp. (a)) of [21] Part IV Proposition 2.3.10 for $D_{\mathrm{SL}(2)}$ (resp. $D_{\mathrm{SL}(2)}^*$) extending the classical period domain. For $D_{\mathrm{SL}(2)}$ of ibid., [21] Part II Theorem 3.2.10 also gives the same structure of $D_{\mathrm{SL}(2)}^{II}$.

Proposition 3.4.8. (1) *The map $D_{\mathrm{SL}(2)}^{II} \rightarrow D_{\mathrm{red},\mathrm{SL}(2)} \times \mathrm{spl}(W)$ is proper. It is an \bar{L} -bundle.*

(2) *The map $D_{\mathrm{SL}(2)}^* \rightarrow D_{\mathrm{red},\mathrm{SL}(2)} \times \mathrm{spl}(W)$ is proper. It is an \bar{L} -bundle.*

For $D_{\mathrm{SL}(2)}^{II}$ (resp. $D_{\mathrm{SL}(2)}^*$), this is a G -MHS version of [21] Part II Theorem 3.5.15 (resp. the situation (a) of [21] Part IV Proposition 2.3.16).

We give a proof of Proposition 3.4.8 assuming Proposition 3.4.7. Let $p \in D_{\mathrm{red},\mathrm{SL}(2)}(\Phi)$, $r := b_{\alpha,\beta}(p) \in D_{\mathrm{red}}$. Take an open neighborhood V of r in D_{red} and a real analytic map $g : V \rightarrow G_{\mathrm{red}}(\mathbf{R})$ such that $g(r) = 1$ and such that $v = g(v)r$ for all $v \in V$. Let $U \subset D_{\mathrm{red},\mathrm{SL}(2)}(\Phi)$ be the inverse of V under $b_{\alpha,\beta} : D_{\mathrm{red},\mathrm{SL}(2)}(\Phi) \rightarrow D_{\mathrm{red}}$ and let \tilde{U} be the inverse image of U under the projection $D_{\mathrm{SL}(2)}(\Phi) \rightarrow D_{\mathrm{red},\mathrm{SL}(2)}(\Phi)$ (resp. $D_{\mathrm{SL}(2)}^*(\Phi) \rightarrow D_{\mathrm{red},\mathrm{SL}(2)}(\Phi)$). Then we have an isomorphism

$$\tilde{U} \xrightarrow{\sim} U \times \mathrm{spl}(W) \times \overline{\mathcal{L}}(r) ; x \mapsto (x_{\mathrm{red}}, \mathrm{spl}_W(x), \mathrm{Ad}(g(b_{\alpha,\beta}(x_{\mathrm{red}})))^{-1} \delta(b_{\alpha,\beta}(x_{\mathrm{red}}))).$$

We next consider the structure $D_{\mathrm{SL}(2)}^I$ of $D_{\mathrm{SL}(2)}$.

Proposition 3.4.9. *Let $\Phi \in \mathcal{W}(G)$.*

(1) *Let α be a splitting of Φ and let β be a distance to Φ -boundary. If $W \notin \Phi$ (resp. $W \in \Phi$), there is a unique map*

$$\mu_{\alpha,\beta} : D_{\mathrm{SL}(2)}(\Phi) \rightarrow D \quad (\text{resp. } D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \rightarrow D)$$

satisfying the following conditions (i) and (ii).

(i) *If $x \in D$ (resp. $x \in D_{\mathrm{nspl}}$),*

$$\mu_{\alpha,\beta}(x) = \alpha(\beta(x))^{-1}x.$$

(ii) *Assume $W \notin \Phi$ (resp. $W \in \Phi$) and let $x = (p, Z)$ be an element of $D_{\mathrm{SL}(2)}(\Phi)$ (resp. $D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi)$). Let $z \in Z$. Then $\mu_{\alpha,\beta}(x)$ is the limit of $\mu_{\alpha,\beta}(\tau_x(t)z)$ where $t \in \mathbf{R}_{>0}^{\Phi(x)}$ and $t \rightarrow (0, \dots, 0) \in \mathbf{R}_{\geq 0}^{\Phi}$.*

(2) *If $W \notin \Phi$ (resp. $W \in \Phi$), the structure of $D_{\mathrm{SL}(2)}(\Phi)$ (resp. $D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi)$) as an object of $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$ induced by the injection*

$$D_{\mathrm{SL}(2)}(\Phi) \text{ (resp. } D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi)) \rightarrow D_{\mathrm{SL}(2)}^{II} \times D ; x \mapsto (x, \mu_{\alpha,\beta}(x))$$

(3.4.3) *is independent of the choice of (α, β) .*

This is a G -MHS version of [21] Part II Proposition 3.2.6 (i), Proposition 3.2.7 (i).

Proposition 3.4.10. *$D_{\mathrm{SL}(2)}$ has a unique structure of an object of $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$ such that for every $\Phi \in \mathcal{W}(G)$ such that $W \notin \Phi$ (resp. $W \in \Phi$), $D_{\mathrm{SL}(2)}(\Phi)$ (resp. $D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi)$) is open in $D_{\mathrm{SL}(2)}$ and the induced structure on this coincides with that in Proposition 3.4.9 (2).*

This generalizes [21] Part II Theorem 3.2.10 (i).

The structure of $D_{\mathrm{SL}(2)}$ in Proposition 3.4.10 as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ is denoted by $D_{\mathrm{SL}(2)}^I$.

The identity map of $D_{\mathrm{SL}(2)}$ is a morphism $D_{\mathrm{SL}(2)}^I \rightarrow D_{\mathrm{SL}(2)}^{II}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$. The log structure with sign of $D_{\mathrm{SL}(2)}^I$ is the inverse image of that of $D_{\mathrm{SL}(2)}^{II}$.

To prove the existence of the global structures Propositions 3.4.7 and 3.4.10, we need the following Proposition 3.4.12 and the local descriptions in Theorems 3.4.15, 3.4.16, and 3.4.18 which are proved by using Proposition 3.4.12.

3.4.11. Assume that G is reductive. Let $p \in D_{\mathrm{SL}(2)}$. We define the subset $U(p)$ of $D_{\mathrm{SL}(2)}$ in the same way as in [24] Section 10.2, as follows. Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit belonging to p . For each $J \subset \Phi(p) = \{1, \dots, n\}$, we define $p_J \in D_{\mathrm{SL}(2)}$ as the class of the $\mathrm{SL}(2)$ -orbit (ρ_J, φ_J) in m -variables, where m is the cardinality of J , defined as follows. $\rho_J(g_1, \dots, g_m) := \rho(h_1, \dots, h_n)$, $\varphi_J(z_1, \dots, z_m) := \varphi(w_1, \dots, w_n)$, where h_j and w_j ($1 \leq j \leq n$) are as follows. Write $J = \{s_1, \dots, s_m\} \subset \{1, \dots, n\}$, $s_1 < \dots < s_m$. If $j \leq s_k$ for some k , define $h_j := g_k$ and $w_j := z_k$ for the smallest integer k with $j \leq s_k$. Otherwise, $h_j := 1$ and $w_j := i$. We have $\Phi(p_J) = J$.

Let

$$U(p) := \bigcup_{J \subset \Phi(p)} G'_J(\mathbf{R}) \cdot p_J \subset D_{\mathrm{SL}(2)}(\Phi(p)),$$

where G'_J is the stabilizer of the weight filtrations $W^{(j)}$ ($j \in J$) in G' .

The following proposition is a generalization of [24] Theorem 10.2.2.

Proposition 3.4.12. *The set $U(p)$ is open in $D_{\mathrm{SL}(2)}$.*

In the case $p \in D$, Proposition 3.4.12 follows from the case where G is reductive of (1) of the following Lemma 3.4.13.

Lemma 3.4.13. (1) *D is a finite disjoint union of $G'(\mathbf{R})G_u(\mathbf{C})$ -orbits which are open and closed.*

(2) *\check{D} is a finite disjoint union of $G'(\mathbf{C})$ -orbits which are open and closed.*

Proof. These are reduced to the case where G is reductive. Assuming that G is reductive, let Z be the center of G . Then these are reduced to the facts that $G'(\mathbf{R})Z(\mathbf{R})$ is an open (and hence closed) subgroup of $G(\mathbf{R})$ of finite index and $G'(\mathbf{C})Z(\mathbf{C})$ is an open (and hence closed) subgroup of $G(\mathbf{C})$ of finite index. \square

3.4.14. Assume that G is reductive. Let $p \in D_{\mathrm{SL}(2)}$, let n be the rank of p , and take a representative (ρ, φ) of p . Let $\mathbf{r} = \varphi(\mathbf{i}) \in D$. We have $\mathrm{Lie}(G'_{\mathbf{R}}) = \bigoplus_{m \in \mathbf{Z}^n} \mathrm{Lie}(G'_{\mathbf{R}})_m$, where

$$\mathrm{Lie}(G'_{\mathbf{R}})_m = \{x \in \mathrm{Lie}(G'_{\mathbf{R}}) \mid \mathrm{Ad}(\tau_p(t))x = \prod_{j=1}^n t_j^{m(j)}x \text{ for all } t \in \mathbf{R}_{>0}^{\Phi}\}.$$

For $x \in \mathrm{Lie}(G'_{\mathbf{R}})$, let $x_m \in \mathrm{Lie}(G'_{\mathbf{R}})_m$ be the m -component of x .

Let $K_{\mathbf{r}}$ be the maximal compact subgroup of $G'(\mathbf{R})$ associated to \mathbf{r} . Let $'K_{\mathbf{r}}$ be the stabilizer of \mathbf{r} in $K_{\mathbf{r}}$. Take \mathbf{R} -subspaces $R \subset \text{Lie}(G'_{\mathbf{R}})$ and $S \subset \text{Lie}(K_{\mathbf{r}})$ such that

$$\text{Lie}(G'_{\mathbf{R}}) = \text{Lie}(\mathbf{R}_{>0}^n) \oplus \text{Lie}(K_{\mathbf{r}}) \oplus R, \quad \text{Lie}(K_{\mathbf{r}}) = \text{Lie}('K_{\mathbf{r}}) \oplus S,$$

where $\mathbf{R}_{>0}^n$ is embedded in $G'(\mathbf{R})$ via τ_p , and such that

$$R = \sum_{m \in \mathbf{Z}^\Phi} R \cap (\text{Lie}(G'_{\mathbf{R}})_m \oplus \text{Lie}(G'_{\mathbf{R}})_{-m}).$$

Such R and S exist.

Let Y be the subset of $\mathbf{R}_{>0}^n \times \text{Lie}(G'_{\mathbf{R}}) \times \text{Lie}(G'_{\mathbf{R}}) \times \text{Lie}(G'_{\mathbf{R}}) \times S$ consisting of all elements (t, f, g, h, k) satisfying the following conditions (i)–(iv). Let $J = \{j \mid t_j = 0\} \subset \{1, \dots, n\}$.

- (i) Let $m \in \mathbf{Z}^\Phi$. Then: $g_m = 0$ unless $m(j) = 0$ for all $j \in J$. $f_m = 0$ unless $m(j) \leq 0$ for all $j \in J$. $h_m = 0$ unless $m(j) \geq 0$ for all $j \in J$.
- (ii) Let t' be any element of $\mathbf{R}_{>0}^n$ such that $t'_j = t_j$ for all $j \in \{1, \dots, n\} \setminus J$. If $m \in \mathbf{Z}^n$ and $m(j) = 0$ for all $j \in J$, then $f_m = (\prod_{j=1}^n (t'_j)^{m(j)}) g_m$ and $h_m = (\prod_{j=1}^n (t'_j)^{m(j)})^{-1} g_m$.
- (iii) We have $g \in R$ and we have $f_m + h_{-m} \in R$ for all $m \in \mathbf{Z}^n$.
- (iv) $\exp(k)\mathbf{r} \in G'_J(\mathbf{R}) \cdot \mathbf{r}$.

Let $Y_0 = \{(t, f, g, h, k) \in Y \mid t \in \mathbf{R}_{>0}^n\}$. Then

$$Y_0 \xrightarrow{\sim} \mathbf{R}_{>0}^n \times R \times S ; (t, f, g, h, k) \mapsto (t, g, k).$$

Hence by Lemma 3.4.13, the map

$$Y_0 \rightarrow D ; (t, f, g, h, k) \mapsto \tau_p(t) \exp(g) \exp(k) \mathbf{r} = \exp(f) \tau_p(t) \exp(k) \mathbf{r}$$

is an open map.

The following Theorems 3.4.15 and 3.4.16 are variants of [21] Part II Theorem 3.4.4 and [21] Part IV Theorem 2.3.14.

Theorem 3.4.15. *Assume that G is reductive. Let the notation be as in 3.4.14. Then there exists an open neighborhood U of $(0, \dots, 0)$ in Y and an open immersion $U \rightarrow D_{\text{SL}(2)}$ which sends $(0, \dots, 0)$ to p such that if U_0 denotes the subset of U consisting of (t, f, g, h, k) such that $t \in \mathbf{R}_{>0}^n$, then it sends $(t, f, g, h, k) \in U_0$ to $\tau_p(t) \exp(g) \exp(k) \mathbf{r}$.*

We return to the general G not necessarily reductive, that is, in the next theorem, G is a (general) algebraic linear group over \mathbf{Q} .

Theorem 3.4.16. *Let $x \in D_{\text{SL}(2)}$ (resp. $D_{\text{SL}(2)}^*$) and let p be the image of x in $D_{\text{red}, \text{SL}(2)}$. Take the spaces Y , U , U_0 in Theorem 3.4.15 for G_{red} and p , and denote them by Y_{red} , U_{red} , and $U_{\text{red}, 0}$, respectively. Let V_{red} be the image of the open immersion $U_{\text{red}} \rightarrow D_{\text{red}, \text{SL}(2)}$ in Theorem 3.4.15. Let $L = \mathcal{L}(\mathbf{r})$. Let V be the inverse image of V_{red} in $D_{\text{SL}(2)}^H$ (resp. $D_{\text{SL}(2)}^*$). Then there is an open immersion $U := U_{\text{red}} \times \text{spl}(W) \times \bar{L} \rightarrow V$ which sends $(t, f, g, h, k, s, \delta) \in U_{\text{red}, 0} \times \text{spl}(W) \times L$ to the element of D whose image in $D_{\text{red}} \times \text{spl}(W) \times \mathcal{L}$ is*

$$(\tau_p(t) \exp(g) \exp(k) \mathbf{r}, s, \text{Ad}(\tau_p(t) \exp(g) \exp(k)) \delta)$$

(resp. $(\tau_p^*(t) \exp(g) \exp(k)\mathbf{r}, s, \text{Ad}(\tau_p^*(t) \exp(g) \exp(k))\delta)$)

such that the diagram

$$\begin{array}{ccc} U & \rightarrow & V \\ \downarrow & & \downarrow \\ U_{\text{red}} & \rightarrow & V_{\text{red}} \end{array}$$

is cartesian.

3.4.17. We next consider the local property of $D_{\text{SL}(2)}^I$. Let $x = (p, Z) \in D_{\text{SL}(2)}$. Take $z \in Z$ and let $s_0 = \text{spl}_W(z) \in \text{spl}(W)$. Then s_0 is independent of the choice of z . Let $\mathbf{r} = z_{\text{red}} \in D_{\text{red}}$. Let $\delta_0 = \delta_W(z) \in L := \mathcal{L}(\mathbf{r})$. Let n be the rank of p .

Let $E = L$ in the case where x is an A -orbit, and let $E = \bar{L} \setminus \{0\}$ in the case where x is a B -orbit.

Take the space Y in Theorem 3.4.14 for G_{red} and p , and denote it by Y_{red} here. In the case where x is a B -orbit, take a real analytic map $\beta_0 : L \setminus \{0\} \rightarrow \mathbf{R}_{>0}$ such that $\beta_0(c \circ \delta) = c\beta_0(\delta)$ for all $c \in \mathbf{R}_{>0}$ and $\delta \in L \setminus \{0\}$. We define a subset X of $Y_{\text{red}} \times \text{Lie}(G_{\mathbf{R},u}) \times \text{Lie}(G_{\mathbf{R},u}) \times E$ as the set of $(t, f, g, h, k, u, v, \delta)$ ($(t, f, g, h, k) \in Y_{\text{red}}$, $u, v \in \text{Lie}(G_{\mathbf{R},u})$, $\delta \in E$) satisfying the following conditions (i)–(iv). Let n be the rank of p and let $J := \{j \mid 1 \leq j \leq n, t_j = 0\}$, $m \in \mathbf{Z}^n$.

- (i) $u_m = 0$ unless $m(j) \leq 0$ for all $j \in J$.
- (ii) $v_m = 0$ unless $m(j) = 0$ for all $j \in J$.
- (iii) If $\delta \in \bar{L} \setminus L$ (this happens only in the case where x is a B -orbit), then $v = 0$.
- (iv) Assume $m(j) = 0$ for all $j \in J$. Let t' be an element of $\mathbf{R}_{>0}^n$ such that $t'_j = t_j$ if $j \in \{1, \dots, n\} \setminus J$. Then if x is an A -orbit, we have $v_m = \sum_{j=1}^n (t'_j)^{-m(j)} u_m$. If x is a B -orbit and $\delta \in L \setminus \{0\}$, we have $v_m = \text{Ad}(\tau_{x,0}(\beta_0(\delta)))^{-1} \prod_{j=1}^n (t'_j)^{-m(j)} u_m$.

The following Theorem 3.4.18 is a variant of [21] Part II Theorem 3.4.6.

Theorem 3.4.18. *There are an open neighborhood V of $(0, \dots, 0)$ in $Y \times \text{Lie}(G_{\mathbf{R},u}) \times \text{Lie}(G_{\mathbf{R},u})$ and an open immersion $U \rightarrow D_{\text{SL}(2)}^I$, where U is the inverse image of V in X , which sends $(0, \dots, \delta_0)$ (resp. $(0, \dots, 0, 0 \circ \delta)$) to x such that if U_0 denotes the subset of U consisting of all $(t, f, g, h, k, u, v, \delta)$ such that $t \in \mathbf{R}_{>0}^{\Phi}$ and $\delta \in L$, it sends $(t, f, g, h, k, u, v, \delta) \in U_0$ to the element of D whose image in $D_{\text{red}} \times \text{spl}(W) \times \mathcal{L}$ is*

$$(\tau_p(t) \exp(g) \exp(k)\mathbf{r}, \exp(u)s_0, \text{Ad}(\tau_p(t) \exp(g) \exp(k))\delta).$$

Remark 3.4.19. In [21] Part II 3.4.5, in the definition of the space $Y^I(p, \mathbf{r}, R, S)$, the last sentence “If $t_0 = 0, \dots$ ” in the condition (6') should be deleted. After this modification, Part II Theorem 3.4.6 becomes correct. In our series of papers [21], Part II Theorem 3.4.6 is used in Part IV 2.7.16, but Part IV 2.7.16 is correct with the modified Part II Theorem 3.4.6.

Another remark concerning Part II 3.4.5 is that the presentation of (5') can be simplified: The condition “ $\exp(v)s_{\mathbf{r}} = s_{\mathbf{r}}$ ” there is equivalent to a simpler one “ $v = 0$ ”.

3.4.20. We can show that our $\text{SL}(2)$ -spaces belong to the category $\mathcal{B}'_{\mathbf{R}}(\text{log})^+$ of nice objects in $\mathcal{B}'_{\mathbf{R}}(\text{log})$ as in [21] Part IV Section 2.7, though the details are omitted.

3.5 Proofs for Section 3.4

We give proofs of the statements in Section 3.4 that have not yet proved.

3.5.1. The proofs of Proposition 3.4.4 for $\Phi \in \mathcal{W}(G)$, Proposition 3.4.5 for $\Phi \in \mathcal{W}(G_{\text{red}})$, and Proposition 3.4.9 for $\Phi \in \mathcal{W}(G)$ are similar to the corresponding parts in [21] Part II and Part IV, which are indicated after each proposition.

3.5.2. The proofs for the remaining results in Section 3.4 are also parallel to the corresponding parts in [21] Part II and Part IV.

That is, by Propositions 3.4.4, 3.4.5, and 3.4.9 for respective Φ already proved, we first endow the Φ -parts of each sets $D_{\text{SL}(2)}$ and $D_{\text{SL}(2)}^*$ with the space structures. Then we consider the following version of Proposition 3.4.12 for Φ -part:

(1) $U(p)$ in 3.4.11 is open in $D_{\text{SL}(2)}(\Phi)$ if $\Phi(p) \subset \Phi$.

We prove this (1) in 3.5.5 after preparations 3.5.3 and Lemma 3.5.4.

3.5.3. Let $\Phi \in \mathcal{W}(G)$. Fix a splitting $\alpha : \mathbf{G}_{m,\mathbf{R}}^\Phi \rightarrow G_{\Phi,\mathbf{R}}$ of Φ and a distance to Φ -boundary $\beta : D \rightarrow \mathbf{R}_{>0}^\Phi$. Let J be a subset of Φ . Let n be the order of Φ and let m be the order of J . Let $\alpha_J : \mathbf{G}_{m,\mathbf{R}}^J \rightarrow G_{J,\mathbf{R}}$ be the splitting of J defined as follows. If we write the inclusion map $J \rightarrow \Phi$ between totally ordered sets as an injective increasing map $\theta : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, then $\alpha_J(t_1, \dots, t_m) = \alpha(t'_1, \dots, t'_n)$, where: If $j \leq \theta(k)$ for some k , define $t'_j := t_{\theta(k)}$ for the smallest integer k such that $j \leq \theta(k)$. Otherwise, $t'_j := 1$.

Let P be the subset of $D_{\text{SL}(2)}$ consisting of all elements p such that $\Phi(p) = J$ and $\tau_p = \alpha_J$. Let Q be the set of $\text{SL}(2)$ -orbits in m variables whose classes belong to P . We define a map

$$\theta_{\alpha,\beta} : P \rightarrow Q$$

as follows. (There is an evident projection $Q \rightarrow P$, but the composition $P \xrightarrow{\theta_{\alpha,\beta}} Q \rightarrow P$ need not be the identity map.)

Let $p \in P$. Take an $\text{SL}(2)$ -orbit (ρ, φ) with class p . Then $\theta_{\alpha,\beta}(p)$ is the $\text{SL}(2)$ -orbit (ρ', φ') whose class p' satisfies $\tau_{p'} = \tau_p = \alpha_J$ such that the N_j of ρ' (denote it by N'_j) for $1 \leq j \leq m$ is defined by using $N_j := (N_j \text{ of } \rho)$ as $N'_j := \text{Ad}(\alpha(\beta(\mathbf{r})))^{-1}(N_j)$, where $\mathbf{r} = \varphi(\mathbf{i})$, and such that φ' is defined as $\varphi'(z) = \alpha(\beta(\mathbf{r}))^{-1}\varphi(z)$. Then (ρ', φ') depends only on p and is independent of the choice of (ρ, φ) .

Lemma 3.5.4. *Let E be a field of characteristic 0 and let V be a finite-dimensional vector space over E on which the Lie algebra $\mathfrak{sl}(2, E)^m$ acts. Denote the action of $X \in \mathfrak{sl}(2, E)^m$ on V by $[X, \cdot]$. For $1 \leq j \leq m$, let $N_j \in \mathfrak{sl}(2, E)^m$ be the element whose j -th component is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and whose k -th components are 0 for all $k \neq j$, and let $Y_j \in \mathfrak{sl}(2, E)^m$ be the element whose j -th component is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and whose k -th components are 0 for all $k \neq j$. Let B_j be the set of all elements v of V such that $[Y_j, v] = -2v$, $[Y_k, v] = 0$ for all $k \neq j$, and $[N_k, v] = 0$ for all $k \neq j$. Let $S_j \in B_j$ for $1 \leq j \leq m$. Then there is an element $v \in V$ such that $[Y_j, v] = 0$ for all j and such that $[N_j, v] = S_j$ for all j .*

Proof. Let A_j be the set of all elements v of V such that $[Y_k, v] = 0$ for all k and $[N_k, v] = 0$ for all $k \neq j$. For a finite-dimensional E -vector space V' on which the Lie algebra $\mathfrak{sl}(2, E)$ acts, if we define $N, Y \in \mathfrak{sl}(2, E)$ in the similar way to the above and we define $V'_a = \{v \in V' \mid [Y, v] = av\}$ for $a \in \mathbf{Z}$, the map $V'_0 \rightarrow V'_{-2} ; v \mapsto [N, v]$ is surjective as is well known. By applying this to $V' = \{v \in V \mid [Y_k, v] = [N_k, v] = 0 \text{ for all } k \neq j\}$, we have that the map $A_j \rightarrow B_j ; v \mapsto [N_j, v]$ is surjective. Take $v_j \in A_j$ such that $[N_j, v_j] = S_j$. Let $v = \sum_j v_j$. Then $[N_j, v] = S_j$ for all j . \square

3.5.5. Assume that G is reductive. We prove 3.5.2 (1). To do so, it is enough to show the following (1).

(1) If $p' \in U(p)$ and if $p'_\lambda \in D_{\mathrm{SL}(2)}(\Phi)$ converges to p' , then $p'_\lambda \in U(p)$ for all sufficiently large λ . (Here $(p'_\lambda)_\lambda$ denotes a directed family.)

Since $U(p') \subset U(p)$, by replacing p' by p , we can reduce (1) to the following:

(2) If $p_\lambda \in D_{\mathrm{SL}(2)}(\Phi)$ converges to p , then $p_\lambda \in U(p)$ for all sufficiently large λ .

Dividing the sequence $(p_\lambda)_\lambda$ into subsequences, we may assume that, for a fixed $J \subset \Phi = \{1, \dots, n\}$, the family of weight filtrations associated to p_λ is J for every λ . Since $D_{\mathrm{SL}(2)}(\Phi(p))$ is open in $D_{\mathrm{SL}(2)}(\Phi)$ (in fact, the former is the inverse image of the open set of $\mathbf{R}_{\geq 0}^\Phi$ consisting of elements whose $\Phi \setminus \Phi(p)$ -components are non-zero), we may assume $J \subset \Phi(p)$. Assume $J \subset \Phi(p)$. Let m be the order of J .

Fix a splitting α of Φ . Let $\alpha_{\Phi(p)} : \mathbf{G}_{m, \mathbf{R}}^{\Phi(p)} \rightarrow G_{\Phi, \mathbf{R}}$ (resp. $\alpha_J : \mathbf{G}_{m, \mathbf{R}}^J \rightarrow G_{J, \mathbf{R}}$) be the splitting of $\Phi(p)$ (resp. J) induced by α defined as in 3.5.3.

We prove the following.

(3) We may assume $\tau_p = \alpha_{\Phi(p)}$ (3.5.3) and $\tau_{p_\lambda} = \tau_{p_J} = \alpha_J$ for all λ .

We prove (3). There exists a unique $u \in (G_{\Phi(p)})_u(\mathbf{R}) = (G'_{\Phi(p)})_u(\mathbf{R})$ such that $\tau_p(t) = u\alpha_{\Phi(p)}(t)u^{-1}$ ($t \in \mathbf{R}_{>0}^{\Phi(p)}$) and there exists a unique $u_\lambda \in (G_J)_u(\mathbf{R}) = (G'_J)_u(\mathbf{R})$ such that $\tau_{p_\lambda}(t) = u_\lambda \alpha_J(t)u_\lambda^{-1}$ ($t \in \mathbf{R}_{>0}^J$). Then $\tau_{p_J}(t) = u\alpha_J(t)u^{-1}$ ($t \in \mathbf{R}_{>0}^J$). For $j \in J$, we have the convergence $\mathrm{spl}_j^{\mathrm{BS}}(p_\lambda) \rightarrow \mathrm{spl}_j^{\mathrm{BS}}(p) = \mathrm{spl}_j^{\mathrm{BS}}(p_J)$. Since τ_{p_λ} gives $(\mathrm{spl}_j^{\mathrm{BS}}(p_\lambda))_{j \in J}$ and τ_{p_J} gives $(\mathrm{spl}_j^{\mathrm{BS}}(p_J))_{j \in J}$, u_λ converges to u . Let $p' = u^{-1}p$ and $p'_\lambda := u_\lambda^{-1}p_\lambda$. Then $\tau_{p'} = \alpha_{\Phi(p)}$ and $\tau_{p'_\lambda} = \tau_{p'_J} = \alpha_J$ and $p'_\lambda \rightarrow p'$. If we can prove that $p'_\lambda \in G'_J(\mathbf{R})p'_J$ for λ sufficiently large, we can obtain $p_\lambda \in G'_J(\mathbf{R})p_J$ for λ sufficiently large. This proves (3).

We now assume $\tau_p = \alpha_{\Phi(p)}$ and $\tau_{p_\lambda} = \tau_{p_J} = \alpha_J$ for all λ .

Fix a distance β to Φ -boundary. Let $(\rho'_J, \varphi'_J) := \theta_{\alpha, \beta}(p_J)$ and $(\rho'_\lambda, \varphi'_\lambda) := \theta_{\alpha, \beta}(p_\lambda)$ (3.5.3). Let $(\rho'_J, \xi'_{1, J})$ and $(\rho'_\lambda, \xi'_{1, \lambda})$ be the elements of the set (iv) in Lemma 3.1.3 corresponding to (ρ'_J, φ'_J) and $(\rho'_\lambda, \varphi'_\lambda)$, respectively.

We prove the following.

(4) $(\rho'_\lambda, \xi'_{1, \lambda})$ converges to $(\rho'_J, \xi'_{1, J})$ for the compact-open topology of the space of continuous homomorphisms $\mathrm{SL}(2, \mathbf{R})^m \times S_{\mathbf{C}/\mathbf{R}}^{(1)}(\mathbf{R}) \rightarrow G(\mathbf{R})$.

We prove (4). For $j \in J$, let $D^{(j)}$ be the subset of D consisting of all F such that for every $V \in \mathrm{Rep}(G)$, $(W^{(j)}, F(V))$ is an \mathbf{R} -mixed Hodge structure. Then we have a continuous map $\delta_{W^{(j)}} : D^{(j)} \rightarrow \mathrm{Lie}(G_{\mathbf{R}})$ (the δ in 1.3.4 for the weight filtration $W^{(j)}$). Let (ρ_J, φ_J) and $(\rho_\lambda, \varphi_\lambda)$ be $\mathrm{SL}(2)$ -orbits in m variables whose classes in $D_{\mathrm{SL}(2)}$ are p_J and p_λ , respectively. Then $\nu_{\alpha, \beta}(p) = \nu_{\alpha, \beta}(p_J) = \alpha\beta(\mathbf{r})^{-1}\mathbf{r}$, where $\mathbf{r} = \varphi_J(\mathbf{i})$. Since $\mathbf{r} = \exp(iN_1 + \dots + iN_j)\varphi_J(\{0\}^j \times \{i\}^{m-j})$, where N_j is the N_j of ρ_J , we have $\nu_{\alpha, \beta}(p_J) \in \bigcap_{j \in J} D^{(j)}$ and $\delta_{W^{(j)}}(\nu_{\alpha, \beta}(p_J)) = N'_1 + \dots + N'_j$ for $1 \leq j \leq m$, where N'_j is the N_j of ρ'_J .

Similarly, $\nu_{\alpha,\beta}(p_\lambda) \in \bigcap_{j \in J} D^{(j)}$ and $\delta_{W^{(j)}}(\nu_{\alpha,\beta}(p_\lambda)) = N'_{1,\lambda} + \cdots + N'_{j,\lambda}$ for $1 \leq j \leq m$, where $N'_{j,\lambda}$ is the N_j of ρ'_λ . Since p_λ converges to p , $\nu_{\alpha,\beta}(p_\lambda)$ converges to $\nu_{\alpha,\beta}(p) = \nu_{\alpha,\beta}(p_J)$. Hence $N'_{j,\lambda}$ converges to N'_j for $1 \leq j \leq m$. Furthermore, $\varphi'_\lambda(\mathbf{i}) = \nu_{\alpha,\beta}(p_\lambda)$ converges to $\varphi'_J(\mathbf{i}) = \nu_{\alpha,\beta}(p_J)$. This proves (4).

We prove the following.

(5) If λ is sufficiently large, there is $g_\lambda \in G'(\mathbf{R})$ which commutes with $\alpha_J(t)$ for all $t \in (\mathbf{R}^\times)^m$ such that $N'_{j,\lambda} = \text{Ad}(g_\lambda)N'_j$ for all $j \in J$ and such that $g_\lambda \rightarrow 1$.

We apply Lemma 3.5.4 to the case $E = \mathbf{R}$, $V = \mathbf{R} \otimes_{\mathbf{Q}} \text{Lie}(G')$, and the action of $\mathfrak{sl}(2, \mathbf{R})^m$ on V is induced by the adjoint action of $\text{SL}(2)_{\mathbf{R}}^m$ on V via ρ_J . Let $H \subset G'_{\mathbf{R}}$ be the centralizer of the image of $\alpha_J : \mathbf{G}_{m,\mathbf{R}}^m \rightarrow G'_{\mathbf{R}}$. Let B_j ($1 \leq j \leq m$) be as in Lemma 3.5.4, let $B = \prod_{j=1}^m B_j$, and let $b = (N'_j)_{1 \leq j \leq m} \in B$. Then $\text{Lie}(H) = \{v \in V \mid [Y_j, v] = 0 \text{ for } 1 \leq j \leq m\}$. The map $H \rightarrow B$; $g \mapsto \text{Ad}(g)b$ induces the map $\text{Lie}(H) \rightarrow T_b(B) = B$, where $T_b(B)$ denotes the tangent space of B at b , and this last map is written as $v \mapsto ([v, N'_j])_{1 \leq j \leq m}$. By Lemma 3.5.4, this last map is surjective. Hence the map $H \rightarrow B$ is smooth at $1 \in H(\mathbf{R})$ as a morphism of algebraic varieties. Hence there is $g_\lambda \in H(\mathbf{R})$, $g_\lambda \rightarrow 1$ such that $\text{Ad}(g_\lambda)N_j = N_{j,\lambda}$. Thus (5) is proved.

Now we complete the proof of 3.5.2 (1). By (5), we may assume that $\rho'_\lambda = \rho'_J$. Consider $\xi'_{1,\lambda}, \xi'_{1,J} : S_{\mathbf{C}/\mathbf{R}}^{(1)} \rightarrow C_{G'}(\rho)$ (Lemma 3.1.3). Here $C_{G'}(\rho)$ denotes the centralizer. We use a result in [26] concerning homomorphisms from compact groups to locally compact groups, applied to homomorphisms from the compact group $S_{\mathbf{C}/\mathbf{R}}^{(1)}(\mathbf{R})$ to the locally compact group $C_{G'}(\rho)(\mathbf{R})$ given by $\xi'_{1,\lambda}, \xi'_{1,J}$. By the above-mentioned result, if λ is sufficiently large, there is an $A \in C_{G'_{\mathbf{R}}}(\rho)(\mathbf{R})$ such that $\xi'_{1,\lambda} = A\xi'_{1,J}A^{-1}$. Hence $(\rho'_\lambda, \xi'_{1,\lambda}) = A(\rho'_J, \xi'_{1,J})A^{-1}$ and hence, by Lemma 3.1.3, $(\rho'_\lambda, \varphi'_\lambda)$ is the twist of (ρ'_J, φ'_J) by A . Hence p_λ is the twist of p_J by an element of $G'_J(\mathbf{R})$. Thus 3.5.2 (1) is proved.

3.5.6. As in [21] Part II Section 3.4, we can reformulate Theorems 3.4.15, 3.4.16, and 3.4.18 in terms of Φ -parts, which will be equivalent to the original statements as soon as the global space structures will be well-defined (Propositions 3.4.7 and 3.4.10), and whose proofs are similar to those of [21] Part II Section 3.4 and Part IV Theorem 2.3.14. Here, the openness of $U(p)$ in $D_{\text{SL}(2)}(\Phi)$ in 3.5.2 (1) is used in connection with the condition (iv) in 3.4.14.

The well-definedness of the global structures as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$ on the $\text{SL}(2)$ -spaces (Proposition 3.4.7, Proposition 3.4.10) follows from the versions of Theorems 3.4.15, 3.4.16 and 3.4.18 in terms of Φ -parts explained above similarly as in the corresponding results in [21] Part II and Part IV indicated after each proposition. That is, first we directly prove that the structure on each Φ -part is independent on the choices of (α, β) . Next, since the intersection of Φ -part and Φ' -part coincides with $\Phi \cap \Phi'$ -part, it is enough to show that the localization map from Φ' -part to Φ -part is an open immersion whenever $\Phi' \subset \Phi$. The last statement is proved by using the version of Theorems 3.4.15, 3.4.16 and 3.4.18 in terms of Φ -parts.

3.6 The fan of parabolic subgroups

3.6.1. For a split torus T over a field, let $X(T) = \text{Hom}(T, \mathbf{G}_m)$ be the group of characters of T and let $X_*(T) = \text{Hom}(\mathbf{G}_m, T)$ be the group of cocharacters of T .

In this Section 3.6, we give a variant 3.6.9 of the classical theory of Weyl fan (3.6.10). Let E be a field of characteristic 0, let \mathcal{G} be a reductive algebraic group over E , let T be a split torus over E , and let $a : T \rightarrow \mathcal{G}$ be a homomorphism. Then in 3.6.9, we will have a bijection between a certain set of parabolic subgroups of \mathcal{G} and the set of all cones of a certain cone decomposition of $\mathbf{R} \otimes X_*(T)$. In the case where E is algebraically closed and T is a maximal torus in \mathcal{G} with the inclusion map $a : T \rightarrow \mathcal{G}$, this is the well-known bijection (3.6.10) between the set of all parabolic subgroups of \mathcal{G} which contain T and the set of all cones of a cone decomposition called Weyl fan. This variant 3.6.9 should be also well-known, and is treated in our previous work [21] Part IV Section 2.6 in a certain situation.

In the next Section 3.7, we will use the results in 3.6 to connect the space of $\mathrm{SL}(2)$ -orbits and the space of Borel–Serre orbits

3.6.2. Let X^* and X_* be finitely generated free abelian groups which are the \mathbf{Z} -duals of each other. We will denote the paring $X_* \times X^* \rightarrow \mathbf{Z}$ by $\langle \cdot, \cdot \rangle$.

Assume that we are given a finite subset R of X^* such that $R = -R$.

In 3.6.3–3.6.5, we will show that we have a fan $\Sigma(R)$ whose support is $\mathbf{R} \otimes X_*$. (Actually, the cones in this fan need not be sharp, and so $\Sigma(R)$ should be called a quasi-fan. But we call it a fan for simplicity.)

For a finite subset S of X^* , let $\langle S \rangle$ be the cone in $\mathbf{R} \otimes X^*$ generated by S and let

$$\sigma(S) = \{y \in \mathbf{R} \otimes X_* \mid \langle y, x \rangle \geq 0 \text{ for all } x \in S\}.$$

We have $\langle S \rangle = \{x \in X^* \mid \langle y, x \rangle \geq 0 \text{ for all } y \in \sigma(S)\}$.

Lemma 3.6.3. *For a subset R' of R , the following two conditions (i) and (ii) are equivalent.*

- (i) *There is $y \in X_*$ such that $R' = \{x \in R \mid \langle y, x \rangle \geq 0\}$.*
- (ii) *The following (ii-1) and (ii-2) are satisfied.*
 - (ii-1) $R = R' \cup (-R')$.
 - (ii-2) $\langle R' \rangle \cap R = R'$.

Proof. The implication (i) \Rightarrow (ii) is clear.

We prove (ii) \Rightarrow (i). Let y be an interior point of $\sigma(R')$. We prove that R' satisfies (i) with this element y . Let $x \in R$ and assume $\langle y, x \rangle \geq 0$. We prove $x \in R'$. Assume $x \notin R'$. Then since $R = R' \cup (-R')$, we have $-x \in R'$. Hence $\langle y, -x \rangle \geq 0$ and hence $\langle y, -x \rangle = 0$. Since y is in the interior of $\sigma(R')$ and $-x \in R'$, this shows that $-x = 0$. Hence $x = -x \in R'$. \square

3.6.4. Let $\Sigma^*(R)$ be the set of all subsets R' of R satisfying the equivalent conditions in 3.6.3.

Note that $R' \in \Sigma^*(R)$ is recovered from $\sigma(R')$ as $R' = \{x \in R \mid \langle y, x \rangle \geq 0 \text{ for all } y \in \sigma(R')\}$.

Let

$$\Sigma(R) = \{\sigma(R') \mid R' \in \Sigma^*(R)\}.$$

We have a bijection $\Sigma^*(R) \rightarrow \Sigma(R)$; $R' \mapsto \sigma(R')$.

Proposition 3.6.5. $\Sigma(R)$ is a rational finite fan whose support is $\mathbf{R} \otimes X_*$.

Proof. We need to prove the following (i), (ii) and (iii).

- (i) If $\sigma \in \Sigma(R)$ and if τ is a face of σ , then $\tau \in \Sigma(R)$.
- (ii) If $\sigma, \tau \in \Sigma(R)$, then $\sigma \cap \tau \in \Sigma(R)$.
- (iii) If $\sigma, \tau \in \Sigma(R)$ and $\tau \subset \sigma$, τ is a face of σ .

Proof of (i). Let $\sigma = \sigma(R')$ ($R' \in \Sigma^*(R)$). Let τ be a face of σ and let $A = \{x \in \langle R' \rangle \mid \langle y, x \rangle = 0 \text{ for all } y \in \tau\}$. Then A is a face of $\langle R' \rangle$ and $\tau = \{y \in \sigma \mid \langle y, x \rangle = 0 \text{ for all } x \in A\}$. Let $S = R' \cap A$. Since the cone $\langle R' \rangle$ is generated by R' , the cone A is generated by S . Hence $\tau = \{y \in \sigma \mid \langle y, x \rangle = 0 \text{ for all } x \in S\} = \sigma(R' \cup (-S)) \in \Sigma(R)$.

Proof of (ii). For $R'_1, R'_2 \in \Sigma^*(R)$, we have $\sigma(R'_1) \cap \sigma(R'_2) = \sigma(R'_3)$, where $R'_3 = \langle R'_1 \cup R'_2 \rangle \cap R \in \Sigma^*(R)$.

Proof of (iii). If $R', R'' \in \Sigma^*(R)$ and if $R' \subset R''$, then $R'' = R' \cup (-S)$ for some subset S of R' , and hence $\sigma(R'')$ is a face of $\sigma(R')$.

We prove that the support of $\Sigma(R)$ is $\mathbf{R} \otimes X_*$. It is sufficient to prove that for each $y \in X_*$, there is $R' \in \Sigma^*(R)$ such that $y \in \sigma(R')$. In fact $R' = \{x \in R \mid \langle y, x \rangle \geq 0\}$ has this property. \square

3.6.6. Now let \mathcal{G} be a reductive algebraic group over a field E of characteristic 0, let T be an E -split torus, and let $a : T \rightarrow \mathcal{G}$ be a homomorphism. Let

$$X^* = X(T) = \text{Hom}(T, \mathbf{G}_m), \quad X_* = X_*(T) = \text{Hom}(\mathbf{G}_m, T) = \text{Hom}(X(T), \mathbf{Z}).$$

Let

$$R = \{\chi \in X^* \mid \text{Lie}(\mathcal{G})_\chi \neq 0\},$$

where $\text{Lie}(\mathcal{G})_\chi$ denotes the part of $\text{Lie}(\mathcal{G})$ on which the adjoint action of T via a is given by χ .

3.6.7. Let the notation be as in 3.6.6. Then we have $R = -R$. This can be seen as follows.

In the case where E is algebraically closed and T is a maximal torus in \mathcal{G} with $a : T \rightarrow \mathcal{G}$ the inclusion map, $R = -R$ is well-known in the theory of root systems.

The general case is reduced to this case by taking an algebraic closure \bar{E} of E and a maximal torus in $\mathcal{G} \otimes_E \bar{E}$ which contains the image of $a : T \otimes_E \bar{E} \rightarrow \mathcal{G} \otimes_E \bar{E}$.

Lemma 3.6.8. Let the notation be as in 3.6.6. Then for a connected closed algebraic subgroup P of \mathcal{G} , the following two conditions (i) and (ii) are equivalent.

- (i) There is $y \in X_*$ such that P is the parabolic subgroup of \mathcal{G} associated to the homomorphism $a \circ y : \mathbf{G}_m \rightarrow \mathcal{G}$ in the sense of 3.3.7.
- (ii) There is $R' \in \Sigma^*(R)$ such that $\text{Lie}(P) = \bigoplus_{\chi \in R'} \text{Lie}(\mathcal{G})_{\chi^{-1}}$. Here we denote the group law of X^* multiplicatively, and so χ^{-1} denotes the inverse of χ .

Proof. Assume that (i) is satisfied. Let $R' \in \Sigma^*(R)$ be the set associated to y as in 3.6.3 (i). Then (ii) is satisfied by this R' .

Conversely assume that (ii) is satisfied. Take $y \in X_*$ which gives R' as in 3.6.3 (i), and let P_1 be the parabolic subgroup of \mathcal{G} associated to $a \circ y$. Then $\text{Lie}(P) = \text{Lie}(P_1)$. Since both P and P_1 are connected, we have $P = P_1$. \square

3.6.9. Let the notation be as in 3.6.6.

Let \mathcal{P} be the set of all parabolic subgroups P of \mathcal{G} satisfying the equivalent conditions in Lemma 3.6.8.

For $P \in \mathcal{P}$, let $R(P) := \{\chi \in X^* \mid \text{Lie}(P)_{\chi^{-1}} \neq 0\}$.

By Lemma 3.6.8, we have a bijection

$$\mathcal{P} \quad \xleftrightarrow{1:1} \quad \Sigma^*(R)$$

which sends $P \in \mathcal{P}$ to $R(P)$ and conversely sends $R' \in \Sigma^*(R)$ to the unique parabolic subgroup P of \mathcal{G} such that $\text{Lie}(P) = \bigoplus_{\chi \in R'} \text{Lie}(\mathcal{G})_{\chi^{-1}}$.

Hence we have the composite bijection

$$\mathcal{P} \quad \xleftrightarrow{1:1} \quad \Sigma(R).$$

For $P \in \mathcal{P}$, we denote the corresponding element of $\Sigma(R)$ by $\sigma(P)$.

3.6.10. Let the notation be as in 3.6.6. Assume that E is algebraically closed, T is a maximal torus in \mathcal{G} , and $a : T \rightarrow \mathcal{G}$ is the inclusion map.

In this case, $\Sigma(R)$ is called the Weyl fan and \mathcal{P} coincides with the set of all parabolic subgroups of \mathcal{G} which contain T .

If $P \in \mathcal{P}$ is a minimal parabolic subgroup, i.e., a Borel subgroup, the open cone of interior points of $\sigma(P)$ is called the dominant Weyl chamber for P ([27] Definition 21.35) and $\Sigma(R) = \bigcup_w$ (faces of $w\sigma(P)$), where w ranges over all elements of the Weyl group (while P is fixed).

See [34] Theorem 8.4.3.

3.6.11. Let the notation be as in 3.6.6. Now fix an isomorphism $T \simeq \mathbf{G}_{m,E}^n$.

Then $X(T)$ is identified with \mathbf{Z}^n . Let $X(T)_+ = \mathbf{N}^n \subset \mathbf{Z}^n = X(T)$. Let $R_+ = R \cap X(T)_+$. Let $\Sigma^*(R)_+$ be the subset of $\Sigma^*(R)$ consisting of all R' such that $R_+ \subset R'$. Let \mathcal{P}_+ be the corresponding subset of \mathcal{P} and let $\Sigma(R)_+ \subset \Sigma(R)$ be the corresponding subset. Then $\Sigma(R)_+$ is a subfan of $\Sigma(R)$. Its support is $X_*(R)_{\mathbf{R},+} = \mathbf{R}_{\geq 0}^n = \sigma(R_+) \subset \mathbf{R}^n = \mathbf{R} \otimes X_*$. (In fact, if $y \in X_*$ and $y \in \sigma(R_+)$, then for $R' := \{x \in X^* \mid \langle y, x \rangle \geq 0\}$, we have $R' \in \Sigma^*(R)_+$ and $y \in \sigma(R')$.)

For $1 \leq j \leq n$, let $W^{(j)}$ be the increasing filtration on the functor $V \mapsto V$ from $\text{Rep}_E(\mathcal{G})$ to the category of E -vector spaces associated to $\mathbf{G}_{m,E} \xrightarrow{j\text{-th}} \mathbf{G}_{m,E}^n \rightarrow \mathcal{G}$ (3.3.1), where j -th means the j -th component. Let $\mathcal{G}_\Phi \subset \mathcal{G}$ be the stabilizer of $\Phi := (W^{(j)})_{1 \leq j \leq n}$, and let \mathcal{G}_Φ° be its connected component containing 1. Then \mathcal{G}_Φ° is the unique connected algebraic subgroup of \mathcal{G} such that $\text{Lie}(\mathcal{G}_\Phi^\circ) = \bigoplus_{\chi \in R_+} \text{Lie}(\mathcal{G})_{\chi^{-1}}$.

For $P \in \mathcal{P}$, $P \in \mathcal{P}_+$ if and only if $\mathcal{G}_\Phi^\circ \subset P$.

3.6.12. This is a complement to 3.6.11. Let \mathcal{G} be a reductive group and assume that we are given increasing filtrations $W^{(j)}$ ($1 \leq j \leq n$) on the functor $V \mapsto V$ from $\text{Rep}_E(\mathcal{G})$ to the category of E -vector spaces such that there is a homomorphism $\mathbf{G}_{m,E}^n \rightarrow \mathcal{G}$ whose j -th $\mathbf{G}_{m,E} \rightarrow \mathcal{G}$ gives $W^{(j)}$ for $1 \leq j \leq n$. Let $\Phi = (W^{(j)})_{1 \leq j \leq n}$ and let \mathcal{G}_Φ be the stabilizer of Φ . Then we have a canonical homomorphism $T := \mathbf{G}_{m,E}^n \rightarrow \mathcal{G}_{\Phi,\text{red}}$ whose every lifting $T \rightarrow \mathcal{G}_\Phi$ gives Φ . The sets R_+ , $\Sigma^*(R)_+$, and \mathcal{P}_+ , and the fan $\Sigma(R)_+$ associated to $\mathbf{G}_{m,E}^n \rightarrow \mathcal{G}$ are independent of the choice of such lifting, for such liftings are conjugates in \mathcal{G}_Φ of each other.

Lemma 3.6.13. *Let the situation and the notation be as in 3.6.11. Then the following three conditions are equivalent.*

- (i) \mathcal{G}_Φ° is a parabolic subgroup of \mathcal{G} .
- (ii) $R_+ \in \Sigma^*(R)$, that is, $R = R_+ \cup (-R_+)$.
- (iii) $\Sigma(R)_+$ coincides with the set of all faces of $\sigma(R_+)$.

Proof. This is clear. \square

3.7 Relation of $D_{\mathrm{SL}(2)}^*$ and D_{BS}

We relate $D_{\mathrm{SL}(2)}^*$ and D_{BS} . To do this, the problem is that $D_{\mathrm{SL}(2)}^*$ does not involve parabolic subgroups though D_{BS} does. We define a modification

$D_{\mathrm{SL}(2)}^{*,W}$, which is something like “ $D_{\mathrm{SL}(2)}^*$ plus parabolic subgroups,”

of $D_{\mathrm{SL}(2)}^*$ and connect $D_{\mathrm{SL}(2)}^*$ and D_{BS} via $D_{\mathrm{SL}(2)}^{*,W}$. More precisely, we define a log modification $D_{\mathrm{SL}(2)}^{*,W} \rightarrow D_{\mathrm{SL}(2)}^*$ associated to cone decompositions related to parabolic subgroups, and define a morphism $D_{\mathrm{SL}(2)}^{*,W} \rightarrow D_{\mathrm{BS}}$.

3.7.1. Assume that G is reductive.

For $\Phi \in \mathcal{W}(G)$ and for a splitting $\alpha : \mathbf{G}_{m,\mathbf{R}}^\Phi \rightarrow G_{\mathbf{R}}$ of Φ , we apply 3.6.12 to the case $E = \mathbf{R}$, $\mathcal{G} = G_{\mathbf{R}}$, $T = \mathbf{G}_{m,\mathbf{R}}^\Phi$, and $a = \alpha$.

Note that for $\Phi \in \mathcal{W}(G)$, we have $X(\mathbf{G}_m)_+ = \mathbf{N}^\Phi \rightarrow M_{>0}/\mathcal{O}_{>0}^\times$ on $D_{\mathrm{SL}(2)}(\Phi)$. The cone decomposition of $X_*(\mathbf{G}_m)_{\mathbf{R},+}$ in 3.6.12 defines a log modification $D_{\mathrm{SL}(2)}^W(\Phi) \rightarrow D_{\mathrm{SL}(2)}(\Phi)$. It is independent of the choice of the splitting α . When Φ moves, these are glued to a log modification $D_{\mathrm{SL}(2)}^W \rightarrow D_{\mathrm{SL}(2)}$. (Here the superscript W respects the Weyl fan. For a log modification in the category $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$, see [21] Part IV 1.4.6.)

In general, for a linear algebraic group G over \mathbf{Q} , we define $D_{\mathrm{SL}(2)}^{*,W}$ to be the fiber product of $D_{\mathrm{SL}(2)}^* \rightarrow D_{\mathrm{red},\mathrm{SL}(2)} \leftarrow D_{\mathrm{red},\mathrm{SL}(2)}^W$.

3.7.2. As a set, $D_{\mathrm{SL}(2)}^{*,W}$ is identified with the set of triples (x, P, Z) , where $p \in D_{\mathrm{red},\mathrm{SL}(2)}$, $x := (p, Z') \in D_{\mathrm{SL}(2)}^*$, P is a parabolic subgroup of G_{red} satisfying the conditions in 3.6.11 and the condition $G_{\mathrm{red},\Phi,u} \subset P_u$ with Φ being the set of weight filtrations associated to p , and Z is a subset of Z' satisfying the following (i). Let $A_{p,P}$ be the inverse image of A_P under $\tau_p : \mathbf{R}_{>0}^\Phi \rightarrow P_{\mathrm{red}}(\mathbf{R})$.

(i) If x is an A -orbit, Z is a $\tau_x^*(A_{p,P})$ -orbit. If x is a B -orbit, Z is a $\tau_x^*(\mathbf{R}_{>0} \times A_{p,P})$ -orbit.

The map $D_{\mathrm{SL}(2)}^{*,W} \rightarrow D_{\mathrm{SL}(2)}^*$ is understood as $(x, P, Z) \mapsto x$.

3.7.3. We have a map

$$D_{\mathrm{SL}(2)}^{*,W} \rightarrow D_{\mathrm{BS}} ; (x, P, Z) \mapsto (P, A_P \circ Z).$$

The fact that this is a morphism is proved by using the local structure theorem (Theorem 3.4.16). The proof is similar to the proof of [21] Part IV Theorem 2.6.22 (1).

Remark 3.7.4. In [21] Part IV Sections 2.5 and 2.6, we considered

$$D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^{\star} \rightarrow D_{\mathrm{SL}(2)}^{\star,-} \leftarrow D_{\mathrm{SL}(2)}^{\star,\mathrm{BS}} \rightarrow D_{\mathrm{BS}}.$$

In that situation, $D_{\mathrm{red},\mathrm{SL}(2)}^{\mathrm{W}}$ in 3.7.1 coincides with $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}$ in [21] Part IV 2.6.3, $D_{\mathrm{SL}(2)}^{\star,\mathrm{W}}$ is the fiber product of $D_{\mathrm{SL}(2)}^{\star} \rightarrow D_{\mathrm{SL}(2)}^{\star,-} \leftarrow D_{\mathrm{SL}(2)}^{\star,\mathrm{BS}}$, and the map $D_{\mathrm{SL}(2)}^{\star,\mathrm{W}} \rightarrow D_{\mathrm{BS}}$ in 3.7.3 coincides with the composition $D_{\mathrm{SL}(2)}^{\star,\mathrm{W}} \rightarrow D_{\mathrm{SL}(2)}^{\star,\mathrm{BS}} \rightarrow D_{\mathrm{BS}}$. The fiber product property can be seen as

$$\begin{aligned} D_{\mathrm{SL}(2)}^{\star} \times_{D_{\mathrm{SL}(2)}^{\star,-}} D_{\mathrm{SL}(2)}^{\star,\mathrm{BS}} &= D_{\mathrm{SL}(2)}^{\star} \times_{D_{\mathrm{SL}(2)}^{\star,-}} (D_{\mathrm{SL}(2)}^{\star,-} \times_{D_{\mathrm{SL}(2)}(\mathrm{gr}^W)} D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}) \\ &= D_{\mathrm{SL}(2)}^{\star} \times_{D_{\mathrm{SL}(2)}(\mathrm{gr}^W)} D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}} = D_{\mathrm{SL}(2)}^{\star} \times_{D_{\mathrm{SL}(2)}(\mathrm{gr}^W)} D_{\mathrm{red},\mathrm{SL}(2)}^{\mathrm{W}} = D_{\mathrm{SL}(2)}^{\star,\mathrm{W}}, \end{aligned}$$

where the first equality is [21] Part IV Proposition 2.6.14.

Remark 3.7.5. (1) In the second line of [21] Part IV 2.6.3, $D_{\mathrm{SL}(2)}^{\star,-}(\mathrm{gr}^W)$ should be $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$. In the last line of loc. cit., $D_{\mathrm{SL}(2)}^{\star}(\mathrm{gr}^W)^{\mathrm{BS}}$ should be $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}$.

(2) In the proof of [21] Part IV Proposition 2.6.9, line 11 from the end of the proof, $L = \mathcal{S}(\sigma) \cup \mathcal{S}(\sigma)^{-1}$ must be corrected as $R(Q) \subset \mathcal{S}(\sigma) \cup \mathcal{S}(\sigma)^{-1}$.

3.8 Case of Shimura varieties

3.8.1. Assume that G is reductive and that $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ satisfies the condition that the Hodge type of $\mathrm{Lie}(G_{\mathbf{R}})$ via h_0 is in $\{(1, -1), (0, 0), (-1, 1)\}$ (as in Shimura data). Then h_0 is \mathbf{R} -polarizable by [12] (Lemma 1.5.3).

We prove

Theorem 3.8.2. *Let the assumption be as in 3.8.1. Then we have an isomorphism*

$$D_{\mathrm{SL}(2)}^{\mathrm{W}} \xrightarrow{\sim} D_{\mathrm{SL}(2)}$$

in $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$. In particular, the identity map of D extends uniquely to a morphism

$$D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$$

of locally ringed spaces with log structures with sign.

3.8.3. Note that for a field E of characteristic 0 and for $n \geq 0$, a finite-dimensional representation of $\mathrm{SL}(2)_E^n$ over E is semisimple and each irreducible representation is isomorphic to $\rho^{(r)} := \mathrm{Sym}^{r(1)}(\rho_1) \otimes \cdots \otimes \mathrm{Sym}^{r(n)}(\rho_n)$ for some $r \in \mathbf{N}^n$, where $\rho_j : \mathrm{SL}(2)_E^n \rightarrow \mathrm{GL}(2)_E$ is the composition of the j -th projection $\mathrm{SL}(2)_E^n \rightarrow \mathrm{SL}(2)_E$ and the inclusion homomorphism $\mathrm{SL}(2)_E \rightarrow \mathrm{GL}(2)_E$. Consider the homomorphism $\mathbf{G}_{m,E}^n \rightarrow \mathrm{SL}(2)_E^n$ whose restriction to the k -th $\mathbf{G}_{m,E}$ ($1 \leq k \leq n$) is

$$t \mapsto (g_1, \dots, g_n), \quad g_j = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad \text{for } 1 \leq j \leq k, \quad g_j = 1 \quad \text{for } k < j \leq n.$$

Write the standard base of E^2 by (e_1, e_2) . Then the action of $\mathbf{G}_{m,E}^n$ on $\bigotimes_{j=1}^n (e_1^{a(j)} e_2^{r(j)-a(j)})$ ($a \in \mathbf{N}^n$, $a \leq r$) via $\rho^{(r)}$ is given by the character

$$\left(\sum_{j=1}^k r(j) - 2 \sum_{j=1}^k a(j) \right)_{1 \leq k \leq n} \in \mathbf{Z}^n = X(\mathbf{G}_{m,E}^n).$$

We will use the following lemma later.

Lemma 3.8.4. *Let $r \in \mathbf{N}^n$ and let $c \in \mathbf{Z}^n = X(\mathbf{G}_{m,E}^n)$ be a character which appears in the representation of $\mathbf{G}_{m,E}^n$ in $\rho^{(r)}$ via the above homomorphism $\mathbf{G}_{m,E}^n \rightarrow \mathrm{SL}(2)_E^n$. Assume that either one of the following conditions (i)–(iii) is satisfied. Then we have either $c \in \mathbf{N}^n$ or $-c \in \mathbf{N}^n$.*

- (i) $r(k) = 0$ for all k .
- (ii) There is k such that $1 \leq k \leq n$, $r(k) > 0$, and $r(j) = 0$ for all $j \neq k$.
- (iii) There are k, l such that $1 \leq k < l \leq n$, $r(k) > 0$, $r(l) = 1$, and $r(j) = 0$ for $j \neq k, l$.

Proof. We consider the character c of the action of $\mathbf{G}_{m,\mathbf{R}}^n$ on $\bigotimes_{j=1}^n (e_1^{a(j)} e_2^{r(j)-a(j)})$ ($0 \leq a(j) \leq r(j)$ for $1 \leq j \leq n$).

In the case (i), $c = 0$.

In the case (ii), $c(j) = 0$ if $j < k$ and $c(j) = r(k) - 2a(k)$ if $j \geq k$.

In the case (iii), $c(j) = 0$ if $j < k$, $c(j) = c(k) = r(k) - 2a(k)$ if $k \leq j < l$, and $c(j) = c(l) = r(k) + r(l) - 2(a(k) + a(l))$ if $j \geq l$. Since $r(l) = 1$ and $a(l) \in \{0, 1\}$, we have $|c(k) - c(l)| = 1$. Hence we have either $\{c(k), c(l)\} \subset \mathbf{N}$ or $\{-c(k), -c(l)\} \subset \mathbf{N}$. Since $c(j) \in \{0, c(k), c(l)\}$ for all j , we have either $c \in \mathbf{N}^n$ or $-c \in \mathbf{N}^n$. \square

3.8.5. Assume that G is reductive. Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit in n variables of rank n for (G, h_0) , and let $V \in \mathrm{Rep}(G)$. Let $r \in \mathbf{N}^n$ and assume that $\rho^{(r)}$ appears in the action of $\mathrm{SL}(2)_\mathbf{R}^n$ on $V_\mathbf{R}$ induced by ρ . Then, by Claim in the proof of 3.1.4, for the Hodge structure of V given by any element of D , there is $p \in \mathbf{Z}$ such that the $(p+b, p-b)$ -Hodge component of V is non-zero for $0 \leq b \leq \sum_{j=1}^n r(j)$.

3.8.6. We prove Theorem 3.8.2. Let $\Phi \in \mathcal{W}(G)$. Let (ρ, φ) be an $\mathrm{SL}(2)$ -orbit in n variables of rank n whose associated family of weight filtrations is Φ . Then by 3.8.5 and by the fact that only the Hodge type $(1, -1), (0, 0), (-1, 1)$ appears in $\mathrm{Lie}(G)$, we have that if $\rho^{(r)}$ appears in the representation $\mathrm{Lie}(G_\mathbf{R})$ of $\mathrm{SL}(2)_\mathbf{R}^n$, then we have $\sum_{j=1}^n r(j) \leq 2$. Hence if $\rho^{(r)}$ appears in $\mathrm{Lie}(G_\mathbf{R})$, the assumption of 3.8.4 for r is satisfied. Hence by 3.8.4, each character of $\mathbf{G}_{m,\mathbf{R}}^n$ which appears in $\mathrm{Lie}(G_\mathbf{R})$ is either in R_+ or $-R_+$, that is, $R = R_+ \cup (-R_+)$. Hence the condition (ii) of 3.6.13 is satisfied for Φ . Hence the condition (iii) of 3.6.13 is satisfied. This proves that $D_{\mathrm{SL}(2)}^W(\Phi) \rightarrow D_{\mathrm{SL}(2)}(\Phi)$ is an isomorphism and hence proves Theorem 3.8.2.

3.8.7. In the classical case of $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathrm{GSp}(g)_\mathbf{R}$ which gives the Siegel upper half space \mathfrak{H}_g of degree g , $D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$ is a homeomorphism ([23] Theorem 6.7).

But even in the case $g = 1$, the real analytic structures of D_{BS} and $D_{\mathrm{SL}(2)}$ are slightly different as is seen in 4.9.1. For the case of some Shimura variety as in 4.9.3, the map $D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$ is not bijective.

3.9 Relation of $D_{\mathrm{SL}(2)}^*$ and $D_{\mathrm{SL}(2)}^{II}$

We define a log modification $D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^*$ associated to a cone decomposition, and define a morphism $D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^{II}$.

3.9.1. Let Φ be a finite set of weight filtrations which has a common splitting.

Note that we have $X(\mathbf{G}_m \times \mathbf{G}_m^\Phi)_+ = \mathbf{N} \times \mathbf{N}^\Phi \rightarrow M_{>0}/\mathcal{O}_{>0}^\times$ on $D_{\mathrm{SL}(2)}^*(\Phi)$. Let $\beta_0^* \in M/\mathcal{O}^\times$ be the image of $(1, 0, \dots, 0)$ and let $\beta_{\mathrm{tot}} \in M/\mathcal{O}^\times$ be the image of $(0, 1, \dots, 1)$. These β_0^* and β_{tot} are glued to global sections of $M_{>0}/\mathcal{O}_{>0}^\times$ on $D_{\mathrm{SL}(2)}^*$ which we still denote by β_0^* and β_{tot} , respectively.

Let $D_{\mathrm{SL}(2)}^{\star,+}(\Phi)$ be the log modification of $D_{\mathrm{SL}(2)}^*(\Phi)$ associated to the cone decomposition of $X_*(\mathbf{G}_m \times \mathbf{G}_m^\Phi)_+ = \mathbf{N} \times \mathbf{N}^\Phi$ consisting of cones

$$\sigma_1 := \{(x_0, x_1, \dots, x_n) \mid x_0 \leq \sum_{j=1}^n x_j\} \text{ and } \sigma_2 := \{(x_0, x_1, \dots, x_n) \mid x_0 \geq \sum_{j=1}^n x_j\}$$

and their faces.

When Φ moves, these are glued to a log modification $D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^*$.

For $j = 1, 2$, let $D_{\mathrm{SL}(2)}^{\star,+}(\sigma_j)$ be the open set of $D_{\mathrm{SL}(2)}^{\star,+}$ whose intersection with $D_{\mathrm{SL}(2)}^{\star,+}(\Phi)$ coincides with its σ_j -part. Then $D_{\mathrm{SL}(2)}^{\star,+}(\sigma_1)$ (resp. $D_{\mathrm{SL}(2)}^{\star,+}(\sigma_2)$) coincides with the set of all points s of $D_{\mathrm{SL}(2)}^{\star,+}$ such that β_{tot} (resp. β_0^*) is divided by β_0^* (resp. β_{tot}) at s .

3.9.2. We define a map $D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^{II}$ as follows (cf. [21] Part IV 2.5.4).

Let x^+ be a point of $D_{\mathrm{SL}(2)}^{\star,+}$ lying over $x \in D_{\mathrm{SL}(2)}^*$. We define the image x^{II} of x^+ in $D_{\mathrm{SL}(2)}^{II}$. There are four cases.

Case 1. Both β_0^* and β_{tot} are trivial at x^+ . That is, $x^+ = x \in D$.

Case 2. β_0^* is strictly divided by β_{tot} at x^+ .

Case 3. β_{tot} is strictly divided by β_0^* at x^+ .

Case 4. β_0^* and β_{tot} coincide at x^+ but are nontrivial.

In Case 1, $x^{II} = x^+ = x \in D$.

In Cases 2–4, write $x = (p, Z)$.

In Case 2, x^{II} is $x = (p, Z)$ regarded as an element (a B -orbit) of $D_{\mathrm{SL}(2)}$.

In Case 3, x^{II} is (p, Z_{spl}) , where $Z_{\mathrm{spl}} := \{\mathrm{spl}_W(z)(z_{\mathrm{red}}) \mid z \in Z\}$. (See 1.3.9 for the notation.)

In Case 4, Z in $x = (p, Z)$ is a $\tau_x^*(\mathbf{R}_{>0} \times \mathbf{R}_{>0}^{\Phi(p)}) = \tau_x(\mathbf{R}_{>0} \times \mathbf{R}_{>0}^{\Phi(p)})$ -orbit, and x^+ is identified with a triple (p, Z, Z') where Z' is a $\tau_x(\{1\} \times \mathbf{R}_{>0}^{\Phi(p)})$ -orbit contained in Z . We define $x^{II} = (p, Z')$.

The proof of the fact that the map just defined is a proper and surjective morphism is similar to that of [21] Part IV Theorem 2.5.5 (1).

Proposition 3.9.3. *There is a unique morphism $D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}} \rightarrow D_{\mathrm{SL}(2)}^{II}$ of $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$ which extends the identity map of D .*

This is a G -MHS version of a part of [21] Part IV Theorem 2.5.5 (1), and proved similarly as follows. The map $D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^*$ is an isomorphism over the open set $D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$ of $D_{\mathrm{SL}(2)}^*$ as is easily seen. Hence the morphism $D_{\mathrm{SL}(2)}^{\star,+} \rightarrow D_{\mathrm{SL}(2)}^{II}$ induces a morphism $D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}} \rightarrow D_{\mathrm{SL}(2)}^{II}$.

Remark 3.9.4. Let $\mathcal{B}'_{\mathbf{R}}(\log)^+$ be the full subcategory of $\mathcal{B}'_{\mathbf{R}}(\log)$ defined in Part IV 2.7.5 of [21]. Then by the method in Section 2.7 of ibid. basing on the local structure theorems (Theorems 3.4.16 and 3.4.18), we can prove the following G -MHS version of Theorem 2.7.14 of ibid. The spaces $D_{\mathrm{SL}(2)}^I$, $D_{\mathrm{SL}(2)}^{II}$, $D_{\mathrm{SL}(2)}^*$, $D_{\mathrm{SL}(2)}^{\star, \mathrm{W}}$, $D_{\mathrm{SL}(2)}^{\star, +}$ belong to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. We do not give the details of the proof.

3.10 Valuative spaces, I

Recall that, for an abelian group L , a submonoid V of L is said to be *valuative* if $V \cup V^{-1} = L$.

3.10.1. We review the associated valuative space.

For an object S of $\mathcal{B}'_{\mathbf{R}}(\log)$, we have a locally ringed space S_{val} endowed with a log structure with sign defined as in [21] Part IV 3.1.13. As a set, S_{val} is the set of triples (s, V, h) , where $s \in S$, V is a valuative submonoid of $(M_S/\mathcal{O}_S^\times)_s^{\mathrm{gp}}$ such that $V \supset (M_S/\mathcal{O}_S^\times)_s$ and such that $V^\times \cap (M_S/\mathcal{O}_S^\times)_s = \{1\}$, and, $\tilde{V}_{>0}$ being the inverse image of V in $M_{S, >0, s}^{\mathrm{gp}}$, h is a homomorphism $(\tilde{V}_{>0})^\times \rightarrow \mathbf{R}_{>0}^{\mathrm{mult}}$ extending the evaluation homomorphism $f \mapsto f(s)$ on $\mathcal{O}_{S, >0, s}^\times$ at s .

[21] Part IV Proposition 3.1.9 explicitly describes the projection $S_{\mathrm{val}} \rightarrow S$ as a projective limit of log modifications of S ([21] Part IV Proposition 1.4.6). It follows that the projection $S_{\mathrm{val}} \rightarrow S$ is proper and surjective (Corollary 3.1.10).

3.10.2. By 3.10.1, we have the following locally ringed spaces with a log structure with sign

$$D_{\mathrm{BS}, \mathrm{val}}, D_{\mathrm{SL}(2), \mathrm{val}}^*, D_{\mathrm{SL}(2), \mathrm{val}}^I, D_{\mathrm{SL}(2), \mathrm{val}}^{II}$$

associated to the objects D_{BS} , $D_{\mathrm{SL}(2)}^*$, $D_{\mathrm{SL}(2)}^I$, $D_{\mathrm{SL}(2)}^{II}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$, respectively.

The underlying sets of $D_{\mathrm{SL}(2), \mathrm{val}}^I$ and $D_{\mathrm{SL}(2), \mathrm{val}}^{II}$ are identified because the log structure with sign of $D_{\mathrm{SL}(2)}^I$ is the inverse image of that of $D_{\mathrm{SL}(2)}^{II}$. We will denote their common underlying set by $D_{\mathrm{SL}(2), \mathrm{val}}$.

Theorem 3.10.3. (1) *There is a unique morphism $D_{\mathrm{SL}(2), \mathrm{val}}^* \rightarrow D_{\mathrm{BS}, \mathrm{val}}$ which extends the identity map of D . This map is injective.*

(2) *There is a unique morphism $D_{\mathrm{SL}(2), \mathrm{val}}^* \rightarrow D_{\mathrm{SL}(2), \mathrm{val}}^{II}$ which extends the identity map of D . It is proper and surjective.*

Proof. (1) The morphism $D_{\mathrm{SL}(2)}^{\star, \mathrm{W}} \rightarrow D_{\mathrm{BS}}$ in Section 3.7 induces the morphism $D_{\mathrm{SL}(2), \mathrm{val}}^* \rightarrow D_{\mathrm{BS}, \mathrm{val}}$ which extends the identity map of D because $D_{\mathrm{SL}(2)}^{\star, \mathrm{W}}$ is a log modification of $D_{\mathrm{SL}(2)}^*$ (3.7.2). The uniqueness is by the density of D in $D_{\mathrm{SL}(2), \mathrm{val}}^*$. The proof of the injectivity is similar to the proof of [21] Part IV Theorem 3.4.4 (1).

(2) The morphism $D_{\mathrm{SL}(2)}^{\star, +} \rightarrow D_{\mathrm{SL}(2)}^{II}$ in Section 3.9 induces the morphism $D_{\mathrm{SL}(2), \mathrm{val}}^* \rightarrow D_{\mathrm{SL}(2), \mathrm{val}}^{II}$ which extends the identity map of D because $D_{\mathrm{SL}(2)}^{\star, +}$ is a log modification of $D_{\mathrm{SL}(2)}^*$ (cf. 3.9.2). The uniqueness is by the same reason as in (1).

This morphism is proper, because both terms are proper over $D_{\mathrm{red}, \mathrm{SL}(2)} \times \mathrm{spl}(W)$ by Proposition 3.4.8. The surjectivity of this map follows from its properness and the fact that the image of this map contains D and hence is dense in $D_{\mathrm{SL}(2), \mathrm{val}}^{II}$. \square

3.10.4. The inverse image of $D_{\text{BS},\text{val}}^{\text{mild}}$ under $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{BS},\text{val}}$ is $D_{\text{SL}(2),\text{val}}^{\star,\text{mild}}$.

Remark 3.10.5. Let $\mathcal{C}_{\mathbf{R}}(\text{val})^+$ be the category defined in Part IV 3.2.5 of [21]. Then by the method in Section 3.2 of ibid., we can prove the following G -MHS version of a statement in 3.3.1 of ibid. The spaces $D_{\text{BS},\text{val}}$, $D_{\text{SL}(2),\text{val}}^I$, $D_{\text{SL}(2),\text{val}}^{II}$, $D_{\text{SL}(2),\text{val}}^{\star}$ belong to $\mathcal{C}_{\mathbf{R}}(\text{val})^+$. We do not give the details of the proof.

3.11 Global properties of $D_{\text{SL}(2)}^I$, $D_{\text{SL}(2)}^{II}$, $D_{\text{SL}(2)}^{\star}$ etc.

Theorem 3.11.1. Let X be one of $D_{\text{SL}(2)}^I$, $D_{\text{SL}(2)}^{II}$, $D_{\text{SL}(2)}^{\star}$, $D_{\text{SL}(2),\text{val}}^I$, $D_{\text{SL}(2),\text{val}}^{II}$, $D_{\text{SL}(2),\text{val}}^{\star}$, $D_{\text{BS},\text{val}}^{\star,+}$, $D_{\text{SL}(2)}^{\star,W}$. Let Γ be a semi-arithmetic subgroup of $G'(\mathbf{Q})$ (1.5.7).

(1) The action of Γ on X is proper and the quotient space $\Gamma \backslash X$ is Hausdorff. In particular, X is Hausdorff.

(2) If Γ is torsion-free, the action of Γ on X is free and the map $X \rightarrow \Gamma \backslash X$ is a local homeomorphism.

The proof, given in 3.11.2 and 3.11.3 below, is similar to that of [21] Part II Theorem 3.5.17 and that of [21] Part IV Theorem 6.1.1. Starting from D_{BS} , we transport various properties along the fundamental diagram in Introduction.

3.11.2. We first prove that the case $\Gamma = \{1\}$ of (1), that is, the space X is Hausdorff.

We have an injective continuous map $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{BS},\text{val}}$ (Theorem 3.10.3 (1)). Since D_{BS} is Hausdorff (Proposition 2.6.2), $D_{\text{BS},\text{val}}$ is Hausdorff. Hence, $D_{\text{SL}(2),\text{val}}^{\star}$ is Hausdorff. Since $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2)}^{\star,W}$, $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2)}^{\star,+}$ and $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2)}^{\star}$ are proper and surjective, $D_{\text{SL}(2)}^{\star,W}$, $D_{\text{SL}(2)}^{\star,+}$ and $D_{\text{SL}(2)}^{\star}$ are Hausdorff. Since the maps $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2),\text{val}}^{II} \rightarrow D_{\text{SL}(2)}^{II}$ are proper and surjective (Theorem 3.10.3 (2)), $D_{\text{SL}(2),\text{val}}^{II}$ and $D_{\text{SL}(2)}^{II}$ are Hausdorff. Since we have a bijective continuous map $D_{\text{SL}(2)}^I \rightarrow D_{\text{SL}(2)}^{II}$, $D_{\text{SL}(2)}^I$ is Hausdorff. Hence $D_{\text{SL}(2),\text{val}}^I$ is also Hausdorff.

3.11.3. We prove Theorem 3.11.1.

(1) We prove the former part, that is, that the action is proper. Since it is valid for $X = D_{\text{BS}}$ (Theorem 2.6.1 (1)), we see that it is valid for $D_{\text{BS},\text{val}}$ and $D_{\text{SL}(2),\text{val}}^{\star}$ by using continuous maps $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{BS},\text{val}} \rightarrow D_{\text{BS}}$, 3.11.2, and the fact 2.3.6 (3.1). Then we see that it is valid also for $D_{\text{SL}(2)}^{\star,W}$, $D_{\text{SL}(2)}^{\star,+}$, $D_{\text{SL}(2)}^{\star}$, $D_{\text{SL}(2),\text{val}}^{II}$ and $D_{\text{SL}(2)}^{II}$ by using the proper and surjective maps $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2)}^{\star,W}$, $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2)}^{\star,+}$, $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2)}^{\star}$, $D_{\text{SL}(2),\text{val}}^{\star} \rightarrow D_{\text{SL}(2),\text{val}} \rightarrow D_{\text{SL}(2)}$ and the fact 2.3.6 (3.2). Then we see it also for $D_{\text{SL}(2)}^I$ and $D_{\text{SL}(2),\text{val}}^I$ by using the continuous maps $D_{\text{SL}(2)}^I \rightarrow D_{\text{SL}(2)}^{II}$ and $D_{\text{SL}(2),\text{val}}^I \rightarrow D_{\text{SL}(2)}^{II}$.

The latter part of (1) follows from the former part by 2.3.6 (1).

(2) We prove the former part, that is, the action is free, by using a similar argument as in 2.6.4. We apply 2.3.6 (4) to $H = \Gamma$, $H_1 = \Gamma_u$, $X_1 = \text{spl}(W)$, $X_2 = D_{\text{red},\text{SL}(2)}$. The action of Γ/Γ_u on $D_{\text{red},\text{SL}(2)}$ is free, by a similar argument in the proof of [23] Lemma 5.7, and the action of Γ_u on $\text{spl}(W) \simeq G_u(\mathbf{R})$ is free. Hence the action of Γ on $\text{spl}(W) \times D_{\text{red},\text{SL}(2)}$ is free. By using the canonical maps from $D_{\text{SL}(2)}^I$, $D_{\text{SL}(2)}^{II}$, $D_{\text{SL}(2)}^{\star}$ to $\text{spl}(W) \times D_{\text{red},\text{SL}(2)}$

together with the related maps in the fundamental diagram, we see that the action of Γ on X is free.

The latter part of (2) follows from the former by 2.3.6 (2), the properness of the action proved in (1), and 3.11.2.

3.11.4. In Theorem 3.11.1, we can use a semi-arithmetic subgroup of $G(\mathbf{Q})$ (not of $G'(\mathbf{Q})$) in the following two situations.

First, if either G is semisimple or the condition (1) in Lemma 1.5.3 is satisfied, 3.11.1 holds for a semi-arithmetic subgroup Γ of $G(\mathbf{Q})$. In fact, $\Gamma \cap G'(\mathbf{Q})$ is of finite index (cf. Proposition 1.5.13 for the latter case). Hence by 2.3.6 (5), we can replace Γ by the semi-arithmetic subgroup $\Gamma \cap G'(\mathbf{Q})$ of $G'(\mathbf{Q})$.

Next

Proposition 3.11.5. *Assume that G is reductive. Let X be one of $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS},\mathrm{val}}$. Let Γ be a semi-arithmetic subgroup of $G(\mathbf{Q})$.*

- (1) *The quotient space $\Gamma \backslash X$ is Hausdorff.*
- (2) *Let Z be the center of G . If the image of Γ in $(G/Z)(\mathbf{Q})$ is torsion-free, the map $X \rightarrow \Gamma \backslash X$ is a local homeomorphism.*

See 4.10.18 for the proof.

4 The space of nilpotent orbits

Let $D = D(G, h_0)$ be as in 1.2.13. We assume that h_0 is \mathbf{R} -polarizable (1.5.2).

In this section, we define and study the toroidal partial compactification $\Gamma \backslash D_\Sigma$ of $\Gamma \backslash D$, an extended period domain consisting of nilpotent orbits. We consider $\Gamma \backslash D_\Sigma$ as the moduli of G -log mixed Hodge structures. It is the G -MHS version of the toroidal partial compactification $\Gamma \backslash D(\Lambda)_\Sigma$ of $\Gamma \backslash D(\Lambda)$ for the classical period domain $D(\Lambda)$ (1.6). This $\Gamma \backslash D(\Lambda)_\Sigma$ is defined and studied in [24] (in the pure case) and in [21] Part III and Part IV (in the mixed case) and is the moduli space of LMH.

4.1 The sets D_Σ and D_Σ^\sharp

4.1.1. A *nilpotent cone* is a subset σ of $\mathrm{Lie}(G'_\mathbf{R})$ satisfying the following (i)–(iii).

- (i) $\sigma = \mathbf{R}_{\geq 0}N_1 + \cdots + \mathbf{R}_{\geq 0}N_n$ for some $N_1, \dots, N_n \in \mathrm{Lie}(G'_\mathbf{R})$.
- (ii) For every $V \in \mathrm{Rep}(G)$, the image of σ under the induced map $\mathrm{Lie}(G_\mathbf{R}) \rightarrow \mathrm{End}_\mathbf{R}(V_\mathbf{R})$ consists of nilpotent operators.
- (iii) We have $[N, N'] = 0$ for $N, N' \in \sigma$.

4.1.2. Recall that for a vector space V over a field E , for an increasing filtration W on V , and for a nilpotent linear map $N : V \rightarrow V$ such that $NW_w \subset W_w$ for all w , an increasing filtration M on V is called the *relative monodromy filtration* of N relative to W if $NM_w \subset M_{w-2}$ for all w and $N^m : \mathrm{gr}_{w+m}^M \mathrm{gr}_w^W \xrightarrow{\sim} \mathrm{gr}_{w-m}^M \mathrm{gr}_w^W$ for all w and all $m \geq 0$.

The relative monodromy filtration M need not exist, but it is unique if it exists.

4.1.3. Let $F \in \check{D}$ and let σ be a nilpotent cone. We say that the pair (σ, F) *generates a nilpotent orbit* if the following conditions (i)–(iii) are satisfied.

- (i) Let N_1, \dots, N_n be as in (i) in 4.1.1. Then $\exp(\sum_{j=1}^n z_j N_j)F \in D$ if $z_j \in \mathbf{C}$ and $\text{Im}(z_j) \gg 0$.
- (ii) $NF^p \subset F^{p-1}$ for all $N \in \sigma$ and $p \in \mathbf{Z}$.
- (iii) For every $N \in \sigma$ and every $V \in \text{Rep}(G)$, the relative monodromy filtration of $N : V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ with respect to $W(V)_{\mathbf{R}}$ exists.

In this case, we also say that (N_1, \dots, N_n, F) *generates a nilpotent orbit*.

Note that the above condition (i) is independent of the choice of (N_1, \dots, N_n) as in (i) in 4.1.1. Note also that it is equivalent to the condition that $\exp(\sum_{j=1}^n iy_j N_j)F \in D$ if $y_j \in \mathbf{R}$ and $y_j \gg 0$.

Proposition 4.1.4. *Let $F \in \check{D}$ and let σ be a nilpotent cone. Then the following conditions (i), (ii), and (iii) are equivalent.*

- (i) (σ, F) *generates a nilpotent orbit in the sense of 4.1.3.*
- (ii) *For every $V \in \text{Rep}(G)$, we have the following (ii-1), (ii-2), (ii-3).*
 - (ii-1) *For each $w \in \mathbf{Z}$, there is a $G'_{\mathbf{R}}$ -invariant \mathbf{R} -bilinear form $\langle \cdot, \cdot \rangle_w : \text{gr}_w^W V_{\mathbf{R}} \times \text{gr}_w^W V_{\mathbf{R}} \rightarrow \mathbf{R}$ such that if $z_j \in \mathbf{C}$ and $\text{Im}(z_j) \gg 0$ ($1 \leq j \leq n$), $(\text{gr}_w^W V_{\mathbf{R}}, \exp(\sum_{j=1}^n z_j N_j)F(\text{gr}_w^W V))$ is a Hodge structure of weight w polarized by $\langle \cdot, \cdot \rangle_w$.*
 - (ii-2) $NF^p(V) \subset F^{p-1}(V)$ *for all $N \in \sigma$ and $p \in \mathbf{Z}$.*
 - (ii-3) *For every $N \in \sigma$, the relative monodromy filtration of $N : V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ with respect to W exists.*
- (iii) *For some faithful $V \in \text{Rep}(G)$, the above conditions (ii-1)–(ii-3) are satisfied.*

Proof. Assume (i). We prove (ii). Consider the continuous map $\mathbf{C}^n \rightarrow \check{D}$; $z \mapsto \exp(\sum_{j=1}^n z_j N_j)F$. For some $c \in \mathbf{R}$, the image of $S := \{z \in \mathbf{C}^n \mid \text{Im}(z_j) \geq c\} \subset \mathbf{C}^n$ under this map is contained in D . Hence it induces a continuous map $S \rightarrow D$. Since D is a disjoint union of $G'(\mathbf{R})G_u(\mathbf{C})$ -orbits which are open and closed and since S is connected, the image of S in D is contained in one $G'(\mathbf{R})G_u(\mathbf{C})$ -orbit D' . Take $F' \in D'$. Let $V \in \text{Rep}(G)$. By the \mathbf{R} -polarizability and Lemma 1.5.3, for each $w \in \mathbf{Z}$, there is a $G'_{\mathbf{R}}$ -invariant \mathbf{R} -bilinear form $\langle \cdot, \cdot \rangle_w : \text{gr}_w^W V_{\mathbf{R}} \times \text{gr}_w^W V_{\mathbf{R}} \rightarrow \mathbf{R}$ which polarizes $F'(\text{gr}_w^W)$. If $z \in S$, $\exp(\sum_{j=1}^n z_j N_j)F = gF'$ for some $g \in G'(\mathbf{R})$. Since $\langle \cdot, \cdot \rangle_w$ is fixed by g , it polarizes $\exp(\sum_{j=1}^n z_j N_j)F$.

The implication (ii) \Rightarrow (iii) is clear.

By 1.4.7 and by [21] Part III 1.2.2.1, 1.2.2.2, we have the implication (iii) \Rightarrow (i). \square

Proposition 4.1.5. *Assume that (σ, Z) generates a nilpotent orbit. Then we have a filtration $M(\tau) \in \mathfrak{W}(G_{\mathbf{R}})$ (3.3.1) for each face τ of σ satisfying the following conditions (i)–(iii). If σ is rational, then all $M(\tau)$ belong to $\mathfrak{W}(G)$.*

- (i) $NM(\tau)_w \subset M(\tau)_w$ *for all $N \in \sigma$ and $w \in \mathbf{Z}$.*
- (ii) $M(0) = W$.
- (iii) *If τ and τ' are faces of σ and if $N \in \sigma$, and if τ' is the smallest face of σ containing τ and N , then $M(\tau')$ is the relative monodromy filtration of N with respect to $M(\tau)$.*

Proof. This follows from Kashiwara [18] 4.4.1 and 5.2.5. \square

4.1.6. A *nilpotent orbit* (resp. *nilpotent i -orbit*) is a pair (σ, Z) of a nilpotent cone σ and an $\exp(\sigma_{\mathbf{C}})$ (resp. $\exp(i \cdot \sigma_{\mathbf{R}})$)-orbit in \check{D} satisfying that, for some $F \in Z$, (σ, F) generates a nilpotent orbit in the sense of 4.1.3. Here $\sigma_{\mathbf{C}}$ (resp. $\sigma_{\mathbf{R}}$) denotes the \mathbf{C} (resp. \mathbf{R})-linear subspace spanned by σ in $\text{Lie}(G'_{\mathbf{C}})$ (resp. $\text{Lie}(G'_{\mathbf{R}})$).

4.1.7. A *weak fan* Σ in $\text{Lie}(G')$ is a nonempty set of sharp rational nilpotent cones in $\text{Lie}(G'_{\mathbf{R}})$ satisfying the following conditions (i) and (ii).

(i) If $\sigma \in \Sigma$ and if σ' is a face of σ , then $\sigma' \in \Sigma$.

(ii) Let $\sigma, \sigma' \in \Sigma$, and assume that σ and σ' have a common interior point and that there is an $F \in \check{D}$ such that both (σ, F) and (σ', F) generate nilpotent orbits. Then $\sigma = \sigma'$.

4.1.8. Let D_{Σ} be the set of all nilpotent orbits (σ, Z) such that $\sigma \in \Sigma$. Then D is naturally embedded in D_{Σ} via $F \mapsto (\{0\}, \{F\})$.

Let D_{Σ}^{\sharp} be the set of all nilpotent i -orbits (σ, Z) such that $\sigma \in \Sigma$. Then D is also naturally embedded in D_{Σ}^{\sharp} via $F \mapsto (\{0\}, \{F\})$.

We have a canonical map $D_{\Sigma}^{\sharp} \rightarrow D_{\Sigma}$; $(\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z)$.

For a rational nilpotent cone σ , we define $D_{\sigma} := D_{\{\text{face of } \sigma\}}$, $D_{\sigma}^{\sharp} := D_{\{\text{face of } \sigma\}}^{\sharp}$.

4.1.9. Let Γ be a subgroup of $G(\mathbf{Q})$ satisfying (SA) (1.5.7).

We say that Σ and Γ are *compatible* if Σ is stable under the adjoint action of Γ . If this is the case, Γ naturally acts on D_{Σ} .

We say that Σ and Γ are *strongly compatible* if they are compatible and if every $\sigma \in \Sigma$ is generated by elements whose \exp in $G(\mathbf{R})$ belong to Γ .

4.2 E_{σ} and the spaces of nilpotent orbits

For Σ and Γ which are strongly compatible, we endow $\Gamma \setminus D_{\Sigma}$ with a structure of a locally ringed space over \mathbf{C} and with a log structure. We endow D_{Σ}^{\sharp} with a topology.

4.2.1. Let Σ and Γ be as in 4.1.7 and in 4.1.9. Assume that they are strongly compatible (4.1.9). Let $\sigma \in \Sigma$.

Let $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$ in $G(\mathbf{R})$. Then $\Gamma(\sigma)$ is an fs monoid and $\Gamma(\sigma)^{\text{gp}} = \Gamma \cap \exp(\sigma_{\mathbf{R}})$ is a finitely generated free abelian group. Let

$$\text{torus}_{\sigma} = \text{torus}_{\sigma, \Gamma} = \mathbf{C}^{\times} \otimes \Gamma(\sigma)^{\text{gp}}.$$

Let $P(\sigma) = \text{Hom}(\Gamma(\sigma), \mathbf{N})$. Let

$$\text{toric}_{\sigma} = \text{toric}_{\sigma, \Gamma} = \text{Hom}(P(\sigma), \mathbf{C}^{\text{mult}}) = \text{Spec}(\mathbf{C}[P(\sigma)])^{\text{an}}.$$

Here $\mathbf{C}^{\text{mult}} = \mathbf{C}$ regarded as a multiplicative monoid. The standard log structure of the toric variety $\text{Spec}(\mathbf{C}[P(\sigma)])$ induces the log structure of the analytic toric space toric_{σ} .

We regard torus_{σ} as an open set of toric_{σ} via the embedding

$$\text{torus}_{\sigma} = \text{Hom}(P(\sigma)^{\text{gp}}, \mathbf{C}^{\times}) \subset \text{toric}_{\sigma}.$$

We have a natural action of torus_σ on toric_σ . We have an exact sequence

$$0 \rightarrow \Gamma(\sigma)^{\text{gp}} \xrightarrow{\log} \sigma_{\mathbf{C}} \xrightarrow{\mathbf{e}} \text{torus}_\sigma \rightarrow 0,$$

where

$$\mathbf{e}(z \otimes \log(\gamma)) = e^{2\pi iz} \otimes \gamma \quad \text{for } z \in \mathbf{C}, \gamma \in \Gamma(\sigma)^{\text{gp}}.$$

For a face τ of σ , let $0_\tau \in \text{toric}_\sigma$ be the homomorphism $P(\sigma) = \text{Hom}(\Gamma(\sigma), \mathbf{N}) \rightarrow \mathbf{C}^{\text{mult}}$ which sends $h \in \text{Hom}(\Gamma(\sigma), \mathbf{N})$ to 1 if $h(\Gamma(\tau)) = 0$ and to 0 otherwise.

Each element q of toric_σ is written in the form $q = \mathbf{e}(a) \cdot 0_\tau$ for $a \in \sigma_{\mathbf{C}}$ and for a face τ of σ . The face τ of σ is determined by q and called the *face associated to q* , and a modulo $\tau_{\mathbf{C}} + \log(\Gamma(\sigma)^{\text{gp}})$ is determined by q . The stalk of M/\mathcal{O}^\times of toric_σ at q is identified with $\text{Hom}(\Gamma(\tau), \mathbf{N})$.

4.2.2. Let the notation be as in 4.2.1. Define

$$|\text{toric}|_\sigma := \text{Hom}(P(\sigma), \mathbf{R}_{\geq 0}^{\text{mult}}) \supset |\text{torus}|_\sigma := \text{Hom}(P(\sigma)^{\text{gp}}, \mathbf{R}_{>0}) = \mathbf{R}_{>0} \otimes \Gamma(\sigma)^{\text{gp}}.$$

Here $\mathbf{R}_{\geq 0}^{\text{mult}}$ denotes the multiplicative monoid $\mathbf{R}_{\geq 0}$. Thus we have $|\text{toric}|_\sigma \subset \text{toric}_\sigma$ and $|\text{torus}|_\sigma \subset \text{torus}_\sigma$.

We have projections $q \mapsto |q|$; $\text{toric}_\sigma \rightarrow |\text{toric}|_\sigma$ and $\text{torus}_\sigma \rightarrow |\text{torus}|_\sigma$ induced by taking the absolute value $|\cdot| : \mathbf{C}^{\text{mult}} \rightarrow \mathbf{R}_{\geq 0}^{\text{mult}}$.

4.2.3. We give additive presentations of $|\text{torus}|_\sigma$ and $|\text{toric}|_\sigma$.

We have an isomorphism of topological groups

$$\sigma_{\mathbf{R}} \simeq |\text{torus}|_\sigma ; b \mapsto \mathbf{e}(ib).$$

We will often identify $|\text{torus}|_\sigma$ with $\sigma_{\mathbf{R}}$ via this isomorphism. This identification is extended to an identification of $|\text{toric}|_\sigma$ and the set of equivalence classes of pairs (τ, b) , where $b \in \sigma_{\mathbf{R}}$ and τ is a face of σ . Here (τ, b) and (τ', b') are equivalent if and only if $\tau' = \tau$ and $b' \equiv b \pmod{\tau_{\mathbf{R}}}$. We identify $\mathbf{e}(ib)0_\tau \in |\text{toric}|_\sigma$ with the class of (τ, b) . The topology of $|\text{toric}|_\sigma$ is understood as follows. Let $x = \text{class}(\tau, b) \in |\text{toric}|_\sigma$. Take a finite set $(N_j)_j$ of generators of the cone τ . Then the following sets $V(U, c)$ form a base of neighborhoods of x . Let U be a neighborhood of b in $\sigma_{\mathbf{R}}$ and let $c \in \mathbf{R}_{>0}$. Then, $V(U, c)$ is the set of class (τ', b') , where τ' is a face of τ and $b' = b'' + \sum_j y_j N_j$ for some $b'' \in U$ and some real numbers $y_j \geq c$.

4.2.4. Note that toric_σ depends on the choice of Γ (actually it depends on $\Gamma(\sigma)$) though the notation toric_σ does not tell this dependence.

However, the topological space $|\text{toric}|_\sigma$ does not depend on Γ as is seen in 4.2.3.

4.2.5. Let

$$\check{E}_\sigma := \text{toric}_\sigma \times \check{D}, \quad \check{E}_\sigma^\sharp := |\text{toric}|_\sigma \times \check{D}.$$

Let E_σ (resp. \tilde{E}_σ) be the set of all $(q, F) \in \check{E}_\sigma$ satisfying the following condition (i) (resp. (i)'). Write $q = \mathbf{e}(a)0_\tau$, where $a \in \sigma_{\mathbf{C}}$ and τ is a face of σ .

(i) The pair $(\tau, \exp(a)F)$ generates a nilpotent orbit.

(i)' We have $NF^p \subset F^{p-1}$ for $N \in \tau$ and $p \in \mathbf{Z}$.

We have $E_\sigma \subset \check{E}_\sigma \subset \check{E}_\sigma$.

Endow the complex analytic space \check{E}_σ with the pullback log structure from toric_σ . Endow E_σ and \check{E}_σ with the strong topologies ([24], 3.1.1) in \check{E}_σ as reviewed below, with the inverse images of the sheaf of holomorphic functions on \check{E}_σ , and with the inverse images of the log structure of \check{E}_σ .

Recall that for a complex analytic space X and a subset S of X , the *strong topology of S in X* is the strongest topology on S such that for every complex analytic space Y and for every morphism $\lambda : Y \rightarrow X$ of complex analytic spaces such that $\lambda(Y) \subset S$, the map $\lambda : Y \rightarrow S$ is continuous. It is stronger than the topology on S as a subspace of X .

Let

$$E_\sigma^\sharp := E_\sigma \cap \check{E}_\sigma^\sharp, \quad \tilde{E}_\sigma^\sharp := \tilde{E}_\sigma \cap \check{E}_\sigma^\sharp,$$

endowed with the topologies as the subspaces of E_σ and \check{E}_σ , respectively. Using the additive presentation of $|\text{toric}|_\sigma$ in 4.2.3, we can identify E_σ^\sharp (resp. \tilde{E}_σ^\sharp) with the set of $(\text{class}(\tau, b), F) \in |\text{toric}|_\sigma \times \check{D} = \check{E}_\sigma^\sharp$ such that $(\tau, \exp(ib)F)$ generates a nilpotent orbit (resp. such that $NF^p \subset F^{p-1}$ for all $N \in \tau$ and $p \in \mathbf{Z}$).

4.2.6. We show that as a topological space, E_σ^\sharp does not depend on Γ . Write E_σ , E_σ^\sharp and $P(\sigma)$ (4.2.1) for Γ as $E_{\Gamma(\sigma)}$, $E_{\Gamma(\sigma)}^\sharp$ and $P(\Gamma(\sigma))$, respectively, to express the dependence on $\Gamma(\sigma)$.

Assume that Γ_1 and Γ_2 are strongly compatible with Σ . Then $\Gamma_1 \cap \Gamma_2$ is strongly compatible with Σ . We show that $E_{(\Gamma_1 \cap \Gamma_2)(\sigma)} \rightarrow E_{\Gamma_j(\sigma)}$ are homeomorphisms for $j = 1, 2$. Replacing $\Gamma_1 \cap \Gamma_2$ by Γ_1 , we may assume $\Gamma_1 \subset \Gamma_2$. Write $\check{E}_{\Gamma_j(\sigma)}$ as Z_j , $E_{\Gamma_j(\sigma)}$ as Y_j , and $E_{\Gamma_j(\sigma)}^\sharp$ as Y_j^\sharp ($j = 1, 2$). Since $P(\Gamma_2(\sigma)) \subset P(\Gamma_1(\sigma))$, and $\mathbf{C}[P(\Gamma_1(\sigma))]$ is a finitely generated $\mathbf{C}[P(\Gamma_2(\sigma))]$ -module, the map $\text{toric}_{\Gamma_1(\sigma)} \rightarrow \text{toric}_{\Gamma_2(\sigma)}$ is proper and surjective and hence the map $Z_1 \rightarrow Z_2$ is proper and surjective.

We prove that for the map $Y_1 \rightarrow Y_2$, the topology of the latter is the image of the topology of the former. Let U be a subset of Y_2 and assume that the inverse image U' of U in Y_1 is open. We prove that U is open. Let S be an analytic space over \mathbf{C} and assume that we have a morphism $S \rightarrow Z_2$ whose image is contained in Y_2 . Our task is to prove that the inverse image of U in S is open. Let S' be the fiber product of $S \rightarrow Z_2 \leftarrow Z_1$ in the category of analytic spaces over \mathbf{C} . Since the set Y_1 is the fiber product of $Y_2 \rightarrow Z_2 \leftarrow Z_1$, the image of $S' \rightarrow Z_1$ is contained in Y_1 . Hence the inverse image of U' in S' , which is the inverse image of U in S' , is open. Since $S' \rightarrow S$ is proper surjective, this proves that the inverse image of U in S is open.

Since Y_j^\sharp is closed in Y_j , this tells that for the bijection $Y_1^\sharp \rightarrow Y_2^\sharp$, the topology of the latter is the image of the topology of the former. Hence the last map is a homeomorphism.

4.2.7. For $\sigma \in \Sigma$, consider the map

$$\varphi_\sigma : E_\sigma \rightarrow \Gamma \setminus D_\Sigma ; (q, F) \mapsto \text{class}(\tau, Z)$$

with $Z = \exp(\tau_C) \exp(a)F$, where τ and $a \in \sigma_C$ are such that $q = \mathbf{e}(a)0_\tau$. We endow $\Gamma \setminus D_\Sigma$ with a structure of a locally ringed space over \mathbf{C} and with a log structure as follows. The topology of $\Gamma \setminus D_\Sigma$ is the strongest topology for which the maps φ_σ are continuous

for all $\sigma \in \Sigma$. The structure sheaf of $\Gamma \setminus D_\Sigma$ consists of functions whose pullbacks to E_σ belong to the structure sheaf of E_σ for all $\sigma \in \Sigma$. The log structure of $\Gamma \setminus D_\Sigma$ is the subsheaf of the structure sheaf of $\Gamma \setminus D_\Sigma$ consisting of functions whose pullbacks to E_σ belong to the log structure of E_σ for all $\sigma \in \Sigma$.

4.2.8. In this topology of $\Gamma \setminus D_\Sigma$, if $(\sigma, Z) \in D_\Sigma$ and if $F \in Z$ and $N_1, \dots, N_n \in \sigma$ generate the cone σ , class $(\sigma, Z) \in \Gamma \setminus D_\Sigma$ is the limit of class $(\exp(\sum_{j=1}^n z_j N_j) F) \in \Gamma \setminus D$ where $z_j \in \mathbf{C}$ and $\text{Im}(z_j)$ tends to ∞ for $1 \leq j \leq n$.

4.2.9. Here we do not need Γ .

For $\sigma \in \Sigma$, consider the surjective map

$$\varphi_\sigma^\sharp : E_\sigma^\sharp \rightarrow D_\sigma^\sharp ; (\text{class } (\tau, b), F) \mapsto \text{class } (\tau, Z)$$

with $Z = \exp(i\tau_{\mathbf{R}}) \exp(ib) F$.

We define the topology of D_Σ^\sharp as the strongest topology such that the composite $E_\sigma^\sharp \rightarrow D_\sigma^\sharp \rightarrow D_\Sigma^\sharp$ also denoted by φ_σ^\sharp are continuous for all $\sigma \in \Sigma$.

4.2.10. In this topology, if $(\sigma, Z) \in D_\Sigma^\sharp$ and if $F \in Z$ and $N_1, \dots, N_n \in \sigma$ generate the cone σ , (σ, Z) is the limit of $\exp(\sum_{j=1}^n iy_j N_j) F \in D$ where $y_j \in \mathbf{R}_{>0}$ and $y_j \rightarrow \infty$ for $1 \leq j \leq n$.

4.2.11. Assume that (Σ, Γ) is strongly compatible.

Since

$$\begin{array}{ccc} E_\sigma^\sharp & \xrightarrow{\varphi_\sigma^\sharp} & D_\Sigma^\sharp \\ \downarrow & & \downarrow \\ E_\sigma & \xrightarrow{\varphi_\sigma} & \Gamma \setminus D_\Sigma \end{array}$$

is commutative, we have that the map $D_\Sigma^\sharp \rightarrow \Gamma \setminus D_\Sigma$ is continuous.

4.2.12. For an fs log analytic space X , by a *strong subspace* of X , we mean a subset of X endowed with the strong topology in X (4.2.5), with the inverse image of the sheaf of holomorphic functions on X , and with the inverse image of the log structure of X .

For example, E_σ is a strong subspace of \check{E}_σ .

Let $\mathcal{B}(\log)$ be the category of locally ringed spaces endowed with a log structure which are locally isomorphic to a strong subspace of an fs log analytic space. See [24] 3.2.4 and [21] Part III 1.1.4.

Both E_σ and \check{E}_σ are objects of $\mathcal{B}(\log)$.

An object of $\mathcal{B}(\log)$ is a *log manifold* if it is locally isomorphic to an open set of a strong subspace S of a log smooth fs log analytic space X satisfying the following condition: There is a finite family of log differential forms $(\omega_j)_j$ on X such that $S = \{x \in X \mid \omega_j(x) = 0 \text{ for all } j\}$. Here $\omega_j(x)$ denotes the pullback of ω_j to the log point x (it is not the germ of ω_j at x). See [24] 3.5.7 and [21] Part III 1.1.5.

Later we will show that E_σ , \check{E}_σ , and $\Gamma \setminus D_\Sigma$ for a strongly compatible (Σ, Γ) with Γ being a neat semi-arithmetic subgroup of $G'(\mathbf{Q})$ (1.5.7), are log manifolds. See Theorem 4.5.7 and Theorem 4.6.1 (4). We will also show that for such Γ , the quotient space $\Gamma \setminus D_\Sigma^\sharp$ is identified with the space $(\Gamma \setminus D_\Sigma)^{\log}$ (Theorem 4.6.1 (5)).

4.3 The space of ratios

We review the space of ratios $S_{[:]}$, and discuss $D_{\Sigma,[:]}$ and $D_{\Sigma,[:]^\sharp}$ which are modified versions of D_Σ and D_Σ^\sharp , respectively.

4.3.1. This is a review of [21] Part IV 4.2. Let S be a locally ringed space over $E = \mathbf{R}$ or \mathbf{C} with an fs log structure M_S satisfying the following conditions (i) and (ii).

(i) For every $s \in S$, the natural homomorphism $E \rightarrow \mathcal{O}_{S,s}/m_s$ is an isomorphism. Here, we denote by m_s the maximal ideal of $\mathcal{O}_{S,s}$.

(ii) For every open set U of S and for every $f \in \mathcal{O}(U)$, the map $U \rightarrow E ; s \mapsto f(s)$ is continuous. Here $f(s)$ is the image of f in $\mathcal{O}_{S,s}/m_s = E$.

Then we have a topological space $S_{[:]}$ over S , called the *space of ratios*, defined as follows.

4.3.2. For a sharp fs monoid \mathcal{S} , let $R(\mathcal{S})$ be the set of all maps $r : (\mathcal{S} \times \mathcal{S}) \setminus \{(1, 1)\} \rightarrow [0, \infty]$ satisfying the following conditions (i)–(iii).

$$(i) \quad r(g, f) = r(f, g)^{-1}.$$

$$(ii) \quad r(f, g)r(g, h) = r(f, h) \text{ if } \{r(f, g), r(g, h)\} \neq \{0, \infty\}.$$

$$(iii) \quad r(fg, h) = r(f, h) + r(g, h).$$

4.3.3. Let σ be the cone $\text{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\text{add}})$, where $\mathbf{R}_{\geq 0}^{\text{add}}$ denotes the additive monoid $\mathbf{R}_{\geq 0}$. Then $R(\mathcal{S})$ is identified with the set of equivalence classes of $(\sigma_j, N_j)_{1 \leq j \leq n}$, where $n \geq 0$, σ_j are faces of σ such that $\{0\} = \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n = \sigma$, and N_j is an element of the interior of σ_j . The equivalence relation is that $(\sigma_j, N_j)_{1 \leq j \leq n} \sim (\sigma'_j, N'_j)_{1 \leq j \leq n'}$ if and only if $n = n'$, $\sigma'_j = \sigma_j$ for all j , and for each j , there is a $c_j > 0$ such that $N'_j \equiv c_j N_j \pmod{\sigma_{j-1, \mathbf{R}}}$.

Such a class $((\sigma_j, N_j)_j)$ is identified with $r \in R(\mathcal{S})$ defined as follows. For $(f, g) \in \mathcal{S} \times \mathcal{S} \setminus \{(1, 1)\}$, take the biggest j such that σ_{j-1} kills f and g . Then $r(f, g) = N_j(f)/N_j(g)$.

Remark 4.3.4. There is an error in a related part 4.1.6 in [21] Part IV. In the definition of the map $R'(\mathcal{S}) \rightarrow R(\mathcal{S})$, “ ∞ ” in (2) and “0” in (3) should be interchanged.

4.3.5. Let S be as in 4.3.1. We define the set $S_{[:]}$ as the set of (s, r) with $s \in S$ and $r \in R((M_S/\mathcal{O}_S^\times)_s)$.

4.3.6. We define the topology of $S_{[:]}$ as the weakest topology for which the following conditions (1) and (2) are satisfied.

(1) The map $S_{[:]} \rightarrow S ; (s, r) \mapsto s$ is continuous.

(2) Let U be an open set of S , let $f, g \in M_S(U)$, and assume that $|f(s)| < 1$ and $|g(s)| < 1$ for all $s \in U$. Here $f(s)$ is the value at s of the image of f in \mathcal{O}_U and $g(s)$ is defined similarly. Let \tilde{U} be the inverse image of U in $S_{[:]}$. Then the following map $r_{f,g} : \tilde{U} \rightarrow [0, \infty]$ is continuous.

Let $s \in U$. If $f, g \in \mathcal{O}_{S,s}^\times$, then $r_{f,g}(s, r) := \log(|f(s)|)/\log(|g(s)|)$. Otherwise, $r_{f,g}(s, r) = r(\bar{f}(s), \bar{g}(s))$, where $\bar{f}(s)$ is the image of f in $(M_S/\mathcal{O}_S^\times)_s$ and $\bar{g}(s)$ is defined similarly.

4.3.7. Assume that we are given a chart $\mathcal{S} \rightarrow M_S$. Let Φ be a set of faces of \mathcal{S} which is totally ordered for the inclusion relation and which contains \mathcal{S} . Let $S_{[:]}(\Phi)$ be the subset of $S_{[:]}$ consisting of all (s, r) such that if $\text{class}((\sigma_j, N_j)_{1 \leq j \leq n})$ corresponds to r as in 4.3.3,

the annihilator of σ_j in \mathcal{S} , which is a face of \mathcal{S} , belongs to Φ for every $1 \leq j \leq n$. Here the annihilator of σ_j in \mathcal{S} means the subset of \mathcal{S} consisting of all elements which are sent to 0 by $\mathcal{S} \rightarrow (M_S/\mathcal{O}_S^\times)_s \xrightarrow{h} \mathbf{R}_{\geq 0}$ for all $h \in \sigma_j$.

Then $S_{[:]}\Phi$ for all above Φ forms an open covering of $S_{[:]}$ (ibid. Part IV 4.2.11).

For a geometric meaning of $S_{[:]}\Phi$, see ibid. Part IV 4.2.11–4.2.19 in the general case, and ibid. Part IV 4.2.20–4.2.22 in the case $S = |\Delta|^n$ with $|\Delta| := \{t \in \mathbf{R} \mid 0 \leq t < 1\}$.

4.3.8. Let $|\text{toric}|_{\sigma,[:]}\subset \text{toric}_{\sigma,[:]}$ be the inverse image of $|\text{toric}|_\sigma$ under the map $\text{toric}_{\sigma,[:]}\rightarrow \text{toric}_\sigma$. We have a projection $\text{toric}_{\sigma,[:]}\rightarrow |\text{toric}|_{\sigma,[:]}$. This is a unique continuous map which is compatible with the projection $\text{toric}_\sigma\rightarrow |\text{toric}|_\sigma$.

4.3.9. We give an additive presentation of $|\text{toric}|_{\sigma,[:]}$.

We can identify $|\text{toric}|_{\sigma,[:]}$ with the set of equivalence classes of $((\sigma_j, N_j)_{1 \leq j \leq n}, b)$, where $n \geq 0$, σ_j are faces of σ such that $\{0\} \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n$ (here σ_n need not coincide with σ), N_j is an element of the interior of σ_j , and $b \in \sigma_{\mathbf{R}}$. The equivalence relation is that $((\sigma_j, N_j)_{1 \leq j \leq n}, b)$ and $((\sigma'_j, N'_j)_{1 \leq j \leq n'}, b')$ are equivalent if and only if $n' = n$, $\sigma'_j = \sigma_j$, $N'_j \equiv c_j N_j \pmod{\sigma_{j-1, \mathbf{R}}}$ for some $c_j \in \mathbf{R}_{>0}$ ($1 \leq j \leq n$), and $b' \equiv b \pmod{\sigma_{n, \mathbf{R}}}$. Here σ_0 denotes $\{0\}$.

The projection $|\text{toric}|_{\sigma,[:]}\rightarrow |\text{toric}|_\sigma$ is understood as class $((\sigma_j, N_j), b) \mapsto \text{class}(\sigma_n, b)$.

4.3.10. Let $x = \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}, b) \in |\text{toric}|_{\sigma,[:]}$. By a *good base* for x , we mean a family of elements $(N_s)_{s \in S}$ of σ_n satisfying the following conditions (i) and (ii).

(i) The index set S is the disjoint union of some subsets S_j ($1 \leq j \leq n$), and the following holds for each j ($1 \leq j \leq n$). Let $S_{\leq j} := \bigsqcup_{k \leq j} S_k \subset S$. Then $N_s \in \sigma_j$ if $s \in S_{\leq j}$, and $(N_s)_{s \in S_{\leq j}}$ is a base of the \mathbf{R} -vector space $\sigma_{j, \mathbf{R}}$.

(ii) There are $a_s \in \mathbf{R}_{>0}$ ($s \in S_j$) such that $N_j \equiv \sum_{s \in S_j} a_s N_s \pmod{\sigma_{j-1, \mathbf{R}}}$ for $1 \leq j \leq n$.

This is a $[:]$ -version of good base discussed in [24] 6.3.

Proposition 4.3.11. *A good base for x exists.*

Proof. For each j , take a simplicial subcone of the cone $(\sigma_j + \sigma_{j-1, \mathbf{R}})/\sigma_{j-1, \mathbf{R}}$ consisting of the interior points except the origin and including the class of N_j . Take a base $(\bar{N}_s)_{s \in S_j}$ of this cone and lift \bar{N}_s ($s \in S_j$) to an element N_s of σ_j for each j . Then these N_s 's form a good base. \square

4.3.12. We describe the topology of $|\text{toric}|_{\sigma,[:]}$ by using the additive presentation 4.3.9.

Let $n \geq 0$, let $\{0\} \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n$ be faces of σ , and let $(N_s)_{s \in S}$ be a finite family of elements of σ_n satisfying the condition (i) in 4.3.10.

Fix an \mathbf{R} -subspace B of $\sigma_{\mathbf{R}}$ such that $\sigma_{\mathbf{R}} = \sigma_{n, \mathbf{R}} \oplus B$.

Let U be the subset of $|\text{toric}|_{\sigma,[:]}$ consisting of classes of $((\sigma'_k, N_k)_{1 \leq k \leq n'}, b)$ satisfying the following conditions. There is an injective increasing map $\theta : \{1, \dots, n'\} \rightarrow \{1, \dots, n\}$ such that $\sigma'_k = \sigma_{\theta(k)}$ for $1 \leq k \leq n'$. For $1 \leq k \leq n'$, if we write $N_k \equiv \sum_s y_s N_s \pmod{\sigma'_{k-1, \mathbf{R}}}$, where s ranges over all elements of S belonging to S_j for some j such that $\theta(k-1) < j \leq \theta(k)$, then $y_s \in \mathbf{R}_{>0}$. (In the case $k = 1$, $\theta(k-1)$ means 0.) If we write $b \equiv b' + \sum_s y_s N_s \pmod{\sigma'_{n', \mathbf{R}}}$, where $b' \in B$ and s ranges over all elements of S belonging to S_j for some j such that $\theta(n') < j$, then $y_s \in \mathbf{R}_{>0}$.

Then U is an open set of $|\text{toric}|_{\sigma,[:]}$.

For each j , choose an element c_j of S_j . Let T be the complement of $\{c_j \mid 1 \leq j \leq n\}$ in S . Then we have a homeomorphism

$$U \simeq \mathbf{R}_{\geq 0}^n \times \mathbf{R}_{>0}^T \times B$$

defined as follows. Let $x' = \text{class}((\sigma'_k, N_k)_{1 \leq k \leq n'}, b) \in U$, and let θ, y_s ($s \in S$), $b' \in B$ be as above. Then the image of x' in $\mathbf{R}_{\geq 0}^n \times \mathbf{R}_{>0}^T \times B$ is $((t_j)_{1 \leq j \leq n}, (y_s/y_{c_j})_{s \in T}, b')$, where t_j are as follows.

If j is in the image of θ , then $t_j = 0$.

If j is not in the image of θ and $j < \theta(n')$, then $t_j = y_{c_{j+1}}/y_{c_j}$.

If $\theta(n') < j$, then $t_j = 1/y_{c_j}$.

Note that if $(N_s)_{s \in S}$ is a good base for $x = \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}, b) \in |\text{toric}|_{\sigma, [:]}$, x belongs to the above U and hence U is an open neighborhood of x .

4.3.13. Let the notation be as in 4.2.1. Let

$$\check{E}_{\sigma, [:]}^\sharp := |\text{toric}|_{\sigma, [:]} \times \check{D} \subset \check{E}_{\sigma, [:]} = \text{toric}_{\sigma, [:]} \times \check{D}.$$

Let

$$E_{\sigma, [:]}^\sharp := E_{\sigma, [:]} \cap \check{E}_{\sigma, [:]}^\sharp, \quad \tilde{E}_{\sigma, [:]}^\sharp := \tilde{E}_{\sigma, [:]} \cap \check{E}_{\sigma, [:]}^\sharp.$$

We endow $E_{\sigma, [:]}^\sharp$ and $\tilde{E}_{\sigma, [:]}^\sharp$ with the topologies as the subspaces of $E_{\sigma, [:]}$ and $\tilde{E}_{\sigma, [:]}$, respectively. These coincide with the topologies defined via the projections $E_{\sigma, [:]} \rightarrow E_{\sigma, [:]}^\sharp$ and $\tilde{E}_{\sigma, [:]} \rightarrow \tilde{E}_{\sigma, [:]}^\sharp$, respectively.

4.3.14. We show that as topological spaces, $|\text{toric}|_{\sigma, [:]}$ and $E_{\sigma, [:]}^\sharp$ are independent of the choice of Γ .

We describe the proof for $E_{\sigma, [:]}^\sharp$. The proof for $|\text{toric}|_{\sigma, [:]}$ is similar. In fact, assume that Γ_j ($j = 1, 2$) are strongly compatible with Σ and assume $\Gamma_1 \subset \Gamma_2$. Since $E_{\Gamma_j(\sigma), [:]}^\sharp$ is identified with $(E_{\Gamma_j(\sigma)}^\sharp)_{[:]}$ (the $[:]$ -space of $E_{\Gamma_j(\sigma)}^\sharp$ which is endowed with the inverse image of the log structure of $E_{\Gamma_j(\sigma)}$), $E_{\Gamma_j(\sigma), [:]}^\sharp$ is proper over $E_{\Gamma_j(\sigma)}^\sharp$. Since $E_{\Gamma_1(\sigma)}^\sharp \rightarrow E_{\Gamma_2(\sigma)}^\sharp$ is a homeomorphism (4.2.6), the map $E_{\Gamma_1(\sigma), [:]}^\sharp \rightarrow E_{\Gamma_2(\sigma), [:]}^\sharp$ is proper continuous and it is bijective. Hence it is a homeomorphism.

4.3.15. Let $D_{\Sigma, [:]}$ (resp. $D_{\Sigma, [:]}^\sharp$) be the set of $(\sigma, Z, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}))$, where $\sigma \in \Sigma$, $(\sigma, Z) \in D_\Sigma$ (resp. $(\sigma, Z) \in D_\Sigma^\sharp$), $n \geq 0$, σ_j are faces of σ such that $\{0\} \subsetneq \sigma_1 \subsetneq \dots \subsetneq \sigma_n = \sigma$, and N_j is an element of the interior of σ_j . The notation class is for the equivalence relation that $(\sigma_j, N_j)_{1 \leq j \leq n} \sim (\sigma'_j, N'_j)_{1 \leq j \leq n'}$ if and only if $n' = n$, $\sigma'_j = \sigma_j$, and $N'_j \equiv a_j N_j \pmod{\sigma_{j-1, \mathbf{R}}}$ for some $a_j \in \mathbf{R}_{>0}$ ($1 \leq j \leq n$).

We have canonical maps

$$D_{\Sigma, [:]} \rightarrow D_\Sigma, \quad D_{\Sigma, [:]}^\sharp \rightarrow D_\Sigma^\sharp$$

by sending the class of $(\sigma, Z, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}))$ to (σ, Z) .

4.3.16. Assume that (Σ, Γ) strongly compatible. We define a topology of $\Gamma \setminus D_{\Sigma, [:]}$.

For $\sigma \in \Sigma$, by 4.3.3, E_σ is identified with the set of triples $(q, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}), F)$, where $q \in \text{toric}_\sigma$ and writing q as $\mathbf{e}(a)0_\tau$, $n \geq 0$, σ_j are faces of τ such that $\{0\} \subsetneq \sigma_1 \subsetneq \dots \subsetneq \sigma_n = \tau$, and N_j is an element of the interior of σ_j . The equivalence relation is defined in the same way as in 4.3.3.

For $\sigma \in \Sigma$, consider the map

$$\varphi_{\sigma, [:]} : E_{\sigma, [:]} \rightarrow \Gamma \setminus D_{\Sigma, [:]} ; (q, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}), F) \mapsto \text{class}(\tau, Z, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n}))$$

with $q = \mathbf{e}(a)0_\tau$ and $Z = \exp(\tau_C) \exp(a)F$.

We endow $\Gamma \setminus D_{\Sigma, [:]}$ with the strongest topology for which the maps $\varphi_{\sigma, [:]}$ are continuous for all $\sigma \in \Sigma$.

In this topology, if $(\sigma, Z, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n})) \in D_{\Sigma, [:]}$ and if $F \in Z$, then $\text{class}(\sigma, Z, (\sigma_j, N_j)_{1 \leq j \leq n}) \in \Gamma \setminus D_\Sigma$ is the limit of $\text{class}(\exp(\sum_{j=1}^n z_j \tilde{N}_j)F) \in \Gamma \setminus D$ where $z_j \in \mathbf{C}$, $\text{Im}(z_j) \rightarrow \infty$ for $1 \leq j \leq n$ and $\text{Im}(z_j)/\text{Im}(z_{j+1}) \rightarrow \infty$ for $1 \leq j \leq n-1$.

4.3.17. Here we do not need Γ .

For $\sigma \in \Sigma$, we have a map

$$\varphi_{\sigma, [:]}^\sharp : E_{\sigma, [:]}^\sharp \rightarrow D_{\Sigma, [:]}^\sharp ; (\text{class}((\sigma_j, N_j)_{1 \leq j \leq n}), b, F) \mapsto (\sigma_n, \exp(i\sigma_n, \mathbf{R} + ib)F, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n})).$$

We define the topology of $D_{\Sigma, [:]}^\sharp$ as the strongest topology for which the maps $\varphi_{\sigma, [:]}^\sharp : E_{\sigma, [:]}^\sharp \rightarrow D_{\Sigma, [:]}^\sharp$ are continuous for all $\sigma \in \Sigma$.

In this topology, if $x = (\sigma, Z, \text{class}((\sigma_j, N_j)_{1 \leq j \leq n})) \in D_{\Sigma, [:]}^\sharp$ and if $F \in Z$ and \tilde{N}_j , then x is the limit of $\exp(\sum_{j=1}^n iy_j N_j)F \in D$ where $y_j \in \mathbf{R}_{>0}$, $y_j/y_{j+1} \rightarrow \infty$ for $1 \leq j \leq n$ ($y_{n+1} := 1$).

4.3.18. The maps $\Gamma \setminus D_{\Sigma, [:]} \rightarrow \Gamma \setminus D_\Sigma$ and $D_{\Sigma, [:]}^\sharp \rightarrow D_\Sigma^\sharp$ are continuous.

4.4 Valuative spaces, II

4.4.1. Let E and S be as in 4.3.1.

We recall that there are two kinds of valuative spaces $S_{\text{val}(E)}$ and $S_{\text{val}(|\cdot|)}$ associated to S ([21] Part IV 3.1.3).

In the case where S is an object of $\mathcal{B}'_{\mathbf{R}}(\log)$, S_{val} in Section 3.10 is identified, as a topological space, with the topological space $S_{\text{val}(|\cdot|)}$. On the other hand, S_{val} for an object S of $\mathcal{B}(\log)$ (4.2.12) defined in [24] 3.6.18, 3.6.23 is a locally ringed space over \mathbf{C} with log structure, and it is $S_{\text{val}(\mathbf{C})}$.

These two kinds of S_{val} played important roles in our previous works [24] and [21].

Our rule of the notation S_{val} in [24] and [21] and in this paper is that if S is an object of $\mathcal{B}(\log)$, then S_{val} means $S_{\text{val}(\mathbf{C})}$, but otherwise, S_{val} means $S_{\text{val}(|\cdot|)}$.

We also consider, for a weak fan Σ , the maps $D_{\Sigma, \text{val}} \rightarrow D_{\Sigma, [:]}$ and $D_{\Sigma, \text{val}}^\sharp \rightarrow D_{\Sigma, [:]}^\sharp$.

4.4.2. Let E and S be as in 4.3.1.

As a set, let $S_{\text{val}(|\cdot|)}$ (resp. $S_{\text{val}(E)}$) be the set of triples (s, V, h) , where $s \in S$, V is a submonoid of $(M_S/\mathcal{O}_S^\times)_s^{\text{gp}}$ such that $(M_S^{\text{gp}}/\mathcal{O}_S^\times)_s = V \cup V^{-1}$, $(M_S/\mathcal{O}_S^\times)_s \subset V$ and

$V^\times \cap (M_S/\mathcal{O}_S^\times)_s = \{1\}$ (here V^\times denotes the subgroup $V \cap V^{-1}$ of $(M_S^{\text{gp}}/\mathcal{O}_S^\times)_s$), and h is a homomorphism from $\{f \in M_{S,s}^{\text{gp}} \mid f \text{ mod } \mathcal{O}_{S,s}^\times \in V^\times\}$ to $\mathbf{R}_{>0}$ (resp. E^\times) such that $h(u) = |u(s)|$ (resp. $h(u) = u(s)$) for all $u \in \mathcal{O}_{S,s}^\times$.

The topologies of $S_{\text{val}(|\cdot|)}$ and $S_{\text{val}(E)}$ are defined to be the weakest topology having the following properties (i) and (ii).

(i) The projection $(s, V, h) \mapsto s$ to S is continuous.

(ii) Let U be an open set of S and let $f \in M_S(U)$. Then the subset $W = \{(s, V, h) \in S_{\text{val}(|\cdot|)}$ (resp. $S_{\text{val}(E)}$) $| s \in U, f \text{ mod } \mathcal{O}_{S,s}^\times \in V\}$ of $S_{\text{val}(|\cdot|)}$ (resp. $S_{\text{val}(E)}$) is open, and the map $W \rightarrow \mathbf{R}_{\geq 0}$ (resp. $W \rightarrow E$), which sends (s, V, h) to $h(f)$ if $f \text{ mod } \mathcal{O}_{S,s}^\times \in V^\times$ and to 0 otherwise, is continuous.

For an object S of $\mathcal{B}(\text{log})$, $S_{\text{val}(E)}$ has a structure of a locally ringed space with log structure given in [24] 3.6.23.

For the understandings of these val spaces using inverse limits for blowing ups along the log structure, see [21] Part IV Section 3.1.

4.4.3. For a sharp fs monoid \mathcal{S} , we recall the set $V(\mathcal{S})$ and the canonical map $V(\mathcal{S}) \rightarrow R(\mathcal{S})$ from [21] Part IV 4.1.7, 4.1.8. Let $R(\mathcal{S})$ be as in 4.3.2 and let $V(\mathcal{S})$ be the set of all valuative submonoids V of \mathcal{S}^{gp} such that $V \supset \mathcal{S}$ and $V^\times \cap \mathcal{S} = \{1\}$. We have a surjective map

$$V(\mathcal{S}) \rightarrow R(\mathcal{S})$$

sending $V \in V(\mathcal{S})$ to the element r_V of $R(\mathcal{S})$ which is the map $\mathcal{S} \times \mathcal{S} \setminus \{(1, 1)\} \rightarrow [0, \infty]$ defined by

$$\begin{aligned} r_V(f, g) &= \sup\{a/b \mid (a, b) \in \mathbf{N}^2 \setminus \{(0, 0)\}, f^b/g^a \in V\} \\ &= \inf\{a/b \mid (a, b) \in \mathbf{N}^2 \setminus \{(0, 0)\}, g^a/f^b \in V\}. \end{aligned}$$

4.4.4. We have proper surjective continuous maps $S_{\text{val}(E)} \rightarrow S_{\text{val}(|\cdot|)} \rightarrow S_{[:]}$ to S . Here the second arrow is defined by 4.4.3.

4.4.5. Define

$$\text{toric}_{\sigma, \text{val}} := (\text{toric}_\sigma)_{\text{val}(\mathbf{C})} \supset |\text{toric}|_{\sigma, \text{val}} := (|\text{toric}|_\sigma)_{\text{val}(|\cdot|)},$$

$$E_{\sigma, \text{val}} := (E_\sigma)_{\text{val}(\mathbf{C})} \supset E_{\sigma, \text{val}}^\sharp := (E_\sigma^\sharp)_{\text{val}(|\cdot|)}.$$

Then $E_{\sigma, \text{val}}^\sharp$ is identified with the inverse image of $|\text{toric}|_{\sigma, \text{val}}$ under $E_{\sigma, \text{val}} \rightarrow \text{toric}_{\sigma, \text{val}}$. We have the projections

$$\text{toric}_{\sigma, \text{val}} \rightarrow |\text{toric}|_{\sigma, \text{val}}, \quad E_{\sigma, \text{val}} \rightarrow E_{\sigma, \text{val}}^\sharp.$$

4.4.6. The topological spaces $|\text{toric}|_{\sigma, \text{val}}$ and $E_{\sigma, \text{val}}^\sharp$ are independent of Γ . The proofs are similar to the proof for $E_{\sigma, [:]}^\sharp$ given in 4.3.14.

4.4.7. Let Σ be a weak fan. We define sets $D_{\Sigma, \text{val}}$ and $D_{\Sigma, \text{val}}^\sharp$.

For $\sigma \in \Sigma$, let $Q(\sigma)$ be the set of all rational linear maps $\sigma_{\mathbf{R}} \rightarrow \mathbf{R}$, and let $\tilde{P}(\sigma)$ be the set of all elements h of $Q(\sigma)$ such that $h(\sigma) \subset \mathbf{R}_{\geq 0}$.

Let $D_{\Sigma, \text{val}}$ (resp. $D_{\Sigma, \text{val}}^\sharp$) be the set of quadruple (σ, Z, V, Z') , where $(\sigma, Z) \in D_\Sigma$ (resp. D_Σ^\sharp), V is a submonoid of $Q(\sigma)$ such that $Q(\sigma) = V \cup (-V)$, $\tilde{P}(\sigma) \subset V$, and $V \cap (-V) \cap \tilde{P}(\sigma) = \{0\}$, and if $A \subset \sigma_{\mathbf{R}} \cap \text{Lie}(G')$ denotes the annihilator of $V \cap (-V)$ in the perfect pairing $\sigma_{\mathbf{R}} \cap \text{Lie}(G') \times Q(\sigma) \rightarrow \mathbf{Q}$ of \mathbf{Q} -vector spaces, Z' is an $\exp(A_{\mathbf{C}})$ (resp. $\exp(iA_{\mathbf{R}})$)-orbit in Z .

The canonical maps $D_{\Sigma, \text{val}} \rightarrow D_\Sigma$, $D_{\Sigma, \text{val}}^\sharp \rightarrow D_\Sigma^\sharp$ are given by $(\sigma, Z, V, Z') \mapsto (\sigma, Z)$.

The shapes of the definitions of $D_{\Sigma, \text{val}}$ and $D_{\Sigma, \text{val}}^\sharp$ seem to be slightly different from those in [24], [21] Part III, but are the same. We hope the presentations of the present definitions are better.

4.4.8. We have canonical maps of sets

$$D_{\Sigma, \text{val}} \rightarrow D_{\Sigma, [:]}, \quad D_{\Sigma, \text{val}}^\sharp \rightarrow D_{\Sigma, [:]}^\sharp$$

sending (σ, Z, V, Z') to $(\sigma, Z, \text{class}((\sigma_j, N_j)_j))$, where $V \mapsto r_V \leftrightarrow \text{class}((\sigma_j, N_j)_j)$ by 4.4.3 and 4.3.3.

4.4.9. Assume that (Σ, Γ) is strongly compatible.

The structure of $\Gamma \setminus D_{\Sigma, \text{val}}$ as a locally ringed space with log structure is defined by using $\varphi_{\sigma, \text{val}} : E_{\sigma, \text{val}} \rightarrow \Gamma \setminus D_{\Sigma, \text{val}}$ ($\sigma \in \Sigma$) analogously as 4.2.7.

4.4.10. Here we do not need Γ .

The topology of $D_{\Sigma, \text{val}}^\sharp$ is defined by using $\varphi_{\sigma, \text{val}}^\sharp : E_{\sigma, \text{val}}^\sharp \rightarrow D_{\Sigma, \text{val}}^\sharp$ ($\sigma \in \Sigma$) analogously as 4.3.17.

4.4.11. The continuity of $\Gamma \setminus D_{\Sigma, \text{val}} \rightarrow \Gamma \setminus D_{\Sigma, [:]}$ and the continuity of $D_{\Sigma, \text{val}}^\sharp \rightarrow D_{\Sigma, [:]}^\sharp$ are clear because the topologies of the val spaces are defined by using $E_{\sigma, \text{val}}$, $E_{\sigma, \text{val}}^\sharp$, and the topologies of $[:]$ -spaces are defined by $E_{\sigma, [:]}$ and $E_{\sigma, [:]}^\sharp$, and $E_{\sigma, \text{val}} \rightarrow E_{\sigma, [:]}$ and $E_{\sigma, \text{val}}^\sharp \rightarrow E_{\sigma, [:]}^\sharp$ are continuous.

4.5 Nilpotent orbits and $\text{SL}(2)$ -orbits

In this section we prove the continuity of the CKS map (Theorem 4.5.5), which is the most important bridge in the fundamental diagram. We prove this together with that E_σ is open in \tilde{E}_σ (Theorem 4.5.6), which implies that E_σ is a log manifold (Theorem 4.5.7).

Note that there are mistakes in the corresponding parts [24] and [21] Part III, which are corrected in [21] Part IV Appendix in a rather complicated way and, unfortunately, the correction itself contains several errors. So in this section 4.5, we present the whole structure of the corrected (and improved) proof in the present context. For more precise explanations about the mistakes and errors in the previous works, see Remark 4.5.31 at the end of this section.

4.5.1. In order to define the map ψ in Theorem 4.5.5 below as a map of sets, we explain how to associate an $\text{SL}(2)$ -orbit to a nilpotent orbit in 4.5.1–4.5.3. This construction is a generalization of [21] Part II 2.4.6 based on the $\text{SL}(2)$ -orbit theorem in many variables for MHS ([20]). The pure case of the construction is essentially due to Cattani, Kaplan and Schmid ([10]).

Let (N_1, \dots, N_n, F) generate a nilpotent orbit (4.1.3). We give the associated triple (ρ, φ, Z) , where (ρ, φ) is an $\mathrm{SL}(2)$ -orbit in n variables for (G_{red}, h_0) and $Z \subset D$ such that $(p, Z) \in D_{\mathrm{SL}(2)}$ for $p = \mathrm{class}(\rho, \varphi) \in D_{\mathrm{red}, \mathrm{SL}(2)}$.

For $0 \leq j \leq n$, we denote by $W^{(j)}$ the functor

$$V \mapsto (V, M(N_{1,V} + \dots + N_{j,V}, W(V)))$$

from $\mathrm{Rep}(G)$ to the category of finite-dimensional \mathbf{Q} -vector spaces endowed with an increasing filtration on its realification. Here $N_{j,V}$ denotes the image of N_j in $\mathrm{End}_{\mathbf{R}}(V_{\mathbf{R}})$ for each j and $M(N_{1,V} + \dots + N_{j,V}, W(V))$ is the relative monodromy filtration of $N_{1,V} + \dots + N_{j,V}$ with respect to $W(V)$. In particular, $W^{(0)} = W$.

The functor $(W^{(n)}, F) : V \mapsto (V, W^{(n)}(V), F(V))$ is a G -MHS. Let $(W^{(n)}, \hat{F}_{(n)})$ be the \mathbf{R} -split G -MHS defined by the canonical splitting of $W^{(n)}$ associated to the G -MHS $(W^{(n)}, F)$ (1.3.5). Then $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$ is a G -MHS. Let $(W^{(n-1)}, \hat{F}_{(n-1)})$ be the associated \mathbf{R} -split G -MHS defined by the canonical splitting of $W^{(n-1)}$ associated to the G -MHS $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$. Then $(W^{(n-2)}, \exp(iN_{n-1})\hat{F}_{(n-1)})$ is a G -MHS. In this way, we obtain inductively an \mathbf{R} -split G -MHS $(W^{(j)}, \hat{F}_{(j)})$ for $0 \leq j \leq n$. We have $\hat{F}_{(j)} \in \tilde{D}$.

We define $\mathbf{r} \in D$ as follows. If all N_j are 0, let $\mathbf{r} = F$. If $N_j \neq 0$ for some j , let k be the smallest such j ($1 \leq j \leq n$), and let $\mathbf{r} = \exp(iN_k)\hat{F}_{(k)}$. (Note that we have used the symbol \mathbf{r}_1 in [21] Part II 2.4.6 instead of \mathbf{r} here. Thus the notation here is not compatible with that in [20] Theorem 0.5 and in [21] Part II 2.4. See the remark after [21] Part II Theorem 2.4.2.)

4.5.2. Now assume that G is reductive. Then there is a unique homomorphism $\tau : \mathbf{G}_{m,\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$ whose j -th component τ_j gives the \mathbf{R} -splitting of $W^{(j)}$ associated to the \mathbf{R} -split G -MHS $(W^{(j)}, \hat{F}_{(j)})$. This τ_j also gives $\mathrm{spl}_{W^{(j)}}^{\mathrm{BS}}(\mathbf{r})$ for $1 \leq j \leq n$. Let $\tau^* : \mathbf{G}_{m,\mathbf{R}}^n \rightarrow G'_{\mathbf{R}} \subset G_{\mathbf{R}}$ be the homomorphism defined by $\tau^*(t) = \tau(t)k_0(\prod_{j=1}^n t_j)^{-1}$.

Consider the direct sum decomposition of $\mathrm{Lie}(G'_{\mathbf{R}})$ by the adjoint action of $\mathbf{G}_{m,\mathbf{R}}^n$ on it via τ^* . For $1 \leq j \leq n$, let \hat{N}_j be the component of N_j in this direct sum decomposition on which the s -th factors of \mathbf{G}_m^n for $1 \leq s < j$ act trivially. Then there is a unique homomorphism $\rho : \mathrm{SL}(2)_{\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$ such that $\tau_j^*(t) = \rho(g_1, \dots, g_n)$, where $g_s = \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix}$ for $1 \leq s \leq j$ and $g_s = 1$ for $j < s \leq n$ and such that the induced map $\mathrm{Lie}(\rho) : \mathfrak{sl}(2, \mathbf{R})^n \rightarrow \mathrm{Lie}(G_{\mathbf{R}})$ sends the j -th $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to \hat{N}_j for $1 \leq j \leq n$. There is a unique holomorphic map $\varphi : \mathbf{P}^1(\mathbf{C})^n \rightarrow D$ such that $\varphi(gz) = \rho(g)\varphi(z)$ for all $g \in \mathrm{SL}(2, \mathbf{C})^n$ and for all $z \in \mathbf{P}^1(\mathbf{C})^n$ and such that $\varphi(\mathbf{i}) = \mathbf{r}$. This holomorphic map φ is also characterized by the properties $\varphi(gz) = \rho(g)\varphi(z)$ for all $g \in \mathrm{SL}(2, \mathbf{C})^n$ and for all $z \in \mathbf{P}^1(\mathbf{C})^n$ and $\varphi(\mathbf{0}) = \hat{F}_{(n)}$. It satisfies $\varphi(\{0\}^j \times \{i\}^{n-j}) = \hat{F}_{(j)}$ for $0 \leq j \leq n$.

We call this (ρ, φ) the $\mathrm{SL}(2)$ -orbit associated to (N_1, \dots, N_n, F) .

See [10] 4.20 for the above facts.

4.5.3. Now G is not necessarily assumed to be reductive. Let $p = \mathrm{class}(\rho, \varphi) \in D_{\mathrm{red}, \mathrm{SL}(2)}$. We define $x = (p, Z) \in D_{\mathrm{SL}(2)}$ as follows. If $N_j \neq 0$ for some j and if $\mathrm{gr}^W(N_k) = 0$ for

$k = \min\{j \mid N_j \neq 0\}$, let x be the unique B -orbit lying over p such that $\mathbf{r} \in Z$. Otherwise, let x be the unique A -orbit lying over p such that $\mathbf{r} \in Z$.

This $x = (p, Z)$ is called the element of $D_{\mathrm{SL}(2)}$ associated to (N_1, \dots, N_n, F) .

The rank of p , the family of weight filtrations $(W_x^{(k)})_{1 \leq k \leq r}$ (resp. $(W_x^{(k)})_{0 \leq k \leq r}$) associated x , and the homomorphism $\tau_x = (\tau_{x,j})_{1 \leq j \leq r}$ (resp. $(\tau_{x,k})_{0 \leq k \leq r}$) associated to x in the case where x is an A (resp. B)-orbit (3.3.2) are described as follows.

(1) Let $W^{(j)}$ ($0 \leq j \leq n$) be as in 4.5.1 and let $\{j \mid 1 \leq j \leq n, W^{(j)} \neq W^{(j-1)}\} = \{s(1), \dots, s(r)\}$ with $1 \leq s(1) < \dots < s(r) \leq n$. Then r is the rank of p . We have $W_x^{(k)} = W^{(s(k))}$ for $1 \leq k \leq r$. In the case where x is a B -orbit, $W_x^{(0)} = W$.

(2) Let $\tau_j : \mathbf{G}_{m,\mathbf{R}} \rightarrow G_{\mathbf{R}}$ ($0 \leq j \leq n$) be the homomorphism corresponding to the splitting of $W^{(j)}$ given by $\hat{F}_{(j)}$ in 4.5.1. Then $\tau_{x,k} = \tau_{s(k)}$ for $1 \leq k \leq r$. In the case where x is a B -orbit, $\tau_{x,0} = \tau_0$.

Let $s(r+1) := n+1$. Then $\tau_j = \tau_{x,k}$ if $1 \leq k \leq r$ and $s(k) \leq j < s(k+1)$, and $\tau_j = \tau_0$ if $0 \leq j < s(1)$. Hence we have:

Assume $N_1 \neq 0$ in the case $n \geq 1$. Then for $(t_j)_{1 \leq j \leq n} \in \mathbf{G}_{m,\mathbf{R}}^n$, we have

$$\tau(t) = \tau_x(t')$$

$(\tau(t) := \prod_{j=1}^n \tau_j(t_j), t' \in \mathbf{G}_{m,\mathbf{R}}^{\{1, \dots, r\}}$ if x is an A -orbit, $t' \in \mathbf{G}_{m,\mathbf{R}}^{\{0, \dots, r\}}$ if x is a B -orbit), where t' is defined as follows. For $1 \leq k \leq r$, $t'_k = \prod_{s(k) \leq j < s(k+1)} t_j$. In the case where x is a B -orbit, $t'_0 = \prod_{1 \leq j < s(1)} t_j$.

Lemma 4.5.4. *There is a unique map $\psi : D_{\Sigma,[:]^\sharp} \rightarrow D_{\mathrm{SL}(2)}$ which sends $(\sigma, Z, \mathrm{class}((\sigma_j, N_j)_{1 \leq j \leq n}))$ to the element of $D_{\mathrm{SL}(2)}$ associated to (N_1, \dots, N_n, F) for $F \in Z$ in 4.5.3.*

This is proved similarly to [21] Part IV 4.5.9.

Theorem 4.5.5. *The map $\psi : D_{\Sigma,[:]^\sharp} \rightarrow D_{\mathrm{SL}(2)}^I$ is continuous.*

This map is the unique continuous extension of the identity map of D . We call this map ψ the *CKS map* respecting the work of Cattani, Kaplan and Schmid on their $\mathrm{SL}(2)$ -orbit theorem in many variables in pure case in [10]. This is the most important map in the fundamental diagram.

For the proof of Theorem 4.5.5, we use the following theorem.

Theorem 4.5.6. *Let $\sigma \in \Sigma$. For the topologies in 4.2.5, E_σ is open in \tilde{E}_σ .*

We will prove Theorem 4.5.6 in Proposition 4.5.8–4.5.17.

From Theorem 4.5.6, we obtain

Theorem 4.5.7. *Let $\sigma \in \Sigma$. Then \tilde{E}_σ (4.2.5) and E_σ are log manifolds.*

Proof. For \tilde{E}_σ , the argument of the proof of [24] Proposition 3.5.10 for the pure standard case also works for the present case (see also [21] Part III 4.1.1).

The result for E_σ follows from this by Theorem 4.5.6. \square

We start to prove Theorem 4.5.6. The following Proposition 4.5.8 is a G -MHS version of [24] Proposition 7.1.1.

Proposition 4.5.8. *Let Σ be a weak fan and let $\sigma \in \Sigma$. Let $\tilde{A}(\sigma)$ be the closed analytic subset of \check{D} defined by $\tilde{A}(\sigma) = \{F \in \check{D} \mid N(F^p V_{\mathbf{C}}) \subset F^{p-1} V_{\mathbf{C}} \text{ for all } N \in \sigma, V \in \text{Rep}(G), p \in \mathbf{Z}\}$, and let $A(\sigma)$ be the subset of \check{D} consisting of all elements F such that $(\sigma, \exp(\sigma_{\mathbf{C}})F)$ is a nilpotent orbit. Then $A(\sigma)$ is an open set of $\tilde{A}(\sigma)$ in the strong topology of $\tilde{A}(\sigma)$ in \check{D} (4.2.5).*

Proof. Assume $F \in A(\sigma)$ and assume that a directed family $F_{\lambda} \in \tilde{A}(\sigma)$ converges to F . We prove that $F_{\lambda} \in A(\sigma)$ for λ sufficiently large. Let $H \in D$. Take a faithful representation $V \in \text{Rep}(G)$ and take \mathbf{R} -polarizations $\langle \cdot, \cdot \rangle_w$ of $H(\text{gr}_w^W V)$ which are stable under G' . By 3.4.13 (2), for λ sufficiently large, the annihilator of $F_{\lambda}^p(\text{gr}_w^W V)$ with respect to $\langle \cdot, \cdot \rangle_w$ is $F^{w+1-p}(\text{gr}_w^W V)$. Take a finite set N_1, \dots, N_n of generators of the cone σ . The proof of [24] Proposition 7.1.1 shows that if λ is sufficiently large, then $\exp(\sum_{j=1}^n z_j N_j)F_{\lambda}(\text{gr}_w^W V) \in G(\mathbf{R})H(\text{gr}_w^W V)$ for every w if $\text{Im}(z_j)$ are sufficiently large. By 1.4.7, $F_{\lambda} \in A(\sigma)$ for λ sufficiently large. \square

4.5.9. The proof of Theorem 4.5.6 consists of complicated inductions. To clarify the idea, we explain here the proof when the cone σ is of rank one.

By the canonical isomorphisms $\Gamma(\sigma) \simeq \mathbf{N}$ and $P(\sigma) \simeq \mathbf{N}$, we have canonical isomorphisms $\text{toric}_{\sigma} \simeq \mathbf{C}$, $|\text{toric}|_{\sigma} \simeq \mathbf{R}_{\geq 0}$. We identify \check{E}_{σ} with $\mathbf{C} \times \check{D}$ and $\check{E}_{\sigma}^{\sharp}$ with $\mathbf{R}_{\geq 0} \times \check{D}$. Since E_{σ} is the inverse image of $\check{E}_{\sigma}^{\sharp}$ under $|\cdot| : \check{E}_{\sigma} \rightarrow \check{E}_{\sigma}^{\sharp}$, it is sufficient to prove that E_{σ}^{\sharp} is open in $\tilde{E}_{\sigma}^{\sharp}$.

Assume that $(q_{\lambda}, F_{\lambda}) \in \tilde{E}_{\sigma}^{\sharp}$ converges to $(0, F) \in E_{\sigma}^{\sharp}$. We prove that $(q_{\lambda}, F_{\lambda}) \in E_{\sigma}^{\sharp}$ for sufficiently large λ . Taking subsequences, we divide it into two cases: the case where $q_{\lambda} = 0$ for all λ and the case $q_{\lambda} \neq 0$ for all λ .

Assume $q_{\lambda} = 0$. This is the case Proposition 4.5.8 for $\text{rank}(\sigma) = 1$.

Assume $q_{\lambda} \neq 0$. Let $q_{\lambda} = e^{-2\pi y_{\lambda}}$. We have to see that, for any sufficiently large λ , $\exp(iy_{\lambda}N)F_{\lambda} \in D$ (which means $(q_{\lambda}, F_{\lambda}) \in E_{\sigma}$ by definition 4.2.5). Here $N \in \sigma$ is such that $\exp(N)$ corresponds $1 \in \mathbf{N}$ via $\Gamma(\sigma) \simeq \mathbf{N}$.

First, applying [24] Proposition 3.1.6 to $S := \tilde{E}_{\sigma} \subset X := \check{E}_{\sigma} = \mathbf{C} \times \check{D}$, $s = (0, F)$, $s_{\lambda} = (q_{\lambda}, F_{\lambda})$, and $A := \{0\} \times \check{D}$, we find $F_{\lambda}^* \in \check{D}$ which is very near to F_{λ} such that $(0, F_{\lambda}^*) \in \tilde{E}_{\sigma}$ and F_{λ}^* converges to F . For the precise meaning of “very near” here, see the condition (i) in Lemma 4.5.13.

By Proposition 4.5.8 again, we may assume that

(1) (N, F_{λ}^*) generates a nilpotent orbit.

Next, let $\tau : \mathbf{G}_{m, \mathbf{R}} \rightarrow G_{\mathbf{R}}$ be the homomorphism corresponding to the splitting of $W^{(1)} := M(N, W)$ given by $\hat{F}_{(1)}$ in 4.5.3, and let $\tau_{\lambda} = \tau(1/\sqrt{y_{\lambda}})$. By (1), $(W^{(1)}, F_{\lambda}^*)$ is a G -MHS. Together with the fact that τ_{λ} splits $W^{(1)}$, we have

(2) $\tau_{\lambda}^{-1}F_{\lambda}^* \rightarrow \hat{F}_{(1)}$.

From this, we have

(3) $\tau_{\lambda}^{-1}\exp(iy_{\lambda}N)F_{\lambda}^* = \exp(iN)\tau_{\lambda}^{-1}F_{\lambda}^* \rightarrow \exp(iN)\hat{F}_{(1)}$.

Since F_{λ} and F_{λ}^* are very near, we have $\tau_{\lambda}^{-1}\exp(iy_{\lambda}N)F_{\lambda} \rightarrow \exp(iN)\hat{F}_{(1)} \in D$. Since D is open in \check{D} , we conclude

(4) $\exp(iy_{\lambda}N)F_{\lambda} \in D$ for sufficiently large λ .

4.5.10. Note that a large part of the following 4.5.10–4.5.17 is a copy from [21] Part IV Appendix (after some modifications explained in Remark 4.5.31). For example, the

statement of Proposition 4.5.14 is almost identical to that of [21] Part IV Proposition A.1.7.

To prove Theorem 4.5.6, since $\tilde{E}_{\sigma,\text{val}} \rightarrow \tilde{E}_\sigma$ is proper and surjective, and $E_{\sigma,\text{val}} \subset \tilde{E}_{\sigma,\text{val}}$ is the inverse image of $E_\sigma \subset \tilde{E}_\sigma$, it is sufficient to prove that $E_{\sigma,\text{val}}$ is open in $\tilde{E}_{\sigma,\text{val}}$. Since we have the continuous projection $\tilde{E}_{\sigma,\text{val}} \rightarrow \tilde{E}_{\sigma,\text{val}}^\sharp$ for which $E_{\sigma,\text{val}}$ is the inverse image of $E_{\sigma,\text{val}}^\sharp$, it is sufficient to prove that $E_{\sigma,\text{val}}^\sharp$ is open in $\tilde{E}_{\sigma,\text{val}}^\sharp$.

Assume that a directed system $w_\lambda = (q_\lambda, F'_\lambda) \in \tilde{E}_{\sigma,\text{val}}^\sharp$ ($\lambda \in L$, where L is a directed ordered set) converges in $\tilde{E}_{\sigma,\text{val}}^\sharp$ to $w = (q, F') \in E_{\sigma,\text{val}}^\sharp$. We prove that $w_\lambda \in E_{\sigma,\text{val}}^\sharp$ for any sufficiently large λ .

4.5.11. We fix notation.

Let $w_{[:]}$ = (t, F') (resp. $w_{\sigma,[:]}$ = (t_λ, F'_λ)) be the image w (resp. w_λ) in $\check{E}_{\sigma,[:]^\sharp} = |\text{toric}|_{\sigma,[:]^\sharp} \times \check{D}$ ($t, t_\lambda \in |\text{toric}|_{\sigma,[:]^\sharp}$, $F', F'_\lambda \in \check{D}$).

We use the additive presentation of $|\text{toric}|_{\sigma,[:]^\sharp}$ in 4.3.9. Let n and σ_n be as those in there for the point t . Take an \mathbf{R} -subspace B of $\sigma_{\mathbf{R}}$ such that $\sigma_{\mathbf{R}} = \sigma_{n,\mathbf{R}} \oplus B$. Take a good base $(N_s)_{s \in S}$ for t and let $S_j \subset S$ be as in the definition of a good base (4.3.10). Write $t = (\text{class}((\sigma_j, N_j)_{1 \leq j \leq n}), b)$, where N_j is an interior point of σ_j and $b \in B$. Let $a_s \in \mathbf{R}_{>0}$ such that $N_j \equiv \sum_{s \in S_j} a_s N_s \pmod{\sigma_{j-1,\mathbf{R}}}$ for all $1 \leq j \leq n$.

Let $F = \exp(ib)F'$. Then $((N_s)_{s \in S}, F)$ generates a nilpotent orbit. We have also the B -component $b_\lambda \in B$ of t_λ . Let $F_\lambda = \exp(ib_\lambda)F'_\lambda$. Then F_λ converges to F in \check{D} .

Take an open neighborhood V of b in $B_{\mathbf{C}}$ such that the map $\text{toric}_{\sigma_n} \times V \rightarrow \text{toric}_\sigma$ induced by the canonical map $\text{toric}_{\sigma_n} \times B_{\mathbf{C}} \rightarrow \text{toric}_\sigma$ is an open immersion. We may assume $b_\lambda \in V \cap B$.

Let $\sigma' \subset \sigma$ be the cone generated by N_s ($s \in S$). Replacing N_s by N_s/r for some integer $r > 0$, we may assume that $\log \Gamma(\sigma') \subset \bigoplus_{s \in S} \mathbf{N} N_s \subset \bigoplus_{s \in S} \mathbf{Z} N_s$. We have the injective homomorphism $\Gamma(\sigma') \hookrightarrow \mathbf{N}^S$. For $s \in S$ we have the s -component $\Gamma(\sigma') \rightarrow \mathbf{N}$, which is an element q_s of $P(\sigma')$, regarded as a holomorphic function on $\text{Spec}(\mathbf{C}[P(\sigma')])^{\text{an}} \times V = \text{toric}_{\sigma'} \times V$.

We use the description of the topology of $|\text{toric}|_{\sigma,[:]^\sharp}$ in 4.3.12 associated to the good base $(N_s)_{s \in S}$ for t . We use the notation there. Let U be the open neighborhood of t . We may assume that all t_λ belong to U .

Since $w_\lambda \rightarrow w$ and since $|\text{toric}|_{\sigma',\text{val}} \times B$ is open in $|\text{toric}|_{\sigma,\text{val}}$, we may assume that all w_λ belong to the open set $\tilde{E}_{\sigma',\text{val}} \times V$ of $\tilde{E}_{\sigma,\text{val}}$.

Note that we have $c_j \in S_j$ and T as in 4.3.12. In the case $q_s(w_\lambda) \neq 0$, $y_{\lambda,s} \in \mathbf{R}_{>0}$ is defined by $|q_s(w_\lambda)| = \exp(-2\pi y_{\lambda,s})$. If $s \in S_j$ and $q_s(w_\lambda) \neq 0$, then $q_{s'}(w_\lambda) \neq 0$ for any $s' \in S_{j'}$ with $j' \geq j$, and in the case $s \neq c_j$, $y_{\lambda,s}/y_{\lambda,c_j}$ gives the s -component of t_λ in $U \simeq \mathbf{R}_{\geq 0}^n \times \mathbf{R}_{\geq 0}^T \times B \rightarrow \mathbf{R}_{\geq 0}^T$. If $s \in S_j$ and if the set $I = \{\lambda \mid q_s(w_\lambda) \neq 0\}$ is cofinal in the directed index set, then for $s' \in S_{j'}$ with $j' > j$ (resp. for $s' \in S_j$) and for $\lambda \in I$, we have $y_{\lambda,s'}/y_{\lambda,s} \rightarrow 0$ (resp. $y_{\lambda,s'}/y_{\lambda,s} \rightarrow a_{s'}/a_s$).

Let $x_{\text{SL}(2)} \in D_{\text{SL}(2)}$ be the element associated to (N_1, \dots, N_n, F) in 4.5.3. We have a homomorphism $\tau : \mathbf{G}_m^n \rightarrow G_{\mathbf{R}}$ as in (2) in 4.5.3. In the following, τ_j denotes its j -th component.

4.5.12. We may assume that there exists an integer m such that $1 \leq m \leq n+1$ and such that $q_s(x_\lambda) = 0$ for any λ and $s \in S_{\leq m-1}$ and $q_s(x_\lambda) \neq 0$ for any λ and $s \in S_{\geq m} := \bigsqcup_{k \geq m} S_k$

($S_{\leq 0}$ and $S_{\geq n+1}$ are defined as the empty set).

For $m \leq j \leq n$, let

$$e_{\lambda, \geq j} = \exp\left(\sum_{s \in S_{\geq j}} iy_{\lambda, s} N_s\right) \in G_{\mathbf{C}},$$

$$\tau_{\lambda, j} = \tau_j \left(\sqrt{y_{\lambda, c_{j+1}} / y_{\lambda, c_j}} \right) \in G_{\mathbf{R}}, \quad \tau_{\lambda, \geq j} = \prod_{k=j}^n \tau_{\lambda, k} \in G_{\mathbf{R}}$$

($y_{\lambda, c_{n+1}}$ denotes 1). Let $\hat{F}_{(j)}$ ($1 \leq j \leq n$) be as in 4.5.1 associated to (N_1, \dots, N_n, F) .

Lemma 4.5.13. *Let the situation and the notation be as above. Let d be a local metric on \check{D} that is compatible with the analytic structure. Let $m \leq j \leq n$ and let $e \geq 0$. Then for any sufficiently large λ , there exist $F_{\lambda}^* \in \check{D}$ satisfying the following (i) and (ii).*

- (i) $y_{\lambda, s}^e d(F_{\lambda}, F_{\lambda}^*) \rightarrow 0$ ($\forall s \in S_j$).
- (ii) (N_s, F_{λ}^*) satisfies Griffiths transversality for any $s \in S_{\leq j}$.

Proof. We apply [24] Proposition 3.1.6 to the following situation: $S := \check{E}_{\sigma'} \times V \subset X := \check{E}_{\sigma'} \times V = \text{toric}_{\sigma'} \times V \times \check{D}$ (4.2.5), where σ' and V are as in 4.5.11, and A is the closed analytic subspace of X defined by $q_s = 0$ for $s \in S_{\leq j}$.

Consider the images of $w_{\lambda} = (q_{\lambda}, F'_{\lambda})$ and $w = (q, F')$ of 4.5.10 in X and S , respectively, and let F_{λ} , F , and $y_{\lambda, s}$ be as in 4.5.11.

Take a metric on a neighborhood of the image of w in X as the direct sum of a local metric on $\text{toric}_{\sigma'} \times V$ and the local metric d on \check{D} . Then we get Lemma 4.5.13. \square

The next is a key proposition of this section.

Proposition 4.5.14. *Let the situation and the assumption be as above. Then the following assertions (A_j) ($m-1 \leq j \leq n$), (B_j) ($m \leq j \leq n$), (C_j) ($m \leq j \leq n$) are true.*

(A_j) (resp. (B_j), resp. (C_j)) for $m \leq j \leq n$: Let $e \geq 1$. Then for any sufficiently large λ , there are $F_{\lambda}^{(j)} \in \check{D}$ satisfying the following (1)–(3).

- (1) $y_{\lambda, c_j}^e d(F_{\lambda}, F_{\lambda}^{(j)}) \rightarrow 0$.
- (2) $((N_s)_{s \in S_{\leq j}}, e_{\lambda, \geq j+1} F_{\lambda}^{(j)})$ generates a nilpotent orbit.
- (3) $\tau_{\lambda, \geq j+1}^{-1} e_{\lambda, \geq j+1} F_{\lambda}^{(j)} \rightarrow \exp(iN_{j+1}) \hat{F}_{(j+1)}$.
(resp. (3) $\tau_{\lambda, \geq j}^{-1} e_{\lambda, \geq j+1} F_{\lambda}^{(j)} \rightarrow \hat{F}_{(j)}$.
resp. (3) $\tau_{\lambda, \geq j}^{-1} e_{\lambda, \geq j} F_{\lambda}^{(j)} \rightarrow \exp(iN_j) \hat{F}_{(j)}$.)

Here (A_n) is formulated by understanding $N_{n+1} = 0$ and $\hat{F}_{(n+1)} = F$.

(A_{m-1}): For any sufficiently large λ , we have the following (2) and (3).

(2) $((N_s)_{s \in S_{\leq m-1}}, e_{\lambda, \geq m} F_{\lambda})$ generates a nilpotent orbit.

(3) $\tau_{\lambda, \geq m}^{-1} e_{\lambda, \geq m} F_{\lambda} \rightarrow \exp(iN_m) \hat{F}_{(m)}$.

Remark 4.5.15. The case $n = 1$ was treated in 4.5.9. The case $n = m = 1$ is the case $q_{\lambda} \neq 0$ for all λ in 4.5.9. The case $n = 1$ and $m = 2$ is the case $q_{\lambda} = 0$ for all λ in 4.5.9. In the case $n = m = 1$, the course of the arguments about (1), (2), (3), (4) in 4.5.9 is understood as (1) = (A₁) \Rightarrow (2) = (B₁) \Rightarrow (3) = (C₁) \Rightarrow (4) = (A₀).

4.5.16. We prove Proposition 4.5.14 by using the downward induction of the form

$$(A_j) \Rightarrow (B_j) \Rightarrow (C_j) \Rightarrow (A_{j-1}). \text{ (Here } m \leq j \leq n\text{.)}$$

$(B_j) \Rightarrow (C_j)$ is clear.

We prove $(A_j) \Rightarrow (B_j)$. By (2), for any sufficiently large λ , $(W^{(j)}, e_{\lambda, \geq j+1} F_\lambda^{(j)})$ is a mixed Hodge structure, where $W^{(j)}$ denotes the relative monodromy filtration of $N_1 + \cdots + N_j$ with respect to W . Since $\tau_{\lambda, j}$ splits $W^{(j)}$, we have (B_j) .

We prove $(C_{j+1}) \Rightarrow (A_j)$ if $m-1 \leq j < n$ and (A_n) .

Let $e \geq 1$. If $m \leq j \leq n$ (resp. $j = m-1$), then, by Lemma 4.5.13, there are $F_\lambda^{(j)} \in \check{D}$ satisfying (1) and (resp. $F_\lambda^{(j)} := F_\lambda$ satisfies) the condition

$$(2') (N_s, F_\lambda^{(j)}) \text{ satisfies Griffiths transversality for any } s \in S_{\leq j}.$$

When $m-1 \leq j < n$, by (C_{j+1}) , there are $F_\lambda^{(j+1)} \in \check{D}$ satisfying (resp. when $j = n$, $F_\lambda^{(j+1)} = F_\lambda$ satisfies) the following.

$$(1'') y_{\lambda, c_{j+1}}^e d(F_\lambda, F_\lambda^{(j+1)}) \rightarrow 0.$$

$$(3'') \tau_{\lambda, \geq j+1}^{-1} e_{\lambda, \geq j+1} F_\lambda^{(j+1)} \rightarrow \exp(iN_{j+1}) \hat{F}_{(j+1)}. \text{ (Recall } \hat{F}_{(n+1)} \text{ means } F\text{.)}$$

By (1'') and (3''), we have

$$(4) \tau_{\lambda, \geq j+1}^{-1} e_{\lambda, \geq j+1} F_\lambda \rightarrow \exp(iN_{j+1}) \hat{F}_{(j+1)}.$$

By (4) and by (1), we have

$$(5) \tau_{\lambda, \geq j+1}^{-1} e_{\lambda, \geq j+1} F_\lambda^{(j)} \rightarrow \exp(iN_{j+1}) \hat{F}_{(j+1)}.$$

For the left-hand side of (5), by (2'), $(N_s, \tau_{\lambda, \geq j+1}^{-1} e_{\lambda, \geq j+1} F_\lambda^{(j)})$ satisfies Griffiths transversality for any $s \in S_{\leq j}$. On the other hand, concerning the right-hand side of (5), $((N_s)_{s \in S_{\leq j}}, \exp(iN_{j+1}) \hat{F}_{(j+1)})$ generates a nilpotent orbit. Hence (5) and Proposition 4.5.8 show that $((N_s)_{s \in S_{\leq j}}, \tau_{\lambda, \geq j+1}^{-1} e_{\lambda, \geq j+1} F_\lambda^{(j)})$ generates a nilpotent orbit for any sufficiently large λ . This proves that $((N_s)_{s \in S_{\leq j}}, e_{\lambda, \geq j+1} F_\lambda^{(j)})$ generates a nilpotent orbit for any sufficiently large λ . By this and by (1) and (5), we have (A_j) .

4.5.17. By (A_{m-1}) (2) of Proposition 4.5.14, w_λ belongs to $E_{\sigma, \text{val}}^\sharp$ if λ is sufficiently large.

This proves that $E_{\sigma, \text{val}}^\sharp$ is open in $\tilde{E}_{\sigma, \text{val}}^\sharp$, and hence proves that E_σ is open in \tilde{E}_σ (4.5.10). The proof of Theorem 4.5.6 is completed.

4.5.18. We next prove Theorem 4.5.5 (the continuity of the CKS map) in 4.5.19–4.5.30.

The outline of the proof of Theorem 4.5.5 is as follows. By using the properties of regular spaces reviewed in 4.5.19 applied to the spaces $D_{\text{SL}(2)}^I(\Phi)$, it becomes enough to discuss the convergence of ordinary points to a boundary point (we do not need to discuss the convergence of boundary points to a boundary point). We reduce the proof of Theorem 4.5.5 to the three convergences in Proposition 4.5.22. These three convergences are proved in 4.5.24, 4.5.25, and 4.5.26–4.5.30, respectively. The structure of these proofs is essentially the same as that in [24], [21] Parts III and IV.

4.5.19. Concerning regular spaces, we first recall two facts.

[24] Definition 6.4.6 ([9], Ch. 1, §8, no. 4, Definition 2). A topological space X is called *regular* if it is Hausdorff and satisfies the following axiom: Given any closed subset F of

X and any point $x \notin F$, there is a neighborhood of x and a neighborhood of F which are disjoint.

([9], Ch. 1, §8, no. 5, Theorem 1.) Let X be a topological space, A a dense subset of X , $f : A \rightarrow Y$ a map from A into a regular space Y . A necessary and sufficient condition for f to extend to a continuous map $\bar{f} : X \rightarrow Y$ is that, for each $x \in X$, $f(y)$ tends to a limit in Y when y tends to x while remaining in A . The continuous extension \bar{f} of f to X is then unique.

4.5.20. To prove that $\psi : D_{\Sigma,[:]^\sharp} \rightarrow D_{\mathrm{SL}(2)}^I$ is continuous, it is sufficient to prove that the composition

$$E_{\sigma, \mathrm{val}}^\sharp \rightarrow E_{\sigma,[:]^\sharp} \rightarrow D_{\Sigma,[:]^\sharp}^\sharp \xrightarrow{\psi} D_{\mathrm{SL}(2)}^I$$

is continuous for each $\sigma \in \Sigma$.

Let the notation and the assumption be as in 4.5.10 and 4.5.11. Assume further that all w_λ belong to $E_{\sigma, \mathrm{val}}^\sharp$. Then it is enough to show that the image of w_λ in $D_{\mathrm{SL}(2)}^I$ converges to the image of w in $D_{\mathrm{SL}(2)}^I$.

If $(\sigma, Z) \in D_\Sigma$, the relative monodromy filtration $M(N', W)$ exists for any $N' \in \sigma$, and $M(N', W)$ depends only on the face σ' of σ such that N' is in an interior point of σ' . We will denote $M(N', W)$ as $W(\sigma')$.

Recall that the image of $w \in E_{\sigma, \mathrm{val}}^\sharp$ in $D_{\Sigma,[:]^\sharp}$ is $(\sigma, Z, \mathrm{class}((\sigma_j, N_j)_{1 \leq j \leq n}))$. Since N_j is in the interior of σ_j for each j , $N_1 + \cdots + N_j$ is in the interior of σ_j , and hence $W^{(j)}$ in 4.5.1 for (N_1, \dots, N_n, F) coincides with $W(\sigma_j)$.

Hence if $w_\lambda \in E_{\sigma, \mathrm{val}}^\sharp$ is near to w , the $W(\sigma_j)$ of w_λ are contained in the set Φ of $W(\sigma_j)$ ($1 \leq j \leq n$), and hence the image of w_λ in $D_{\mathrm{SL}(2)}^I$ is contained in $D_{\mathrm{SL}(2)}^I(\Phi)$. Since $D_{\mathrm{SL}(2)}^I(\Phi)$ is regular by Proposition 3.4.10, we can use the above regular point method, and hence in the proof of the fact that the image of w_λ in $D_{\mathrm{SL}(2)}^I$ converges to the image of w in $D_{\mathrm{SL}(2)}^I$, we may assume that $w_\lambda \in |\mathrm{torus}|_\sigma \times D$.

4.5.21. To state Proposition 4.5.22 on three kinds of convergences, which are used to prove the continuity of the CKS map, we recall that we are in the following situation. We keep the notation in 4.5.11 and assume $w_\lambda \in |\mathrm{torus}|_\sigma \times D$. So the $|\mathrm{torus}|_\sigma$ -component of w_λ is written as $(\sum_{s \in S} y_{\lambda,s} N_s) + b_\lambda$.

For each $1 \leq j \leq n$, we have $c_j \in S_j$ and we have $N_j \equiv \sum_{s \in S_j} a_s N_s \pmod{\sigma_{j-1, \mathbf{R}}}$.

Note that (N_1, \dots, N_n, F) generates a nilpotent orbit. Let $\mathbf{r} \in D$ be the point associated to (N_1, \dots, N_n, F) in 4.5.1.

By 4.5.19, for the proof of Theorem 4.5.5, it is sufficient to prove that $\exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda$ converges to the class in $D_{\mathrm{SL}(2)}^I$ of the $\mathrm{SL}(2)$ -orbit associated to (N_1, \dots, N_n, F) . As will be explained in 4.5.23 below, this is reduced to the following proposition.

Proposition 4.5.22. *We have the following convergences (1)–(3) as $\lambda \rightarrow \infty$.*

$$(1) \tau \left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \dots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}} \right) \exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda \rightarrow \mathbf{r} \text{ in } D, \text{ where } y_{\lambda,c_{n+1}} = 1.$$

$$(2) \mathrm{spl}_W \left(\exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda \right) \rightarrow \mathrm{spl}_W(\mathbf{r}) \text{ in } \mathrm{spl}(W).$$

(3) $(\exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda)_{\mathrm{red}}$ converges to the class in $D_{\mathrm{red}, \mathrm{SL}(2)}$ of the $\mathrm{SL}(2)$ -orbit associated to (N_1, \dots, N_n, F) .

4.5.23. We explain that this Proposition 4.5.22 shows that $x_\lambda := \exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda$ converges to the class $x_{\mathrm{SL}(2)}$ in $D_{\mathrm{SL}(2)}^I$ of the $\mathrm{SL}(2)$ -orbit associated to (N_1, \dots, N_n, F) and hence proves Theorem 4.5.5.

By (2) and (3) of Proposition 4.5.22 and by Proposition 3.4.5 and Proposition 3.4.9, for the proof of $x_\lambda \rightarrow x_{\mathrm{SL}(2)}$, it is sufficient to prove the following (1) and (2). Let $p = x_{\mathrm{SL}(2),\mathrm{red}} \in D_{\mathrm{red},\mathrm{SL}(2)}$.

(1) Let β be a distance to $\Phi(p)$ -boundary. Then if $x_{\mathrm{SL}(2)}$ is an A -orbit (resp. B -orbit), $\mathrm{Ad}(\tau_p(\beta(x_\lambda)))^{-1}\delta(x_\lambda)$ converges to $\mathrm{Ad}(\tau_p(\beta(\mathbf{r})))^{-1}\delta(\mathbf{r})$ (resp. $0 \circ \mathrm{Ad}(\tau_p(\beta(\mathbf{r})))^{-1}\delta(\mathbf{r})$) in $\bar{\mathcal{L}}$.

(2) Let β be a distance to $\Phi(x_{\mathrm{SL}(2)})$ -boundary. Then $\tau_{x_{\mathrm{SL}(2)}}(\beta(x_\lambda))^{-1}x_\lambda$ converges to $\tau_{x_{\mathrm{SL}(2)}}(\beta(\mathbf{r}))^{-1}\mathbf{r}$ in D .

We prove that these (1) and (2) follow from (1) of Proposition 4.5.22.

Let $t_\lambda = \left(\sqrt{\frac{y_{\lambda,c_2}}{y_{\lambda,c_1}}}, \dots, \sqrt{\frac{y_{\lambda,c_{n+1}}}{y_{\lambda,c_n}}} \right) \in \mathbf{R}_{\geq 0}^n$. We have $\tau(t_\lambda) = \tau_{x_{\mathrm{SL}(2)}}(t'_\lambda)$, where $t'_\lambda \in \mathbf{R}_{>0}^{\Phi(x_{\mathrm{SL}(2)})}$ is as in 4.5.3 (2). Since t_λ converges to $\mathbf{0}$ in $\mathbf{R}_{\geq 0}^n$, t'_λ converges to $\mathbf{0}$ in $\mathbf{R}_{\geq 0}^{\Phi(x_{\mathrm{SL}(2)})}$.

By (1) of Proposition 4.5.22, we have

$$(*) \quad x_\lambda = \tau_{x_{\mathrm{SL}(2)}}(t'_\lambda)u_\lambda\mathbf{r}, \quad u_\lambda \in G(\mathbf{R})G_u(\mathbf{C}), \quad u_\lambda \rightarrow 1.$$

If $x_{\mathrm{SL}(2)}$ is a B -orbit, since $\mathbf{r} \in D_{\mathrm{nspl}}$, $u_\lambda\mathbf{r} \in D_{\mathrm{nspl}}$ if λ is sufficiently large and hence by (*), $x_\lambda \in D_{\mathrm{nspl}}$ if λ is sufficiently large.

We prove (1). By applying β to $(*)_{\mathrm{red}}$, we obtain that if $x_{\mathrm{SL}(2)}$ is an A -orbit (resp. B -orbit), $\mathrm{Ad}(\tau_p(\beta(x_{\lambda,\mathrm{red}})))^{-1}\delta(x_\lambda)$ is equal to $\tau_p(\beta(u_\lambda\mathbf{r}_{\mathrm{red}}))^{-1}\delta(u_\lambda\mathbf{r})$ (resp. $t'_{\lambda,0} \circ \mathrm{Ad}(\tau_p(\beta(u_\lambda\mathbf{r}_{\mathrm{red}})))^{-1}\delta(u_\lambda\mathbf{r})$) and this converges to $\mathrm{Ad}(\tau_p(\beta(\mathbf{r}_{\mathrm{red}})))^{-1}\delta(\mathbf{r})$ (resp. $0 \circ \mathrm{Ad}(\tau_p(\beta(\mathbf{r}_{\mathrm{red}})))^{-1}\delta(\mathbf{r})$).

We prove (2). By applying β to (*), we have $\tau_{x_{\mathrm{SL}(2)}}(\beta(x_\lambda))^{-1}x_\lambda = \tau_{x_{\mathrm{SL}(2)}}(\beta(u_\lambda\mathbf{r}))^{-1}u_\lambda\mathbf{r}$, and this converges to $\tau_{x_{\mathrm{SL}(2)}}(\beta(\mathbf{r}))^{-1}\mathbf{r}$.

4.5.24. We start to prove Proposition 4.5.22.

Proposition 4.5.22 (1) follows from (A_0) of the case $m = 1$ of Proposition 4.5.14.

We prepare the proofs of Proposition 4.5.22 (2) and (3).

For $1 \leq j \leq n$, let \check{D}_j be the subset of \check{D} consisting of all $F' \in \check{D}$ such that $((N_s)_{s \in S_{\leq j}}, F')$ generates a nilpotent orbit. We have $F \in \check{D}_n$, $F_\lambda \in \check{D}$, and we know from 4.5.11, Proposition 4.5.14 and Theorem 4.5.6 that the following five conditions are satisfied.

- (1) F_λ converges to F in \check{D} .
- (2) $y_{\lambda,s} \rightarrow \infty$ for any $s \in S$.
- (3) If $1 \leq j < n$, $s \in S_{\leq j}$ and $t \in S_{\geq j+1}$, then $\frac{y_{\lambda,s}}{y_{\lambda,t}} \rightarrow \infty$.
- (4) If $1 \leq j \leq n$ and $s, t \in S_j$, then $\frac{y_{\lambda,s}}{y_{\lambda,t}} \rightarrow \frac{a_s}{a_t}$.
- (5) For $1 \leq j \leq n$ and $e \geq 0$, there exist $F_\lambda^* \in \check{D}$ ($\lambda \in L$) satisfying the following conditions (5.1) and (5.2).
 - (5.1) $\exp(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t) F_\lambda^* \in \check{D}_j$ (λ : sufficiently large),
 - (5.2) $y_{\lambda,s}^e d(F_\lambda, F_\lambda^*) \rightarrow 0$ ($\forall s \in S_j$).

Here d is a metric on a neighborhood of F in \check{D} which is compatible with the analytic topology of \check{D} .

4.5.25. Proof of Proposition 4.5.22 (2). (This is almost a copy from [21] Part III 3.3.7.)

We prove the following assertion (A_j) by downward induction on j . (Note that (A_0) is what we want to prove.)

(A_j) : Proposition 4.5.22 (2) is true in the case where $\exp\left(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t\right) F_\lambda \in \check{D}_j$ for all λ . Here \check{D}_j for $1 \leq j \leq n$ are defined in 4.5.24, and $\check{D}_0 := D$.

Proof. Let $0 \leq j \leq n$.

Let $x_\lambda = \exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda$, $\tau_{\lambda,>j} = \prod_{j < k \leq n} \tau_k\left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}\right)$, and $x_{\lambda,j} = \tau_{\lambda,>j} x_\lambda$.

Then, by [20] 10.3, we have

$$x_{\lambda,j} = \tau_{\lambda,>j} x_\lambda = \exp\left(\sum_{s \in S_{\leq j}} i \frac{y_{\lambda,s}}{y_{\lambda,c_{j+1}}} N_s\right) U_{\lambda,j},$$

$$\text{where } U_{\lambda,j} := \tau_{\lambda,>j} \exp\left(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t\right) F_\lambda.$$

Assume $\exp\left(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t\right) F_\lambda \in \check{D}_j$. Then $(N'_1, \dots, N'_j, U_{\lambda,j})$ generates a nilpotent orbit, where $N'_k = \sum_{s \in S_k} \frac{y_{\lambda,s}}{y_{\lambda,c_k}} N_s$ ($1 \leq k \leq j$). Let s_λ be the associated limit splitting. By [20] Theorem 0.5 (2) and ibid. Proposition 10.8 (1), there is a convergent power series u_λ whose values are in $G(\mathbf{R})$, whose constant term is 1 and whose coefficients depend on $U_{\lambda,j}$ and $y_{\lambda,s}/y_{\lambda,c_k}$ ($1 \leq k \leq j$, $s \in S_k$) real analytically such that $\text{spl}_W(x_{\lambda,j}) = u_\lambda\left(\frac{y_{\lambda,c_2}}{y_{\lambda,c_1}}, \dots, \frac{y_{\lambda,c_{j+1}}}{y_{\lambda,c_j}}\right) s_\lambda$. Since s_λ also depends real analytically on $U_{\lambda,j}$, we have

(1) $\text{spl}_W(x_{\lambda,j})$ converges to $\text{spl}_W(\mathbf{r})$.

This already showed (A_n) .

Next, assume $j < n$ and assume that (A_{j+1}) is true. We prove that (A_j) is true.

Choose a sufficiently big $e > 0$ depending on $\tau_{j+1}, \dots, \tau_n$.

Take F_λ^* as in 4.5.24 (5). Define x_λ^* and $x_{\lambda,j}^*$ similarly as x_λ and $x_{\lambda,j}$, respectively.

Then we have

(2) $\text{spl}_W(x_{\lambda,j}^*)$ converges to $\text{spl}_W(\mathbf{r})$, and $y_{\lambda,c_{j+1}}^e d(\text{spl}_W(x_{\lambda,j}), \text{spl}_W(x_{\lambda,j}^*)) \rightarrow 0$.

By downward induction hypothesis on j , we have

(3) $\text{spl}_W(x_\lambda^*) = \tau_{\lambda,>j}^{-1} \text{spl}_W(x_{\lambda,j}^*) \tau_{\lambda,>j}(\text{gr}^W)$ converges to $\text{spl}_W(\mathbf{r})$.

By (1)–(3), we have $\text{spl}_W(x_\lambda) = \tau_{\lambda,>j}^{-1} \text{spl}_W(x_{\lambda,j}) \tau_{\lambda,>j}(\text{gr}^W)$ also converges to $\text{spl}_W(\mathbf{r})$.

We next prove (3) of Proposition 4.5.22. We assume that G is reductive in 4.5.26–4.5.30.

4.5.26. (Cf. [24] 6.2.1.) Let $N_j \in \mathfrak{g}_\mathbf{R}$ ($1 \leq j \leq n$) be mutually commuting nilpotent elements, let $F \in \check{D}$, and assume that (N_1, \dots, N_n, F) generates a nilpotent orbit. Let (ρ, φ) be the associated $\text{SL}(2)$ -orbit in n variables and let $\mathbf{r} = \varphi(\mathbf{i})$ as in 4.5.21. Then, from [24] 6.1.5 (recall that G is reductive now), we see that there are $c_h \in \mathfrak{g}_\mathbf{R}^-$, $k_h \in \mathfrak{g}_\mathbf{R}^+$ ($h \in \mathbf{N}^n$), where $\mathfrak{g}^\pm = \mathfrak{g}^{\pm, \mathbf{r}}$ are the (± 1) -eigen subspaces of $\mathfrak{g}_\mathbf{R}$ under the Cartan involution associated to $K_\mathbf{r}$, respectively, such that $\sum_{h \in \mathbf{N}^n} c_h \prod_{j=1}^n t_j^{h(j)}$ and $\sum_{h \in \mathbf{N}^n} k_h \prod_{j=1}^n t_j^{h(j)}$ ($t_j \in$

C) converge when $|t_j|$ are sufficiently small, that $c_0 = k_0 = 0$ and that, for $y_j/y_{j+1} \gg 0$ ($1 \leq j \leq n, y_{n+1} = 1$),

$$(1) \quad \begin{aligned} & \tau \left(\sqrt{\frac{y_1}{y_2}}, \dots, \sqrt{\frac{y_n}{y_{n+1}}} \right) \exp \left(\sum_{1 \leq j \leq n} iy_j N_j \right) F \\ &= \exp \left(\sum_{h \in \mathbf{N}^n} c_h \prod_{j=1}^n (y_j/y_{j+1})^{-h(j)/2} \right) \cdot \exp \left(\sum_{h \in \mathbf{N}^n} k_h \prod_{j=1}^n (y_j/y_{j+1})^{-h(j)/2} \right) \cdot \mathbf{r}. \end{aligned}$$

Note that these c_h ($h \in \mathbf{N}^n$) are uniquely determined (whereas the k_h are not).

Proposition 4.5.27. (Cf. [24] Proposition 6.2.2.) *We use the notation in 4.5.26. For $v \in \mathfrak{g}_{\mathbf{R}}$ and $e \in \mathbf{Z}^n$, let $v(e)$ be the component of v on which $\text{Ad}(\tau(t))$ ($t = (t_j)_{1 \leq j \leq n} \in \mathbf{G}_{m,\mathbf{R}}^n$) acts as $\prod_{j=1}^n t_j^{e(j)}$. We denote $|e| = (|e(j)|)_{1 \leq j \leq n}$. Let $h \in \mathbf{N}^n \setminus \{0\}$, $e \in \mathbf{Z}^n$.*

Then, $c_h(e) = 0$ unless $|e| < h$ for the product order in \mathbf{N}^n , i.e., $|e(j)| \leq h(j)$ ($1 \leq j \leq n$) and $|e| \neq h$.

Proof. This proposition is proved in the same way as [24] 6.2.4–6.2.6. \square

4.5.28. The following arguments are almost copied from the proof [24] 6.4.4 of [24] Proposition 6.4.1.

We prove the following assertion (C_j) by a downward induction on j . Let $0 \leq j \leq n$, and let $\check{D}_0 := D$ (this is the correction of a typo “ $\check{D}_0 := \check{D}$ ” in [24] 6.4.4).

(C_j) *Assume that $\exp(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t) F_{\lambda} \in \check{D}_j$ for any λ . Then, for a sufficiently large λ , we have*

$$\begin{aligned} & \tau \left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \dots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}} \right) \exp \left(\sum_{s \in S} iy_{\lambda,s} N_s \right) F_{\lambda} \\ &= \exp \left(\sum_{h \in \mathbf{N}^j} b_{\lambda,h} \prod_{1 \leq k \leq j} \sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}^{-h(k)} \right) k_{\lambda} \cdot \mathbf{r}, \end{aligned}$$

where $b_{\lambda,h} \in \mathfrak{g}_{\mathbf{R}}^-$ ($\mathfrak{g}_{\mathbf{R}}^-$ denotes the (-1) -eigenspace of $\mathfrak{g}_{\mathbf{R}}$ under the Cartan involution associated to $K_{\mathbf{R}}$), $k_{\lambda} \in K_{\mathbf{R}}$, and $\sum_{h \in \mathbf{N}^j} b_{\lambda,h} \prod_{1 \leq k \leq j} x_k^{h(k)}$ ($x_k \in \mathbf{C}$) absolutely converges when $|x_k|$ ($1 \leq k \leq j$) are sufficiently small, which satisfy the following three conditions. Here $y_{c_{n+1}} := 1$.

- (1) $k_{\lambda} \rightarrow 1$.
- (2) $|(\text{Ad} \circ \tau_k)_{1 \leq k \leq j}\text{-weight of } b_{\lambda,h}| \leq h$ for the product order in \mathbf{N}^j .
- (3) For each $h \in \mathbf{N}^j$, $\text{Ad} \left(\prod_{k \geq j+1} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right)^{\nu} (b_{\lambda,h})$ converges for $\nu = 0, \pm 1$. Moreover, if $h = 0$ then it converges to 0.

Note that by the local structure theorem of $D_{\text{SL}(2)}$ in the reductive group case (Theorem 3.4.15), Proposition 4.5.22 (3) is equivalent to (C_0) .

Lemma 4.5.29. (Cf. [24] Lemma 6.4.3.) Let the notation be as above. Fix j such that $1 \leq j \leq n$, and assume that $\exp(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t) F_\lambda \in \check{D}_j$ for any λ . Put

$$N_{\lambda,k} := \sum_{s \in S_k} \frac{y_{\lambda,s}}{y_{\lambda,c_k}} N_s \quad (1 \leq k \leq j),$$

$$U_\lambda := \left(\prod_{l \geq j+1} \tau_l \left(\sqrt{\frac{y_{\lambda,c_l}}{y_{\lambda,c_{l+1}}}} \right) \right) \exp(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t) F_\lambda.$$

For each λ , let $(\rho_\lambda, \varphi_\lambda)$ be the $\mathrm{SL}(2)$ -orbit in j variables associated to $((N_{\lambda,k})_{1 \leq k \leq j}, U_\lambda)$, and let $\mathbf{r}_\lambda := \varphi_\lambda(\mathbf{i})$. Then, \mathbf{r}_λ converges to \mathbf{r} .

The proof of [24] Lemma 6.4.3 also works for this Lemma 4.5.29, which uses the important fact that an $\mathrm{SL}(2)$ -orbit moves real analytically if the monodromy weight filtration is constant ([10], see also [24] 6.1.6).

4.5.30. We prove (C_j) . By [20] 10.3, we have

$$\begin{aligned} & \tau \left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \dots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}} \right) \exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda \\ &= \left(\prod_{k \leq j} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right) \exp(\sum_{s \in S_{\leq j}} i \frac{y_{\lambda,s}}{y_{\lambda,c_{j+1}}} N_s) U_\lambda \\ &= \left(\prod_{k \leq j} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right) \exp(\sum_{k \leq j} i \frac{y_{\lambda,c_k}}{y_{\lambda,c_{j+1}}} N_{\lambda,k}) U_\lambda. \end{aligned}$$

Then, by 4.5.26 at \mathbf{r}_λ (see Lemma 4.5.29 for \mathbf{r}_λ),

$$\begin{aligned} & \tau \left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \dots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}} \right) \exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda \\ &= \left(\prod_{k \leq j} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right) \left(\prod_{k \leq j} \tau_{\lambda,k} \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right)^{-1} f_\lambda k_{1,\lambda} \cdot \mathbf{r}_\lambda, \quad \text{where} \\ & f_\lambda := \exp \left(\sum_{h \in \mathbf{N}^j} a_{\lambda,h} \prod_{k \leq j} \sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}^{-h(k)} \right), \quad a_{\lambda,h} \in \mathfrak{g}_\mathbf{R}^{-,\mathbf{r}_\lambda}, \\ & k_{1,\lambda} \in K_{\mathbf{r}_\lambda}, \quad k_{1,\lambda} \rightarrow 1. \end{aligned}$$

Here $\mathfrak{g}_\mathbf{R}^{-,\mathbf{r}_\lambda}$ denotes the (-1) -eigenspace of $\mathfrak{g}_\mathbf{R}$ under the Cartan involution associated to the maximal compact subgroup $K_{\mathbf{r}_\lambda}$ of $G_\mathbf{R}$.

Claim 1. We can write

$$\begin{aligned} \mathbf{r}_\lambda &= g_\lambda k_{2,\lambda} \cdot \mathbf{r}, \quad \tau_{\lambda,k} = \mathrm{Int}(g_\lambda) \circ \tau_k, \\ g_\lambda &\in (G^\circ)_{W^{(1)}, \dots, W^{(j)}, \mathbf{R}}, \quad k_{2,\lambda} \in K_\mathbf{r}, \quad g_\lambda \rightarrow 1, \quad k_{2,\lambda} \rightarrow 1. \end{aligned}$$

This claim is proved as exactly in the same way as in [24]. We obtain a proof of this claim from the corresponding proof in [24] by replacing “ $\tilde{\rho}_{\lambda,k}, \tilde{\rho}_k$ ” by $\tau_{\lambda,k}, \tau_k$, respectively, and by replacing “Lemma 6.4.3” by Lemma 4.5.29. The key point of the proof is the fact that $\tau_{\lambda,k}$ and τ_λ are the Borel–Serre liftings at \mathbf{r}_λ and \mathbf{r} , respectively, of the common homomorphism $\mathbf{G}_m \rightarrow P/P_u$, where P is a \mathbf{Q} -parabolic subgroup of G containing $(G^\circ)_{W^{(1)}, \dots, W^{(j)}, \mathbf{R}}$.

By Claim 1, we have

$$(4) \quad \begin{aligned} & \tau \left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \dots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}} \right) \exp \left(\sum_{s \in S} iy_{\lambda,s} N_s \right) F_{\lambda} \\ &= \text{Int} \left(\prod_{k \leq j} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right) (g_{\lambda}) \text{Int}(g_{\lambda})^{-1}(f_{\lambda}) \text{Int}(g_{\lambda})^{-1}(k_{1,\lambda}) k_{2,\lambda} \cdot \mathbf{r}. \end{aligned}$$

Here $\text{Int}(g_{\lambda})^{-1}(k_{1,\lambda}) \in K_{\mathbf{r}}$ and, concerning $a_{\lambda,h} \in \mathfrak{g}_{\mathbf{R}}^{-,r_{\lambda}}$ in the definition of f_{λ} , we have $\text{Ad}(g_{\lambda})^{-1}(a_{\lambda,h}) \in \mathfrak{g}_{\mathbf{R}}^- = \mathfrak{g}_{\mathbf{R}}^{-,r}$. Furthermore, if we write

$$(5) \quad \begin{aligned} g_{\lambda} &= \exp \left(\sum_{h \in \mathbf{N}^j} g_{\lambda,-h} \right), \quad g_{\lambda,-h} \in \text{Lie}(G_{W^{(1)}, \dots, W^{(j)}, \mathbf{R}}), \\ & \text{((Ad} \circ \tau_k)_{1 \leq k \leq j} \text{-weight of } g_{\lambda,-h}) = -h, \quad g_{\lambda,-h} \rightarrow 0, \end{aligned}$$

we have

$$(6) \quad \text{Int} \left(\prod_{k \leq j} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right) (g_{\lambda}) = \exp \left(\sum_{h \in \mathbf{N}^j} g_{\lambda,-h} \prod_{k \leq j} \sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}^{-h(k)} \right).$$

By Proposition 4.5.27,

$$(7) \quad |(\text{Ad} \circ \tau_k)_{1 \leq k \leq j} \text{-weight of } \text{Ad}(g_{\lambda})^{-1}(a_{\lambda,h})| < h.$$

From (4)–(7), we obtain

$$(8) \quad \begin{aligned} & \tau \left(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \dots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}} \right) \exp \left(\sum_{s \in S} iy_{\lambda,s} N_s \right) F_{\lambda} \\ &= \exp \left(\sum_{h \in \mathbf{N}^j} b_{\lambda,h} \prod_{k \leq j} \sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}}^{-h(k)} \right) k_{\lambda} \cdot \mathbf{r}, \quad \text{where} \\ & k_{\lambda} := \text{Int}(g_{\lambda})^{-1}(k_{1,\lambda}) k_{2,\lambda} \in K_{\mathbf{r}}, \quad k_{\lambda} \rightarrow 1, \\ & b_{\lambda,h} \in \mathfrak{g}_{\mathbf{R}}^-, \quad b_{\lambda,h} \text{ converges for each } h, \quad b_{\lambda,h}(\pm h) \rightarrow 0, \\ & |(\text{Ad} \circ \tau_k)_{1 \leq k \leq j} \text{-weight of } b_{\lambda,h}| \leq h. \end{aligned}$$

Here $b_{\lambda,h}(\pm h)$ denotes the parts of $b_{\lambda,h}$ of weight $\pm h$ with respect to $(\text{Ad} \circ \tau_k)_{1 \leq k \leq j}$.

In the case $j = n$, (8) already completes the proof of (C_j) .

If $0 \leq j < n$, it remains to prove (3) of (C_j) . This is shown by downward induction on j , as we have mentioned. First we show the following.

Claim 2. *To prove (C_j) (3), we may assume $\exp \left(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t \right) F_{\lambda} \in \check{D}_{j+1}$ for any λ .*

The following is a simplification of the proof of the generalization of [24] 6.4.4 Claim 2 by an observation that we can take $y_{\lambda,t}^* = y_{\lambda,t}$ there, so that the condition ibid. (11) holds trivially.

We prove Claim 2. By the condition (5) of 4.5.24 for $j+1$, there exist $e \geq 0$, $F_{\lambda}^* \in \check{D}$ satisfying the following (9)–(12).

$$(9) \quad \exp \left(\sum_{t \in S_{\geq j+1}} iy_{\lambda,t} N_t \right) F_{\lambda}^* \in \check{D}_{j+1}.$$

$$(10) \quad y_{\lambda,c_{j+1}}^{2e} d(F_{\lambda}, F_{\lambda}^*) \rightarrow 0.$$

(11) Let $\alpha_\lambda = \left(\prod_{k \geq j+1} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right) \right) \exp \left(\sum_{t \in S_{\geq j+1}} i y_{\lambda,t} N_t \right)$. (Note $U_\lambda = \alpha_\lambda F_\lambda$.) Then,

$$y_{\lambda,c_{j+1}}^{-e} \text{Ad}(\alpha_\lambda) \rightarrow 0$$

in the space of linear endomorphisms of \mathfrak{g}_C .

(12) Let $\beta_\lambda = \prod_{k \geq j+1} \tau_k \left(\sqrt{\frac{y_{\lambda,c_k}}{y_{\lambda,c_{k+1}}}} \right)$. Then,

$$y_{\lambda,c_{j+1}}^{-e} \text{Ad}(\beta_\lambda)^\nu \rightarrow 0 \quad \text{for } \nu = 0, \pm 1$$

in the space of linear endomorphisms of \mathfrak{g}_C .

Define $b_{\lambda,h}^*$ ($h \in \mathbf{N}^j$) just as $b_{\lambda,h}$ by replacing F_λ by F_λ^* . To prove Claim 2, it is sufficient to prove

$$\text{Ad}(\beta_\lambda)^\nu(b_{\lambda,h}) - \text{Ad}(\beta_\lambda)^\nu(b_{\lambda,h}^*) \rightarrow 0 \quad \text{for } \nu = 0, \pm 1.$$

The left-hand side of this is equal to

$$y_{\lambda,c_{j+1}}^{-e} \text{Ad}(\beta_\lambda)^\nu(y_{\lambda,c_{j+1}}^e(b_{\lambda,h} - b_{\lambda,h}^*)).$$

Hence, by (12), it is sufficient to prove

$$y_{\lambda,c_{j+1}}^e(b_{\lambda,h} - b_{\lambda,h}^*) \rightarrow 0.$$

Since $b_{\lambda,h}$ is a real analytic function in $((N_{\lambda,k})_{1 \leq k \leq j}, U_\lambda)$ ([24], 6.1.6), it is sufficient to prove that

$$(13) \quad y_{\lambda,c_{j+1}}^e d(U_\lambda, U_\lambda^*) \rightarrow 0,$$

where $U_\lambda^* = \alpha_\lambda F_\lambda^*$. By (10), we can write

$$F_\lambda = \exp(x_\lambda) F_\lambda^* \quad \text{with} \quad x_\lambda \in \mathfrak{g}_C, \quad y_{\lambda,c_{j+1}}^{2e} x_\lambda \rightarrow 0.$$

We have

$$U_\lambda = \alpha_\lambda F_\lambda = \exp(\text{Ad}(\alpha_\lambda)(x_\lambda)) U_\lambda^*.$$

By (11) and by

$$y_{\lambda,c_{j+1}}^e \text{Ad}(\alpha_\lambda)(x_\lambda) = (y_{\lambda,c_{j+1}}^{-e} \text{Ad}(\alpha_\lambda))(y_{\lambda,c_{j+1}}^{2e} x_\lambda) \rightarrow 0,$$

we obtain (13). Thus Claim 2 is proved.

By Claim 2, we assume $\exp \left(\sum_{t \in S_{\geq j+1}} i y_{\lambda,t} N_t \right) F_\lambda \in \check{D}_{j+1}$ for any λ . Then, by 4.5.26 at \mathbf{r} , we have elements $b_{\lambda,h'}$ for $h' \in \mathbf{N}^{j+1}$ and we have, for $h \in \mathbf{N}^j$,

$$(14) \quad b_{\lambda,h} = \sum_{k=0}^{\infty} b_{\lambda,(h,k)} \sqrt{\frac{y_{\lambda,c_{j+1}}}{y_{\lambda,c_{j+2}}}}^{-k}.$$

Using (8), applied to $b_{\lambda,(h,k)}$ replacing j by $j+1$, we have

(15) $|(Ad \circ \tau_{j+1})\text{-weight of } b_{\lambda, (h, k)}| \leq k$. If $h = 0$, the parts of $b_{\lambda, (h, k)}$ of $(Ad \circ \tau_{j+1})\text{-weight } \pm k$ converge to 0.

By the hypothesis of induction,

(16) $Ad\left(\prod_{k \geq j+2} \tau_k\left(\sqrt{\frac{y_{\lambda, c_k}}{y_{\lambda, c_{k+1}}}}\right)\right)^\nu (b_{\lambda, (h, k)})$ converges for $\nu = 0, \pm 1$.

By (14)–(16), we have (3) of (C_j) . \square

The proof of Proposition 4.5.22 (3) is completed.

Remark 4.5.31. As mentioned at the top of this section, [24] and [21] Part III contain some mistakes which are corrected as in [21] Part IV Appendix. But still the corrections contain some errors. We review the outline of the mistakes of [24] in (1) below, and explain the errors of [21] Part IV Appendix in (2) below.

(1) The essence of the errors are in 6.4.12 and 7.1.2 (3) of [24], called the errors (1) and (2) in [21] Part IV A.1.1, respectively. Both errors base on a misuse of [24] Proposition 3.1.6: In both situations, the authors thought that they could show the existence of real numbers $y_{\lambda, t}^*$ satisfying some conditions including the convergence

$$y_{\lambda, s}^e |y_{\lambda, t} - y_{\lambda, t}^*| \rightarrow 0,$$

where e is a fixed nonnegative integer. But, [24] Proposition 3.1.6 (applied in the way explained there) gives only those satisfying

$$y_{\lambda, s}^e (q_{\lambda, t} - q_{\lambda, t}^*) \rightarrow 0,$$

which is not enough.

We use [24] Proposition 3.1.6 correctly in 4.5.9 and Lemma 4.5.13 in the present paper. Then, we generalize (and correct, see (2) below) in this section (Section 4.5) an alternative argument [21] Part IV A.1.3–A.1.9 explained in [21] Part IV A.1.2, which shows that, as is said in [21] Part IV A.1.11, in fact we can take just $y_{\lambda, s}$, $y_{\lambda, t}$ as $y_{\lambda, s}^*$, $y_{\lambda, t}^*$ satisfying all conditions, that is, the stronger statement including $y_{\lambda, s}^e |y_{\lambda, t} - y_{\lambda, t}^*| = 0$.

(2) The correction in [21] Part IV Appendix contains several errors including some critical typos. For example,

1. In A.1.2, “problem (1)” and “problem (2)” should be interchanged.
2. In A.1.5, the definition of N_1, \dots, N_{m-1} is missing.
3. In the last line of A.1.5, X should be replaced by some log blowing-up of \tilde{E}_σ and S should be the corresponding subspace coming from \tilde{E}_σ (not E_σ ; the tilde was missed).
4. In A.1.8, line 3 : (A_n) does not follow from Lemma A.1.6 if $m - 1 = n$ because, then, Lemma A.1.6 is an empty statement.
5. In A.1.8, line 4: $j \leq n$ should be $j < n$.
6. In A.1.10 and A.3.4, all “ $m = 0$ ” should be “ $m = 1$.”
7. In A.3.4, all “A.1.6” should be “A.1.7.”

The proof given in this section (Section 4.5), in particular, in Proposition 4.5.14 and Theorem 4.5.6, corrects and generalizes, in the present context, that in [21] Part IV A.1.3–A.1.9.

4.6 Properties of the spaces of nilpotent orbits

The aim of this Section 4.6 is to prove the following theorem.

Theorem 4.6.1. *Let Σ be a weak fan in $\text{Lie}(G')$ (4.1.7) and let Γ be a semi-arithmetic subgroup (1.5.7) of $G'(\mathbf{Q})$.*

In (1) and (2) below, we assume that Γ is compatible with Σ . In (3)–(5) below, we assume that Γ is strongly compatible with Σ .

- (1) *Let X be one of D_Σ^\sharp , $D_{\Sigma,[\cdot]}^\sharp$, $D_{\Sigma,\text{val}}^\sharp$. Then the action of Γ on X is proper, and the quotient space $\Gamma \backslash X$ is Hausdorff.*
- (2) *Let X be as in (1). Assume that Γ is torsion-free. Then the action of Γ on X is free, and the projection $X \rightarrow \Gamma \backslash X$ is a local homeomorphism.*
- (3) *The quotient space $\Gamma \backslash D_\Sigma$ is Hausdorff.*
- (4) *Assume that Γ is neat. Then $\Gamma \backslash D_\Sigma$ is a log manifold (4.2.12). In particular, it belongs to $\mathcal{B}(\log)$. For each $\sigma \in \Sigma$, the map $\Gamma(\sigma)^{\text{gp}} \backslash D_\sigma \rightarrow \Gamma \backslash D_\Sigma$ is locally an isomorphism of log manifolds.*
- (5) *Assume that Γ is neat. Then there is a homeomorphism*

$$(\Gamma \backslash D_\Sigma)^{\log} \simeq \Gamma \backslash D_\Sigma^\sharp$$

over $\Gamma \backslash D_\Sigma$.

See 4.1.9 for the compatibility and the strong compatibility of Γ and Σ .

Remark 4.6.2. In Theorem 4.6.1, we can use a semi-arithmetic subgroup of $G(\mathbf{Q})$ (not of $G'(\mathbf{Q})$) in the following situation (1) and also in the following situation (2).

(1) If either G is semisimple or the condition (1) in Lemma 1.5.3 is satisfied, Theorem 4.6.1 holds for a semi-arithmetic subgroup Γ of $G(\mathbf{Q})$. In fact, $\Gamma \cap G'(\mathbf{Q})$ is of finite index in Γ (see Proposition 1.5.13 for the latter case). Hence by 2.3.6 (5), we can replace Γ by the semi-arithmetic subgroup $\Gamma \cap G'(\mathbf{Q})$ of $G'(\mathbf{Q})$.

(2) Assume that G is reductive. Then Theorem 4.6.1 remains true for a semi-arithmetic subgroup of $G(\mathbf{Q})$ if we make the following modifications $(*)$ and $(**)$ below. Let $\bar{\Gamma}$ be the image of Γ in $(G/Z)(\mathbf{Q})$, where Z denotes the center of G .

$(*)$ In (2) (resp. (4) and (5)), the torsion free (resp. neat) property is assumed for $\bar{\Gamma}$, not for Γ .

$(**)$ In (1) (resp. (2)), the proper action (resp. free action) is stated for $\bar{\Gamma}$, not for Γ .

For the proof, see 4.10.18.

Remark 4.6.3. In [25] Theorem 6.1 and in its proof, the conclusions of Theorem 4.6.1 except the parts on $X = D_{\Sigma,[\cdot]}^\sharp$, $D_{\Sigma,\text{val}}^\sharp$ in (1) and (2) for the Mumford–Tate domain D (and its extensions) associated to a polarized pure Hodge structure H are proved with G being the (semisimple) Mumford–Tate group associated to H (which is isomorphic to M/Z in 1.6.6), Γ being an arithmetic subgroup of $G(\mathbf{Q})$ which is in the connected component of $G(\mathbf{R})$ containing 1, and Σ being a fan in $\text{Lie}(G)$ which is strongly compatible with Γ . The proof in [25] is a reduction to the case ([24] 4.1.1 Theorem A) of the extended period domains of classical period domains for pure Hodge structures. Our proof of Theorem 4.6.1 bases on our studies of the space of Borel–Serre orbits, the space of $\text{SL}(2)$ -orbits, and

the CKS map, which were not considered in [25], and provides alternative proofs of their results. (Minor remarks: Precisely, their D and its extensions are connected components of our period domain $D(G, \bar{h}_0)$, where $\bar{h}_0: S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ is the homomorphism associated to H , and its extensions, respectively, and their results are deduced from Theorem 4.6.1. Conversely, under the above assumption, their method can treat $\Gamma \backslash D(M, h_0)$ (h_0 is as in 1.6.6) and its extensions (which are open and closed subspaces of $\Gamma \backslash D(G, \bar{h}_0)$ and its extensions, as is seen by the latter half of 4.10). See 4.10.19 for some details. Additionally, their Γ is assumed to be neat, though it is easy to reduce the non-neat case to this case.)

4.6.4. The relation of this theorem (Theorem 4.6.1) with the results and their proofs in the former parts is as follows.

The above (3), (4), and (5) of Theorem 4.6.1 is the G -MHS versions of Theorem 2.5.5, Theorem 2.5.2 plus Theorem 2.5.4, and Theorem 2.5.6, respectively, of [21] Part III for the extended period domains of classical period domains (which are the mixed Hodge structure versions of (v), (ii) plus (iv), and (vi), respectively, of [24] 4.1.1 Theorem A for pure Hodge structures). In the case of the extended period domains of classical period domains, these (3)–(5) of Theorem 4.6.1 are proved in Section 4 of [21] Part III.

In there, also are proved the portion of (1) where $X = D_{\Sigma}^{\sharp}, D_{\Sigma, \text{val}}^{\sharp}$ and the portion of (2) where $X = D_{\Sigma}^{\sharp}, D_{\Sigma, \text{val}}^{\sharp}$ and Γ is neat. The space $X = D_{\Sigma, [:]}^{\sharp}$ was not considered in [21] Part III, and the portion of (1) where $X = D_{\Sigma, [:]}^{\sharp}$ and the portion of (2) where $X = D_{\Sigma, [:]}^{\sharp}$ and Γ is neat follow from [21] Part IV Theorem 6.1.1.

Most of this Section 4.6 is devoted to the proof of Theorem 4.6.1 which is completed in 4.6.26. Almost descriptions in the proof in this section are some abridged versions of arguments in [21] Part III Section 4 etc. We describe the main steps in the proof, but often omit the details if the arguments are the same as those in [21] Part III Section 4 etc.

We start to explain the proof of Theorem 4.6.1.

Let Σ be a weak fan.

Proposition 4.6.5. *Let $\sigma, \sigma' \in \Sigma$. Let*

$$w_{\lambda} = (y_{\lambda}, F_{\lambda}) \in \{(y, F) \in \sigma_{\mathbf{R}} \times \check{D} \mid \exp(iy)F \in D\} = (|\text{torus}|_{\sigma} \times \check{D}) \cap E_{\sigma}^{\sharp}$$

and

$$w'_{\lambda} = (y'_{\lambda}, F'_{\lambda}) \in \{(y, F) \in \sigma'_{\mathbf{R}} \times \check{D} \mid \exp(iy)F \in D\} = (|\text{torus}|_{\sigma'} \times \check{D}) \cap E_{\sigma'}^{\sharp}$$

be directed families with the same index set. Let $*$ be $[:]$ or val . Assume that w_{λ} converges to α in $E_{\sigma, *}^{\sharp}$, w'_{λ} converges to α' in $E_{\sigma', *}^{\sharp}$, and $\exp(iy_{\lambda})F_{\lambda} = \exp(iy'_{\lambda})F'_{\lambda}$ in D for all λ . Then:

- (1) The images of α and α' in $D_{\Sigma, *}^{\sharp}$ coincide.
- (2) $y_{\lambda} - y'_{\lambda}$ converges in $\text{Lie}(G'_{\mathbf{R}})$.

Proof. In the case $* = \text{val}$, this is a G -MHS version of [21] Part III Proposition 4.2.3. The proof of the latter works for the proof of Proposition 4.6.5 for $* = \text{val}$ and also for $* = [:]$. A key point of the proof is the continuity of the CKS map $D_{\Sigma, *}^{\sharp} \rightarrow D_{\text{SL}(2)}^I$. \square

4.6.6. Let $\sigma \in \Sigma$. We consider the continuous actions of $i\sigma_{\mathbf{R}} (\subset \sigma_{\mathbf{C}})$ on E_{σ}^{\sharp} , $E_{\sigma,[:]^{\sharp}}$, $E_{\sigma, \text{val}}^{\sharp}$. For $a \in \sigma_{\mathbf{R}}$, ia sends the class of (τ, b, F) in E_{σ}^{\sharp} (resp. $((\sigma_j, N_j)_j, b, F)$ in $E_{\sigma,[:]^{\sharp}}$, resp. (τ, V, b, F) in $E_{\sigma, \text{val}}^{\sharp}$) in the additive presentation to the class of $(\tau, b + a, \exp(-ia)F)$ (resp. $((\sigma_j, N_j)_j, b + a, \exp(-ia)F)$, resp. $(\tau, V, b + a, \exp(-ia)F)$).

It may seem strange that we regard the above action as an action of ia ($a \in \sigma_{\mathbf{R}}$), not of a , but we do so because, in the situation 4.6.15 below, this action of $i\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} will be compatible with the action of $\sigma_{\mathbf{C}} (\supset i\sigma_{\mathbf{R}})$ on E_{σ} .

We have $i\sigma_{\mathbf{R}} \setminus E_{\sigma}^{\sharp} = D_{\sigma}^{\sharp}$, $i\sigma_{\mathbf{R}} \setminus E_{\sigma,[:]^{\sharp}} = D_{\sigma,[:]^{\sharp}}$, and $i\sigma_{\mathbf{R}} \setminus E_{\sigma, \text{val}}^{\sharp} = D_{\sigma, \text{val}}^{\sharp}$.

Proposition 4.6.7. Let $\sigma \in \Sigma$. Then $E_{\sigma}^{\sharp} \rightarrow D_{\sigma}^{\sharp}$, $E_{\sigma,[:]^{\sharp}} \rightarrow D_{\sigma,[:]^{\sharp}}$, and $E_{\sigma, \text{val}}^{\sharp} \rightarrow D_{\sigma, \text{val}}^{\sharp}$ are $i\sigma_{\mathbf{R}}$ -torsors in the category of topological spaces.

To prove Proposition 4.6.7, we use the following Lemma 4.6.8 and Lemma 4.6.9.

Lemma 4.6.8. Let H be a topological group, X a topological space, and assume that we have a continuous action $H \times X \rightarrow X$. Assume the following (i)–(iii).

(i) This action is free set-theoretically.

(ii) This action is proper topologically.

(iii) For each point $x \in X$, there exist a subset S of X which contains x and an open neighborhood U of 1 in H such that $U \times S \rightarrow X$, $(h, s) \mapsto hs$, induces a homeomorphism from $U \times S$ onto an open set of X .

Then $X \rightarrow H \setminus X$ is an H -torsor in the category of topological spaces.

See [24] Lemma 7.3.3 for the proof.

Lemma 4.6.9. Assume that a Hausdorff topological group H acts on a Hausdorff topological space X continuously and freely. Let X' be a dense subset of X . Then, the following two conditions (i) and (ii) are equivalent.

(i) The action of H on X is proper.

(ii) Let $(h_{\lambda}, x_{\lambda})_{\lambda}$ be a directed family of elements of $H \times X'$ such that $(x_{\lambda})_{\lambda}$ and $(h_{\lambda}x_{\lambda})_{\lambda}$ converge in X . Then $(h_{\lambda})_{\lambda}$ converges in H .

See [24] Lemma 7.2.7 for the proof.

4.6.10. We prove Proposition 4.6.7. We check that the actions of $i\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} , $E_{\sigma,[:]^{\sharp}}$, $E_{\sigma, \text{val}}^{\sharp}$ satisfy the conditions (i), (ii), (iii) of Lemma 4.6.8, in Claim 1, Claim 2, Claim 3 below, respectively, which completes the proof.

Claim 1. The actions of $i\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} , $E_{\sigma,[:]^{\sharp}}$, $E_{\sigma, \text{val}}^{\sharp}$ are free.

Proof of Claim 1. This is a G -MHS version of [21] Part III Proposition 4.2.2 (ii). The proof of the latter works for the proof of Claim 1.

Claim 2. The actions of $i\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} , $E_{\sigma,[:]^{\sharp}}$, $E_{\sigma, \text{val}}^{\sharp}$ are proper.

Proof of Claim 2. For $E_{\sigma,[:]^{\sharp}}$ and $E_{\sigma, \text{val}}^{\sharp}$, this follows from (2) of Proposition 4.6.5 by Lemma 4.6.9. The result for E_{σ}^{\sharp} follows from this by 2.3.6 (3.2).

Finally we have to check

Claim 3. For $X = E_{\sigma}^{\sharp}$, $E_{\sigma,[:]^{\sharp}}$, $E_{\sigma, \text{val}}^{\sharp}$ and for each $x = (q, F') \in X$, there are a subspace S of X passing through x and an open neighborhood U of 0 in $i\sigma_{\mathbf{R}}$ such that the induced map $U \times S \rightarrow X$ is an open immersion.

Proof of Claim 3. The construction of S at the end of the proof of [21] Part III Proposition 4.4.3 works here.

Proposition 4.6.11. *Let $\sigma \in \Sigma$. Then the inclusion maps $D_\sigma^\sharp \rightarrow D_\Sigma^\sharp$, $D_{\sigma,[:]^\sharp} \rightarrow D_{\Sigma,[:]^\sharp}^\sharp$, and $D_{\sigma, \text{val}}^\sharp \rightarrow D_{\Sigma, \text{val}}^\sharp$ are continuous open maps.*

Proof. This is a G -MHS version of [21] Part III Theorem 4.3.1. The proof of the latter works for the proof of Proposition 4.6.11. \square

Proposition 4.6.12. *The canonical continuous surjections $D_{\Sigma, \text{val}}^\sharp \rightarrow D_{\Sigma,[:]^\sharp}^\sharp$ and $D_{\Sigma,[:]^\sharp}^\sharp \rightarrow D_\Sigma^\sharp$ are proper.*

Proof. This is a G -MHS version of [21] Part III Proposition 4.3.2, Part IV 4.4.3, and Part IV 4.4.6. By Proposition 4.6.7 and Proposition 4.6.11, this is reduced to the fact that the maps $E_{\sigma, \text{val}}^\sharp \rightarrow E_{\sigma,[:]^\sharp}^\sharp$ and $E_{\sigma,[:]^\sharp}^\sharp \rightarrow E_\sigma^\sharp$ are proper, and hence to the fact that $|\text{toric}|_{\sigma, \text{val}} \rightarrow |\text{toric}|_{\sigma,[:]^\sharp}$ and $|\text{toric}|_{\sigma,[:]^\sharp} \rightarrow |\text{toric}|_\sigma$ are proper. \square

Proposition 4.6.13. *The spaces D_Σ^\sharp , $D_{\Sigma,[:]^\sharp}^\sharp$, and $D_{\Sigma, \text{val}}^\sharp$ are Hausdorff. (This is the case $\Gamma = \{1\}$ of (1) of Theorem 4.6.1.)*

Proof. Let $*$ be $[:]$ or val . By Proposition 4.6.12 and by the case $H = \{1\}$ of (3.2) of 2.3.6, the Hausdorffness of D_Σ^\sharp follows from that of $D_{\Sigma,*}^\sharp$.

To see that $D_{\Sigma,*}^\sharp$ is Hausdorff, by Proposition 4.6.11, it is enough to show the following.

(1) Let $\sigma, \sigma' \in \Sigma$ and let $\beta \in D_{\sigma,*}^\sharp$ and $\beta' \in D_{\sigma',*}^\sharp$. Assume that $x_\lambda \in D$ converges to β in $D_{\sigma,*}^\sharp$ and to β' in $D_{\sigma',*}^\sharp$. Then $\beta = \beta'$ in $D_{\Sigma,*}^\sharp$.

We prove (1). By Proposition 4.6.7, there exist an open neighborhood U of β in $D_{\sigma,*}^\sharp$ (resp. U' of β' in $D_{\sigma',*}^\sharp$) and a continuous section $s_\sigma : U \rightarrow E_{\sigma,*}^\sharp$ (resp. $s_{\sigma'} : U' \rightarrow E_{\sigma',*}^\sharp$) of the projection $E_{\sigma,*}^\sharp \rightarrow D_{\sigma,*}^\sharp$ (resp. $E_{\sigma',*}^\sharp \rightarrow D_{\sigma',*}^\sharp$). Let $w_\lambda = s_\sigma(x_\lambda)$, $w'_\lambda = s_{\sigma'}(x_\lambda)$, $\alpha = s_\sigma(\beta)$, and $\alpha' = s_{\sigma'}(\beta')$. Then we can apply Proposition 4.6.5 (1) and conclude $\beta = \beta'$. \square

4.6.14. We prove (1) and (2) of Theorem 4.6.1. Let Γ be a semi-arithmetic subgroup of $G'(\mathbf{Q})$ which is compatible (4.1.9) with Σ . Under this assumption, the action of Γ on $D_{\text{SL}(2)}^I$ is proper by Theorem 3.11.1. Together with Theorem 4.5.5 and the fact that $D_{\Sigma,[:]^\sharp}^\sharp$ is Hausdorff by Proposition 4.6.13, we see that the action of Γ on $D_{\Sigma,[:]^\sharp}^\sharp$ is proper (2.3.6 (3.1)). Hence, again by Proposition 4.6.13 and 2.3.6 (3.1), the action of Γ on $D_{\Sigma, \text{val}}^\sharp$ is proper. Since $D_{\Sigma,[:]^\sharp}^\sharp \rightarrow D_\Sigma^\sharp$ is proper and surjective, the action of Γ on D_Σ^\sharp is also proper (2.3.6 (3.2)). By 2.3.6 (1), the quotient spaces by these proper actions are Hausdorff. Thus we proved (1) of Theorem 4.6.1.

We prove (2) of Theorem 4.6.1. Assume that Γ is torsion-free. Then the action of Γ on D_Σ^\sharp is free. In fact, by using a neat subgroup of Γ of finite index, this is reduced to the case where Γ is neat. Then it becomes the G -MHS version of [21] Part III Theorem 4.3.5 (i) whose proof also works in the present situation. By 2.3.6 (2), we have the stated local homeomorphism.

In 4.6.15–4.6.26, we assume that Γ is a semi-arithmetic subgroup of $G'(\mathbf{Q})$ and is strongly compatible with Σ .

4.6.15. Let $\sigma \in \Sigma$. We consider the action

$$\sigma_{\mathbf{C}} \times E_{\sigma} \rightarrow E_{\sigma}; (a, (q, F)) \mapsto (\mathbf{e}(a)q, \exp(-a)F)$$

of $\sigma_{\mathbf{C}}$ on E_{σ} , where $a \in \sigma_{\mathbf{C}}$, $q \in \text{toric}_{\sigma}$, $F \in \check{D}$ and $(q, F) \in E_{\sigma}$. This is an action in the category of log manifolds (we endow $\sigma_{\mathbf{C}}$ with the trivial log structure). This action is compatible with the action of $i\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} in 4.6.6. We have $\sigma_{\mathbf{C}} \setminus E_{\sigma} = \Gamma(\sigma)^{\text{gp}} \setminus D_{\sigma}$.

The following Proposition 4.6.16 and Proposition 4.6.17 are proved together.

Proposition 4.6.16. *In the category of locally ringed spaces over \mathbf{C} with log structures, $E_{\sigma} \rightarrow \Gamma(\sigma)^{\text{gp}} \setminus D_{\sigma}$ is a $\sigma_{\mathbf{C}}$ -torsor.*

Proposition 4.6.17. *The space $\Gamma(\sigma)^{\text{gp}} \setminus D_{\sigma}$ is a log manifold.*

Lemma 4.6.18. *Let H be a complex analytic group, X a log manifold, and assume that we have an action $H \times X \rightarrow X$ in the category of log manifolds (we regard H as having the trivial log structure). Assume the following (i)–(iii).*

(i) *This action is free set-theoretically.*

(ii) *This action is proper topologically.*

(iii) *For each point $x \in X$, there exist a log manifold S and a morphism $\iota : S \rightarrow X$ of log manifolds whose image contains x and an open neighborhood U of 1 in H such that $U \times S \rightarrow X$, $(h, s) \mapsto h\iota(s)$, induces an isomorphism of log manifolds from $U \times S$ onto an open set of X .*

Then:

(1) *The quotient topological space $H \setminus X$ has a unique structure of a log manifold such that, for an open set V of $H \setminus X$, $\mathcal{O}_{H \setminus X}(V)$ (resp. $M_{H \setminus X}(V)$) is the set of all functions on V whose pullbacks to the inverse image V' of V in X belong to $\mathcal{O}_X(V')$ (resp. $M_X(V')$). (Here M_* denotes the log structure of $*$.)*

(2) *$X \rightarrow H \setminus X$ is an H -torsor in the category of log manifolds.*

See [24] Lemma 7.3.3 for the proof.

4.6.19. We prove Proposition 4.6.16 and Proposition 4.6.17. By Lemma 4.6.18, it is sufficient to prove the following Claim 1–Claim 3. We prove these claims one by one, which completes the proofs.

Claim 1. The action of $\sigma_{\mathbf{C}}$ on E_{σ} is free.

Proof of Claim 1. This is a G -MHS version of [21] Part III 4.2.2 (i) whose proof also works as a proof in the present situation.

Claim 2. The action of $\sigma_{\mathbf{C}}$ on E_{σ} is proper.

Proof of Claim 2. This is a G -MHS version of [21] Part III 4.4.2 whose proof also works as a proof in the present situation. (The properness of the action of $i\sigma_{\mathbf{R}}$ on E_{σ}^{\sharp} in Claim 2 in 4.6.10 is used in the proof through the projection $E_{\sigma} \rightarrow E_{\sigma}^{\sharp}$.)

Claim 3. For each point $x \in E_{\sigma}$, there exist a log manifold S and a morphism $\iota : S \rightarrow E_{\sigma}$ of log manifolds whose image contains x and an open neighborhood U of 0 in

σ_C such that $U \times S \rightarrow E_\sigma$, $(h, s) \rightarrow h\iota(s)$, induces an isomorphism of log manifolds from $U \times S$ onto an open set of E_σ .

Proof of Claim 3. The construction of S at the end of the proof of [21] Part III 4.4.3 works here.

4.6.20. We have the special case $(\Gamma(\sigma)^{gp} \setminus D_\sigma)^{log} \simeq \Gamma(\sigma)^{gp} \setminus D_\sigma^\sharp$ of (5) of Theorem 4.6.1 as follows.

The map $E_\sigma \rightarrow \Gamma(\sigma)^{gp} \setminus D_\sigma$ induces a continuous map $(E_\sigma)^{log} \rightarrow (\Gamma(\sigma)^{gp} \setminus D_\sigma)^{log}$. On the other hand, the map $|\text{toric}|_\sigma = \text{Hom}(P(\sigma), \mathbf{R}_{\geq 0}^{\text{mult}}) \rightarrow (\text{toric}_\sigma)^{log} = \text{Hom}(P(\sigma), \mathbf{R}_{\geq 0}^{\text{mult}} \times \mathbf{S}^1)$ ($P(\sigma)$ is as in 4.2.1 and $\mathbf{S}^1 = \{u \in \mathbf{C}^\times \mid |u| = 1\}$) induces a continuous map $E_\sigma^\sharp \rightarrow (E_\sigma)^{log}$. The composition $E_\sigma^\sharp \rightarrow (E_\sigma)^{log} \rightarrow (\Gamma(\sigma)^{gp} \setminus D_\sigma)^{log}$ induces $\Gamma(\sigma)^{gp} \setminus D_\sigma^\sharp \rightarrow (\Gamma(\sigma)^{gp} \setminus D_\sigma)^{log}$. The last map is a homeomorphism by Proposition 4.6.16 as in [21] Part III 4.4.4.

Proposition 4.6.21. *Assume that Γ is neat. Let $x = (\sigma, Z) \in D_\Sigma$, $\gamma \in \Gamma$, and assume $\gamma x = x$. Then $\gamma \in \Gamma(\sigma)^{gp}$.*

Proof. This is a G -MHS version of [21] Part III 4.3.5 (ii), and the proof of the latter works for the proof of Proposition 4.6.21. \square

Lemma 4.6.22. *Let X be a topological space with a continuous action of a discrete group Γ , let Y be a set with an action of Γ , and let $f : X \rightarrow Y$ be a Γ -equivariant surjective map. Let Γ_0 be a subgroup of Γ . We introduce the quotient topologies of X on $\Gamma_0 \setminus Y$ and on $\Gamma \setminus Y$. Let V be an open set of $\Gamma_0 \setminus Y$ and let U be the inverse image of V in $\Gamma_0 \setminus X$. We assume moreover the three conditions (i)–(iii) below. Then, the map $V \rightarrow \Gamma \setminus Y$ is a local homeomorphism.*

- (i) $X \rightarrow \Gamma \setminus X$ is a local homeomorphism and $\Gamma \setminus X$ is Hausdorff.
- (ii) The map $U \rightarrow V$ is proper.
- (iii) If $x \in X$ and $\gamma \in \Gamma$, and if the images of γx and x in $\Gamma_0 \setminus Y$ are contained in V and they coincide, then $\gamma \in \Gamma_0$.

Proof. This is [24] Lemma 7.4.7. \square

4.6.23. Assume that Γ is neat. We prove that the map $\Gamma(\sigma)^{gp} \setminus D_\sigma \rightarrow \Gamma \setminus D_\Sigma$ is a local homeomorphism.

We use Lemma 4.6.22 for $X = D_\Sigma^\sharp$, $Y = D_\Sigma$, $\Gamma = \Gamma$, $\Gamma_0 = \Gamma(\sigma)^{gp}$, $V = \Gamma(\sigma)^{gp} \setminus D_\sigma$ and $U = \Gamma(\sigma)^{gp} \setminus D_\sigma^\sharp$. The (1) and (2) of Theorem 4.6.1 show that the condition (i) in Lemma 4.6.22 is satisfied. By 4.6.20, the condition (ii) in Lemma 4.6.22 is satisfied. Proposition 4.6.21 shows that the condition (iii) in Lemma 4.6.22 is satisfied.

4.6.24. We obtain (4) of Theorem 4.6.1 by Proposition 4.6.17 and by 4.6.23.

We obtain (5) of Theorem 4.6.1 by 4.6.20 and by 4.6.23.

Proposition 4.6.25. *The map $\Gamma \setminus D_\Sigma^\sharp \rightarrow \Gamma \setminus D_\Sigma$ is proper and surjective.*

Proof. Replacing Γ by its neat subgroup of finite index, we may assume that Γ is neat. Then this follows from (5) of Theorem 4.6.1. \square

4.6.26. By Proposition 4.6.25 and by (1) of Theorem 4.6.1, we obtain (3) of Theorem 4.6.1.

The proof of Theorem 4.6.1 is completed.

Proposition 4.6.27. *Let the assumption be as in (4) of Theorem 4.6.1. Then we have canonical homeomorphisms*

$$\begin{aligned} (\Gamma \setminus D_\Sigma)_{[:]}) &\simeq \Gamma \setminus D_{\Sigma,[:]}, \quad (\Gamma \setminus D_\Sigma)_{\text{val}} \simeq \Gamma \setminus D_{\Sigma, \text{val}}, \\ ((\Gamma \setminus D_\Sigma)^{\log})_{[:]}) &\simeq \Gamma \setminus D_{\Sigma,[:]}, \quad \text{and } ((\Gamma \setminus D_\Sigma)^{\log})_{\text{val}} \simeq \Gamma \setminus D_{\Sigma, \text{val}}^{\sharp}. \end{aligned}$$

Here topologies of the spaces on the right hand side of these homeomorphisms are as in 4.3.16, 4.4.9, and (1) of Theorem 4.6.1.

Proof. This follows from (4) and (5) of Theorem 4.6.1. \square

Remark 4.6.28. The conclusion of Proposition 4.6.27 holds if G is reductive, Γ is a semi-arithmetic subgroup of $G(\mathbf{Q})$, and if the image of Γ in $(G/Z)(\mathbf{Q})$ is neat, where Z is the center of G . See 4.10.18 for the proof.

4.7 Valuative spaces, III

The spaces $S_{[:]}$ of ratios are endowed with new log structures and the associated valuative spaces $S_{[\text{val}]}$ are extensively studied in [21] Part IV Section 4. In this section, we review this subject and obtain a space $D_{\Sigma, [\text{val}]}^{\sharp}$ over $D_{\Sigma, [:]}^{\sharp}$.

4.7.1. Let E , S and M_S be as in 4.3.1. Let $S_{[:]}$ be the topological space defined in 4.3.5 and 4.3.6.

We review the definition of the *new log structure on $S_{[:]}$* ([21] Part IV 4.3.3). We endow $S_{[:]}$ with the sheaf $\mathcal{O}_{S_{[:]}}$ of all \mathbf{R} -valued continuous functions. Assume that we are given a chart $\mathcal{S} \rightarrow M_S$ with \mathcal{S} being a sharp fs monoid such that $|f(s)| < 1$ for any $f \in \mathcal{S} \setminus \{1\}$ and any $s \in S$. Let $\Phi = \{\mathcal{S}^{(j)} \mid 0 \leq j \leq n\}$ with $\mathcal{S} = \mathcal{S}^{(0)} \supsetneq \mathcal{S}^{(1)} \supsetneq \cdots \supsetneq \mathcal{S}^{(n)}$ be as in 4.3.7. Take $q_j \in \mathcal{S}^{(j-1)} \setminus \mathcal{S}^{(j)}$ for $1 \leq j \leq n$. We consider the log structure on $S_{[:]}\Phi$ (4.3.7) associated to a chart

$$\mathbf{N}^n \rightarrow \mathcal{O}_{S_{[:]}}; m \mapsto \left(\prod_{j=1}^{n-1} r(q_{j+1}, q_j)^{m(j)/2} \right) \cdot (-1/\log(|q_n|))^{m(n)/2}.$$

These log structures on $S_{[:]}\Phi$ are glued to an fs log structure $M_{S_{[:]}}^{\text{new}} \rightarrow \mathcal{O}_{S_{[:]}}$ on $S_{[:]}$ which is independent of any choices.

We denote by $S_{[\text{val}]}$ the valuative space $(S_{[:]})_{\text{val}}$ associated to $S_{[:]}$ endowed with this new log structure. We have a proper and surjective map $S_{[\text{val}]} \rightarrow S_{[:]}$ ([21] Part IV Corollary 3.1.10).

4.7.2. Let Σ be a weak fan in $\text{Lie}(G')$. Let $\sigma \in \Sigma$.

There is a log structure on E_{σ}^{\sharp} which is the inverse image of that of E_{σ} but it depends on Γ (precisely speaking, on $\Gamma(\sigma)^{\text{gp}}$).

We define the *new log structure* $M^{\text{new}} = M_{D_{\Sigma,[:]}}^{\text{new}}$ of $D_{\Sigma,[:]}$ by using the new log structure $M_{S[:]}$ for $S = E_{\sigma}^{\sharp}$ in 4.7.1 as follows.

Let U be an open set of $D_{\Sigma,[:]}$. An \mathbf{R} -valued continuous function on U belongs to the new log structure of $D_{\Sigma,[:]}$ if its pullback on $E_{\sigma,[:]}$ belongs to the new log structure of $E_{\sigma,[:]}$ for all $\sigma \in \Sigma$.

Locally on $D_{\Sigma,[:]}$, for $\sigma \in \Sigma$, for an open set U of $D_{\sigma,[:]}$, and for a continuous section $s : U \rightarrow E_{\sigma,[:]}$ of $E_{\sigma,[:]}^{\sharp} \rightarrow D_{\sigma,[:]}$ given on U , the restriction of this new log structure to U coincides with the inverse image of the new log structure of $E_{\sigma,[:]}$ by s .

Hence the new log structure M^{new} on $D_{\Sigma,[:]}$ is an fs log structure which is independent of the choice of Γ . Denote by $D_{\Sigma, [\text{val}]}^{\sharp}$ the valuative space associated to the new log structure M^{new} of $D_{\Sigma,[:]}$.

4.7.3. Consider the log structure of $D_{\text{SL}(2)}^I$ which is defined in Proposition 3.4.10 by using distance to the boundary. Similarly as in [21] Part IV 4.5.12, we can prove that the continuous map $D_{\Sigma,[:]}^{\sharp} \rightarrow D_{\text{SL}(2)}^I$ (Theorem 4.5.5) respects these log structures. (In [21] Part IV 4.5.12, the reference 4.3.3 is a little ambiguous. The precise meaning is as above.)

Thus the map $D_{\Sigma,[:]}^{\sharp} \rightarrow D_{\text{SL}(2)}^I$ induces the continuous map $D_{\Sigma, [\text{val}]}^{\sharp} \rightarrow D_{\text{SL}(2), \text{val}}^I$ of associated valuative spaces (cf. [21] Part IV 4.5.13).

Proposition 4.7.4. *Let the notation and the assumption be as in (1) and (2) of Theorem 4.6.1. Then the conclusions of (1) and (2) of Theorem 4.6.1 and their variants in Remark 4.6.2 are true also for $X = D_{\Sigma, [\text{val}]}^{\sharp}$.*

Proof. This follows from the corresponding results for $X = D_{\Sigma,[:]}$ because the map $D_{\Sigma, [\text{val}]}^{\sharp} \rightarrow D_{\Sigma,[:]}$ is separated. \square

4.8 Mild degeneration

We consider the G -MHS version of [21] Part IV Section 5.1 in which we studied mild degenerations.

4.8.1. We define the *mild part* D_{Σ}^{mild} of D_{Σ} as the part of points (σ, Z) which satisfy the following condition.

(C) For each N in the cone σ , there is an \mathbf{R} -splitting of W (which can depend on N) that is compatible with N .

For the other spaces of nilpotent orbits D_{Σ}^{\sharp} , $D_{\Sigma,[:]}$, $D_{\Sigma,[:]}$, $D_{\Sigma, \text{val}}$, and so on, we define their mild parts $D_{\Sigma}^{\sharp, \text{mild}}$, $D_{\Sigma,[:]}$, $D_{\Sigma,[:]}$, $D_{\Sigma, \text{val}}$, and so on as the inverse images of D_{Σ}^{mild} .

4.8.2. Let $D_{\text{SL}(2)}^{\diamond}$ be the subset of $D_{\text{SL}(2)}^{\star, \text{mild}} \times \mathcal{L}$ consisting of all elements (p, Z, δ) $((p, Z) \in D_{\text{SL}(2)}^{\star, \text{mild}}$ with $p \in D_{\text{red}, \text{SL}(2)}$ and $Z \subset D$ (3.2.2), $\delta \in \mathcal{L} = W_{-2}\text{Lie}(G_{u, \mathbf{R}})$) satisfying the following conditions (i) and (ii).

(i) Let n be the rank of p , and let $\mathbf{0} := (0, \dots, 0) \in \mathbf{Z}^n$. Then the $\text{Ad}(\tau_p^{\star})$ -weights of δ are $\leq \mathbf{0}$.

(ii) For every $x \in Z$, $\delta_W(x)$ coincides with the component of δ of $\text{Ad}(\tau_p^*)$ -weight $\mathbf{0}$.

We define the structure of $D_{\text{SL}(2)}^\diamond$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by the embedding $D_{\text{SL}(2)}^\diamond \hookrightarrow D_{\text{SL}(2)}^{\star, \text{mild}} \times \mathcal{L}$.

We regard D as a subspace of $D_{\text{SL}(2)}^\diamond$ via the embedding $x \mapsto (x, \delta_W(x))$.

This is the G -MHS version of [21] Part IV 5.1.8, 5.1.9.

Proposition 4.8.3. *The canonical map $D_{\text{SL}(2)}^\star \rightarrow D_{\text{red}, \text{SL}(2)} \times \text{spl}(W)$ induces a bijection from $D_{\text{SL}(2)}^\diamond$ to the subset of $D_{\text{red}, \text{SL}(2)} \times \text{spl}(W) \times \mathcal{L}$ consisting of (p, s, δ) satisfying the following conditions (i) and (ii).*

(i) *The $\text{Ad}(\tau_p^*)$ -weights of δ are $\leq \mathbf{0}$.*

(ii) *Let (ρ, φ) be an $\text{SL}(2)$ -orbit for G_{red} which represents p . Then the component of δ of $\text{Ad}(\tau_p^*)$ -weight $\mathbf{0}$ is of Hodge type $(\leq -1, \leq -1)$ with respect to $\varphi(\mathbf{i})$.*

This is a G -MHS version of [21] Part IV Proposition 5.1.11 and is proved in the same way.

Theorem 4.8.4. *The identity map of D extends uniquely to a continuous map $D_{\Sigma, [:]}^{\sharp, \text{mild}} \rightarrow D_{\text{SL}(2)}^\diamond$. The last map is compatible with the new log structure of $D_{\Sigma, [:]}^{\sharp, \text{mild}}$ and the log structure of $D_{\text{SL}(2)}^\diamond$, and induces a continuous map between the associated valuative spaces $D_{\Sigma, [\text{val}]}^{\sharp, \text{mild}} \rightarrow D_{\text{SL}(2), \text{val}}^\diamond$.*

This is a G -MHS version of [21] Part IV Theorem 5.1.10 and is proved in the same way.

4.8.5. For mild degenerations, we can replace the upper right part of the fundamental diagram in Introduction by the following commutative diagram (maps respect structures of the spaces) which contain the space $D_{\text{SL}(2)}^\diamond$ and its associated valuative space $D_{\text{SL}(2), \text{val}}^\diamond$.

$$\begin{array}{ccccccc}
 D_{\Sigma, [\text{val}]}^{\sharp, \text{mild}} & \xrightarrow{\psi} & D_{\text{SL}(2), \text{val}}^\diamond & \rightarrow & D_{\text{SL}(2), \text{val}}^{\star, \text{mild}} & \xrightarrow{\eta^*} & D_{\text{BS}, \text{val}}^{\text{mild}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 D_{\Sigma, [:]}^{\sharp, \text{mild}} & \xrightarrow{\psi} & D_{\text{SL}(2)}^\diamond & \rightarrow & D_{\text{SL}(2)}^{\star, \text{mild}} & & D_{\text{BS}}^{\text{mild}} \\
 \downarrow & & & & \downarrow & & \\
 \Gamma \setminus D_{\Sigma}^{\text{mild}} & \leftarrow & D_{\Sigma}^{\sharp, \text{mild}} & & D_{\text{SL}(2)} & &
 \end{array}$$

Proposition 4.8.6. *The conclusions of Theorem 3.11.1 and their variants in 3.11.4 and Proposition 3.11.5 are true also for $X = D_{\text{SL}(2)}^\diamond$.*

Proof. This follows from the corresponding results for $X = D_{\text{SL}(2)}^\star$ because the map $D_{\text{SL}(2)}^\diamond \rightarrow D_{\text{SL}(2)}^\star$ is continuous and $D_{\text{SL}(2)}^\diamond$ is Hausdorff. \square

4.9 The fundamental diagram in examples

We explain what our fundamental diagram tells in special examples.

In particular, by using the fundamental diagram, we give a complement to the work of Goresky and Tai [15] on the relation of the toroidal compactification and the reductive Borel–Serre compactification (see 4.9.4).

In 4.9.1, 4.9.3, 4.9.4, 4.9.5, we consider cases in which G is reductive. For a reductive G , the main part of the fundamental diagram in Introduction becomes

$$\begin{array}{ccccc} D_{\Sigma, [\text{val}]}^\sharp & \xrightarrow{\psi} & D_{\text{SL}(2), \text{val}} & \xrightarrow{\eta} & D_{\text{BS}, \text{val}} \\ \downarrow & & \downarrow & & \downarrow \\ D_{\Sigma, [:]}^\sharp & \xrightarrow{\psi} & D_{\text{SL}(2)} & & D_{\text{BS}} \\ \downarrow & & & & \\ \Gamma \setminus D_\Sigma & \leftarrow & D_\Sigma^\sharp & & \end{array},$$

and we consider this part.

4.9.1. Example. Let $G = \text{GL}(2)$ and $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G$ be the standard one (cf. 1.6.5).

Then $D = D(G, h)$ is the complex analytic manifold $\mathfrak{H}^\pm = \{\tau \mid \text{Im}(\tau) \neq 0\} = \mathbf{C} \setminus \mathbf{R}$ on which $G(\mathbf{R}) = \text{GL}(2, \mathbf{R})$ acts naturally.

We describe the space D_{BS} of Borel–Serre orbits. Let

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \text{GL}(2, \mathbf{Q}) = G.$$

All parabolic subgroups of G other than G are conjugate to P under $G(\mathbf{Q})$, and hence the whole D_{BS} can be understood as the union of the following picture of the open set $D_{\text{BS}}(P)$. Let $P_1 := P \cap G'$. For $iy \in D$ ($y \in \mathbf{R}^\times$), the Borel–Serre lifting of $S_{P_1} = P_1/P_{1,u}$ is the group of matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ such that $ab = 1$. The adjoint action of this matrix on $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Lie}(P_{1,u})$ is the multiplication by ab^{-1} , and hence the fundamental root sends this matrix to $a^{-1}b$. Hence the isomorphism $A_P \simeq \mathbf{R}_{>0}$ given by the fundamental root sends $r \in \mathbf{R}_{>0}$ to the matrix $\begin{pmatrix} 1/\sqrt{r} & 0 \\ 0 & \sqrt{r} \end{pmatrix}$. Hence the Borel–Serre action of $r \in \mathbf{R}_{>0} \simeq A_P$ sends $x + iy$ to $x + ir^{-1}y$. From this, we have a commutative diagram of spaces

$$\begin{array}{ccc} D & \simeq & \mathbf{R} \times \mathbf{R}_{>0} \times \{\pm 1\} \\ \cap & & \cap \\ D_{\text{BS}}(P) & \simeq & \mathbf{R} \times \mathbf{R}_{\geq 0} \times \{\pm 1\}, \end{array}$$

in which the upper horizontal arrow sends $x + iy$ ($x \in \mathbf{R}$, $y \in \mathbf{R}^\times$) to $(x, 1/|y|, \text{sgn}(y))$ and the lower isomorphism preserves the structure of real analytic manifolds with corners. We have $D_{\text{BS}, \text{val}} = D_{\text{BS}}$.

Next we describe the space $D_{\text{SL}(2)}$ of $\text{SL}(2)$ -orbits. Let $W' \in \mathfrak{W}(G)$ be the filtration associated to the homomorphism

$$\alpha : \mathbf{G}_m \rightarrow G ; t \mapsto \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix},$$

and let $\Phi = \{W'\}$. Since $D_{\text{SL}(2)} = \bigcup_{g \in G(\mathbf{Q})} gD_{\text{SL}(2)}(\Phi)$, the whole $D_{\text{SL}(2)}$ can be understood as the union of the following picture of the open set $D_{\text{SL}(2)}(\Phi)$. We have the

distance $\beta : D \rightarrow \mathbf{R}_{>0}$ to Φ -boundary defined as $x + iy \mapsto 1/\sqrt{|y|}$. Then the injective real analytic map

$$\nu_{\alpha,\beta} : D_{\mathrm{SL}(2)}(\Phi) \hookrightarrow \mathbf{R}_{\geq 0} \times D \times \mathrm{spl}(W')$$

in Proposition 3.4.4 sends $x + iy$ ($x \in \mathbf{R}$, $y \in \mathbf{R}^\times$) to $(1/\sqrt{|y|}, x|y|^{-1} + i \cdot \mathrm{sgn}(y), x)$, where we identify $\mathrm{spl}(W')$ with \mathbf{R} by sending $x \in \mathbf{R}$ to $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot s$ with $s \in \mathrm{spl}(W')$ given by α . The closure of $\nu_{\alpha,\beta}(D)$ in the target space is $C = \{(r, z, x) \mid z = xr^2 \pm i\}$. Since D is dense in $D_{\mathrm{SL}(2)}(\Phi)$, $\nu_{\alpha,\beta}$ induces an injective map $D_{\mathrm{SL}(2)}(\Phi) \rightarrow C$. Let $p \in D_{\mathrm{SL}(2)}(\Phi)$ be the class of the $\mathrm{SL}(2)$ -orbit in 3.1.2. Then $\nu_{\alpha,\beta}$ sends $\begin{pmatrix} \epsilon & x \\ 0 & 1 \end{pmatrix} \cdot p \in D_{\mathrm{SL}(2)}(\Phi)$ for $\epsilon \in \{\pm 1\}$ to $(0, i\epsilon, x)$. This proves the surjectivity of $D_{\mathrm{SL}(2)}(\Phi) \rightarrow C$ and hence we have an isomorphism $D_{\mathrm{SL}(2)}(\Phi) \xrightarrow{\sim} C$ of objects $\mathcal{B}'_{\mathbf{R}}(\log)$. From this, we have a commutative diagram of spaces

$$\begin{array}{ccc} D & \simeq & \mathbf{R} \times \mathbf{R}_{>0} \times \{\pm 1\} \\ \cap & & \cap \\ D_{\mathrm{SL}(2)}(\Phi) & \simeq & \mathbf{R} \times \mathbf{R}_{\geq 0} \times \{\pm 1\}, \end{array}$$

in which the upper horizontal arrow sends $x + iy$ ($x \in \mathbf{R}$, $y \in \mathbf{R}^\times$) to $(x, 1/\sqrt{|y|}, \mathrm{sgn}(y))$ and the lower isomorphism preserves the real analytic structure and the log structure with sign. The image of $p \in D_{\mathrm{SL}(2)}(\Phi)$ under the lower horizontal arrow is $(0, 0, 1)$. We have $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{SL}(2)}$.

By Theorem 3.8.2, the identity map of D extends uniquely to a morphism $D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$. This induces $D_{\mathrm{SL}(2)}(\Phi) \rightarrow D_{\mathrm{BS}}(P)$ for which the following diagram is commutative.

$$\begin{array}{ccc} D_{\mathrm{SL}(2)}(\Phi) & \xrightarrow{\sim} & \mathbf{R} \times \mathbf{R}_{\geq 0} \times \{\pm 1\} \\ \downarrow & & \downarrow \\ D_{\mathrm{BS}}(P) & \xrightarrow{\sim} & \mathbf{R} \times \mathbf{R}_{\geq 0} \times \{\pm 1\} \end{array}$$

Here the right vertical arrow is $(x, r, \epsilon) \mapsto (x, r^2, \epsilon)$. Thus the map $D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$ is a homeomorphism, but their real analytic structures are slightly different.

Next we describe the spaces of nilpotent orbits for the fan Σ consisting of all cones $\mathbf{R}_{\geq 0}N$ with $N \in \mathrm{Lie}(G') = \mathfrak{sl}(2, \mathbf{Q})$ such that $N^2 = 0$ as a $(2, 2)$ -matrix. Then for a congruence subgroup Γ of $\mathrm{SL}(2, \mathbf{Z})$, $\Gamma \backslash D_\Sigma$ is a compactified modular curve. We have $D_{\Sigma,[\mathrm{val}]}^\sharp = D_{\Sigma,[:]}^\sharp = D_\Sigma^\sharp$, and the CKS map induces a homeomorphism $D_\Sigma^\sharp = D_{\Sigma,[:]}^\sharp \xrightarrow{\sim} D_{\mathrm{SL}(2)}$.

Define $\sigma, \tau \in \Sigma$ by

$$\sigma := \mathbf{R}_{\geq 0}N, \quad \tau := \mathbf{R}_{\geq 0}(-N) \quad \text{with} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The CKS map induces a homeomorphism $D_\sigma^\sharp \cup D_\tau^\sharp \xrightarrow{\sim} D_{\mathrm{SL}(2)}(\Phi)$. For $a \in \mathbf{R}$, this map sends the nilpotent i -orbit $(\sigma, a + i\mathbf{R}) \in D_\sigma^\sharp$ (resp. $(\tau, a + i\mathbf{R}) \in D_\tau^\sharp$) to the element of $D_{\mathrm{SL}(2)}(\Phi)$ corresponding to the element $(a, 0, 1)$ (resp. $(a, 0, -1)$) of $\mathbf{R} \times \mathbf{R}_{\geq 0} \times \{\pm 1\}$. Here $a + i\mathbf{R}$ is regarded as a subset of \check{D} by identifying \check{D} with $\mathbf{P}^1(\mathbf{C}) \supset a + i\mathbf{R}$.

The part about D_{BS} and $D_{\mathrm{SL}(2)}$ as topological spaces in this 4.9.1 is essentially described in [23] 6.2.

4.9.2. Example. Let $G = \begin{pmatrix} \mathbf{G}_m & \mathbf{G}_a \\ 0 & 1 \end{pmatrix}$ and let $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\text{red}, \mathbf{R}} = \mathbf{G}_{m, \mathbf{R}}$ be the homomorphism which induces $S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^\times \rightarrow G_{\text{red}}(\mathbf{R}) : z \mapsto z^{-1}\bar{z}^{-1}$ ($z \in \mathbf{C}^\times$). Let $x \in D = D(G, h)$ be the point defined by the homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ which induces

$$S_{\mathbf{C}/\mathbf{R}}(\mathbf{R}) = \mathbf{C}^\times \rightarrow G(\mathbf{R}) = \begin{pmatrix} \mathbf{R}^\times & \mathbf{R} \\ 0 & 1 \end{pmatrix} ; z \mapsto \begin{pmatrix} z^{-1}\bar{z}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad (z \in \mathbf{C}^\times).$$

Then we have an isomorphism of complex analytic manifolds

$$\mathbf{C} \xrightarrow{\sim} D ; c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \cdot x.$$

Let p be the unique element of D_{red} . The real analytic isomorphism $D \xrightarrow{\sim} \text{spl}(W) \times \mathcal{L}(p)$ in Proposition 1.3.6 sends $c = a + ib \in \mathbf{C} = D$ ($a, b \in \mathbf{R}$) to $(s(a), \delta(b))$, where

$$s(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{spl}_W(x), \quad \delta(b) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in W_{-2}\text{Lie}(G_{\mathbf{R}}) = \text{gr}_{-2}^W \text{Lie}(G_{\mathbf{R}}).$$

Let P be the unique parabolic subgroup of G_{red} , that is, G_{red} itself. Since $A_P = \{1\}$, $B_P = \mathbf{R}_{>0}$. By 2.4.4, the Borel–Serre action of $t \in B_P$ on $D = \mathbf{C}$ sends $c = a + ib$ ($a, b \in \mathbf{R}$) to $a + it^{-2}b$. As topological spaces with sheaves of real analytic spaces and with log structures with sign, all the spaces D_{BS} , $D_{\text{BS}, \text{val}}$, $D_{\text{SL}(2)}^*$, $D_{\text{SL}(2), \text{val}}^*$, $D_{\text{SL}(2)}^{II}$, $D_{\text{SL}(2), \text{val}}^{II}$, $D_{\text{SL}(2)}^I$, $D_{\text{SL}(2), \text{val}}^I$ coincide with the real analytic manifold with corners $\mathbf{R} \times [-\infty, \infty] \supset \mathbf{R} \times \mathbf{R} \simeq D$, where $\mathbf{R} \times \mathbf{R} \simeq D$ sends $(a, b) \in \mathbf{R} \times \mathbf{R}$ to $a + ib \in \mathbf{C} = D$. For $a \in \mathbf{R}$, $(a, \infty) \in \mathbf{R} \times [-\infty, \infty]$ corresponds to the element $(p, a + i\mathbf{R}_{>0})$ of $D_{\text{SL}(2)}$ and to the element $(P, a + i\mathbf{R}_{>0})$ of D_{BS} , and $(a, -\infty) \in \mathbf{R} \times [-\infty, \infty]$ corresponds to the element $(p, a + i\mathbf{R}_{<0})$ of $D_{\text{SL}(2)}$ and to the element $(P, a + i\mathbf{R}_{<0})$ of D_{BS} .

Define

$$\sigma := \mathbf{R}_{\geq 0}N, \quad \tau := \mathbf{R}_{\geq 0}(-N) \quad \text{with} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Lie}(G'),$$

$$\Sigma := \{\{0\}, \sigma, \tau\}, \quad \Gamma := \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}.$$

Then Γ and Σ are strongly compatible. We have isomorphisms of complex analytic manifolds

$$\Gamma \setminus D \simeq \mathbf{C}^\times, \quad \Gamma \setminus D_\Sigma \simeq \mathbf{P}^1(\mathbf{C}),$$

where the class of $c \in \mathbf{C} = D$ in $\Gamma \setminus D$ is identified with $\exp(2\pi i c) \in \mathbf{C}^\times$. The point $0 \in \mathbf{P}^1(\mathbf{C})$ corresponds to the class of the nilpotent orbit (σ, \mathbf{C}) and the point $\infty \in \mathbf{P}^1(\mathbf{C})$ corresponds to the class of the nilpotent orbit (τ, \mathbf{C}) . We have $D_{\Sigma, [\text{val}]}^\sharp = D_{\Sigma, [:]}^\sharp = D_\Sigma^\sharp$, and the CKS map induces a homeomorphism $D_\Sigma^\sharp = D_{\Sigma, [:]}^\sharp \xrightarrow{\sim} D_{\text{SL}(2)}^I$. For $a \in \mathbf{R}$, this map sends the nilpotent i -orbit $(\sigma, a + i\mathbf{R}) \in D_\sigma^\sharp$ to the element of $D_{\text{SL}(2)}$ corresponding to $(a, \infty) \in \mathbf{R} \times [-\infty, \infty]$, and the nilpotent i -orbit $(\tau, a + i\mathbf{R}) \in D_\tau^\sharp$ to the element of $D_{\text{SL}(2)}$ corresponding to $(a, -\infty) \in \mathbf{R} \times [-\infty, \infty]$. Here $a + i\mathbf{R}$ is regarded as a subset of $\check{D} = D = \mathbf{C}$.

4.9.3. Example. We omit the details in this 4.9.3.

Let $G = \mathrm{GL}(2) \times \mathrm{GL}(2)$ and let $h : S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathrm{GL}(2)_{\mathbf{R}} \times \mathrm{GL}(2)_{\mathbf{R}}$ be the diagonal embedding of the homomorphism in 4.9.1. This h is \mathbf{R} -polarizable. We have

$$D = \mathfrak{H}^\pm \times \mathfrak{H}^\pm.$$

Let Σ be the fan consisting of all cones of the form $\mathbf{R}_{\geq 0}N_1 \times \mathbf{R}_{\geq 0}N_2 \subset \mathrm{Lie}(G'_{\mathbf{R}}) = \mathfrak{sl}(2, \mathbf{R}) \times \mathfrak{sl}(2, \mathbf{R})$, where N_j is an element of the j -th $\mathfrak{sl}(2, \mathbf{Q})$ in $\mathrm{Lie}(G')$ such that $N_j^2 = 0$ as a $(2, 2)$ -matrix ($j = 1, 2$). Then in the category of topological spaces, the fundamental diagram presents

$$\begin{array}{ccccc} D_{\Sigma, [\mathrm{val}]}^\sharp & \xrightarrow{\sim} & D_{\mathrm{SL}(2), \mathrm{val}} & \xrightarrow{\sim} & D_{\mathrm{BS}, \mathrm{val}} \\ \downarrow & & \downarrow & & \downarrow \\ D_{\Sigma, [:]}^\sharp & \xrightarrow{\sim} & D_{\mathrm{SL}(2)} & & \downarrow \\ \downarrow & & \searrow & & \\ D_\Sigma^\sharp & \xrightarrow{\sim} & & & D_{\mathrm{BS}}. \end{array}$$

Furthermore, D_Σ^\sharp is canonically isomorphic to the product of two copies of D_Σ^\sharp of 4.9.1, and D_{BS} is canonically isomorphic to the product of two copies of D_{BS} of 4.9.1 (see Proposition 4.10.6).

The proper surjective map $D_{\Sigma, [:]}^\sharp \rightarrow D_\Sigma^\sharp$ is not injective as the following property of the convergences show, and hence the map $D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$ is not injective.

Consider the point $p = (iy_1, iy_2) \in \mathfrak{H} \times \mathfrak{H} \subset D$ ($y_1, y_2 \in \mathbf{R}_{>0}$). In the following (1), (2), (3), we give examples of the convergence of p in D_Σ^\sharp , $D_{\Sigma, [:]}^\sharp$, and $D_{\Sigma, [\mathrm{val}]}^\sharp$, respectively, to show how the topologies of these three spaces are different.

(1) If $y_j \rightarrow \infty$ for $j = 1, 2$, p converges to the class of the nilpotent i -orbit (σ, Z) , where $\sigma = \mathbf{R}_{\geq 0}N_1 \times \mathbf{R}_{\geq 0}N_2$ with N_j the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the j -th $\mathfrak{sl}(2, \mathbf{Q})$ in $\mathrm{Lie}(G')$ ($j = 1, 2$) and Z is the $\exp(i\sigma_{\mathbf{R}})$ orbit which passes $(i, i) \in D$.

(2) If $y_1, y_2 \rightarrow \infty$ and $y_1/y_2 \rightarrow \infty$, then p converges in $D_{\Sigma, [:]}^\sharp$ to a point a , and if $y_1, y_2 \rightarrow \infty$ and $y_1/y_2 \rightarrow 1$, then p converges in $D_{\Sigma, [:]}^\sharp$ to a point b , and $a \neq b$. These $a, b \in D_{\Sigma, [:]}^\sharp$ lie over the above class of (σ, Z) in D_Σ^\sharp .

(3) If $y_2 \rightarrow \infty$ and $y_1/y_2^2 \rightarrow \infty$, then p converges in $D_{\Sigma, [\mathrm{val}]}^\sharp$ to a point c , and if $y_2 \rightarrow \infty$ and $y_1/y_2^3 \rightarrow \infty$, p converges in $D_{\Sigma, [:]}^\sharp$ to a point d , and $c \neq d$. These $c, d \in D_{\Sigma, [\mathrm{val}]}^\sharp$ lie over the point $a \in D_{\Sigma, [:]}^\sharp$.

4.9.4. Shimura varieties.

Assume that G is reductive and that $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ satisfies the condition that the Hodge type of $\mathrm{Lie}(G_{\mathbf{R}})$ via h_0 is in $\{(1, -1), (0, 0), (-1, 1)\}$ (as in 3.8.1).

Then for an arithmetic subgroup (that is, a subgroup satisfying the condition (A) in 1.5.7) Γ of $G'(\mathbf{Q})$, there is a fan Σ which is strongly compatible with Γ such that $\Gamma \backslash D_\Sigma$ is compact. This compact space is called a Mumford (or toroidal) compactification of $\Gamma \backslash D$.

As in Theorem 3.8.2, we have a morphism $D_{\mathrm{SL}(2)} \rightarrow D_{\mathrm{BS}}$ which extends the identity morphism of D . We can prove $D_{\mathrm{SL}(2), \mathrm{val}} \xrightarrow{\sim} D_{\mathrm{BS}, \mathrm{val}}$, but we do not give the proof here.

As an application of the fundamental diagram, we have the following complement to the work [15] of Goresky and Tai on the relation of toroidal compactifications and reductive Borel–Serre compactification.

The reductive Borel–Serre space, which we denote by D_{BS}^b here, is defined to be the quotient of D_{BS} by the following equivalence relation. For $p_1 = (P_1, Z_1), p_2 = (P_2, Z_2) \in D_{\text{BS}}$, $p_1 \sim p_2$ if and only if $P_1 = P_2$ and $P_{1,u}Z_1 = P_{2,u}Z_2$, where $(\cdot)_u$ denotes the unipotent radical. The quotient $\Gamma \backslash D_{\text{BS}}^b$ is compact, and is called the reductive Borel–Serre compactification of $\Gamma \backslash D$.

Let Γ be a neat arithmetic subgroup of $G'(\mathbf{Q})$ and let Σ be strongly compactible with Γ such that $\Gamma \backslash D_\Sigma$ is compact.

Then concerning the relation of the compactifications $\Gamma \backslash D_\Sigma$ and $\Gamma \backslash D_{\text{BS}}^b$ of $\Gamma \backslash D$, Goresky and Tai obtained the following result. The identity map of $\Gamma \backslash D$ extends to a “continuous map modulo homotopy” from $\Gamma \backslash D_\Sigma \rightarrow \Gamma \backslash D_{\text{BS}}^b$ if we replace Σ by a sufficiently finer subdivision. Precisely speaking, if we replace Σ by a sufficiently finer subdivision, there are a compact topological space T which contains $\Gamma \backslash D$ as a dense open subspace, and continuous surjective maps $f : T \rightarrow \Gamma \backslash D_\Sigma$ and $g : T \rightarrow \Gamma \backslash D_{\text{BS}}^b$ such that f is a homotopy equivalence and such that f and g induce the identity map of $\Gamma \backslash D$.

We have the following result. The map $\Gamma \backslash D_{\Sigma,[\cdot]} \rightarrow \Gamma \backslash D_\Sigma$ is proper surjective and a weak homotopy equivalence, and we have the continuous surjective map $\Gamma \backslash D_{\Sigma,[\cdot]} \rightarrow \Gamma \backslash D_{\text{BS}}^b$ induced by the continuous maps $D_{\Sigma,[\cdot]}^\sharp \rightarrow D_{\text{SL}(2)} \rightarrow D_{\text{BS}}$ by passing to the quotients. That is, compared to [15], we do not need a subdivision of Σ here and we present a standard space $\Gamma \backslash D_{\Sigma,[\cdot]}$ which connects $\Gamma \backslash D_\Sigma$ and $\Gamma \backslash D_{\text{BS}}^b$. This gives an alternative proof for the existence of the canonical maps $H^m(\Gamma \backslash D_{\text{BS}}^b, A) \rightarrow H^m(\Gamma \backslash D_\Sigma, A)$ with A being an abelian group for any m , A being a group for $m = 1$, and A being a set for $m = 0$, obtained by Goresky–Tai. We plan to discuss the details in a forthcoming paper.

4.9.5. An example of $D = D(G, h)$ with G reductive for which the identity map of D does not extend to a continuous map $D_{\text{SL}(2)} \rightarrow D_{\text{BS}}$ is given in 4.10.10 basing on [23] and [24].

Remark 4.9.6. The spaces $D_{\Sigma,\text{val}}^\sharp$ and $D_{\Sigma,[\cdot]}^\sharp$ in the fundamental diagram play similar roles in our work. The former appears in [24] and in [21] Part II, Part III and so on of our series of papers, and the latter appears in [21] Part IV and in this part. Both have canonical proper continuous maps to D_Σ^\sharp and also continuous maps (the CKS maps) to $D_{\text{SL}(2)}$. In this Part V, we are using $D_{\Sigma,[\cdot]}^\sharp$ (and the related space $\Gamma \backslash D_{\Sigma,[\cdot]}$) more than $D_{\Sigma,\text{val}}^\sharp$ (and the related space $\Gamma \backslash D_{\Sigma,\text{val}}$). The advantages of the space $D_{\Sigma,[\cdot]}^\sharp$ and $\Gamma \backslash D_{\Sigma,[\cdot]}$ are:

(1) The space $D_{\Sigma,[\cdot]}^\sharp$ naturally produces the space $D_{\Sigma,[\text{val}]}^\sharp$ from which we can go to $D_{\text{SL}(2),\text{val}}^I$ (and in the case G is reductive, to $D_{\text{BS},\text{val}}$ and D_{BS}).

(2) The definition of the CKS map by using $D_{\Sigma,[\cdot]}^\sharp$ is simpler and more natural than that by using $D_{\Sigma,\text{val}}^\sharp$. The convergences of ratios such as $y_j/y_{j+1} \rightarrow a_j$ ($a_j \in \mathbf{R}_{>0}$), $y_j/y_{j+1} \rightarrow \infty$ appear in $\text{SL}(2)$ -orbit theorem and these are the convergences in the space of ratios, and so the relation to the $\text{SL}(2)$ -orbit theorem of $D_{\Sigma,[\cdot]}^\sharp$ seems stronger than that of $D_{\Sigma,\text{val}}^\sharp$.

(3) The spaces $D_{\Sigma,[:]^\sharp}$ and $\Gamma \setminus D_{\Sigma,[:]}$ have the application described in 4.9.4.

On the other hand, the space $\Gamma \setminus D_{\Sigma, \text{val}}$ has the sheaf of holomorphic functions which the space $\Gamma \setminus D_{\Sigma,[:]}$ does not have.

4.10 Functoriality in G

4.10.1. Assume that we are given a homomorphism $f : G_1 \rightarrow G_2$. We describe how we can relate the period domains and extended period domains for G_1 and those for G_2 .

We have to introduce some conditions for this functoriality.

We assume $f(G_{1,u}) \subset G_{2,u}$ and hence f induces $f_{\text{red}} : G_{1,\text{red}} \rightarrow G_{2,\text{red}}$. We assume that we are given $h_1 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{1,\text{red},\mathbf{R}}$ as in 1.2.13, and we further assume that the induced homomorphism $h_2 := f_{\text{red}} \circ h_1 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{2,\text{red},\mathbf{R}}$ satisfies the condition that the composition $\mathbf{G}_{m,\mathbf{R}} \rightarrow S_{\mathbf{C}/\mathbf{R}} \xrightarrow{h_2} G_{2,\text{red},\mathbf{R}}$ is central.

Then we have holomorphic maps

$$D(G_1, h_1) \rightarrow D(G_2, h_2), \quad D(G_{1,\text{red}}, h_1) \rightarrow D(G_{2,\text{red}}, h_2).$$

In 4.10.2–4.10.4, we further assume that h_1 and h_2 are \mathbf{R} -polarizable.

4.10.2. We first consider the spaces of Borel–Serre orbits.

Assume the the following condition (i) is satisfied.

(i) The map $\text{Lie}(G'_1) \rightarrow \text{Lie}(G'_2)$ is surjective.

Here as usual, $(-)'$ denote the commutator groups. Then we have a morphism of real analytic manifolds with corners

$$D(G_1, h_1)_{\text{BS}}^{\text{mild}} \rightarrow D(G_2, h_2)_{\text{BS}}^{\text{mild}} ; (P_1, Z_1) \mapsto (P_2, Z_2),$$

where P_1 is a parabolic subgroup of $G_{1,\text{red}}$, Z_1 is an A_{P_1} -orbit in $D(G_1, h_1)$, P_2 is the algebraic subgroup of G_2 generated by the image of P_1 and the center of G_2° , which is a parabolic subgroup of G_2 , and Z_2 is the unique A_{P_2} -orbit in $D(G_2, h_2)$ which contains the image of Z_1 .

Assume that the above condition (i) and the following condition (ii) is satisfied.

(ii) The map $\text{Lie}(G_{1,u}) \rightarrow \text{Lie}(G_{2,u})$ is injective.

Then we have a morphism of real analytic manifolds with corners

$$D(G_1, h_1)_{\text{BS}} \rightarrow D(G_2, h_2)_{\text{BS}} ; (P_1, Z_1) \mapsto (P_2, Z_2),$$

where P_1 and P_2 are as above, Z_2 is as above if Z_1 is an A_{P_1} -orbit, and in the case where Z_1 is a B_{P_1} -orbit, Z_2 is the unique B_{P_2} -orbit which contains the image of Z_1 .

Here in the case where Z_1 is a B_{P_1} -orbit, the image of Z_1 in $D(G_2, h_2)$ does not meet $D(G_2, h_2)_{\text{spl}}$ by the injectivity of $\text{Lie}(G_{1,u}) \rightarrow \text{Lie}(G_{2,u})$.

4.10.3. Next we consider the spaces of $\text{SL}(2)$ -orbits.

In the case G_1 and G_2 are reductive, we have a morphism $D(G_1, h_1)_{\text{SL}(2)} \rightarrow D(G_2, h_2)_{\text{SL}(2)}$ of locally ringed spaces with log structures with sign, which sends the class of an $\text{SL}(2)$ -orbit (ρ_1, φ_1) ($\rho_1 : \text{SL}(2)_{\mathbf{R}}^n \rightarrow G_{\mathbf{R}}$, $\varphi_1 : \mathfrak{H}^n \rightarrow D(G_2, h_2)$) to the class of (ρ_2, φ_2) , where ρ_2

is the composition $\mathrm{SL}(2)_{\mathbf{R}}^n \xrightarrow{\rho_1} G_{1,\mathbf{R}} \rightarrow G_{2,\mathbf{R}}$ and φ_2 is the composition $\mathfrak{H}^n \rightarrow D(G_1, h_1) \rightarrow D(G_2, h_2)$.

In general, we have morphisms

$$\begin{aligned} D(G_1, h_1)_{\mathrm{SL}(2)}^A &\rightarrow D(G_2, h_2)_{\mathrm{SL}(2)}^A \quad (\text{for the structures } I, II), \\ D(G_1, h_1)_{\mathrm{SL}(2)}^{\star, \text{mild}} &\rightarrow D(G_2, h_2)_{\mathrm{SL}(2)}^{\star, \text{mild}}, \end{aligned}$$

where $(-)^A$ denotes the part consisting of A -orbits, which sends (p_1, Z_1) to (p_2, Z_2) ($p_1 \in D(G_1, h_1)_{\mathrm{red}, \mathrm{SL}(2)}$, Z_1 is an A -orbit in $D(G_1, h_1)$, p_2 is the image of p_1 in $D(G_2, h_2)_{\mathrm{red}, \mathrm{SL}(2)}$ and Z_2 is the unique A -orbit in $D(G_2, h_2)$ containing the image of Z_1).

If $\mathrm{Lie}(G_{1,u}) \rightarrow \mathrm{Lie}(G_{2,u})$ is injective, we have morphisms

$$\begin{aligned} D(G_1, h_1)_{\mathrm{SL}(2)} &\rightarrow D(G_2, h_2)_{\mathrm{SL}(2)} \quad (\text{for the structures } I, II), \\ D(G_1, h_1)_{\mathrm{SL}(2)}^{\star} &\rightarrow D(G_2, h_2)_{\mathrm{SL}(2)}^{\star}, \end{aligned}$$

which sends (p_1, Z_1) to (p_2, Z_2) ($p_1 \in D(G_1, h_1)_{\mathrm{red}, \mathrm{SL}(2)}$, p_2 is the image of p_1 in $D(G_2, h_2)_{\mathrm{red}, \mathrm{SL}(2)}$, either Z_1 is an A -orbit in $D(G_1, h_1)$ and Z_2 is the unique A -orbit in $D(G_2, h_2)$ containing the image of Z_1 , or Z_1 is a B -orbit in $D(G_1, h_1)$ and Z_2 is the unique B -orbit in $D(G_2, h_2)$ containing the image of Z_1).

4.10.4. Lastly, we consider the spaces of nilpotent orbits.

Assume that we are given Σ_1 for G_1 . Assume that the images of elements of Σ_1 in $\mathrm{Lie}(G'_{2,\mathbf{R}})$ form a weak fan Σ_2 in $\mathrm{Lie}(G'_2)$. (For example, this is satisfied if $\mathrm{Lie}(G'_1) \rightarrow \mathrm{Lie}(G'_2)$ is injective.) Then we have a map

$$D(G_1, h_1)_{\Sigma_1} \rightarrow D(G_2, h_2)_{\Sigma_2},$$

a continuous map

$$D(G_1, h_1)_{\Sigma_1, *}^{\sharp} \rightarrow D(G_2, h_2)_{\Sigma_2, *}^{\sharp} \quad \text{for } * = [:, \mathrm{val}, [\mathrm{val}]],$$

and a morphism of log manifolds

$$\Gamma_1 \setminus D(G_1, h_1)_{\Sigma_1} \rightarrow \Gamma_2 \setminus D(G_2, h_2)_{\Sigma_2},$$

where $\Gamma_j \subset G'_j(\mathbf{Q})$ ($j = 1, 2$) are neat semi-arithmetic subgroups such that Γ_j is strongly compatible with Σ_j and the image of Γ_1 in $G_2(\mathbf{Q})$ is contained in Γ_2 .

Proposition 4.10.5. *Let G_1 and G_2 be linear algebraic groups over \mathbf{Q} and let $h_j : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{j,\mathbf{R}}$ ($j = 1, 2$) be homomorphisms as in 1.2.13. Let $h = (h_1, h_2) : S_{\mathbf{C}/\mathbf{R}} \rightarrow (G_1 \times G_2)_{\mathbf{R}}$. Then h satisfies the condition as in 1.2.13. The morphisms $D(G_1 \times G_2, h) \rightarrow D(G_j, h_j)$ associated to the projections $G_1 \times G_2 \rightarrow G_j$ ($j = 1, 2$) induce an isomorphism*

$$D(G_1 \times G_2, h) \xrightarrow{\sim} D(G_1, h_1) \times D(G_2, h_2).$$

Proof. In the case where G_1 and G_2 are reductive, this is proved as

$$D(G_1 \times G_2, h) = \{(G_1 \times G_2)(\mathbf{R})\text{-conjugate of } h\}$$

$$= \prod_{j=1}^2 \{G_j(\mathbf{R})\text{-conjugate of } h_j\} = D(G_1, h_1) \times D(G_2, h_2).$$

The general case is reduced to the reductive case by Proposition 1.3.6 using $(G_1 \times G_2)_u(\mathbf{R}) = G_{1,u}(\mathbf{R}) \times G_{2,u}(\mathbf{R})$ (this proves that $\text{spl}(W)$ for $G_1 \times G_2$ is the product of $\text{spl}(W)$ for G_j) and $\mathcal{L}(p) = \mathcal{L}(p_1) \times \mathcal{L}(p_2)$ for $p_j \in D(G_j, h_j)_{\text{red}}$ and $p = (p_1, p_2) \in D(G_1 \times G_2, h)_{\text{red}}$. \square

Proposition 4.10.6. *Let the notation be as in Proposition 4.10.5 and assume that h_j ($j = 1, 2$) are \mathbf{R} -polarizable.*

Then h is \mathbf{R} -polarizable and we have:

(1) *The canonical maps $D(G_1 \times G_2, h)_{\text{BS}}^{\text{mild}} \rightarrow D(G_j, h_j)_{\text{BS}}^{\text{mild}}$ ($j = 1, 2$) induce an isomorphism of real analytic manifolds with corners*

$$D(G_1 \times G_2, h)_{\text{BS}}^{\text{mild}} \xrightarrow{\sim} D(G_1, h_1)_{\text{BS}}^{\text{mild}} \times D(G_2, h_2)_{\text{BS}}^{\text{mild}}.$$

In particular, if G_1 and G_2 are reductive, we have

$$D(G_1 \times G_2, h)_{\text{BS}} \xrightarrow{\sim} D(G_1, h_1)_{\text{BS}} \times D(G_2, h_2)_{\text{BS}}.$$

(2) *Assume that Σ_j for G_j are given ($j = 1, 2$). Let $\Sigma = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i\}$. Then*

$$D(G_1 \times G_2, h)_{\Sigma_1 \times \Sigma_2}^{\sharp} \xrightarrow{\sim} D(G_1, h_1)_{\Sigma_1}^{\sharp} \times D(G_2, h_2)_{\Sigma_2}^{\sharp}$$

as topological spaces, and for neat semi-arithmetic subgroups Γ_j of $G'_j(\mathbf{Q})$ which are strongly compatible with Σ_j ($j = 1, 2$), we have an isomorphism of log manifolds

$$(\Gamma_1 \times \Gamma_2) \setminus D(G_1 \times G_2, h)_{\Sigma} \xrightarrow{\sim} \Gamma_1 \setminus D(G_1, h_1)_{\Sigma_1} \times \Gamma_2 \setminus D(G_2, h_2)_{\Sigma_2}.$$

Proof. (1) Parabolic subgroups of $G_1 \times G_2$ are $P_1 \times P_2$ for parabolic subgroups P_j of G_j . (This fact is deduced from the surjectivity of the map $\text{Hom}(\mathbf{G}_m, \mathcal{G}) \rightarrow \{\text{parabolic subgroups of } \mathcal{G}\}$ in 3.3.7 and from $\text{Hom}(\mathbf{G}_m, G_1 \times G_2) = \text{Hom}(\mathbf{G}_m, G_1) \times \text{Hom}(\mathbf{G}_m, G_2)$.) We have $A_{P_1 \times P_2} = A_{P_1} \times A_{P_2}$. Hence the converse map is given by $((P_1, Z_1), (P_2, Z_2)) \mapsto (P_1 \times P_2, Z_1 \times Z_2)$.

(2) The converse map is given by $((\sigma_1, Z_1), (\sigma_2, Z_2)) \mapsto (\sigma_1 \times \sigma_2, Z_1 \times Z_2)$. \square

Remark 4.10.7. On the other hand, even if G is reductive, $D(G_1 \times G_2, h)_{\text{SL}(2)}$ need not be the product of $D(G_j, h_j)_{\text{SL}(2)}$ ($j = 1, 2$) as examples in 4.9.1 and 4.9.3 show.

In 4.10.8–4.10.10, we give examples of $G_1 \rightarrow G_2$ whose associated morphisms of spaces of Borel–Serre orbits do not exist, looking at examples of convergence and divergence in the extended period domains.

4.10.8. Example. Let (G_1, h_1) be the (G, h) of 4.9.2 and let $G_1 = G \times G$, $h_1 = (h, h) : S_{\mathbf{C}/\mathbf{R}} \rightarrow G \times G$, let $G_2 = G$, let $G_1 \rightarrow G_2$ be the second projection. Then the induced morphism $D(G_1, h_1) \rightarrow D(G_2, h_2)$ is understood as $\mathbf{C}^2 \rightarrow \mathbf{C}$; $(c_1, c_2) \mapsto c_2$. This morphism does not extend to a continuous map $D(G_1, h_1)_{\text{BS}} \rightarrow D(G_2, h_2)_{\text{BS}}$. Note that in this case, $\text{Lie}(G_{1,u}) \rightarrow \text{Lie}(G_{2,u})$ is not injective. Let $P_1 = G_{1,\text{red}}$, the unique parabolic

subgroup of $G_{1,\text{red}}$. For $\theta \in \mathbf{R}$, let $Z(\theta)$ be the B_{P_1} -orbit $\{(ir \cos(\theta), ir \sin(\theta)) \mid r \in \mathbf{R}_{>0}\}$ in $\mathbf{C}^2 = D(G_1, h_1)$. Then when $\theta \rightarrow 0$ and $r \rightarrow \infty$, $(ir \cos(\theta), ir \sin(\theta)) \in \mathbf{C}^2 = D(G_1, h_1)$ converges to $(P_1, Z(0)) \in D(G_1, h_1)_{\text{BS}}$, but the image $ir \sin(\theta) \in \mathbf{C} = D(G_2, h_2)$ does not converge in $D(G_2, h_2)_{\text{BS}}$.

The point is that for finite dimensional graded \mathbf{R} -vector spaces V_1 and V_2 of weight ≤ -1 and for a linear map $f : V_1 \rightarrow V_2$ which is compatible with the gradings, the following three conditions are equivalent. (i) Either f is injective or $V_2 = 0$. (ii) f extends to a continuous map $\overline{V}_1 \rightarrow \overline{V}_2$. If these conditions are satisfied, f extends to a morphism $\overline{V}_1 \rightarrow \overline{V}_2$ of real analytic manifolds with corners uniquely.

4.10.9. Example. Let $G_1 = \text{GL}(2) \times \text{GL}(2)$, $G_2 = \text{GSp}(4)$, and let $f : G_1 \rightarrow G_2$ be the natural embedding. Let $h_1 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{1,\mathbf{R}}$ be as in 4.9.3 and let $h_2 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{2,\mathbf{R}}$ be $f \circ h_1$. Then both h_1 and h_2 are \mathbf{R} -polarizable. We show that the canonical map $D(G_1, h_1) \rightarrow D(G_2, h_2)$, which we also denote by f , does not extend to a continuous map $D(G_1, h_1)_{\text{BS}} \rightarrow D(G_2, h_2)_{\text{BS}}$.

In fact, we have:

(1) When $y_1, y_2 \rightarrow \infty$, $(iy_1, iy_2) \in \mathfrak{H}^\pm \times \mathfrak{H}^\pm = D(G_1, h_1)$ converges in $D(G_1, h_1)_{\text{BS}}$.

On the other hand, the identity map of $D(G_2, h_2)$ extends to a homeomorphism $D(G_2, h_2)_{\text{SL}(2)} \xrightarrow{\sim} D(G_2, h_2)_{\text{BS}}$ ([23] Theorem 6.7). We have

(2) When $y_2, y_1/y_2 \rightarrow \infty$, $f(iy_1, iy_2)$ converges in $D(G_2, h_2)_{\text{SL}(2)}$ to a point a , and when $y_1, y_2/y_1 \rightarrow \infty$, $f(iy_1, iy_2)$ converges in $D(G_2, h_2)$ to a point b , and $a \neq b$.

Hence in $D(G_2, h_2)_{\text{BS}}$, $f(iy_1, iy_2)$ for $y_1, y_2 \rightarrow \infty$ with $y_1/y_2 \rightarrow \infty$ and that with $y_2/y_1 \rightarrow \infty$ have different limits.

4.10.10. Example. Let $G_1 = \text{GL}(2) \times \text{GL}(2)$, $G_2 = \text{GSp}(6)$, and let $f : G_1 \rightarrow G_2$ be the homomorphism $(g_1, g_2) \mapsto g_1 \otimes \text{Sym}^2(g_2)$. Let $h_1 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{1,\mathbf{R}}$ be as in 4.9.3 and let $h_2 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{2,\mathbf{R}}$ be $f \circ h_1$. Both h_1 and h_2 are \mathbf{R} -polarizable.

We explain the following (1) and (2).

(1) The canonical map $D(G_1, h_1) \rightarrow D(G_2, h_2)$, which we also denote by f , does not extend to a continuous map $D(G_1, h_1)_{\text{BS}} \rightarrow D(G_2, h_2)_{\text{BS}}$.

(2) The identity map of $D(G_2, h_2)$ does not extend to a continuous map $D(G_2, h_2)_{\text{SL}(2)} \rightarrow D(G_2, h_2)_{\text{BS}}$.

These (1) and (2) follow from (3)–(6) below. Note that $D(G_1, h_1) = \mathfrak{H}^\pm \times \mathfrak{H}^\pm$.

(3) When $y_1, y_2 \rightarrow \infty$, (iy_1, iy_2) converges in $D(G_1, h_1)_{\text{BS}}$.

(4) When $y_2, y_2/y_1 \rightarrow \infty$, $f(iy_1, iy_2)$ converges in $D(G_2, h_2)_{\text{SL}(2)}$.

Let P and Q be the parabolic subgroup of G_2 associated to the following homomorphisms $\mu : \mathbf{G}_m \rightarrow G_2$ and $\nu : \mathbf{G}_m \rightarrow G_2$, respectively.

$$\mu(t) := f\left(\begin{pmatrix} t^{-3} & 0 \\ 0 & t^3 \end{pmatrix}, \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right), \quad \nu(t) := f\left(\begin{pmatrix} t^{-3} & 0 \\ 0 & t^3 \end{pmatrix}, \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix}\right).$$

Then $P \neq Q$ because the adjoint action of \mathbf{G}_m on $\text{Lie}(G_2)$ defined by μ (resp. ν) multiplies

$$N = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

by t^{-2} (resp. t^2).

(5) When $y \rightarrow \infty$, $f(iy^3, iy)$ converges in $D(G_2, h_2)_{\text{BS}}$ to a point whose associated parabolic subgroup of G_2 is P .

(6) When $y \rightarrow \infty$, $f(iy^3, iy^2)$ converges in $D(G_2, h_2)_{\text{BS}}$ to a point whose associated parabolic subgroup of G_2 is Q .

(These (5) and (6) are essentially contained in [23] Proposition 6.10 and also in [24] Section 12.4.)

In the rest of this Section 4.10, we assume that G is reductive and let Z be the center of G . We consider the case $G_1 = G$ and $G_2 = G/Z$. We assume that $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$ is \mathbf{R} -polarizable. We will see that the extended period domains for G are understood from those of the semisimple group G/Z .

Lemma 4.10.11. *Let \bar{h}_0 be the homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow (G/Z)_{\mathbf{R}}$ induced by $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathbf{R}}$. Then \bar{h}_0 is \mathbf{R} -polarizable.*

Proof. This is because $\text{Lie}((G/Z)') = \text{Lie}(G')$ and hence $\text{Ad}(\bar{h}_0(i))$ on $\text{Lie}((G/Z)'_{\mathbf{R}})$ is a Cartan involution. \square

Proposition 4.10.12. *The complex analytic manifold $D(G, h_0)$ is an open and closed submanifold of the complex analytic manifold $D(G/Z, \bar{h}_0)$.*

Proof. We have a canonical morphism $D(G, h_0) \rightarrow D(G/Z, \bar{h}_0)$ of complex analytic manifolds. Hence, to prove Proposition 4.10.12, it is sufficient to prove that via this map, $D(G, h_0)$ is an open and closed real analytic submanifold of $D(G/Z, \bar{h}_0)$.

We first prove

Claim. Let $g \in G(\mathbf{C})$. Then $\text{Int}(g)(h_0) = h_0$ in $\text{Hom}(S_{\mathbf{C}/\mathbf{R}, \mathbf{C}}, G_{\mathbf{C}})$ if and only if $\text{Int}(g)(\bar{h}_0) = \bar{h}_0$ in $\text{Hom}(S_{\mathbf{C}/\mathbf{R}, \mathbf{C}}, (G/Z)_{\mathbf{C}})$. Here $\text{Int}(g)$ denotes the inner-automorphism given by g .

Proof of Claim. Assume $\text{Int}(g)(\bar{h}_0) = \bar{h}_0$. Then there is a homomorphism $z : S_{\mathbf{C}/\mathbf{R}, \mathbf{C}} \rightarrow Z_{\mathbf{C}}$ such that $\text{Int}(g)(h_0(s)) = z(s)h_0(s)$ for all $s \in S_{\mathbf{C}/\mathbf{R}, \mathbf{C}}$. Since $z(s) = \text{Int}(g)(h_0(s))h_0(s)^{-1}$ belongs to the commutator subgroup $G'_{\mathbf{C}}$ of $G_{\mathbf{C}}$ and $G'_{\mathbf{C}} \cap Z_{\mathbf{C}}$ is finite, the image of $z : S_{\mathbf{C}/\mathbf{R}, \mathbf{C}} \rightarrow Z_{\mathbf{C}}$ is finite. Since $S_{\mathbf{C}/\mathbf{R}, \mathbf{C}}$ is connected, z is the trivial homomorphism. Claim is proved.

Let $N = \{g \in G(\mathbf{C}) \mid \text{Int}(g)(h_0) = h_0\} = \{g \in G(\mathbf{C}) \mid \text{Int}(g)(\bar{h}_0) = \bar{h}_0\}$ and let $\bar{N} = N/Z(\mathbf{C})$. Since $G(\mathbf{R})/Z(\mathbf{R})$ is an open and closed real analytic submanifold of $(G/Z)(\mathbf{R})$, $D(G, h_0) = G(\mathbf{R})/(G(\mathbf{R}) \cap N) = (G(\mathbf{R})/Z(\mathbf{R}))/((G(\mathbf{R})/Z(\mathbf{R})) \cap \bar{N})$ is an open and closed real analytic submanifold of $(G/Z)(\mathbf{R})/((G/Z)(\mathbf{R}) \cap \bar{N}) = D(G/Z, \bar{h}_0)$ as a real analytic manifold. \square

4.10.13. Example. The map $D(G, h_0) \rightarrow D(G/Z, \bar{h}_0)$ for a reductive G need not be bijective. Let $(\mathbf{Z}/4\mathbf{Z})(1)$ be the algebraic group of 4-th roots of 1 over \mathbf{R} and let G be the semi-direct product of $S_{\mathbf{C}/\mathbf{R}}$ and $\mathbf{Z}/4\mathbf{Z}(1)$ in which $S_{\mathbf{C}/\mathbf{R}}$ is the normal subgroup and the action of the generator of $\mathbf{Z}/4\mathbf{Z}(1)$ on $S_{\mathbf{C}/\mathbf{R}}$ via the inner-automorphism is $z \mapsto z^{-1}$. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G$ be the inclusion map. Then $G(\mathbf{R}) = \mathbf{C}^\times \times \{\pm 1\}$ (here $\{\pm 1\} \subset \mathbf{Z}/4\mathbf{Z}(1)$) and $D(G, h_0)$ is a one-point set. On the other hand, G/Z is the semi-direct product of $S_{\mathbf{C}/\mathbf{R}}/\{\pm 1\}$ and $\mathbf{Z}/4\mathbf{Z}(1)/\{\pm 1\} \simeq \{\pm 1\}$ in which $S_{\mathbf{C}/\mathbf{R}}/\{\pm 1\}$ is normal and the generator

of $\mathbf{Z}/4\mathbf{Z}(1)/\{\pm 1\}$ acts on it via the inner-automorphism by $z \mapsto z^{-1}$. Hence $D(G/Z, \bar{h}_0)$ consists of two points.

Both $\check{D}(G, h_0)$ and $\check{D}(G/Z, \bar{h}_0)$ consist of two points.

Proposition 4.10.14. *As a real analytic manifold with corners, $D(G, h_0)_{\text{BS}}$ is canonically isomorphic to an open and closed subspace of $D(G/Z, \bar{h}_0)_{\text{BS}}$.*

Proof. By Proposition 4.10.12, we have that $Y := D(G/Z, \bar{h}_0)$ is the disjoint union $Y_1 \coprod Y_2$ of open closed subspaces Y_1 and Y_2 of Y such that the map $X := D(G, h_0) \rightarrow D(G/Z, \bar{h}_0)$ induces an isomorphism $X \xrightarrow{\sim} Y_1$. As is easily seen, Y_{BS} is the disjoint union $Y_{\text{BS},1} \coprod Y_{\text{BS},2}$, where $Y_{\text{BS},j}$ ($j = 1, 2$) is the open and closed subset of Y_{BS} consisting of all elements (P, Z) such that $Z \subset Y_j$. The morphism $X \rightarrow Y$ induces a morphism $X_{\text{BS}} \rightarrow Y_{\text{BS}}$ and this induces a morphism $X_{\text{BS}} \rightarrow Y_{\text{BS},1}$. We show that the last morphism is an isomorphism.

We have a bijection $P \mapsto P/Z$ from the set of all parabolic subgroups of G to that of G/Z . It is sufficient to prove that we have an isomorphism $X_{\text{BS}}(P) \xrightarrow{\sim} Y_{\text{BS},1}(P/Z) := Y_{\text{BS},1} \cap Y_{\text{BS}}(P/Z)$. Since $A_P \xrightarrow{\sim} A_{P/Z}$ and $\bar{A}_P \xrightarrow{\sim} \bar{A}_{P/Z}$, we have

$$X_{\text{BS}}(P) = D(G, h_0) \times^{A_P} \bar{A}_P \xrightarrow{\sim} Y_1 \times^{A_{P/Z}} \bar{A}_{P/Z} = Y_{\text{BS},1}(P/Z).$$

□

Proposition 4.10.15. *As a locally ringed space with log structure with sign, $D(G, h_0)_{\text{SL}(2)}$ (resp. $D(G, h_0)_{\text{BS, val}}$, resp. $D(G, h_0)_{\text{SL}(2), \text{val}}$) is canonically isomorphic to an open and closed subspace of $D(G/Z, \bar{h}_0)_{\text{SL}(2)}$ (resp. $D(G/Z, \bar{h}_0)_{\text{BS, val}}$, resp. $D(G/Z, \bar{h}_0)_{\text{SL}(2), \text{val}}$).*

Proof. We use the notation in the proof of Proposition 4.10.14.

The case of $D_{\text{SL}(2)}$ is proved as follows. As is easily seen, $Y_{\text{SL}(2)}$ is the disjoint union of $Y_{\text{SL}(2),1} \coprod Y_{\text{SL}(2),2}$, where $Y_{\text{SL}(2),j}$ ($j = 1, 2$) is the open and closed subspace of $Y_{\text{SL}(2)}$ consisting of all elements whose torus orbits are contained in Y_j . The map $X_{\text{SL}(2)} \rightarrow Y_{\text{SL}(2)}$ induces a morphism $X_{\text{SL}(2)} \rightarrow Y_{\text{SL}(2),1}$. We show that the last morphism is an isomorphism. We use

Claim. For any field $E \supset \mathbf{Q}$, the map $\text{Hom}(\text{SL}(2)_E^n, G_E) \rightarrow \text{Hom}(\text{SL}(2)_E^n, (G/Z)_E)$ is a bijection.

We prove Claim. We first prove the injectivity. Assume that $h_1, h_2 \in \text{Hom}(\text{SL}(2)_E^n, G_E)$ have the same image in $\text{Hom}(\text{SL}(2)_E^n, (G/Z)_E)$. Then there is a homomorphism $a : \text{SL}(2)_E^n \rightarrow Z_E$ such that $h_2 = ah_1$. But a is trivial because $\text{SL}(2)^n = [\text{SL}(2)^n, \text{SL}(2)^n]$. Next we prove the surjectivity. Let $h \in \text{Hom}(\text{SL}(2)_E^n, (G/Z)_E)$. Then the image of h is contained in the commutator subgroup $G'_E/(G' \cap Z)_E$ of $(G/Z)_E$, where $G' = [G, G]$. Since $G' \rightarrow G'/(G' \cap Z)$ is an isogeny and since $\text{SL}(2)^n$ is simply connected, this homomorphism $\text{SL}(2)_E^n \rightarrow G'_E/(G' \cap Z)_E$ comes from a homomorphism $\text{SL}(2)_E^n \rightarrow G'_E$. Claim is proved.

By Claim and by using the description of the set of $\text{SL}(2)$ -orbits in (ii) or (iii) in Lemma 3.1.3, we see that the map $X_{\text{SL}(2)} \rightarrow Y_{\text{SL}(2),1}$ is bijective. It remains to compare the real analytic structures and the log structures with sign. By using the local descriptions Proposition 3.4.4 of these structures, it is sufficient to prove that the canonical map from space $\text{spl}(W')$ in Proposition 3.4.4 for (G, h_0) to the corresponding space $\text{spl}(\bar{W}')$

for $(G/Z, \bar{h}_0)$ is an isomorphism of real analytic manifolds. This map is identified with the map $G_{\mathbf{R}, W', u} \rightarrow G_{\mathbf{R}, \bar{W}', u}$ and is identified with the isomorphism $\text{Lie}(G_{\mathbf{R}, W', u}) \xrightarrow{\sim} \text{Lie}(G_{\mathbf{R}, \bar{W}', u})$. This completes the proof for $D_{\text{SL}(2)}$.

The proof for $D_{\text{BS, val}}$ (resp. $D_{\text{SL}(2), \text{val}}$) is similar to that for D_{BS} (Proposition 4.10.14) (resp. for $D_{\text{SL}(2)}$). \square

Proposition 4.10.16. *Let Σ be a weak fan in $\text{Lie}(G')$. Then $D(G, h_0)_{\Sigma}^{\sharp}$ (resp. $D(G, h_0)_{\Sigma, [:]}^{\sharp}$, $D(G, h_0)_{\Sigma, \text{val}}^{\sharp}$, $D(G, h_0)_{\Sigma, [\text{val}]}^{\sharp}$) is canonically homeomorphic to an open and closed subset of $D(G/Z, \bar{h}_0)_{\Sigma}^{\sharp}$ (resp. $D(G/Z, h_0)_{\Sigma, [:]}^{\sharp}$, $D(G/Z, h_0)_{\Sigma, \text{val}}^{\sharp}$, $D(G/Z, h_0)_{\Sigma, [\text{val}]}^{\sharp}$).*

Here we denote the image of Σ under the isomorphism $\text{Lie}(G'_{\mathbf{R}}) \xrightarrow{\sim} \text{Lie}((G/Z)'_{\mathbf{R}})$ by the same letter Σ .

Proof. We use the notation in the proof of Proposition 4.10.15.

For $j = 1, 2$, let $Y_{\Sigma, [:], j}^{\sharp}$ be the inverse image of $Y_{\text{SL}(2), [:], j}$ under the CKS map $Y_{\Sigma, [:]}^{\sharp} \rightarrow Y_{\text{SL}(2)}$. Then $Y_{\Sigma, [:]}^{\sharp}$ is the disjoint union of an open and closed subsets $Y_{\Sigma, [:], 1}^{\sharp}$ and $Y_{\Sigma, [:], 2}^{\sharp}$. Since the map $Y_{\Sigma, [:]}^{\sharp} \rightarrow Y_{\Sigma}^{\sharp}$ is proper and all fibers of this map are connected, Y_{Σ}^{\sharp} is the disjoint union of the open and closed subsets $Y_{\Sigma, 1}^{\sharp}$ and $Y_{\Sigma, 2}^{\sharp}$, where $Y_{\Sigma, j}^{\sharp}$ ($j = 1, 2$) denotes the image of $Y_{\Sigma, [:], j}^{\sharp}$ in Y_{Σ}^{\sharp} . The set $Y_{\Sigma, j}^{\sharp}$ is the subset of Y_{Σ}^{\sharp} consisting of nilpotent i -orbits (σ, Z) such that there is an $F \in Z$ having the property that if N_1, \dots, N_n generate σ , then $\exp(\sum_{k=1}^n iy_k N_k)F \in Y_j$ if $y_k \geq 0$ for all k . From this description, we see that the map $X_{\Sigma}^{\sharp} \rightarrow Y_{\Sigma, 1}^{\sharp}$ is bijective. The coincidence of the topologies can be seen by the fact that for each $\sigma \in \Sigma$, both X_{σ}^{\sharp} and $Y_{\sigma, 1}^{\sharp}$ have the quotient topologies of the topology of $\{(q, F) \in |\text{toric}|_{\sigma} \times X \mid (q, F) \text{ belongs to } E_{\sigma} \text{ of } (G, h_0)\} \simeq \{(q, F) \in |\text{toric}|_{\sigma} \times Y_1 \mid (q, F) \text{ belongs to } E_{\sigma} \text{ of } (G/Z, \bar{h}_0)\}$ (this fact follows from Proposition 4.6.7). This proves Proposition 4.10.16 for D_{Σ}^{\sharp} . The proofs for $D_{\Sigma, [:]}^{\sharp}$, $D_{\Sigma, \text{val}}^{\sharp}$ and $D_{\Sigma, [\text{val}]}^{\sharp}$ are similar. \square

Proposition 4.10.17. *Let Σ be a weak fan in $\text{Lie}(G')$, and let Γ be a semi-arithmetic subgroup of $G(\mathbf{Q})$ which is strongly compatible with Σ and such that the image $\bar{\Gamma}$ of Γ in $(G/Z)(\mathbf{Q})$ is neat. Then as a locally ringed space with log structure, $\Gamma \setminus D(G, h_0)_{\Sigma}$ is canonically isomorphic to an open and closed subspace of $\bar{\Gamma} \setminus D(G/Z, \bar{h}_0)_{\Sigma}$.*

Proof. We use the notation in the proof of Proposition 4.10.16. For $j = 1, 2$, let $Y_{\Sigma, j}$ be the subset of Y_{Σ} consisting of nilpotent orbits (σ, Z) such that for some $F \in Z$, if N_k ($1 \leq k \leq n$) generate σ , then $\exp(\sum_{k=1}^n z_k N_k)F \in Y_j$ if $\text{Im}(z_k) \geq 0$ for all k . By the similar descriptions of $Y_{\Sigma, j}^{\sharp}$ for $j = 1, 2$ in the proof of Proposition 4.10.16, we have that Y_{Σ} is the disjoint union of $Y_{\Sigma, j}$ for $j = 1, 2$. From the bijectivity of $X_{\Sigma}^{\sharp} \rightarrow Y_{\Sigma, 1}^{\sharp}$, we obtain the bijectivity of $X_{\Sigma} \rightarrow Y_{\Sigma, 1}$. Hence the map $\Gamma \setminus X_{\Sigma} \rightarrow \bar{\Gamma} \setminus Y_{\Sigma, 1}$ is bijective. The coincidence of the sheaf of rings of holomorphic functions and the coincidence of the log structure can be seen by the following facts (i) and (ii) concerning both $\Gamma(\sigma)^{\text{gp}} \setminus X_{\sigma}$ and $\bar{\Gamma}(\sigma)^{\text{gp}} \setminus Y_{\sigma, 1}$ for $\sigma \in \Sigma$.

(i) A subset U is open if and only if for any fs log analytic space S and for any morphism $S \rightarrow \text{toric}_{\sigma} \times X \simeq \text{toric}_{\sigma} \times Y_1$ whose image is contained in $\{(q, F) \in \text{toric}_{\sigma} \times X \mid (q, F) \in E_{\sigma} \text{ of } (G, h_0)\} \simeq \{(q, F) \in \text{toric}_{\sigma} \times Y_1 \mid (q, F) \in E_{\sigma} \text{ of } (G/Z, \bar{h}_0)\}$, the inverse image of U in S is open.

(ii) For an open set U and for a function $f : U \rightarrow \mathbf{C}$, f belongs to $\mathcal{O}(U)$ (resp. $M(U)$) if and only if for any S and $S \rightarrow \text{toric}_\sigma \times X \simeq \text{toric}_\sigma \times Y_1$ as in (i) such that \mathcal{O}_S is a sheaf of reduced rings (i.e., rings without non-zero nilpotent elements), the pullback of f on the inverse image of U in S belongs to $\mathcal{O}_S(U)$ (resp. $M_S(U)$).

These (i) and (ii) follow from Proposition 4.6.16. \square

4.10.18. We prove Proposition 2.6.6, Proposition 3.11.5, (2) of Remark 4.6.2, and Remark 4.6.28. If Γ is a semi-arithmetic subgroup of $G(\mathbf{Q})$, the image $\bar{\Gamma}$ of Γ in $(G/Z)(\mathbf{Q})$ is a semi-arithmetic subgroup, and $\bar{\Gamma} \cap (G/Z)'(\mathbf{Q})$ is of finite index in $\bar{\Gamma}$. By 2.3.6 (5) and by Propositions 4.10.14, 4.10.15, 4.10.16, 4.10.17, we can replace G by G/Z and replace Γ by the semi-arithmetic subgroup $\bar{\Gamma} \cap (G/Z)'(\mathbf{Q})$ of $(G/Z)'(\mathbf{Q})$. Thus Proposition 2.6.6, Proposition 3.11.5, (2) of Remark 4.6.2, and Remark 4.6.28 are reduced to Theorem 2.6.1, Theorem 3.11.1, Theorem 4.6.1, and Proposition 4.6.27, respectively.

4.10.19. We describe some details of the relation of this paper with the work [25] explained in Remark 4.6.3.

Let $H, M, h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow M_{\mathbf{R}}, \bar{h}_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow (M/Z)_{\mathbf{R}}$ be as in 1.6.6. Let Γ be a neat arithmetic subgroup of $(M/Z)(\mathbf{Q})$ which is contained in the connected component of $(M/Z)(\mathbf{R})$ containing 1 and let Σ be a fan in $\text{Lie}((M/Z)') = \text{Lie}(M')$ which is strongly compatible with Γ . Then Γ acts on $D(M, h_0)$ and on $D(M, h_0)_\Sigma$. The work [25] shows that $\Gamma \backslash D(M, h_0)_\Sigma$ is a logarithmic manifold. The method in [25] is to use the inclusion map $D(M, h_0) \rightarrow D(\Lambda)$ (Λ is as in 1.6.6) and to use the work [24] on the toroidal partial compactification for $D(\Lambda)$.

This result can be deduced also from Theorem 4.6.1 for $G = M/Z$ as follows. Take a neat semi-arithmetic subgroup Γ_1 of $M'(\mathbf{Q})$ whose image Γ_2 in $(M/Z)(\mathbf{Q})$ is a normal subgroup of Γ of finite index. Then by Proposition 4.10.17, $\Gamma_1 \backslash D(M, h_0)_\Sigma$ is an open and closed subspace of $\Gamma_2 \backslash D(M/Z, \bar{h}_0)_\Sigma$. By taking the quotients by Γ/Γ_2 , we have that $\Gamma \backslash D(M, h_0)_\Sigma$ is an open and closed subspace of $\Gamma \backslash D(M/Z, \bar{h}_0)_\Sigma$, which is a logarithmic manifold by Theorem 4.6.1 and by the part of Remark 4.6.2 (1) for semisimple algebraic groups, and hence is a logarithmic manifold.

4.11 G -log mixed Hodge structures

We consider the G -MHS version of the notion log mixed Hodge structure.

In this Section 4.11, Γ denotes a semi-arithmetic subgroup (1.5.7) of $G(\mathbf{Q})$.

4.11.1. Let S be an object of the category $\mathcal{B}(\log)$ (4.2.12). Recall that the topological space S^{\log} is endowed with a proper surjective continuous map $\tau : S^{\log} \rightarrow S$ and a sheaf of rings \mathcal{O}_S^{\log} over $\tau^{-1}(\mathcal{O}_S)$. A log \mathbf{Q} -mixed Hodge structure on S is a triple $(H_{\mathbf{Q}}, W, H_{\mathcal{O}})$, where $H_{\mathbf{Q}}$ is a locally constant sheaf on S^{\log} of finite-dimensional \mathbf{Q} -vector spaces, W is an increasing filtration on $H_{\mathbf{Q}}$, $H_{\mathcal{O}}$ is a vector bundle on S endowed with an isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Q}} H_{\mathbf{Q}} \simeq \mathcal{O}_S^{\log} \otimes_{\tau^{-1}(\mathcal{O}_S)} \tau^{-1}(H_{\mathcal{O}})$ and with a decreasing filtration F , satisfying certain conditions (see [21] Part III 1.3). We denote by $\text{LMH}(S)$ the category of log \mathbf{Q} -mixed Hodge structures over S .

A G -log mixed Hodge structure (G -LMH, for short) over S is an exact \otimes -functor from $\text{Rep}(G)$ to $\text{LMH}(S)$.

A G -LMH on S with a Γ -level structure is a G -LMH H over S endowed with a global section of the quotient sheaf $\Gamma \setminus \mathcal{I}$, where \mathcal{I} is the following sheaf on S^{\log} . For an open set U of S^{\log} , $\mathcal{I}(U)$ is the set of all isomorphisms $H_{\mathbf{Q}}|_U \xrightarrow{\sim} \text{id}$ of \otimes -functors from $\text{Rep}(G)$ to the category of local systems of \mathbf{Q} -modules over U .

4.11.2. Let Σ be a weak fan in $\text{Lie}(G')$ which is strongly compatible with Γ (4.1.9). A G -LMH H over S with a Γ -level structure λ is said to be of type (h_0, Σ) if for any $s \in S$, any $t \in s^{\log}$, and any \otimes -isomorphism $\tilde{\lambda}_t : H_{\mathbf{Q},t} \simeq \text{id}$ which belongs to λ_t , there is a $\sigma \in \Sigma$ satisfying the following (i) and (ii).

- (i) The logarithm of the action of $\text{Hom}((M_S/\mathcal{O}_S^{\times})_s, \mathbf{N}) \subset \pi_1(s^{\log})$ on $H_{\mathbf{Q},t}$ is contained, via $\tilde{\lambda}_t$, in $\sigma \subset \text{Lie}(G_{\mathbf{R}})$.
- (ii) Let $a : \mathcal{O}_{S,t}^{\log} \rightarrow \mathbf{C}$ be a ring homomorphism which induces the evaluation $\mathcal{O}_{S,s} \rightarrow \mathbf{C}$ at s and consider the element $F : V \mapsto \tilde{\lambda}_t a(H(V))$ of Y (1.4.1). Then this element belongs to \check{D} and (σ, F) generates a nilpotent orbit (4.1.3).

Remark 4.11.3. The definition of the type (h_0, Σ) in [22] 4.2.2 should be modified as above because “the smallest cone satisfying (i)” in the condition (ii) there may not be well-defined when Σ is not a fan.

4.11.4. If (H, λ) is a G -LMH with a Γ -level structure of type (h_0, Σ) , we have a map $S \rightarrow \Gamma \setminus D_{\Sigma}$, called the *period map* associated to (H, λ) , which sends $s \in S$ to the class of the nilpotent orbit $(\sigma, Z) \in D_{\Sigma}$. Here σ is the smallest cone of Σ satisfying (i) and (ii) in 4.11.2, which exists by a variant of [21] Part III Lemma 2.2.4 (see also Appendix of this paper), and Z is the associated $\exp(\sigma_{\mathbf{C}})$ -orbit obtained in (ii) in 4.11.2.

4.11.5. Let S be an object of $\mathcal{B}(\log)$. Let S° be the underlying locally ringed space over \mathbf{C} of S with the trivial log structure. By a G -MHS on S with a Γ -level structure, we mean a G -LMH on S° with a Γ -level structure. By a G -MHS on S with a Γ -level structure of type h_0 , we mean a G -LMH on S° with a Γ -level structure of type (h_0, Σ) , where Σ is the fan consisting of the one cone $\{0\}$.

4.12 Moduli of G -log mixed Hodge structures and period maps

We show that $\Gamma \setminus D_{\Sigma}$ is a moduli space of G -LMH.

In this Section 4.12, let Γ be a subgroup of $G(\mathbf{Q})$, and assume that either one of the following two conditions is satisfied.

- (i) Γ is a neat semi-arithmetic subgroup (2.6.3, 1.5.7) of $G'(\mathbf{Q})$.
- (ii) G is reductive, Γ is a semi-arithmetic subgroup of $G(\mathbf{Q})$, and the image of Γ in $(G/Z)(\mathbf{Q})$ is neat, where Z is the center of G .

Note first the following.

Proposition 4.12.1. *The complex analytic manifold $\Gamma \setminus D$ represents the functor*

$$S \mapsto \{\text{isomorphism class of } G\text{-MHS on } S \text{ with a } \Gamma\text{-level structure of type } h_0\}$$

from $\mathcal{B}(\log)$ to the category of sets.

The main result here is the following.

Theorem 4.12.2. *Let Σ be as in 4.11.2. Then $\Gamma \setminus D_\Sigma$ represents the functor*

$$S \mapsto \{\text{isomorphism class of } G\text{-LMH on } S \text{ with a } \Gamma\text{-level structure of type } (h_0, \Sigma)\}.$$

from $\mathcal{B}(\log)$ to the category of sets.

4.12.3. The proof of Theorem 4.12.2 is similar to the proof of [21] Part III Theorem 2.6.6.

The first part of the proof is to understand the functor which E_σ represents. Then take the quotient $\Gamma(\sigma) \setminus D_\sigma$ of E_σ by $\sigma_{\mathbf{C}}$.

4.12.4. There is a variant of Theorem 4.12.2 for G -LMH with adelic level structure.

Let G_1 be a closed algebraic subgroup of G and let K be an open compact subgroup of $G_1(\mathbf{A}_{\mathbf{Q}}^\infty)$, where $\mathbf{A}_{\mathbf{Q}}^\infty$ is the adele ring of \mathbf{Q} without the ∞ -component. We show that under certain assumptions, the space

$$G_1(\mathbf{Q}) \setminus (D \times G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K)$$

is a moduli space of G -MHS with K -level structure and its toroidal partial compactification is a moduli space of G -LMH with K -level structure.

For each $g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$, let $\Gamma(g) = G_1(\mathbf{Q}) \cap \tilde{g}K\tilde{g}^{-1}$, where \tilde{g} denotes a lifting of g to $G_1(\mathbf{A}_{\mathbf{Q}}^\infty)$. Then $\Gamma(g)$ is an arithmetic subgroup of $G_1(\mathbf{Q})$. We have $\Gamma(\gamma g) = \gamma\Gamma(g)\gamma^{-1}$ for $\gamma \in G_1(\mathbf{Q})$ and $g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$.

Let R be a representative of $G_1(\mathbf{Q}) \setminus G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$ in $G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$.

We assume that for every $g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$ (equivalently, for each $g \in R$), the subgroup of $\Gamma(g)$ of $G(\mathbf{Q})$ is neat and satisfies either one of the conditions (i) and (ii) at the beginning of Section 4.12.

We also assume that for each $g \in R$, we are given a weak fan $\Sigma(g)$ in $\text{Lie}(G')$ which is strongly compatible with $\Gamma(g)$. For each $g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$, define $\Sigma(g) := \text{Ad}(\gamma)\Sigma(g_0)$, where $g = \gamma g_0$ with $\gamma \in G_1(\mathbf{Q})$ and $g_0 \in R$ (then $\Sigma(g)$ is independent of the choices of such γ and g_0). We have $\Sigma(\gamma g) = \text{Ad}(\gamma)\Sigma(g)$ for all $\gamma \in G_1(\mathbf{Q})$ and $g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$.

Then we have a log manifold

$$G_1(\mathbf{Q}) \setminus \coprod_{g \in G_1(\mathbf{A}^\infty)/K} D_{\Sigma(g)} = G_1(\mathbf{Q}) \setminus \left(\bigcup_{g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K} D_{\Sigma(g)} \times \{g\} \right) / K = \coprod_{g \in R} \Gamma(g) \setminus D_{\Sigma(g)}$$

(here the action of $\gamma \in G_1(\mathbf{Q})$ sends an element (x, g) of $\coprod_{g \in G_1(\mathbf{A}^\infty)/K} D_{\Sigma(g)}$ with $x \in D_{\Sigma(g)}$ and $g \in G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K$ to the element $(\gamma x, \gamma g)$). This log manifold contains

$$G_1(\mathbf{Q}) \setminus (D \times G_1(\mathbf{A}_{\mathbf{Q}}^\infty)/K) = \coprod_{g \in R} \Gamma(g) \setminus D$$

as an open set.

By Theorem 4.12.2, this log manifold represents the functor

$$S \mapsto \{\text{isomorphism class of } G\text{-LMH on } S \text{ with a } K\text{-level structure of type } (h_0, \Sigma)\}$$

and the above open set represents the functor

$$S \mapsto \{\text{isomorphism class of } G\text{-MHS on } S \text{ with a } K\text{-level structure of type } h_0\}.$$

Here Σ denotes the family $(\Sigma(g))_g$. A K -level structure on a G -LMH (or its special case G -MHS) H on S means a global section of the quotient sheaf $K \setminus \mathcal{J}_1$, where \mathcal{J}_1 is the following sheaf on S^{\log} . Let \mathcal{J} be the following sheaf on S^{\log} . For an open set U of S^{\log} , $\mathcal{J}(U)$ is the set of all isomorphisms $H_{\mathbf{Q}}|_U \otimes_{\mathbf{Q}} \mathbf{A}_{\mathbf{Q}}^{\infty} \xrightarrow{\sim} \text{id} \otimes_{\mathbf{Q}} \mathbf{A}_{\mathbf{Q}}^{\infty}$ of \otimes -functors from $\text{Rep}(G)$ to the category of sheaves of $\mathbf{A}_{\mathbf{Q}}^{\infty}$ -modules over U . Then $G_1(\mathbf{A}_{\mathbf{Q}}^{\infty})$ acts on \mathcal{J} and we have a canonical injective morphism $\mathcal{I} \rightarrow \mathcal{J}$ for \mathcal{I} as in 4.11.1. Let $\mathcal{J}_1 = G_1(\mathbf{A}_{\mathbf{Q}}^{\infty})\mathcal{I} \subset \mathcal{J}$. For a G -LMH (resp. G -MHS) H on S with a K -level structure, we say that H is of type (h_0, Σ) if for each $s \in S$, if the K -level structure at the point of S^{\log} lying over $s \in S$ is $g^{-1}\lambda$ with $g \in G_1(\mathbf{A}_{\mathbf{Q}}^{\infty})$ and $\lambda \in \mathcal{I}$, then H with the $\Gamma(g)$ -level structure λ is of type $(h_0, \Sigma(g))$ (resp. h_0). The period morphism from S to this log manifold (resp. the above open set) associated to H with this K -level structure of type (h_0, Σ) (resp. h_0) is as follows. On an open neighborhood of s , taking the above g in R , it is the period map from S to $\Gamma(g) \setminus D_{\Sigma(g)}$ (resp. $\Gamma(g) \setminus D$) associated to (H, λ) .

Next we consider extensions of the associated period maps in Theorem 4.12.5 and Theorem 4.12.6.

Theorem 4.12.5. *Let S be a connected, log smooth, fs log analytic space, and let U be the open subspace of S consisting of all points of S at which the log structure of S is trivial. Let (H, λ) be a G -MHS on U with a Γ -level structure of type h_0 (4.11.5). Let $\varphi : U \rightarrow \Gamma \setminus D$ be the associated period map. Assume that (H, λ) extends to a G -LMH on S with a Γ -level structure (4.11.1). Then:*

(1) *For any point $s \in S$, there exist an open neighborhood V of s , a log modification V' of V ([24] 3.6.12), a subgroup Γ_1 of Γ , and a fan (we do not need a weak fan here) Σ in $\text{Lie}(G')$ which is strongly compatible with Γ_1 such that the period map $\varphi|_{U \cap V}$ lifts to a morphism $U \cap V \rightarrow \Gamma_1 \setminus D$ which extends uniquely to a morphism $V' \rightarrow \Gamma_1 \setminus D_{\Sigma}$ of log manifolds. Furthermore, we can take a commutative group Γ_1 .*

$$\begin{array}{ccccc} U & \supset & U \cap V & \subset & V' \\ \varphi \downarrow & & \downarrow & & \downarrow \\ \Gamma \setminus D & \longleftarrow & \Gamma_1 \setminus D & \subset & \Gamma_1 \setminus D_{\Sigma}. \end{array}$$

(2) *Assume that $S \setminus U$ is a smooth divisor. Then we can take $V = V' = S$ and $\Gamma_1 = \Gamma$ in (1). That is, we have a commutative diagram*

$$\begin{array}{ccc} U & \subset & S \\ \varphi \downarrow & & \downarrow \\ \Gamma \setminus D & \subset & \Gamma \setminus D_{\Sigma}. \end{array}$$

(3) *Assume that Γ is commutative. Then we can take $\Gamma_1 = \Gamma$ in (1).*

(4) *Assume that Γ is commutative and that the following condition (i) is satisfied.*

(i) *There is a finite family $(S_j)_{1 \leq j \leq n}$ of connected locally closed analytic subspaces of S such that $S = \bigcup_{j=1}^n S_j$ as a set and such that, for each j , the inverse image of the sheaf $M_S/\mathcal{O}_S^{\times}$ on S_j is locally constant.*

Then we can take $\Gamma_1 = \Gamma$ and $V = S$ in (1).

This is the G -MHS version of [21] Part III Theorem 7.5.1 in mixed Hodge case and of [24] Theorem 4.3.1 in pure Hodge case. The proof goes exactly in the same way as in the pure case treated in [24].

Theorem 4.12.6. *Let the notation S , U , (H, λ) and the assumptions be as in Theorem 4.12.5. Let $\varphi : U \rightarrow \Gamma \setminus D$ be the associated period map. Let $S_{[:]^\log} = S^\log \times_S S_{[:]}$ and let $S_{[\text{val}]^\log} = S^\log \times_S S_{[\text{val}]}$, and regard U as open sets of these spaces. Then:*

- (1) *The map $\varphi : U \rightarrow \Gamma \setminus D$ extends uniquely to continuous maps*

$$S_{[:]^\log} \rightarrow \Gamma \setminus D_{\text{SL}(2)}^I, \quad S_{[\text{val}]^\log} \rightarrow \Gamma \setminus D_{\text{SL}(2), \text{val}}^I.$$

(2) *Assume that the complement $S \setminus U$ of U is a smooth divisor on S . Then the map $\varphi : U \rightarrow \Gamma \setminus D$ extends uniquely to a continuous map*

$$S^\log \rightarrow \Gamma \setminus D_{\text{SL}(2), \text{val}}^*$$

and hence extends uniquely to a continuous map $S^\log \rightarrow \Gamma \setminus D_{\text{BS}, \text{val}}$ and to $S^\log \rightarrow \Gamma \setminus D_{\text{BS}}$.

This is the G -MHS version of [21] Part IV Theorem 6.3.1. The proof goes exactly in the same way as there. In the proof of loc. cit., there are typos, i.e., the two $S_{[:]}$ in the middle of the proof should be changed to $S_{[:]^\log}$.

4.13 Infinitesimal study

In this section, we consider the logarithmic tangent bundle of the moduli space of G -LMH. We will prove

Proposition 4.13.1. *Let $Z = \Gamma \setminus D_\Sigma$ be as in Theorem 4.12.2 and let $(H_{\text{univ}}, \lambda)$ be the universal object on Z . Let θ_Z be the logarithmic tangent bundle of Z (i.e., the \mathcal{O}_Z -dual of the sheaf ω_Z^1 of differential forms with log poles). Then, we have a canonical isomorphism of \mathcal{O}_Z -modules*

$$\theta_Z \simeq H_{\text{univ}}(\text{Lie}(G))_{\mathcal{O}} / F^0 H_{\text{univ}}(\text{Lie}(G))_{\mathcal{O}}.$$

This is an analogue of [24] Proposition 4.4.3 and proved in a similar way as below.

4.13.2. In 4.13.2–4.13.12, let S be an object of $\mathcal{B}(\log)$ and assume that S is log smooth (this means that S is locally a strong subspace (4.2.12) of a log smooth fs log analytic space).

Let H be a G -LMH on S . We will construct a commutative diagram of \mathcal{O}_S -modules

$$\begin{array}{ccc} \theta_S & \xrightarrow{\sim} & \tilde{\theta}_S \\ \downarrow & & \downarrow \\ H(\text{Lie}(G))_{\mathcal{O}} / F^0 & \xrightarrow{\sim} & \mathcal{E}(H) \xrightarrow{\sim} \tilde{\mathcal{E}}(H) \end{array}$$

whose horizontal arrows are isomorphisms. Here θ_S is the logarithmic tangent bundle of S and the sheaves $\tilde{\theta}_S$, $\mathcal{E}(H)$ and $\tilde{\mathcal{E}}(H)$ are defined below.

Furthermore, in the case $S = \Gamma \setminus D_\Sigma$, we will show that the right vertical arrow is an isomorphism. This will give the isomorphism in Proposition 4.13.1.

4.13.3. Define the sheaf $\tilde{\theta}_S$ as follows.

Let U be an open set of S , and let $\tilde{U} = U[T]/(T^2) = (U, \mathcal{O}_U[T]/(T^2))$. Then $\tilde{\theta}_S(U)$ is the set of all morphisms $\tilde{U} \rightarrow S$ which extend the inclusion morphism $U \rightarrow S$.

We have a canonical isomorphism $\theta_S \xrightarrow{\sim} \tilde{\theta}_S$.

4.13.4. We define the sheaf $\mathcal{E}(H)$.

A section of $\mathcal{E}(H)$ is a collection of an \mathcal{O}_S -homomorphism $\delta_{p,V} : F^p H(V)_\mathcal{O} \rightarrow H(V)_\mathcal{O}/F^p$ for $p \in \mathbf{Z}$ and $V \in \text{Rep}(G)$ satisfying the following conditions (i)–(iii).

(i) Functoriality in V . For a morphism $V_1 \rightarrow V_2$ in $\text{Rep}(G)$, the square

$$\begin{array}{ccc} F^p H(V_1)_\mathcal{O} & \rightarrow & H(V_1)_\mathcal{O}/F^p \\ \downarrow & & \downarrow \\ F^p H(V_2)_\mathcal{O} & \rightarrow & H(V_2)_\mathcal{O}/F^p \end{array}$$

is commutative.

(ii) The diagram

$$\begin{array}{ccc} F^{p+1} H(V)_\mathcal{O} & \xrightarrow{\delta_{p+1,V}} & H(V)_\mathcal{O}/F^{p+1} \\ \downarrow & & \downarrow \\ F^p H(V)_\mathcal{O} & \xrightarrow{\delta_{p,V}} & H(V)_\mathcal{O}/F^p \end{array}$$

is commutative for every p .

(iii) For $V_1, V_2 \in \text{Rep}(G)$ and for $p, q \in \mathbf{Z}$, the following diagram is commutative.

$$\begin{array}{ccc} F^p H(V_1)_\mathcal{O} \otimes F^q H(V_2)_\mathcal{O} & \rightarrow & (H(V_1)_\mathcal{O}/F^p \otimes F^q H(V_2)_\mathcal{O}) \oplus (F^p H(V_1)_\mathcal{O} \otimes H(V_2)_\mathcal{O}/F^q) \\ \downarrow & & \downarrow \\ F^{p+q} H(V_1 \otimes V_2)_\mathcal{O} & \rightarrow & H(V_1 \otimes V_2)_\mathcal{O}/F^{p+q} \end{array}$$

Here the vertical arrows are the evident ones. The upper horizontal row is $x \otimes y \mapsto (\delta_{p,V_1}(x) \otimes y, x \otimes \delta_{q,V_2}(y))$. The lower horizontal arrow is $\delta_{p+q, V_1 \otimes V_2}$.

A section of $\mathcal{E}(H)$ on an open set U of S is defined in the same way by replacing S by U .

We have the evident homomorphism $H(\text{Lie}(G))_\mathcal{O}/F^0 \rightarrow \mathcal{E}(H)$.

4.13.5. We define the sheaf $\tilde{\mathcal{E}}(H)$.

Let U be an open set of S , and let \tilde{U} be as in 4.13.3. Then $\tilde{\mathcal{E}}(H)(U)$ is the set of all isomorphism classes of G -LMH \tilde{H} on \tilde{U} whose pullbacks to U coincide with the restriction of H to U .

4.13.6. We define the map $\theta_S \rightarrow \mathcal{E}(H)$.

Note that for $V \in \text{Rep}(G)$, $H(V)_\mathcal{O} = \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Q}} H(V)_{\mathbf{Q}})$. Then, $d \otimes 1_{H(V)_{\mathbf{C}}} : \mathcal{O}_S^{\log} \otimes_{\mathbf{C}} H(V)_{\mathbf{C}} \rightarrow \omega_S^{1,\log} \otimes_{\mathbf{C}} H(V)_{\mathbf{C}}$ induces a connection

$$\nabla : \mathcal{H}(V)_\mathcal{O} \rightarrow \omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{H}(V)_\mathcal{O}.$$

We define a map

$$\theta_S \rightarrow \bigoplus_p \mathcal{H}om(F^p H(V)_\mathcal{O}, H(V)_\mathcal{O}/F^p)$$

by assigning $\delta \in \theta_S$ the element of $\bigoplus_p \mathcal{H}om(F^p H(V)_\mathcal{O}, H(V)_\mathcal{O}/F^p)$ induced by the composite map

$$H(V)_\mathcal{O} \xrightarrow{\nabla} \omega_S^1 \otimes_{\mathcal{O}_S} H(V)_\mathcal{O} \xrightarrow{\delta} H(V)_\mathcal{O}.$$

Note that this map is not \mathcal{O}_S -linear but that it induces an \mathcal{O}_S -linear map $F^p H(V)_\mathcal{O} \rightarrow H(V)_\mathcal{O}/F^p$. Thus, we have a homomorphism of \mathcal{O}_S -modules

$$\theta_S \rightarrow \mathcal{E}(H).$$

4.13.7. We have a map $\tilde{\theta}_S \rightarrow \tilde{\mathcal{E}}(H)$ which sends a morphism $f : \tilde{U} \rightarrow S$ to the class of f^*H .

4.13.8. We have a canonical isomorphism $\mathcal{E}(H) \rightarrow \tilde{\mathcal{E}}(H)$ defied as follows.

Let $(\delta_{p,V})_{p,V}$ be a section of $\mathcal{E}(H)$ on U . Define the corresponding G -LMH \tilde{H} on \tilde{U} as follows. We identify the topological spaces U^{\log} and $(\tilde{U})^{\log}$. We define $\tilde{H}(V)_\mathbf{Q}$ with the weight filtration and the level structure as the same as the restriction of $H(V)_\mathbf{Q}$ to U^{\log} . Since $\mathcal{O}_{\tilde{U}}^{\log} = \mathcal{O}_{\tilde{U}} \otimes_{\mathcal{O}_U} \mathcal{O}_U^{\log}$, $\tilde{H}(V)_\mathcal{O} = \tau_*(\mathcal{O}_{\tilde{U}}^{\log} \otimes \tilde{H}(V)_\mathbf{Q})$ is identified with $\mathcal{O}_{\tilde{U}} \otimes_{\mathcal{O}_U} H(V)_\mathcal{O}|_U$. We define the p -th Hodge filter on $\tilde{H}(V)_\mathcal{O}$ as the $\mathcal{O}_{\tilde{U}}$ -submodule of $\tilde{H}_\mathcal{O} = \mathcal{O}_{\tilde{U}} \otimes_{\mathcal{O}_U} H(V)_\mathcal{O}|_U$ generated by $x + Ty$, where $x \in F^p H(V)_\mathcal{O}|_U$ and $y \in H(V)_\mathcal{O}|_U$ such that $y \bmod F^p H(V)_\mathcal{O}|_U$ coincides with $\delta_{p,V}(x)$. It is easy to see that this $\mathcal{E}(H) \rightarrow \tilde{\mathcal{E}}(H)$ is an isomorphism.

4.13.9. The diagram

$$\begin{array}{ccc} \theta_S & \xrightarrow{\sim} & \tilde{\theta}_S \\ \downarrow & & \downarrow \\ \mathcal{E}(H) & \xrightarrow{\sim} & \tilde{\mathcal{E}}(H) \end{array}$$

is commutative.

4.13.10. By Tannaka duality (see below), $H(\text{Lie}(G))_\mathcal{O}$ is identified with the collection of $\delta_V \in \mathcal{E}nd_{\mathcal{O}_S}(H(V)_\mathcal{O})$ for $V \in \text{Rep}(G)$ satisfying the following conditions (i) and (ii).

(i) It is functorial in V . That is, for a morphism $h : V_1 \rightarrow V_2$ in $\text{Rep}(G)$, we have $h \circ \delta_{V_1} = \delta_{V_2} \circ h$.

(ii) For all $V_1, V_2 \in \text{Rep}(G)$, $\delta_{V_1 \otimes V_2}$ coincides with $\delta_{V_1} \otimes 1 + 1 \otimes \delta_{V_2}$.

Here Tannaka duality is used as follows. By [30], locally on S , the functors $\text{Rep}(G) \ni V \mapsto \mathcal{O}_S \otimes_{\mathbf{Q}} V$ and $\text{Rep}(G) \ni V \mapsto H(V)_\mathcal{O}$ are isomorphic as \otimes -functors. By this and by the Tannaka duality ([30]) applied to the functor $\text{Rep}(G) \ni V \mapsto \mathcal{O}_S[T]/(T^2) \otimes_{\mathbf{Q}} V$, we have the above understanding of $H(\text{Lie}(G))_\mathcal{O}$.

Lemma 4.13.11. *The map $H(\text{Lie}(G))_\mathcal{O}/F^0 H(\text{Lie}(G))_\mathcal{O} \rightarrow \mathcal{E}(H)$ is an isomorphism.*

Proof. By [30] Ch. IV 2.4, the \otimes -functors with filtrations $V \mapsto H(V)_\mathcal{O}$ and $V \mapsto \text{gr}_F(H(V)_\mathcal{O})$ are isomorphic locally on S . Hence we may assume that they are isomorphic. Fix an isomorphism. Then via it, $H(\text{Lie}(G))_\mathcal{O}/F^0$ is identified with $\bigoplus_{p \leq 0} \text{gr}_F^p(H(\text{Lie}(G))_\mathcal{O})$ and $\mathcal{E}(H)$ is identified with the sheaf of collections $\delta_V : \text{gr}_F(H(V)) \rightarrow \text{gr}_F(H(V))$ ($V \in \text{Rep}(G)$) satisfying the following conditions (i)–(iii).

(i) $V \mapsto \delta_V$ is functorial in V .

(ii) $\delta_{V_1 \otimes V_2} = \delta_{V_1} \otimes 1 + 1 \otimes \delta_{V_2}$ for all $V_1, V_2 \in \text{Rep}(G)$.

(iii) $\text{Image}(\delta_V) \subset \bigoplus_{p<0} \text{gr}_F^p(H(V)_{\mathcal{O}})$.

By 4.13.10, the sheaf of those $(\delta_V)_V$ satisfying (i) and (ii) (not necessarily satisfying (iii)) is isomorphic to $\text{gr}_F(H(\text{Lie}(G))_{\mathcal{O}})$. This proves Lemma 4.13.11. \square

4.13.12. If $S = \Gamma \setminus D_{\Sigma}$ and $H = H_{\text{univ}}$, $\tilde{\theta}_S \rightarrow \tilde{\mathcal{E}}(S)$ is an isomorphism because S is the moduli space.

Consequently, we have $\theta_S \xrightarrow{\sim} \mathcal{E}(H) \simeq H(\text{Lie}(G))_{\mathcal{O}}/F^0$. This completes the proof of Proposition 4.13.1.

4.13.13. Let $Z = \Gamma \setminus D_{\Sigma}$ be as in Proposition 4.13.1. We define $\theta_Z^1 = \text{gr}_F^{-1} H_{\text{univ}}(\text{Lie}(G))_{\mathcal{O}}$.

4.13.14. Let $Z = \Gamma \setminus D_{\Sigma}$ be as above. Let S be a logarithmically smooth object of $\mathcal{B}(\log)$, let (H, λ) be a G -LMH on S of type (h_0, Σ) , and let $\varphi : S \rightarrow Z$ be the corresponding period map. Then H is the pullback of H_{univ} on Z by φ , and the map $\theta_S \rightarrow \mathcal{E}(H) \simeq H(\text{Lie}(G))_{\mathcal{O}}/F^0$ is identified with the canonical map $\theta_S \rightarrow \varphi^*(\theta_Z)$ and $\text{gr}_F^{-1}(H(\text{Lie}(G))_{\mathcal{O}})$ is identified with $\varphi^*(\theta_Z^1)$.

The connection of $H(V)_{\mathcal{O}}$ satisfies the Griffiths transversality for every $V \in \text{Rep}(G)$ if and only if the map $\theta_S \rightarrow \varphi^*\theta_Z$ factors through $\varphi^*\theta_Z^1$.

4.14 Generalizations, I

In this section and the next, we give two generalizations of the theory in this paper.

4.14.1. Let E be a subfield of \mathbf{R} . We have the following generalization whose case $E = \mathbf{Q}$ is the theory explained so far, as far as Γ is not involved.

Let \mathcal{G} be a linear algebraic group over E . Assume that we are given a homomorphism $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathcal{G}_{\text{red}} \otimes_E \mathbf{R}$ of algebraic groups over \mathbf{R} such that the composition $\mathbf{G}_{m,\mathbf{R}} \rightarrow S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathcal{G}_{\text{red}} \otimes_E \mathbf{R}$ comes from a homomorphism $k_0 : \mathbf{G}_{m,E} \rightarrow \mathcal{G}_{\text{red}}$ whose image is contained in the center of \mathcal{G}_{red} such that for some (and hence for every) lifting $\tilde{k}_0 : \mathbf{G}_{m,E} \rightarrow \mathcal{G}$ of k_0 , the adjoint action of $\mathbf{G}_{m,E}$ on $\text{Lie}(\mathcal{G}_u)$ via \tilde{k}_0 is of weights ≤ -1 . Let $\text{Rep}_E(\mathcal{G})$ be the category of finite-dimensional representations of \mathcal{G} over E . Note that every $V \in \text{Rep}_E(\mathcal{G})$ has a \mathcal{G} -stable weight filtration $W_{\bullet}V$.

Let $D = D(\mathcal{G}, h_0)$ be the set of isomorphism classes of exact \otimes -functors $H : \text{Rep}_E(\mathcal{G}) \rightarrow \text{EMHS}$ over E which keeps the underlying E -vector spaces and their weight filtrations satisfying the following condition. The homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathcal{G}_{\text{red}} \otimes_E \mathbf{R}$ associated to the restriction of H to $\text{Rep}_E(\mathcal{G}_{\text{red}})$ is $\mathcal{G}_{\text{red}}(\mathbf{R})$ -conjugate to h_0 .

By the method of Section 1.4, D is regarded as a complex analytic manifold.

4.14.2. Define D_{BS} in the same way as in Section 2 except that for a parabolic subgroup P of \mathcal{G}_{red} (defined over E), S_P is defined this time to be the maximal E -split torus in the center of P_{red} and $A_P := \text{Hom}(X(S_P)^+, \mathbf{R}_{>0}^{\text{mult}})$. ($X(S_P)^+$ is defined in the same way as in 2.4.2.) We have the Borel–Serre action of A_P on D and the Borel–Serre action of $\mathbf{R}_{>0} \times A_P$ on D_{nspl} .

Thus D_{BS} is defined to be the set of all pairs (P, Z) , where P is a parabolic subgroup of \mathcal{G}_{red} and Z is either an A_P -orbit in D or a B_P -orbit in D_{nspl} for the Borel–Serre action.

By the method of Section 2.5, D_{BS} is regarded as a real analytic manifold with corners.

4.14.3. Define the objects $D_{\mathrm{SL}(2)}^*$, $D_{\mathrm{SL}(2)}^I$, and $D_{\mathrm{SL}(2)}^{II}$ of $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$ in the same way as in Section 3 except that we change the condition of the rationality of the weight filtrations $W_{\bullet}^{(j)}$ to the E -rationality.

4.14.4. We define the topological spaces D_{Σ}^{\sharp} , $D_{\Sigma,[:]}^{\sharp}$ and $D_{\Sigma,[\mathrm{val}]}^{\sharp}$ as explained in Section 4 except that the rationality of cones in Σ are replaced by E -rationality.

4.14.5. We have the following fundamental diagram (without Γ):

$$\begin{array}{ccccccc} D_{\Sigma,[\mathrm{val}]}^{\sharp} & \rightarrow & D_{\mathrm{SL}(2),\mathrm{val}} & \leftarrow & D_{\mathrm{SL}(2),\mathrm{val}}^* & \rightarrow & D_{\mathrm{BS},\mathrm{val}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D_{\Sigma}^{\sharp} & \leftarrow & D_{\Sigma,[:]}^{\sharp} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}}, \\ \end{array}$$

where the arrows respect the structures (the structure of $D_{\mathrm{SL}(2)}^*$ as an object of $\mathcal{B}'_{\mathbf{R}}(\mathrm{log})$, etc.) of these spaces.

These extended period domains are Hausdorff spaces.

4.14.6. Assume that $\mathcal{G} = G \otimes_{\mathbf{Q}} E$ for a linear algebraic group G over \mathbf{Q} and that $k_0 : \mathbf{G}_{m,E} \rightarrow \mathcal{G}_{\mathrm{red}}$ comes from $\mathbf{G}_m \rightarrow G_{\mathrm{red}}$. Then D is the same as the period domain $D(G, h_0)$ for G , and the above extended period domains contain the ones for G .

4.14.7. In this generalization, however, we can not have a nice theory of the quotients by Γ .

We show an example in which $\mathcal{G} = G \otimes_{\mathbf{Q}} E$, where G is a reductive algebraic group over \mathbf{Q} , Γ is a semi-arithmetic subgroup of $G'(\mathbf{Q})$, and $\Gamma \backslash D_{\mathrm{BS}}$, $\Gamma \backslash D_{\mathrm{SL}(2)}$, and $\Gamma \backslash D_{\Sigma}$ for some E -rational fan Σ which is strongly compatible with Γ , are not Hausdorff.

Let L be a totally real field of degree > 2 and let E be a subfield of \mathbf{R} which contains all conjugates of L . Let $G := \mathrm{Res}_{L/\mathbf{Q}}(\mathrm{GL}(2)_L)$, $\mathcal{G} := G \otimes_{\mathbf{Q}} E = \prod_{j=1}^n \nu_j^*(\mathrm{GL}(2)_L) = \mathrm{GL}(2)_E^n$, where $n = [L : \mathbf{Q}]$ and ν_1, \dots, ν_n are all the different field homomorphisms $L \rightarrow E$. Let $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathcal{G} \otimes_k \mathbf{R} = \mathrm{GL}(2)_E^n$ be the homomorphism $z \mapsto (\langle z \rangle, \dots, \langle z \rangle)$ (1.6.5). Then D is canonically isomorphic to \mathfrak{H}^n , the n -fold product of the upper half plane, and $G(\mathbf{Q}) = \mathrm{GL}(2, L)$ acts on it via (ν_1, \dots, ν_n) .

Let Γ be a subgroup of $\mathrm{SL}(2, O_L)$ of finite index. We show that the quotient spaces $\Gamma \backslash D_{\mathrm{BS}}$, $\Gamma \backslash D_{\mathrm{SL}(2)}$, and $\Gamma \backslash D_{\Sigma}$ are not Hausdorff, where Σ is the set of all nilpotent cones in $\mathrm{Lie}(\mathcal{G}_{\mathbf{R}}) = \mathfrak{gl}(2, \mathbf{R})^n$ of the form $\bigoplus_{j=1}^n \mathbf{R} N_j$ with N_j a nilpotent $(2, 2)$ -matrix over E .

First we show that $\Gamma \backslash D_{\mathrm{BS}}$ is not Hausdorff. For $z \in \mathfrak{H}$, let $p_{\mathrm{BS}}(z) \in D_{\mathrm{BS}}$ be the limit of $(iy_1, \dots, iy_{n-1}, z) \in D = \mathfrak{H}^n$, where $y_j \in \mathbf{R}_{>0}$ and $y_j \rightarrow \infty$. That is, $p_{\mathrm{BS}}(z)$ is the A_P -orbit containing (i, \dots, i, z) , where P is the E -parabolic subgroup $\prod_{j=1}^n P_j$ of $\mathcal{G} = \mathrm{GL}(2)_E^n$ with $P_j = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathrm{GL}(2)_E$ for $1 \leq j \leq n-1$ and $P_n = \mathrm{GL}(2)_E$. Let $S = \{p_{\mathrm{BS}}(ia) \mid a \in \mathbf{R}_{>0}\}$. Then the diagonal matrix (u, u^{-1}) with $u \in O_L^{\times}$ in $\mathrm{SL}(2, L)$ acts on S as $(i\infty, \dots, i\infty, ia) \mapsto (i\infty, \dots, i\infty, ia\nu_n(u)^2)$. But $\{\nu_n(\gamma)^2 \mid \gamma \in \Gamma_1\} \backslash \mathbf{R}_{>0}$ for a subgroup Γ_1 of O_L^{\times} of finite index is not Hausdorff.

Next we show that $\Gamma \backslash D_{\mathrm{SL}(2)}$ is not Hausdorff. For $z \in \mathfrak{H}$, let $p_{\mathrm{SL}(2)}(z)$ be the limit point of $(iy_1, \dots, iy_{n-1}, z)$, where $y_j \in \mathbf{R}_{>0}$, $y_j/y_{j+1} \rightarrow \infty$ ($1 \leq j \leq n-1$) with $y_n := 1$. That is, $p_{\mathrm{SL}(2)}(z)$ is the class of (ρ, \mathbf{r}) , where $\rho : \mathrm{SL}(2)_{\mathbf{R}}^{n-1} \rightarrow \mathcal{G}_{\mathbf{R}} = \mathrm{GL}(2)_{\mathbf{R}}^n$ is the

homomorphism $(g_q, \dots, g_{n-1}) \mapsto (g_1, \dots, g_{n-1}, 1)$ and $\mathbf{r} = (i, \dots, i, z)$. Let S be the set $\{p_{\mathrm{SL}(2)}(ia) \mid a \in \mathbf{R}_{>0}\}$. The rest of the proof is similar to the above.

Lastly, we show that $\Gamma \backslash D_\Sigma$ is not Hausdorff. For $z \in \mathfrak{H}$, let $p_\Sigma(z) \in \Gamma \backslash D_\Sigma$ be the limit point of $(z_1, \dots, z_{n-1}, z) \bmod \Gamma$, where $z_j \in \mathfrak{H}$ and $\mathrm{Im}(z_j) \rightarrow \infty$. That is, $p_{\mathrm{SL}(2)}(z)$ is the class mod Γ of the nilpotent orbit (σ, Z) , where $\sigma = \sigma_1 \times \dots \times \sigma_n \subset \mathfrak{gl}(2, \mathbf{R})^n$ with $\sigma_j = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ for $1 \leq j \leq n-1$ and $\sigma_n = \{0\}$ and Z is the $\exp(\sigma_C)$ -orbit in \check{D} which contains $\mathfrak{H}^{n-1} \times \{z\}$. Let $S = \{p_\Sigma(z) \mid z \in \mathfrak{H}\}$. The rest of the proof is similar to the above (we use the fact that for a subgroup Γ_1 of O_L^\times of finite index, $\{\nu_n(\gamma)^2 \mid \gamma \in \Gamma_1\} \backslash \mathfrak{H}$ is not Hausdorff).

4.15 Generalizations, II

Here we give a generalization which contains partial toroidal compactifications of higher Albanese manifolds treated in [22]. This generalization is used in [19] for applications to number theory.

4.15.1. Let \mathcal{G} be a normal algebraic subgroup of G and let $\mathcal{Q} = G/\mathcal{G}$. We fix an element b of $D(G, h_0)$. Let $D(G, h_0, \mathcal{G}, b) \subset D(G, h_0)$ be the inverse image in $D(G, h_0)$ of the image of b in $D(\mathcal{Q}, h_{0,\mathcal{Q}})$, where $h_{0,\mathcal{Q}}$ denotes the composite homomorphism $S_{\mathbf{C}/\mathbf{R}} \xrightarrow{h_0} G_{\mathrm{red}, \mathbf{R}} \rightarrow \mathcal{Q}_{\mathrm{red}, \mathbf{R}}$.

Proposition 4.15.2. *The morphism $D(G, h_0) \rightarrow D(\mathcal{Q}, h_{0,\mathcal{Q}})$ is smooth.*

This follows from the surjectivity of the map of tangent spaces.

Corollary 4.15.3. *$D(G, h_0, \mathcal{G}, b)$ is smooth.*

4.15.4. We consider the quotient space $\Gamma \backslash D(G, h_0, \mathcal{G}, b)$ and its toroidal partial compactification for a neat semi-arithmetic subgroup Γ of $\mathcal{G}'(\mathbf{Q})$.

4.15.5. *Example 1.* A higher Albanese manifold is regarded as an example of $\Gamma \backslash D(G, h_0, \mathcal{G}, b)$ as is explained in [22].

4.15.6. *Example 2.* Let H_0 be a \mathbf{Z} -MHS, that is, a \mathbf{Q} -MHS endowed with a \mathbf{Z} -lattice $H_{0,\mathbf{Z}}$ in $H_{0,\mathbf{Q}}$. Assume that we have a polarization $p_w : \mathrm{gr}_w^W H_0 \otimes \mathrm{gr}_w^W H_0 \rightarrow \mathbf{Q}(-w)$ in the sense of Deligne for each $w \in \mathbf{Z}$. Then for G , \mathcal{G} , h_0 and b as below, $\Gamma \backslash D(G, h_0, \mathcal{G}, b)$ is identified with

(*) the set of all isomorphism classes of \mathbf{Z} -MHS H endowed with isomorphisms $\mathrm{gr}_w^W H \simeq \mathrm{gr}_w^W H_0$ of \mathbf{Z} -HS for all w .

Let

$$G = \{(g, t) \in \mathrm{Aut}(H_{0,\mathbf{Q}}, W) \times \mathbf{G}_m \mid p_w(gx \otimes gy) = t^w p_w(x, y) \text{ for all } w \in \mathbf{Z}, x, y \in \mathrm{gr}_w^W H_{0,\mathbf{Q}}\},$$

$$\mathcal{G} = G_u,$$

and hence

$$\mathcal{Q} = G/\mathcal{G} = G_{\mathrm{red}} =$$

$$\{(g, t) \in \left(\prod_w \mathrm{Aut}(\mathrm{gr}_w^W H_{0, \mathbf{Q}})\right) \times \mathbf{G}_m \mid p_w(g_w x \otimes g_w y) = t^w p_w(x \otimes y) \text{ for all } w \in \mathbf{Z}, x, y \in \mathrm{gr}_w^W H_{0, \mathbf{Q}}\}.$$

Then we have $h_0 : S_{\mathbf{C}/\mathbf{R}} \rightarrow G_{\mathrm{red}, \mathbf{R}} = \mathcal{Q}_{\mathbf{R}}$ and $b \in D(G, h_0)$ corresponding to H_0 as follows. We have the homomorphism h_0 which is associated to the Hodge structure $\mathrm{gr}^W H_0$. Then the condition (1) in Lemma 1.5.3 is satisfied. Let \mathcal{C}_0 (resp. $\mathcal{C}_{0, \mathrm{red}}$) be the smallest full subcategory of \mathbf{QMHS} (resp. \mathbf{QHS}) which contains H_0 and $\mathbf{Q}(1)$ (resp. $\mathrm{gr}^W H_0$ and $\mathbf{Q}(1)$) and which is stable under taking \oplus , \otimes , duals and subquotients. Let \mathcal{T}_0 be the Tannakian group of \mathcal{C}_0 associated to the fiber functor $H \mapsto H_{\mathbf{Q}}$. Then $\mathcal{T}_{0, \mathrm{red}}$ is identified with the Tannakian group of $\mathcal{C}_{0, \mathrm{red}}$ associated to the fiber functor $H \mapsto H_{\mathbf{Q}}$. We have a homomorphism $S_{\mathbf{C}/\mathbf{R}} \rightarrow \mathcal{T}_{0, \mathrm{red}, \mathbf{R}}$ associated to $\mathrm{gr}^W H_0$, which we denote also by h_0 . We have a canonical homomorphism $\mathcal{T}_0 \rightarrow G$ and this induces a morphism $D(\mathcal{T}_0, h_0) \rightarrow D(G, h_0)$. We have the canonical element of $D(\mathcal{T}_0, h_0)$ which sends $V \in \mathrm{Rep}(\mathcal{T}_0)$ to the corresponding \mathbf{QMHS} . Let b be the image of this canonical element under $D(\mathcal{T}_0, h_0) \rightarrow D(G, h_0)$.

Let

$$\Gamma = \{g \in \mathrm{Aut}(H_{0, \mathbf{Z}}, W) \mid \mathrm{gr}_w^W(g) = 1 \text{ for all } w \in \mathbf{Z}\}.$$

We prove that $\Gamma \setminus D(G, h_0, \mathcal{G}, b)$ is identified with the above set (*). For an element $H \in D(G, h_0, \mathcal{G}, b)$, $H(V)$ for $V = H_{0, \mathbf{Q}} \in \mathrm{Rep}(G)$ with the \mathbf{Z} -lattice $H_{0, \mathbf{Z}}$ gives an element of the set (*). This gives a map $\Gamma \setminus D(G, h_0, \mathcal{G}, b) \rightarrow (*)$. Conversely let H be a \mathbf{ZMHS} with $\mathrm{gr}_w^W H \simeq \mathrm{gr}_w^W H_0$ for all w . Let \mathcal{C} be the smallest full subcategory of \mathbf{QMHS} which contains H and $\mathbf{Q}(1)$ and which is stable under taking \oplus , \otimes , duals and subquotients. Let \mathcal{T} be the Tannakian group of \mathcal{C} associated to the fiber functor $H' \mapsto H'_{\mathbf{Q}}$. We have $\mathcal{T}_{\mathrm{red}} = \mathcal{T}_{0, \mathrm{red}}$ and we have a canonical element $D(\mathcal{T}, h_0)$. We have a canonical homomorphism $\mathcal{T} \rightarrow G$, and the image of this canonical element in $D(G, h_0)$ belongs to $D(G, h_0, \mathcal{G}, b)$. This gives the converse map $(*) \rightarrow \Gamma \setminus D(G, h_0, \mathcal{G}, b)$.

4.15.7. Toroidal partial compactifications of $\Gamma \setminus D(G, h_0, \mathcal{G}, b)$ are obtained by using a weak fan Σ in $\mathrm{Lie}(G)$ which is strongly compatible with Γ such that $\sigma \subset \mathrm{Lie}(\mathcal{G}_{\mathbf{R}})$ for all $\sigma \in \Sigma$. In fact, for such a Σ , the map $D(G, h_0) \rightarrow D(\mathcal{Q}, h_0, \mathcal{Q})$ induces a morphism $\Gamma \setminus D(G, h_0)_{\Sigma} \rightarrow D(\mathcal{Q}, h_0, \mathcal{Q})$ of log manifolds. Let $D(G, h_0, \mathcal{G}, b)_{\Sigma}$ be the inverse image of $b_{\mathcal{Q}} \in D(\mathcal{Q}, h_0, \mathcal{Q})$ in $D(G, h_0)_{\Sigma}$. The fiber $\Gamma \setminus D(G, h_0, \mathcal{G}, b)_{\Sigma}$ of $b_{\mathcal{Q}}$ in $\Gamma \setminus D(G, h_0)_{\Sigma}$ is our toroidal partial compactification of $\Gamma \setminus D(G, h_0, \mathcal{G}, b)$.

Proposition 4.15.8. *The space $\Gamma \setminus D(G, b_0, \mathcal{G}, b)_{\Sigma}$ is a log manifold which represents the following functor on $\mathcal{B}(\mathrm{log})$: It sends $S \in \mathcal{B}(\mathrm{log})$ to the set of all morphisms $S \rightarrow D(G, h_0)_{\Sigma}$ such that the composition $S \rightarrow D(G, h_0)_{\Sigma} \rightarrow D(\mathcal{Q}, (h_0)_{\mathcal{Q}})$ is the constant function $b_{\mathcal{Q}}$.*

Proof. Let $X = \Gamma \setminus D(G, h_0)_{\Sigma}$, $Y = D(\mathcal{Q}, b_{\mathcal{Q}})$, and let $x \in X$. Then the proof of Claim 3 in 4.6.19 shows that there are an open neighborhood U of x in X , a log smooth fs log analytic space Z over \mathbf{C} , and log differential forms $\omega_1, \dots, \omega_n$ on Z such that the morphism $U \rightarrow Y$ factors as $U \rightarrow Z \rightarrow Y$ satisfying the following conditions (i) and (ii). Let $Z' = \{z \in Z \mid \omega_j(z) = 0 \text{ for } 1 \leq j \leq n\}$. Here $\omega_j(z)$ denotes the log differential form on the log point z obtained from ω_j (4.2.12).

(i) U is isomorphic over Z to an open subspace of Z' for the strong topology of Z' in Z .

(ii) The morphism $Z \rightarrow Y$ is log smooth.

Hence the fiber Z_b of $b_{\mathcal{Q}}$ in Z is log smooth, and the fiber of $b_{\mathcal{Q}}$ in U is an open subspace (for the strong topology) of $\{z \in Z_b \mid \omega_j(z) = 0 \text{ for } 1 \leq j \leq n\}$ and hence is a log manifold. \square

4.15.9. In Example 1 in 4.15.5, $\Gamma \setminus D(G, h_0, \mathcal{G}, b)_{\Sigma}$ is the toroidal partial compactification of the higher Albanese manifold discussed in [22] Section 5.

4.15.10. In Example 2 in 4.15.6, $\Gamma \setminus D(G, h_0, \mathcal{G}, b)_{\Sigma}$ represents the functor on $\mathcal{B}(\log)$ which sends $S \in \mathcal{B}(\log)$ to the set of all isomorphism classes of \mathbf{Z} -LMH H of type (h_0, Σ) on S endowed with isomorphisms $\text{gr}_w^W H \simeq \text{gr}_w^W H_0$ of \mathbf{Z} -LMH for all $w \in \mathbf{Z}$ (this tells that $\text{gr}_w^W H$ are constant \mathbf{Z} -HS for all w). See [21] Part III Section 5 for a more general moduli space of \mathbf{Z} -LMH on S with given graded quotients for the weight filtration.

4.15.11. On this toroidal partial compactification $\Gamma \setminus D(G, h_0, \mathcal{G}, b)_{\Sigma}$, $\mathcal{H}_{\text{univ}}(\text{Lie}(\mathcal{G}))_{\mathcal{O}}/F^0$ is the log tangent bundle (this is a generalization of Proposition 4.13.1).

A Appendix

Here we give a complement to [21] Part III.

In the proof of $(2) \Rightarrow (2)''$ in [21] Part III Lemma 2.2.4, it is stated that “ (σ_0, F) generates a nilpotent orbit because both (τ, F) and (σ, F) generate nilpotent orbits.” It turns to be valid, though not trivial, and follows from the following proposition.

Proposition A.1. *Let $\Lambda = (H_0, W, (\langle \cdot, \cdot \rangle_w)_w, (h^{p,q})_{p,q})$ be as in [21] Part III 2.1.1. Let σ be a nilpotent cone, let σ_0 be its face, let N be an interior of σ_0 , and let $F \in \check{D}$. Assume that both (σ, F) and (N, F) generate nilpotent orbits. Then (σ_0, F) also generates a nilpotent orbit.*

First, the admissibility and the Griffiths transversality for (σ_0, F) follow from those for (σ, F) . Thus the problem is reduced to the following proposition by forgetting σ . (Notation is changed.)

Proposition A.2. *Let σ be an admissible nilpotent cone, let N_0 be an interior of σ , and let $F \in \check{D}$ such that F satisfies the Griffiths transversality with respect to σ . Assume that (N_0, F) generates a nilpotent orbit. Then (σ, F) also generates a nilpotent orbit.*

Remark A.2.1. The proof below shows that the assumption of admissibility is weakened to that the weight filtration with respect to an interior is constant. (Cf. the first and the second lines of [10] p.505.)

We prove Proposition A.2. Let N_1, \dots, N_n generate σ . Then the conclusion is equivalent to that $\exp(\sum iy_j N_j)F \in D$ for any $y_j \gg 0$. Since this condition can be checked on each gr_w^W , we can reduce to the pure case. In the rest of this proof, we assume that we are in the situation of pure weight k .

Let M be the weight filtration of one (and hence for any) interior of σ . By Remark in Part III 1.3.2 (which is seen by [10] (4.66)), the conclusion of Proposition A.2 is equivalent

to that $(H_0, \langle \cdot, \cdot \rangle_0, M[-k], F)$ is a mixed Hodge structure polarized in the sense of [10] (2.26) by any interior N of σ . By the assumption that (N_0, F) generates a nilpotent orbit, $(H_0, M[-k], F)$ is at least a mixed Hodge structure. Hence for an interior N , the condition for N is equivalent to that for any $l \geq 0$, the primitive part $P_{k+l} := \text{Ker}(N^{l+1} : \text{gr}_{k+l}^M \rightarrow \text{gr}_{k-l-2}^M)$ for N is a Hodge structure polarized by $S_N := \langle \cdot, N^l \cdot \rangle_0$. Note here that by the assumption, each gr_w^M is already a Hodge structure, and by Griffiths transversality, any interior N is a homomorphism of Hodge structures so that the primitive part P_{k+l} for any N is a Hodge structure (of weight $k+l$) whose Hodge numbers are independent of N (determined by those for gr^M).

Consider the Hodge decomposition of the Hodge structure on P_{k+l} for N and denote its (p, q) -component by $P_N^{p,q}$. Since the dimension of $P_N^{p,q}$ is constant with respect to N (as noted in the above) and the pair of the subspace $P_N^{p,q}$ of $(\text{gr}_{k+l}^M)_{\mathbf{C}}$ and the induced Hermitian form $S_N^{p,q}$ by $S_N(\cdot, \bar{\cdot})$ on $P_N^{p,q}$ varies continuously with respect to N , if we prove that $S_N^{p,q}$ is always nondegenerate, the positivity for one N inherits to all N .

We prove the nondegeneracy. Assume that $S_N^{p,q}$ degenerates for some N, p, q . Then there is a nonzero vector $v \in P_N^{p,q}$ such that $S_N(\cdot, \bar{v})$ is zero on $P_N^{p,q}$. Since the Hodge components are orthogonal with respect to $S_N(\cdot, \bar{\cdot})$, $S_N(\cdot, \bar{v})$ is zero on the whole P_{k+l} for this N . Further, consider the decomposition of gr_{k+l}^M into the images of primitive parts of weights $\geq k+l$. Then this decomposition is orthogonal with respect to $S_N = \langle \cdot, N^l \cdot \rangle_0$ ([32] Lemma (6.4)). Hence $\langle \cdot, N^l \bar{v} \rangle_0$ is zero on $\text{gr}_{k+l, \mathbf{C}}^M$. Since M_{k+l} is the orthogonal complement to M_{k-l-1} (the same lemma of ibid.), $N^l \bar{v} = 0$ in $\text{gr}_{k-l, \mathbf{C}}^M$ and $v = 0$ in $\text{gr}_{k+l, \mathbf{C}}^M$, a contradiction.

Correction to [21] Part IV

There are some typos in the proof of Theorem 6.1.1 in [21] Part IV. Here we give corrections to them.

In the line 3 of the proof of loc. cit., $D_{\Sigma, [\text{val}]} \rightarrow D_{\Sigma, [:]} \rightarrow \Gamma \setminus D_{\Sigma}$ should be changed to $D_{\Sigma, [\text{val}]}^{\sharp} \rightarrow D_{\Sigma, [:]}^{\sharp} \rightarrow D_{\Sigma}^{\sharp}$.

In the lines 4, 5 of the proof of loc. cit., $D_{\Sigma, [\text{val}]}$, $D_{\Sigma, [:]}$, and $\Gamma \setminus D_{\Sigma}$ should be changed to $D_{\Sigma, [\text{val}]}^{\sharp}$, $D_{\Sigma, [:]}^{\sharp}$, and D_{Σ}^{\sharp} , respectively.

There are some typos in the proof of Theorem 6.1.3 in [21] Part IV.

In the line 2, $D_{\text{SL}(2)}^I$ should be $\Gamma \setminus D_{\text{SL}(2)}^I$.

In the lines 4, 5, $S_{[:]}$ should be $S_{[:]}^{\log}$.

Correction to [22]

Here we give a correction to [22] 6.1.2. The construction there does not necessarily give an MHS on $\text{Lie}(\mathcal{G}_{\Gamma})$.

Change the part

We define the weight filtration on $\text{Lie}(\mathcal{G}_{\Gamma})$ (resp. the Hodge filtration on $\text{Lie}(\mathcal{G}_{\Gamma})_{\mathbf{C}}$) as the image of that of $\text{Lie}(\mathcal{G}_{\Gamma_r})$ (resp. $\text{Lie}(\mathcal{G}_{\Gamma_r})_{\mathbf{C}}$) (2.2.4, 2.3). This gives a structure of an MHS on $\text{Lie}(\mathcal{G}_{\Gamma})$ which is independent of the choice of r .

into

Hereafter we assume that $\text{Lie}(\mathcal{G}_\Gamma)$ has an MHS which is a quotient of MHS on $\text{Lie}(\mathcal{G}_{\Gamma_r})$ induced from the canonical variation of MHS in 2.2.4. Note that this MHS on $\text{Lie}(\mathcal{G}_\Gamma)$ is independent of the choice of r .

List of notation

Period domains

$D = D(G, h_0)$	1.1.4, 1.2.13
D_{red}	1.3.3
\check{D}	1.4.1

Extended period domains

D_{BS}	2.4.5
$D_{\text{SL}(2)}$	3.1.10 (case of reductive groups), 3.2.1 (general case)
Structures of $D_{\text{SL}(2)}$:	$D_{\text{SL}(2)}^I$ 3.4.10, $D_{\text{SL}(2)}^{II}$ 3.4.7
$D_{\text{SL}(2)}^*$	3.2.1
$D_{\text{SL}(2)}^{*,W}$	3.7.1
$D_{\text{SL}(2)}^{*,+}$	3.9.1
$D_{\text{BS},\text{val}}, D_{\text{SL}(2),\text{val}}^*, D_{\text{SL}(2),\text{val}}^I, D_{\text{SL}(2),\text{val}}^{II}$	3.10.2
$D_\Sigma, D_\Sigma^\sharp$	4.1.8
$D_{\Sigma,[::]}, D_{\Sigma,[::]}^\sharp$	4.3.15
$D_{\Sigma,\text{val}}, D_{\Sigma,\text{val}}^\sharp$	4.4.7
$D_{\Sigma,[\text{val}]}^\sharp$	4.7.2
$D_{\text{SL}(2)}^\diamond$	4.8.2

Categories

$\mathcal{B}'_{\mathbf{R}}(\log)$	3.4.1
$\mathcal{B}(\log)$	4.2.12

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Kazuya KATO
 Department of mathematics
 University of Chicago

CLASSIFYING SPACES OF DEGENERATING MIXED HODGE STRUCTURES, V

Chicago, Illinois, 60637, USA
kkato@math.uchicago.edu

Chikara NAKAYAMA
Department of Economics
Hitotsubashi University
2-1 Naka, Kunitachi, Tokyo 186-8601, Japan
c.nakayama@r.hit-u.ac.jp

Sampei USUI
Graduate School of Science
Osaka University
Toyonaka, Osaka, 560-0043, Japan
usui@math.sci.osaka-u.ac.jp