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LÉVY'S FUNCTIONAL ANALYSIS IN TERMS OF AN INFINITE DIMENSIONAL BROWNIAN MOTION II

YOSHIHEI HASEGAWA

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0. Introduction

In the previous paper (Hasegawa [7])[1], we have reformulated some aspects of Lévy's infinite dimensional potential theory (cf. Lévy [1]) in terms of our infinite dimensional Brownian motion $\boldsymbol{B}=(\Omega, B(t), P^x)$ on an infinite dimensional sequence space $\boldsymbol{E}$. There we have noticed that in the finite dimensional space R^n such objects as the Laplacian and the volume-element are determined in terms of the standard Riemannian metric ds_n^2, and called this fact the mutual compatibility in the finite dimensional potential theory. Then it induces various linkages among the objects in R^n, e.g., Green-Stokes' formula describes the one among the Laplacian, normal derivatives, the volume-element and surface-elements.

We have constructed our infinite dimensional potential theory by a limiting procedure of the finite dimensional one. Then the above-mentioned mutual linkages should be inherited by the corresponding objects in our infinite dimensional potential theory. Further these objects and linkages should be described by the limiting procedure. In fact the semi-norm $||\cdot||_\infty$ on $\boldsymbol{E}$, the standard Gaussian white noise μ on the unit sphere $S_\infty=\{x\in\boldsymbol{E};\ ||x||_\infty=1\}$ and the infinite dimensional Laplacian Δ_∞ have been constructed from the Riemannian metrics ds_n^2 on R^n, the uniform probability measures μ_{n-1} on the unit spheres $S_{n-1}=\{x=(x_1, \cdots, x_n)\in R^n;\ x_1^2+\cdots+x_n^2=n\}$ of R^n and the Laplacians Δ_n on R^n, respectively, by this procedure, (see I, §§1.1, §§1.2 and also Hida and Nomoto [8]). Moreover we have introduced the Dirichlet solution $f(x)$ on the unit ball $D_\infty=\{x\in\boldsymbol{E};\ ||x||_\infty<1\}$ for a boundary function $\psi(\xi)$ on the unit sphere S_∞, and obtained the following linkage among the objects, $f(x)$, $\psi(\xi)$, $||\cdot||_\infty$, μ, (see I, Th. 3.1):

$$(0.1)\qquad f(x)=\int_{S_\infty}\psi(x+\sqrt{1-||x||_\infty^2}\cdot\xi)\mu(d\xi)\,,\quad x\in D_\infty\,.$$

The purpose of this paper is now to construct the Dirichlet solution $f(x)$

1) Without special mentions we shall use the terminologies in Hasegawa [7], which will be simply quoted as I.

together with the description of the linkage (0.1) by the limiting procedure. First, by dualizing a result of Hida and Nomoto [8], we shall obtain orthogonal projections ρ_n; $L^2(S_\infty, \mu) \to L^2(S_n, \mu_n)$. Second we shall define projections π_n; $\boldsymbol{E} \to R^{n+1}$ as follows:

$$\pi_n x = \frac{||x||_\infty}{||x||_{n+1}}(x_1, \cdots, x_{n+1}) \in R^{n+1}$$

for $x=(x_1, \cdots, x_{n+1}, \cdots) \in \boldsymbol{E}$, $||x||_\infty > 0$, $||x||_{n+1} > 0$. Then, for each element $\psi \in L^2(S_\infty, \mu)$ we have

$$\text{(0.2)} \qquad \lim_{n \uparrow \infty} (\rho_n \psi)(\pi_n \xi) = \psi(\xi) \quad \text{in} \quad L^2(S_\infty, \mu),$$

(see Theorem 1.4). Next, for a tame boundary function $\psi \in L^2(S_\infty, \mu)$ we denote by $f_n(x)$ the Dirichlet solution on the ball $D_{n+1} = \{x=(x_1, \cdots, x_{n+1}) \in R^{n+1}; x_1^2 + \cdots + x_{n+1}^2 < n+1\}$ for the boundary function $\rho_n \psi$ on S_n. Then we have the following

Construction Theorem.

$$\text{(0.3)} \qquad \lim_{n \uparrow \infty} f_n(\pi_n x) = f(x)$$

for $x \in D_\infty$ *such that* $\lim_{n \uparrow \infty} ||x||_n = ||x||_\infty > 0$, *provided* ψ *assumes some integrability condition*, (see Theorem 2.8).

Now we notice the mutual singularity of harmonic measures $\mathcal{P} = \{\mu_x; x \in D_\infty\}$, $\mu_x(d\zeta) = \mu(\{\xi \in S_\infty; x + \sqrt{1 - ||x||_\infty^2}\xi \in d\zeta\})$, (see I, Th. 3.3). Hence, the above-mentioned Construction Theorem, which corresponds with the orthogonal projections $\{\rho_n; n \geqslant 1\}$ defined in association with only one harmonic measure $\mu_0 = \mu$, cannot be extended to general boundary functions $\psi(\xi)$. In this stage we therefore cannot help restricting the boundary functions ψ to the tame ones. In spite of this restriction, peculiar phenomena can be seen in the Dirichlet solution $f(x)$. Actually, our infinite dimensional Laplacian Δ_∞ acts on $f(x)$ in the form:

$$\text{(0.4)} \qquad \Delta_\infty f = \frac{1}{r} \cdot \frac{\partial}{\partial r} f + \sum_k \frac{\partial^2}{\partial x_k^2} f, \quad r = ||x||_\infty,$$

which is of different feature from the finite dimensional Laplacians.

Our proof of this theorem will heavily owe to uniform asymptotic estimates of the Gegenbauer polynomials $C_k^\nu(x/\sqrt{2\nu})$ as $k, \nu \to \infty$, (see Propositions 3.1, 3.2).

1. Projectively consistent construction of the multiple Wiener integrals

1.1. Projectively consistent construction of the standard Gaussian white noise.

In this subsection we shall reformulate Hida and Nomoto's results [8] in a slightly different manner from theirs.

First we introduce to the $(N+1)$-dimensional Euclidean space R^{N+1} the norm r_{N+1}:

$$(1.1)\qquad r_{N+1}(x)=\left(\frac{1}{N+1}\sum_{n=1}^{N+1}x_n^2\right)^{1/2}\qquad\text{for}\quad x=(x_1,\cdots,x_{N+1})\in R^{N+1},$$

and set

$$(1.2)\qquad S_N=\{x\in R^{N+1};\ r_{N+1}(x)=1\}\ .$$

DEFINITION 1. An open subset $\mathring{S}_N$

$$(1.3)\qquad \mathring{S}_N=\{\xi=(\xi_1,\cdots,\xi_{N+1})\in S_N;\ \xi_1\neq 0\quad\text{or}\quad \xi_2<0\}$$

of the sphere S_N is called the N-dimensional unit pre-sphere.

The polar expression of points $\xi=(\xi_1,\cdots,\xi_{N+1})\in\mathring{S}_N$:

$$(1.4)\qquad \begin{cases}\xi_1=\sqrt{N+1}\prod_{i=1}^{N}\sin\theta_i\\ \xi_k=\sqrt{N+1}\cos\theta_{k-1}\prod_{i=k}^{N}\sin\theta_i\,,\quad (k=2,\cdots,N)\,,\\ \xi_{N+1}=\sqrt{N+1}\cos\theta_N\end{cases}$$

induces a homeomorphism of $\mathring{S}_N$ onto a set Π_N:

$$(1.5)\qquad \Pi_N=\{(\theta_1,\cdots,\theta_N);\ 0<\theta_1<2\pi,\ 0<\theta_i<\pi,\ i=2,\cdots,N\}\ ,$$

and $\{\theta_1,\cdots,\theta_N\}$ are called the Euler angles on $\mathring{S}_N$. Then the restriction μ_N of the rotation-invariant probability measure on S_N to the measurable space $(\mathring{S}_N,\mathring{\mathcal{S}}_N)$ can be expressed as follows:

$$(1.6)\qquad \mu_N(d\theta_1,\cdots,d\theta_N)=\frac{\Gamma((N+1)/2)}{2\pi^{(N+1)/2}}\prod_{j=2}^{N}(\sin\theta_j)^{j-1}d\theta_1\cdots d\theta_N\,,$$

where $\mathring{\mathcal{S}}_N$ denotes the topological σ-algebra of $\mathring{S}_N$. Then the family $\{(\mathring{S}_N,\mathring{\mathcal{S}}_N,\mu_N);\ N\geqslant 1\}$ constitutes a topological stochastic family in the sense of S. Bochner [4] with the aid of projections $\pi_{N,M}$, $(N>N)$ defined by:

$$(1.7)\qquad \pi_{N,M};\ \mathring{S}_N\ni\xi=(\xi_1,\cdots,\xi_{N+1})\to\sqrt{\frac{M+1}{\xi_1^2+\cdots+\xi_{M+1}^2}}(\xi_1,\cdots,\xi_{M+1})\in\mathring{S}_M\,,$$

or, equivalently in terms of the Euler angles $(\theta_1,\cdots,\theta_N)$ of the point ξ:

$$(1.8)\qquad \pi_{N,M};\ \Pi_N \ni (\theta_1, \cdots, \theta_N) \rightarrow (\theta_1, \cdots, \theta_M) \in \Pi_M\,.$$

Now we set the following

DEFINITION 2. A measurable subset $\mathring{S}_\infty$

$$(1.9)\qquad \mathring{S}_\infty = \{(\xi_1, \cdots, \xi_N, \cdots) \in \boldsymbol{E};\ \lim_{N\uparrow\infty} \frac{1}{N}\sum_{n=1}^{N} \xi_n^2 = 1,\ \xi_1 \neq 0 \ \text{ or } \ \xi_2 < 0\}$$

of the infinite dimensional unit sphere $S_\infty = \{x \in E;\ ||x||_\infty = 1\}$ is called the infinite dimensional unit pre-sphere.

We denote by $\mathring{\mathcal{S}}_\infty$ the restriction of the σ-algebra $\mathcal{S}_\infty$, (see I, Def. 4) on the unit sphere S_∞ to the unit pre-sphere $\mathring{S}_\infty$.

DEFINITION 3. We define a measurable projection π_N, $(N \geqslant 1)$ of the unit pre-sphere $\mathring{S}_\infty$ onto the N-dimensional unit pre-sphere $\mathring{S}_N$ as follows:

$$(1.10)\qquad \pi_N \xi = \frac{1}{||\xi||_{N+1}} (\xi_1, \cdots, \xi_{N+1}) \quad \text{for} \quad \xi = (\xi_1, \cdots, \xi_{N+1}, \cdots) \in \mathring{S}_\infty\,,$$

(see I, (1.4) for $||\cdot||_{N+1}$).

Since $\pi_M \xi = \pi_{N,M}(\pi_N \xi)$ for $\xi \in \mathring{S}_\infty$ for a pair of integers N, M, $(N > M \geqslant 1)$, we can define an Euler angle $\theta_k(\xi)$, $(k \geqslant 1)$ on $\mathring{S}_\infty$ as follows:

$$\theta_k(\xi) = \theta_k^N(\pi_N \xi)\,, \qquad (k \leqslant N)\,,$$

where $\{\theta_1^N, \cdots, \theta_N^N\}$ denote the Euler angles on $\mathring{S}_N$. Then the functions $\{\theta_k(\xi); k \geqslant 1\}$ are $\mathring{\mathcal{S}}_\infty$-measurable. Conversely for a point $\xi = (\xi_1, \cdots, \xi_n, \cdots) \in \mathring{S}_\infty$ with the Euler angles $\{\theta_n; n \geqslant 1\}$, we have

$$(1.11)\qquad \begin{cases} \xi_1 = \lim\limits_{k\uparrow\infty} \sqrt{k+1} \sin\theta_1 \sin\theta_2 \cdots \sin\theta_k\,, \\ \xi_n = \lim\limits_{k\uparrow\infty} \sqrt{k+1} \cos\theta_{n-1} \sin\theta_n \cdots \sin\theta_k\,, \quad (n \geqslant 2)\,. \end{cases}$$

Hence we have the following equality:

$$(1.12)\qquad \mathring{\mathcal{S}}_\infty = \sigma(\theta_n;\ n \geqslant 1)\,.$$

The standard Gaussian white noise μ which is defined as the distribution on S_∞ of a sequence of mutually independent Gaussian random variables subject to $N(0, 1)$, (see I, Def. 6) can be regarded as a probability measure on the measurable space $(\mathring{S}_\infty, \mathring{\mathcal{S}}_\infty)$.

Now we are in position to state the following

Proposition 1.1 (cf. Hida and Nomoto [8]).

1) $\bigcup\limits_{n=1}^{\infty} \pi_n^{-1}(\mathring{\mathcal{S}}_n)$ *generates the* σ-*algebra* $\mathring{\mathcal{S}}_\infty$.

2)

$$\mu_n(A) = \mu(\pi_n^{-1}(A)) \quad \textit{for elements} \quad A \in \mathring{S}_n . \tag{1.13}$$

3)

$$\mu_n(\pi_{n,m}^{-1}(A)) = \mu_m(A) , \quad (n>m) \quad \textit{for elements} \quad A \in \mathring{S}_m . \tag{1.14}$$

Proof. The first assertion is immediate by the equality (1.12), and the third one comes from the formulas (1.6), (1.8). Put $\tilde{\mu}_n(A) = \mu(\pi_n^{-1}(A))$ for measurable subsets A of S_n. Then by (1.10) and the definition of μ, we can see the rotation-invariance of the probability measure $\tilde{\mu}_n$ on S_n. Hence we have the second assertion. (Q.E.D.)

1.2. Projectively consistent construction of the multiple Wiener integrals $L^2(S_\infty, \mu)$.

In this subsection, we shall dualize Proposition 1.1. To begin with, for integers j, k, m, $(j \geq 2,\ m \geq k \geq 0)$ we put

$$D_{j,k,m}(\theta) = A_{j,k,m} C_n^{\nu}(\cos\theta)\,(\sin\theta)^k . \tag{1.15}$$

Here $\nu = k + (j-1)/2$, $n = m-k$, and C_n^{ν} denotes the Gegenbauer polynomial and the positive constant $A_{j,k,m}$ is chosen so as to hold

$$\int_0^{\pi} D_{j,k,m}^2(\theta)\,(\sin\theta)^{j-1} d\theta = \int_0^{\pi} (\sin\theta)^{j-1} d\theta .$$

For a sequence $K = (k_1, k_2, \cdots, k_n)$ of integers, $(0 \leq |k_1| \leq k_2 \leq \cdots \leq k_n)$, we introduce a function $\Xi_K^n(\theta_1, \cdots, \theta_n)$ on Π_n as follows:

$$\Xi_K^n(\theta_1, \cdots, \theta_n) = e^{ik_1\theta_1} \prod_{j=2}^{n} D_{j,|k_{j-1}|,k_j}(\theta_j) , \tag{1.16}$$

which can be regarded as a function on the unit pre-sphere $\mathring{S}_n$ through the polar expression (1.4). Then we have a *C.O.N.S.* $\{\Xi_K^n; K\}$ in the complex Hilbert space $\mathcal{H}_n = L^2(S_n, \mu_n)$, which is called the canonical basis in $\mathcal{H}_n$, (cf. Vilenkin [12], p. 468).

Here we pause to prepare a representation-theoretic proposition concerning the rotation group $SO(n+1)$. We denote by L^{n+1} the quasi-regular representation of $SO(n+1)$ on $\mathcal{H}_n$, and introduce the following subspaces $\mathcal{H}_{n,p}$, $\mathcal{H}_{n,p,q}$, $(p \geq q)$ of $\mathcal{H}_n$:

$$\mathcal{H}_{n,p} = [\Xi_K^n; K = (k_1, \cdots, k_n),\ 0 \leq |k_1| \leq \cdots \leq k_n = p] , \quad (n \geq 2) , \tag{1.17}$$

$$\mathcal{H}_{n,p,q} = [\Xi_K^n; K = (k_1, \cdots, k_n),\ 0 \leq |k_1| \leq \cdots \leq k_{n-1} = q \leq k_n = p] , \quad (n \geq 3) . \tag{1.18}$$

Here the bracket [ ] denotes the closed linear hull of vectors in the bracket. In the usual manner we identify the elements of the group $SO(n)$ with the ones of the subgroup H_n of $SO(n+1)$:

$$H_n = \{h \in SO(n+1);\ he_{n+1} = e_{n+1}\}\ , \tag{1.19}$$

where $e_{n+1}=(0, \cdots, 0, \sqrt{n+1}) \in S_n$. On the other hand, the subspaces $\mathcal{H}_{n,p}$ are the eigenspaces of the spherical Laplacian $\bar{\Delta}_n$ on S_n. Therefore it is reasonable to require the orthogonal projection ρ; $\mathcal{H}_n \to \mathcal{H}_{n-1}$, $(n \geqslant 3)$ to satisfy the following conditions:

(C1: Eigenspaces-preserving property)

$$\rho(\mathcal{H}_{n,p}) = \mathcal{H}_{n-1,p}\ , \quad (p \geqslant 0)\ , \tag{1.20}$$

(C2: Commutativity of the representations of the group $SO(n)$)

$$L^n(h)\rho = \rho L^{n+1}(h) \qquad \text{for} \quad h \in H_n \subset SO(n+1)\ , \tag{1.21}$$

under the identification of $SO(n)$ and H_n.

Then we have the following

Proposition 1.2. *The orthogonal projection* ρ; $\mathcal{H}_n \to \mathcal{H}_{n-1}$, $(n \geqslant 3)$ *satisfying the conditions* (C1), (C2) *is identical with the following one, up to multiplication constant:*
For an index $K=(k_1, \cdots, k_n)$, $(0 \leqslant |k_1| \leqslant \cdots \leqslant k_n)$,

$$\rho \Xi_K^n = \begin{cases} \Xi_{K_{n-1}}^{n-1} & \text{if} \quad k_{n-1} = k_n\ , \\ 0 & \text{if} \quad k_{n-1} \neq k_n\ , \end{cases} \tag{1.22}$$

where $K_{n-1}=(k_1, \cdots, k_{n-1})$.

Proof. Because of the irreducibility of the representation L^{n+1} of $SO(n+1)$ on each subspace $\mathcal{H}_{n,p}$, $(p \geqslant 0)$ and the unitary equivalence of the two representations $\{H_n, L^{n+1}, \mathcal{H}_{n,p,q}\}$ and $\{SO(n), L^n, \mathcal{H}_{n-1,q}\}$, $(p \geqslant q)$, (see Vilenkin [12], p. 451), we obtain this proposition with the aid of Shur's lemma. (Q.E.D.)

According to Proposition 1.2 we have therefore the following

DEFINITION 4. We define an orthogonal projection $\rho_{n,m}$, $(n > m \geqslant 2)$ of the Hilbert space $\mathcal{H}_n$ onto the another space $\mathcal{H}_m$ as follows: For an index $K=(k_1, \cdots, k_n)$ $(0 \leqslant |k_1| \leqslant k_2 \leqslant \cdots \leqslant k_n)$,

$$\rho_{n,m} \Xi_K^n = \begin{cases} \Xi_{K_m}^m & \text{if} \quad k_m = k_{m+1} = \cdots = k_n\ , \\ 0 & \text{otherwise,} \end{cases} \tag{1.23}$$

where $K_m=(k_1, \cdots, k_m)$.

Thus the projective system $\{\mathcal{H}_n, \rho_{n,m}\}$ has been defined. Now we shall construct a $C.O.N.S.$ in the complex Hilbert space $(\mathcal{H}=L^2(S_\infty, \mu), \|\cdot\|)$ according to the canonical bases of the Hilbert spaces $\mathcal{H}_n$. First for an infinite sequence K of integers

$$(1.24)\quad K=(k_1, \cdots, k_p, \cdots),\ (0\leqslant|k_1|\leqslant\cdots\leqslant k_p=k_{p+1}=\cdots, \text{ with an integer } p),$$

we define a homogeneous harmonic polynomial $\Xi_K(x_1, \cdots, x_{p+1})$ of degree $|K|$ on R^{p+1} as follows:

$$(1.25)\qquad \Xi_K(x)=\Xi^p_{K_p}(\theta_1, \cdots, \theta_p)\sqrt{\frac{\Gamma((p+1)/2)2^{-|K|}}{\Gamma(|K|+(p+1)/2)}}\cdot r^{|K|}.$$

Here $K_p=(k_1, \cdots, k_p)$, $|K|=k_p$ for the integer p, $\{\theta_1, \cdots, \theta_p\}$ denote the Euler angles of the point $x=(x_1, \cdots, x_{p+1})\in R^{p+1}$, and $r=x_{p+1}/\cos\theta_p=(x_1^2+\cdots+x_{p+1}^2)^{1/2}$. We denote by the same notation Ξ_K the lift-up of the function Ξ_K on R^{p+1} to the space $\boldsymbol{E}$:

$$(1.26)\qquad \Xi_K(x)=\Xi_K(x_1, \cdots, x_{p+1})\qquad \text{for}\quad x=(x_1, \cdots, x_{p+1}, \cdots)\in\boldsymbol{E}.$$

REMARK. The functions $\Xi_K(x)$ are harmonic on the space $\boldsymbol{E}$ in our sense, (see I, Proposition 3.8).

Proposition 1.3 (cf. Hida and Nomoto [8]). *The family $\{\Xi_K;\ K\in\mathcal{K}\}$ constitues a C.O.N.S. of the Hilbert space $\mathcal{H}$, where $\mathcal{K}$ is the family of infinite sequences of the above-mentioned type.*

Proof. First we put

$$(1.27)\qquad Z_{n,p}(\xi)=\sqrt{n+1}\sin\theta_{p+1}\cdots\sin\theta_n,$$

where $\{\theta_1, \cdots, \theta_n, \cdots\}$ are the Euler angles of $\xi=(\xi_1, \cdots, \xi_n, \cdots)\in\mathring{S}_\infty$. Then, by Proposition 1.1 and (1.6) we have

$$(1.28)\qquad E[Z^k_{n,p}(\xi)]\to 2^{k/2}\cdot\frac{\Gamma((k+p+1)/2)}{\Gamma((p+1)/2)}\qquad \text{as}\quad n\to\infty,\quad (p\geqslant 1,\ k\geqslant 1),$$

and by (1.11) we have

$$(1.29)\qquad Z_{n,p}(\xi)\to r=(\xi_1^2+\cdots+\xi_{p+1}^2)^{1/2}\qquad \text{as}\quad n\to\infty,\quad a.s.\ (\mu).$$

Now we notice the following expression of $\Xi^n_{K_n}$ for the subindex $K_n=(k_1, \cdots, k_n)$ of K, (1.24), $(n>p)$:

(1.30)
$$\Xi^n_{K_n}(\pi_n\xi)=\Xi^p_{K_p}(\theta_1, \cdots, \theta_p)\sqrt{\frac{\Gamma((p+1)/2)2^{-|K|}}{\Gamma(|K|+(p+1)/2)}}\sqrt{\frac{\Gamma(|K|+(n+1)/2)}{\Gamma((n+1)/2)((n+1)/2)^{|K|}}}\cdot Z^{|K|}_{n,p}(\xi).$$

Consequently the family $\{\Xi^n_{K_n}(\pi_n\xi)\cdot\overline{\Xi^n_{K'_n}(\pi_n\xi)};\ n\}$ for any pair of $K, K'\in\mathcal{K}$ is uniformly integrable, (the bar denotes the complex conjugate). Therefore the system $\{\Xi_K; K\}$ is orthonormal in $\mathcal{H}$. Since the family $\{\Xi^n_K; K\}$ constitutes a *C.O.N.S.* of the Hilbert space $\mathcal{H}_n$, we can see the completeness of the system $\{\Xi_K; K\in\mathcal{K}\}$ in the Hilbert space $\mathcal{H}$. (Q.E.D.)

DEFINITION 5. The restriction of the function Ξ_K to the unit sphere S_∞ is is called the infinite dimensional spherical harmonics of degree $|K|$ on the unit sphere S_∞.

Now we are in position to obtain the following

DEFINITION 6. We denote by ρ_n, $(n\geqslant 2)$ the following orthogonal projection of the Hilbert space $\mathcal{H}=L^2(S_\infty, \mu)$ onto the Hilbert space $\mathcal{H}_n=L^2(S_n, \mu_n)$:

$$(1.31)\qquad \rho_n\Xi_K = \begin{cases} \Xi^n_{K_n} & \text{if} \quad k_n = k_{n+1} = k_{n+2} = \cdots, \\ 0 & \text{otherwise,} \end{cases}$$

where $K=(k_1, \cdots, k_n, \cdots)$, $(0\leqslant |k_1|\leqslant\cdots\leqslant k_n\leqslant\cdots)$, and $K_n=(k_1, \cdots, k_n)$. Then, dualizing Proposition 1.1, we have the following

Theorem 1.4.

1) *For an element $f\in\mathcal{H}$, it holds that*

$$(1.32)\qquad \lim_{n\uparrow\infty} (\rho_n f)(\pi_n\xi) = f(\xi) \qquad in \quad \mathcal{H}.$$

2) *Let $\{f_n\in\mathcal{H}_n; n\geqslant 2\}$ be a projectively consistent sequence, that is, $\rho_{n,m}f_n=f_m$, $(n>m)$. Then there exists a unique function $f\in\mathcal{H}$ such that $\rho_n f=f_n$, $(n\geqslant 2)$, if and only if $\{\|f_n\|_n; n\geqslant 2\}$ is bounded, where $\|\cdot\|_n$ denotes the norm of $\mathcal{H}_n$.*

Proof. The first assertion can be obtained in a quite similar manner to the proof of Proposition 1.3. To see the second assertion, we put

$$(1.33)\qquad \hat{\mathcal{H}}_p = [\Xi_K; K\in\hat{\mathcal{K}}_p], \quad \text{(the closed linear hull),}$$

where $\hat{\mathcal{K}}_p=\{K=(k_1, \cdots, k_p, \cdots)\in\mathcal{K};\ 0\leqslant|k_1|\leqslant\cdots\leqslant k_{p-1}<k_p=k_{p+1}=\cdots\}$. Moreover we define an isometric inclusion $\hat{\rho}_n$ of $\mathcal{H}_n$ into $\mathcal{H}$ as follows: $\hat{\rho}_n\Xi^n_K=\Xi_{K'}$, where we put $K'=(k_1, \cdots, k_n, k_n, k_n, \cdots)$ for $K=(k_1, \cdots, k_n)$, $(0\leqslant|k_1|\leqslant k_2\leqslant\cdots\leqslant k_n)$. Now we assume the boundedness of the set $\{\|f_n\|_n; n\geqslant 2\}$ for the projectively consistent sequence $\{f_n; n\geqslant 2\}$ and set $\hat{\rho}_n f_n=g_n\in\mathcal{H}$. Then, from the projective consistency of the sequence $\{f_n; n\geqslant 2\}$ we obtain $h_n=g_n-g_{n-1}\in\hat{\mathcal{H}}_n$, $(n\geqslant 3)$ and the boundedness of the set $\{g_n; n\geqslant 2\}$. Since $\{\hat{\mathcal{H}}_p\}$ are mutually orthogonal subspaces, we have an element $g_\infty=\lim_{n\uparrow\infty} g_n$ in $\mathcal{H}$, and $\rho_n g_\infty=f_n$, for $\rho_n g_m=f_n$, $(n<m)$. (Q.E.D.)

REMARK. $\rho_n f$, $(n\geqslant 2)$ is real for a real function $f\in\mathcal{H}$.

Finally we have the following

Proposition 1.5.

$$[\Xi_K;\ |K|=k]=[H_K;\ |K|=k]\,,\quad k\geqslant 0\,, \tag{1.34}$$

where H_K denote Fourier-Hermite polynomials, (see I, Def. 7).

Proof. It is easy to see that the function Ξ_K, (1.26) is included in the domain of the infinitesimal generator A of the contraction semi-group $\{\hat{T}_t;\, t\geqslant 0\}$, (see I, (2.22)) and

$$\begin{aligned}(A\Xi_K)(\xi) &= \frac{1}{2}\sum_{n=1}^{p+1}\left(\frac{\partial^2}{\partial\xi_n^2}-\xi_n\frac{\partial}{\partial\xi_n}\right)\Xi_K(\xi_1,\cdots,\xi_{p+1})\\ &= -\frac{1}{2}r\cdot\frac{\partial}{\partial r}\Xi_K(\xi_1,\cdots,\xi_{p+1}) = -\frac{|K|}{2}\Xi_K(\xi)\,.\end{aligned}$$

Hence $\{\Xi_K;\ |K|=k\}$ are eigenfunctions of the eigenvalue $-k/2$. Therefore the uniqueness of eigenspaces of a self-adjoint operator gives the assertion (1.34) with the aid of Proposition 1.3 and Proposition 2.4 in I. (Q.E.D.)

2. Finite dimensional construction of the Dirichlet solutions on the infinite dimensional unit ball

2.1. Finite dimensional construction of the Dirichlet solutions for the boundary functions Ξ_K.

In this subsection we shall construct the Dirichlet solution on the unit ball $D_\infty=\{x\in E;\, \|x\|_\infty<1\}$ for the boundary function Ξ_K according to "la méthode du passage du fini à l'infini".

In §§ 1.2 we have introduced the condition (C1: Eigenspaces-preserving property) concerning the orthogonal projection $\rho;\, \mathcal{H}_n\to\mathcal{H}_{n-1}$. Here we shall give a condition connected with the Dirichlet problems, which induces the condition (C1). We denote by $\mathcal{P}_n\psi$ the Dirichlet solution on an $(n+1)$-dimensional unit ball $D_{n+1}=\{x\in R^{n+1};\, r_{n+1}(x)<1\}$ for a boundary function $\psi\in\mathcal{H}_n=L^2(S_n,\mu_n)$.

We are now ready to set the following

DEFINITION 7. An orthogonal projection $\rho;\, \mathcal{H}_n\to\mathcal{H}_{n-1}$, $(n\geqslant 3)$ is said to be Poisson kernel-preserving, if the following holds:

$$\rho((\mathcal{P}_n\psi)_r)=(\mathcal{P}_{n-1}(\rho\psi))_r \qquad \text{for } r\in[0,1)\,,\ \psi\in\mathcal{H}_n\,, \tag{2.1}$$

where $\quad (\mathcal{P}_k\varphi)_r(\xi)=(\mathcal{P}_k\varphi)(r\xi) \qquad \text{for } \xi\in S_k\,,\ \varphi\in\mathcal{H}_k\,.$

Then we have the following

Proposition 2.1. *If the orthogonal projection ρ; $\mathcal{H}_n \to \mathcal{H}_{n-1}$, $(n \geqslant 3)$ is surjective and Poisson kernel-preserving, it holds that*

$$\rho(\mathcal{H}_{n,p}) = \mathcal{H}_{n-1,p} \qquad \text{for} \quad p = 0, 1, 2, \cdots . \tag{C1}$$

Now, we notice that the Dirichlet solution on the unit ball D_∞ for the boundary function Ξ_K is just identical with $\Xi_K(x)$ on D_∞. We shall construct this solution Ξ_K by the finite dimensional Dirichlet solutions. First we modify the projections π_n; $S_\infty \to S_n$.

DEFINITION 8.

$$\pi_n x = \frac{||x||_\infty}{||x||_{n+1}} (x_1, \cdots, x_{n+1}) \in R^{n+1} \tag{2.2}$$

for $x = (x_1, \cdots, x_{n+1}, \cdots) \in \boldsymbol{E}$ such that $||x||_\infty > 0$, $||x||_{n+1} > 0$.

Then we notice $r_{n+1}(\pi_n x) = ||x||_\infty$ and the Euler angles of $\pi_n x$ and the ones of x agree with each other. We are now ready to state the finite dimensional construction theorem in the case of the spherical harmonics Ξ_K.

Theorem 2.2. *Let $f_n(x)$ be the Dirichlet solution on the $(n+1)$-dimensional unit ball D_{n+1} for the boundary function $\rho_n \Xi_K$, $(n \geqslant 2)$. Then it holds that*

$$\lim_{n \uparrow \infty} f_n(\pi_n x) = \Xi_K(x) \tag{2.3}$$

at points $x = (x_1, \cdots, x_{p+1}, \cdots) \in \mathring{D}_\infty$:

$$\mathring{D}_\infty = \{x \in D_\infty;\ \lim_{n \uparrow \infty} ||x||_n = ||x||_\infty > 0\} . \tag{2.4}$$

Proof. Noticing (1.4), (1.30), (2.2), we have the following formula for sufficiently large n:

$$f_n(\pi_n x) = \Xi^p_{K_p}(\theta_1, \cdots, \theta_p) \sqrt{\frac{\Gamma((p+1)/2) 2^{-|K|}}{\Gamma(|K| + (p+1)/2)}} \sqrt{\frac{\Gamma(|K| + (n+1)/2)}{\Gamma((n+1)/2)((n+1)/2)^{|K|}}} \times \left(\frac{||x||_\infty}{||x||_{n+1}} \cdot \frac{x_{p+1}}{\cos \theta_p} \right)^{|K|} .$$

Hence by Stirling's formula we have the assertion (2.3). (Q.E.D.)

Consequently we have the following

Corollary 2.3. *For a polynomial $\tilde{\psi}(x_1, \cdots, x_p)$ on R^p, we set*

$$\psi(\xi) = \tilde{\psi}(\xi_1, \cdots, \xi_p) \qquad \textit{for} \quad \xi = (\xi_1, \cdots, \xi_p, \cdots) \in S_\infty . \tag{2.5}$$

Then it holds that

$$\lim_{n \uparrow \infty} f_n(\pi_n x) = f(x) \qquad \textit{for} \quad x = (x_1, \cdots, x_p, \cdots) \in \mathring{D}_\infty , \tag{2.6}$$

where f_n and f denote the Dirichlet solution on D_{n+1} for the boundary function $\rho_n\psi$, $(n\geqslant 2)$, and the one on D_∞ for the boundary function ψ, respectively.

2.2. Finite dimensional construction of the Dirichlet solutions for general tame boundary functions.

In this subsection, we shall extend Theorem 2.2 to the case of more general tame boundary functions.

Let $(\mathcal{L}^2, \|\cdot\|)$ be the Hilbert space consisting of functions $\tilde{\psi}(u)$, $u=(u_1, \cdots, u_p)$, $(p\geqslant 1)$ on R^p such that

$$(2.7) \qquad \langle\tilde{\psi}, \tilde{\psi}\rangle = \int_{R_p} |\tilde{\psi}(u)|^2 \exp(-u^2/2)/(2\pi)^{p/2} du < \infty .$$

Here we set $u^2=u_1^2+\cdots+u_p^2$, $du=du_1\cdots du_p$ for $u=(u_1, \cdots, u_p)\in R^p$, and in the sequel we shall use these abbreviations without confusion. Take a real function $\tilde{\psi}(u_1, \cdots, u_p)\in\mathcal{L}^2$ and denote by $\psi(\xi)$ the lift-up of $\tilde{\psi}$ to the unit sphere S_∞:

$$(2.8) \qquad \psi(\xi) = \tilde{\psi}(\xi_1, \cdots, \xi_p) \qquad \text{for} \quad \xi = (\xi_1, \cdots, \xi_p, \cdots)\in S_\infty .$$

Now we shall show that $\rho_n\psi\in\mathcal{H}_n$ has a continuous version on S_n, $(n\geqslant 2)$. First we have to show the following addition formula.

Lemma 2.4. *For vectors $\xi=(\xi_1, \cdots, \xi_{n+1})$, $\zeta=(\zeta_1, \cdots, \zeta_{n+1})\in S_n$, $(n\geqslant 2)$, it holds that*

$$(2.9) \qquad \sum_{|K|=k} \Xi_K^n(\xi)\cdot\overline{\Xi_K^n(\zeta)} = \frac{2k+n-1}{n-1} C_k^\nu\Big(\frac{\xi\cdot\zeta}{n+1}\Big) .$$

Here $\xi\cdot\zeta=\xi_1\zeta_1+\cdots+\xi_{n+1}\zeta_{n+1}$, $\nu=(n-1)/2$ and $|K|=k_n$ for $K=(k_1, \cdots, k_n)$, $(0\leqslant |k_1|\leqslant\cdots\leqslant k_n)$.

Proof. Since $\{\Xi_K^n; K\}$ constitutes a *C.O.N.S.* in $\mathcal{H}_n$, and the Dirichlet solutions on D_{n+1} for the boundary functions $\Xi_K^n(\xi)$ are given by $r^{|K|}\cdot\Xi_K^n(\xi)$, $r\in[0, 1)$, $\xi\in S_n$, we have the following equality:

$$(2.10) \qquad \sum_{k=0}^{\infty} r^k \sum_{|K|=k} \Xi_K^n(\xi)\cdot\overline{\Xi_K^n(\zeta)} = (1-r^2)\cdot\Big(1+r^2-2r\frac{\xi\cdot\zeta}{n+1}\Big)^{-\nu-1} .$$

On the other hand, the generating function (see Vilenkin [12], p. 492) of the Gegenbauer polynomials gives

$$(2.11) \qquad (1-r^2)\cdot\Big(1+r^2-2r\frac{\xi\cdot\zeta}{n+1}\Big)^{-\nu-1} = \sum_{k=0}^{\infty} \frac{2k+n-1}{n-1} r^k C_k^\nu\Big(\frac{\xi\cdot\zeta}{n+1}\Big) .$$

Comparing the coefficients of r^k in these power series, we have therefore the asked formula (2.9). (Q.E.D.)

By using this proposition, we have the following

Proposition 2.5. *The function* $\rho_n\psi\in L^2(S_n, \mu_n)$, $(n\geqslant 2)$ *has a continuous version on the sphere* S_n.

Proof. In the space $L^2(S_n, \mu_n)$, $(n+1\geqslant p)$ Proposition 1.3 gives

$$(2.12)\qquad \rho_n\psi(\xi)=\sum_{k=0}^{\infty}\sum_{\substack{|K|=k\\ K\in\mathcal{K}_n}}\int\tilde{\psi}(\hat{u})\overline{\Xi}_K(u)\frac{\exp(-u^2/2)}{(2\pi)^{(n+1)/2}}\,du\cdot\Xi^n_{K_n}(\xi),$$

where $K_n=(k_1, \cdots, k_n)$ for $K=(k_1, \cdots, k_n, \cdots)\in\mathcal{K}$,

$$\mathcal{K}_m=\{K=(k_1, \cdots, k_m, \cdots)\in\mathcal{K};\ 0\leqslant|k_1|\leqslant\cdots\leqslant k_m=k_{m+1}=\cdots\},\quad (m\geqslant 1)$$

and $\hat{u}=(u_1, \cdots, u_p)$ for $u=(u_1, \cdots, u_{n+1})\in R^{n+1}$. By (1.25) and Lemma 2.4, we have therefore the following

$$(2.13)\qquad \rho_n\psi(\xi)=\sum_{k=0}^{\infty}\psi_k^{(n)}(\xi),$$

where

$$(2.14)\qquad \psi_k^{(n)}(\xi)=\int\tilde{\psi}(\hat{u})C_k(u)|u|^k\exp(-u^2/2)/(2\pi)^{(n+1)/2}du,$$

$$(2.15)\qquad C_k(u)=\frac{2k+n-1}{n-1}\sqrt{\frac{\Gamma((n+1)/2)2^{-k}}{\Gamma(k+(n+1)/2)}}C_k^{\nu}\left(\frac{\xi\cdot u}{|u|\sqrt{n+1}}\right).$$

On the other hand, we have

$$|C_k(u)|\leqslant d_k=\frac{2k+n-1}{n-1}\sqrt{\frac{\Gamma((n+1)/2)2^{-k}}{\Gamma(k+(n+1)/2)}}\cdot\frac{\Gamma(n+k-1)}{\Gamma(k+1)\Gamma(n-1)},$$

and the power series $g(z)=\sum_{k=0}^{\infty}d_kz^k$, ($z$; complex numbers) is an entire function of order 2, (cf. Boas [3], pp. 8, 9), that is, for any positive number ε, there exists a positive constant M_ε such that

$$|g(z)|\leqslant M_\varepsilon e^{(2+\varepsilon)|z|}\qquad\text{for all complex numbers}\quad z.$$

Hence by the Schwarz inequality the series (2.13) converges uniformly and absolutely on the sphere S_n. Therefore $\rho_n\psi\in\mathcal{H}_n$ has a continuous version on S_n. Also in the case of $p\geqslant n+1$, this proposition can be proved in a quite similar way to the case of $n+1\geqslant p$. (Q.E.D.)

Proposition 2.6. *It holds that in the space* $L^2(S_n, \mu_n)$, $(1+n\geqslant p)$:

$$(2.16)\qquad \sum_{|K|=k}\frac{t^K}{\sqrt{K!}}(\rho_nH_K)(\xi)=\frac{2k+n-1}{n-1}\sqrt{\frac{\Gamma((n+1)/2)2^{-k}}{\Gamma(k+(n+1)/2)}}|t|^kC_k^{\nu}\left(\frac{t\cdot\xi}{\sqrt{n+1}|t|}\right)$$

for $\xi=(\xi_1, \cdots, \xi_p, \cdots, \xi_{n+1})\in S_n$, $(\nu=(n-1)/2)$.

Here $K!=k_1!\cdots k_p!$, $|K|=k_1+\cdots+k_p$ *for* $K=(k_1, \cdots, k_p)$, $(k_j\geqslant 0, j\geqslant 1)$, *and* $t^K=$

$t_1^{k_1}\cdots t_p^{k_p}$, $t\cdot\xi=t_1\xi_1+\cdots+t_p\xi_p$ for $t=(t_1,\cdots,t_p)$, and H_K denote Fourier-Hermite polynomials (see I, (2.23), (2.25)).

Proof. Since the function $\Xi_K(x)$ is a tame homogeneous harmonic function in our sense (see I, Prop. 3.8), the density formula (see I, (3.9)) gives

$$\int_{S_\infty} \exp(s\tilde{t}\cdot\xi - s^2\tilde{t}^2/2)\cdot\Xi_K(\xi)\mu(d\xi) = s^{|K|}\Xi_K(\tilde{t}), \tag{2.17}$$

where $s\in(-\infty,+\infty)$, $\tilde{t}=(t_1,\cdots,t_p,0,0,0,\cdots)\in l^2$.

Then by Proposition 1.3 and the generating function of the Hermite polynomials (see I, (2.24)), we have the following in the space $\mathcal{H}$:

$$\sum_{|K|=k}\frac{t^K}{\sqrt{K!}}H_K(\xi) = \sum_{|K|=k}\Xi_K(\tilde{t})\cdot\overline{\Xi_K}(\xi). \tag{2.18}$$

Hence, applying the orthogonal projection ρ_n to the both sides of (2.18), and next using Lemma 2.4, we have the asked formula (2.16), (Q.E.D.)

Now we give another expansion of $\tilde{\psi}$ in $\mathcal{L}^2$:

$$\tilde{\psi}(u) = \sum_{k=0}^\infty \tilde{\psi}_k(u), \quad u=(u_1,\cdots,u_p)\in R^p, \tag{2.19}$$

where

$$\tilde{\psi}_k(u) = \sum_{|K|=k}\langle\tilde{\psi}, H_K\rangle H_K(u), \quad K=(k_1,\cdots,k_p), \tag{2.20}$$

and denote by ψ_k the lift-up of $\tilde{\psi}_k$ to S_∞:

$$\psi_k(\xi) = \tilde{\psi}_k(\xi_1,\cdots,\xi_p) \qquad \text{for } \xi=(\xi_1,\cdots,\xi_p,\cdots)\in S_\infty. \tag{2.21}$$

Then, we have the following

Proposition 2.7. *Denoting by f_k^n the Dirichlet solution on D_{n+1} for the boundary function $\rho_n\psi_k\in\mathcal{H}_n$, we have:*

$$f_k^n(\pi_n x) = \frac{(i\|x\|_\infty)^k}{(2\pi)^p}\sqrt{\frac{(k+\nu)\Gamma(\nu)2^{-k}}{\nu\Gamma(k+\nu)}}\int_{R^p}\tilde{\psi}(u)e^{-u^2/2}du \times\int_{R^p} e^{-(t+iu)^2/2}|t|^k C_k^\nu\left(\frac{\hat{x}\cdot t}{\|x\|_{n+1}\sqrt{n+1}|t|}\right)dt \tag{2.22}$$

for $x=(x_1,\cdots,x_p,\cdots)\in D_\infty$ such that $\|x\|_\infty>0$, $\|x\|_{n+1}>0$, where $\hat{x}=(x_1,\cdots,x_p)$ for the point x and $(t+iu)^2=(t_1+iu_1)^2+\cdots+(t_p+iu_p)^2$ for $t=(t_1,\cdots,t_p)$, $u=(u_1,\cdots,u_p)\in R^p$.

Proof. First the integral representation of the Hermite polynomials and the formula (2.18) give the following in $\mathcal{H}_n$, $(n+1\geqq p)$:

$$(2.23)\qquad \rho_n\psi_k(\xi) = \int i^k \frac{e^{-u^2/2}}{(2\pi)^p}\tilde{\psi}(u)du \int e^{-(t+iu)^2/2} \sum_{\substack{|K|=k\\ K\in\mathcal{K}_n}} \Xi_K(\tilde{t})\cdot\overline{\Xi^n_{K_n}}(\xi)dt\,.$$

Observing the Dirichlet solution on D_{n+1} for the boundary function Ξ^n_K on S_n is given by $r^{|K|}\Xi^n_K(\xi)$ at $r\xi\in D_{n+1}$, $r\in[0, 1)$, $\xi\in S_n$, we have therefore the asked formula (2.22) with the aid of Lemma (2.4). (Q.E.D.)

We are now in position to state our Construction Theorem of the Dirichlet solutions for the tame boundary functions ψ.

Theorem 2.8 (Construction Theorem). *Let the real function $\tilde{\psi}(u)\in\mathcal{L}^2$ satisfy an additional condition*

$$(2.24)\qquad \int|\tilde{\psi}(u)|du<\infty\,.$$

Then it holds that

$$(2.25)\qquad \lim_{n\uparrow\infty} f_n(\pi_n x) = f(x) \qquad \text{for points}\quad x\in\mathring{D}_\infty\,.$$

Here f_n and f denote the Dirichlet solution on the ball D_{n+1} for the continuous boundary function $\rho_n\psi$ on the sphere S_n and the one on the ball D_∞ for the tame boundary function ψ on the unit sphere S_∞ respectively:

$$(2.26)\qquad f_n(x) = \sum_{k=0}^{\infty} f^n_k(x) \qquad on\quad D_{n+1}\,,\quad n\geqslant 2\,,$$

$$(2.27)\qquad f(x) = \int\tilde{\psi}(u)\exp\left[-(u-\hat{x})^2/(2(1-\|x\|^2_\infty))\right]/(2\pi(1-\|x\|^2_\infty))^{p/2}du\,,$$

for $x=(x_1, \cdots, x_p, \cdots)\in D_\infty$ and $u=(u_1, \cdots, u_p)\in R^p$, (see I, Prop. 3.5).

REMARK. We have $\psi^{(n)}_k=\rho_n\psi_k$, $(k\geqslant 0)$ by Proposition 1.5. Hence Proposition 2.5 shows that the right-hand side of (2.26) converges uniformly and absolutely on the closed ball $\overline{D_{n+1}}$.

The proof of this theorem will break down into several parts.

3. Proof of Construction Theorem

3.1. Uniform asymptotic estimates of the Gegenbauer polynomials.

In this subsection, we shall show several estimates of the Gegenbauer polynomials $C^\nu_k(x/\sqrt{2\nu})$ as $k, \nu\to\infty$. In the course of these estimations the method in Iwano's paper [10] on asymptotic solutions of ordinary differential equations is very instructive to us.

Now, observing that $(x_1, \cdots, x_p)/\|x\|_n$ are bounded for sufficiently large n and a fixed point $x=(x_1, \cdots, x_p, \cdots)\in\mathring{D}_\infty$, we consider the following function $y(z)$ in the complex domain $D_1=\{z\in \boldsymbol{C};\ |z|<R\}$, ($R>0$, fixed, $\sqrt{2\nu}>R$) of the complex plane $\boldsymbol{C}$:

$$(3.1)\qquad y(z)=\left(1-\frac{z^2}{2\nu}\right)^{(2\nu+1)/4}C_k^\nu\left(\frac{z}{\sqrt{2\nu}}\right),$$

where we use the principal branch of the logarithmic function. Then we have the following ordinary differential equation:

$$(3.2)\qquad Y'(z)=\begin{pmatrix}0 & 1\\ q(z) & 0\end{pmatrix}Y(z),$$

where

$$(3.3)\qquad Y(z)=\begin{pmatrix}y(z)\\ y'(z)\end{pmatrix},$$

and

$$(3.4)\qquad q(z)=-\left\{\frac{(k+\nu)^2}{2\nu-z^2}+\frac{\nu+2\nu^2-2\nu^3+(z^2/4)}{(2\nu-z^2)^2}\right\}.$$

First we shall obtain estimates in the

Case 1 ("Tail part": ν/k is bounded).

We introduce two parameters λ, μ defined by:

$$(3.5)\qquad \mu=\nu/k,\quad \lambda=\nu^{-1/2},$$

and two functions $\omega(\mu)$, $r(z,\lambda,\mu)$ as follows:

$$(3.6)\qquad \omega(\mu)=\sqrt{(1/2)+\mu},$$

$$(3.7)\qquad r(z,\lambda,\mu)=\frac{(1+\mu)^2z^2}{2(2-\lambda^2z^2)}+\frac{\mu^2(4\lambda^2+8+\lambda^4z^2+2\lambda^2z^4-8z^2)}{4(2-\lambda^2z^2)^2}.$$

Using these functions, we set

$$(3.8)\qquad P_1=\begin{pmatrix}1 & 0\\ 0 & 1/(\lambda\mu)\end{pmatrix}\cdot\begin{pmatrix}1 & 1\\ i\omega & -i\omega\end{pmatrix},$$

$$(3.9)\qquad A_1=i\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\frac{ir\lambda^2}{2\omega}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}.$$

Then the equaton (3.2) turns out to be

$$(3.10)\qquad \lambda\mu Y_1'(z)=A_1(z)Y_1(z),$$

where

$$(3.11)\qquad Y(z)=P_1(z)Y_1(z).$$

Next we set

$$(3.12)\qquad P_2=\begin{pmatrix}1 & q\\ \tilde{q} & 1\end{pmatrix}$$

with the aid of the solutions $q(z)$, $\tilde{q}(z)$ of the following non-linear differential equations

$$(3.13)\qquad \lambda\mu q'(z)=2i\omega q(z)+\frac{ir\lambda^2}{2\omega}(1+q(z))^2\,,$$

$$(3.14)\qquad \lambda\mu\tilde{q}'(z)=-2i\omega\tilde{q}(z)-\frac{ir\lambda^2}{2\omega}(1+\tilde{q}(z))^2\,.$$

Then the equation (3.10) turns out easily to be

$$(3.15)\qquad \lambda\mu Y_2'(z)=A_2(z)Y_2(z)\,.$$

Here

$$(3.16)\qquad Y_1(z)=P_2(z)Y_2(z)\,,$$

$$(3.17)\qquad A_2(z)=\begin{pmatrix}a_1(z) & 0\\ 0 & a_2(z)\end{pmatrix},$$

$$(3.18)\qquad a_1(z)=i\omega+ir\lambda^2(1+\tilde{q}(z))/(2\omega)\,,$$

$$(3.19)\qquad a_2(z)=-i\omega-ir\lambda^2(1+q(z))/(2\omega)\,.$$

Now we have to show the existence of solutions of the equations (3.13), (3.14) and the invertibility of the matrix $P_2(z)$. First we seek the formal solution $q(z,\lambda,\mu)=\sum_{k=0}^{\infty}q_k(z,\mu)\cdot\lambda^k$ of the equation (3.13). Then we have $q_0=0$, $q_1=0$ and

$$(3.20)\qquad q_2(z,\mu)=-r_0(z,\mu)/(4\omega^2)\,,$$

where

$$(3.21)\qquad r_0(z,\mu)=(1+\mu)^2z^2/4+\mu^2(1-z^2)/2\,,$$

and we set another function $r_1(z,\lambda,\mu)$ so as to hold

$$(3.22)\qquad r(z,\lambda,\mu)=r_0(z,\mu)+r_1(z,\lambda,\mu)\cdot\lambda^2\,.$$

Next, for the sake of simple application of Schauder-Tychonoff's fixed point theorem (cf. Dunford and Schwartz [5], p. 456), we shall seek a solution $q(z,\lambda,\mu)$ of the equation (3.13) holomorphic in the three variables (z,λ,μ) in a suitable complex domain. We denote by Σ the open lozenge in the z-plane with four vertexes $a^{(1)}=-ia$, $a^{(2)}=b$, $a^{(3)}=ia$, $a^{(4)}=-b$, $(a>0,\ b=a\cdot\tan\gamma,\ 0<\gamma<(\pi/2))$ and included in the domain D_1, and set

$$(3.23)\qquad \Lambda=\{(\lambda,\mu)\in \boldsymbol{C}^2;\ 0<|\lambda|<\lambda_0,\quad 0<|\mu|<\mu_0\,,\quad |\arg\lambda|<\alpha_0\,,\quad |\arg\mu|<\beta_0\}\,,$$

($0<\alpha_0$, $\beta_0<(\pi/2)$, λ_0, $\mu_0>0$). Now, in order to solve the equation (3.13) on $\Sigma\times\Lambda$ by the method of constant variation, we put

$$(3.24)\qquad q(z)-q_2(z)\cdot\lambda^2=v(z)=w(z)\exp(D(z)),$$

where

$$(3.25)\qquad D(z)=2i\omega(z-a^{(1)})/(\lambda\mu),\quad (z,\lambda,\mu)\in\Sigma\times\Lambda.$$

Then the equation (3.13) turns out to be

$$(3.26)\qquad \lambda\mu w'(z)=g(z,w(z)\cdot\exp(D(z)))\cdot\exp(-D(z)),$$

where

$$(3.27)\qquad g(z,v)=-\lambda^3h(z)+ir\lambda^2(1+q_2\lambda^2)v(z)/\omega+ir\lambda^2v^2(z)/(2\omega),$$

$$(3.28)\qquad h=-\frac{r_0'\mu}{4\omega^2}+\frac{i\lambda}{\omega}\left\{\frac{r_0^2}{4\omega^2}-\frac{r_0^3\lambda^2}{32\omega^4}-\frac{r_1}{2}+\frac{r_0r_1\lambda^2}{4\omega^2}-\frac{r_0^2r_1\lambda^4}{32\omega^4}\right\}.$$

Now we introduce the following complete locally convex topological vector space X with the uniform convergence topology on compact subsets of $\Sigma\times\Lambda$:

$$(3.29)\qquad X=\{w(z,\lambda,\mu);\ w(z,\lambda,\mu)\ \text{ is holomorphic in }\ \Sigma\times\Lambda\},$$

and denote by $\mathcal{F}$ the following non-empty convex compact subset of X:

$$(3.30)\qquad \mathcal{F}=\{w\in X;\ |w(z,\lambda,\mu)|\leqslant K\cdot e^{-\mathcal{R}e D(z)}\cdot|\lambda|^3\ \text{ on }\ \Sigma\times\Lambda\},\quad (K>0),$$

(see Hörmander [9], p. 26). Next we define a mapping S from $\mathcal{F}$ to X as follows:

$$(3.31)\qquad (Sw)(z_0,\lambda,\mu)=\int_{a^{(1)}}^{z_0}\frac{1}{\lambda\mu}g(z,w(z)\cdot e^{D(z)})e^{-D(z)}dz.$$

Here we can choose the positive constant K so as to hold $S(\mathcal{F})\subset\mathcal{F}$. Indeed, first we notice for some positive constant M,

$$(3.32)\qquad |g(z,w\cdot e^D)|\leqslant M|\lambda|^2(|\lambda|+|w|+|w|^2)\qquad\text{on }\ \Sigma\times\Lambda,$$

and also the following

$$(3.33)\qquad \int_0^s\exp(-\mathcal{R}e\,D(z(t)))dt=\frac{|\lambda\mu|(\exp(-\mathcal{R}e\,D(z(s)))-1)}{2|\omega|\cos(\psi-\theta)},$$

where $z(t)=a^{(1)}+t\cdot e^{i((\pi/2)-\psi)}$, $(-\gamma<\psi<\gamma,\ s>0)$,

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{|\mu|\sin\beta}{(1/2)+|\mu|\cos\beta}\right)-\alpha-\beta,\quad(\alpha=\arg\lambda,\ \beta=\arg\mu).$$

Hence we have the following estimate for $w\in\mathcal{F}$

$$(3.34)\qquad |(Sw)(z)| \leqslant \frac{M(1+K|\lambda|^2+K^2|\lambda|^5)}{2|\omega|\cos(\psi-\theta)} \cdot |\lambda|^3(e^{-\mathcal{R}e\, D(z)}-1),$$

which shows the existence of the asked positive constant K. Moreover we can easily see the continuity of the mapping S. Therefore Schauder-Tychonoff's fixed point theorem shows the existence of a solution $w(z, \lambda, \mu)\in\mathcal{F}$ of the equation (3.26). Thus we obtain the following solution of the equation (3.13):

$$(3.35)\qquad \begin{aligned} &q(z, \lambda, \mu) = q_2(z, \mu)\lambda^2+v(z, \lambda, \mu), \\ &|v(z, \lambda, \mu)| \leqslant K|\lambda|^3 \quad \text{on} \quad \Sigma\times\Lambda. \end{aligned}$$

It is easily seen that the function

$$(3.36)\qquad \tilde{q}(z, \lambda, \mu) = \overline{q(\bar{z}, \bar{\lambda}, \bar{\mu})}$$

on $\Sigma\times\Lambda$ is a solution of the equation (3.14), ($\bar{z}$ denotes the complex conjugate of z). Therefore it holds that

$$(3.37)\qquad Y_2(z) = B(z)Y_2(0),$$

with

$$(3.38)\qquad B(z) = \begin{pmatrix} b_1(z) & 0 \\ 0 & b_2(z) \end{pmatrix},$$

where

$$b_1(z) = \exp\left(\int_0^z \frac{a_1(z)}{\lambda\mu}dz\right), \quad b_2(z) = \exp\left(\int_0^z \frac{a_2(z)}{\lambda\mu}dz\right).$$

Now, we set

$$(3.39)\qquad P_1(z)P_2(z)B(z)(P_1(0)P_2(0))^{-1} = P(z) = \begin{pmatrix} p_{1,1}(z) & p_{1,2}(z) \\ p_{2,1}(z) & p_{2,2}(z) \end{pmatrix}.$$

Then we have

$$(3.40)\qquad y(z) = p_{1,1}(z)y(0)+p_{1,2}(z)y'(0)$$

and

$$(3.41)\qquad y(0) = C_k^\nu(0), \quad y'(0) = (C_k^\nu)'(0)/\sqrt{2\nu}.$$

Noticing the definition (3.1) of the function $y(z)$, we consequently obtain the following

Proposition 3.1. *For given positive numbers R, μ_0, there exist positive numbers ν_0, M_1, K_1 for which it holds that*

$$(3.42)\qquad |C_k^\nu(x/\sqrt{2\nu})| \leqslant \frac{M_1}{\sqrt{k(2+\gamma)}}\left(\frac{(2+\gamma)^{2+\gamma}}{4\gamma^\gamma}\right)^{\nu/2} e^{K_1 k/\nu^3}, \quad (k\text{: even}),$$

$$(3.43) \qquad |C_k^\nu(x/\sqrt{2\nu})| \leqslant \frac{M_1}{\sqrt{k\gamma}}\left(\frac{(2+\gamma)^{2+\gamma}}{4\gamma^\gamma}\right)^{\nu/2} e^{K_1 k/\nu^3}, \qquad (k\text{: odd}),$$

for $(\nu/k)\leqslant\mu_0$, $\nu\geqslant\nu_0$ *and* $-R\leqslant x\leqslant R$. *Here we set*

$$(3.44) \qquad \gamma = k/\nu .$$

Next we shall obtain estimates in the

Case 2 ("Middle part": k/ν is bounded).
In this case we can go ahead in a quite similar way to the case 1. Therefore we list up only a series of formulas different from the ones in the case 1, and give them corresponding numbers.

$$(3.5)' \qquad \mu = (k/\nu)^{1/2}, \quad \lambda = k^{-1/2}.$$

$$(3,6)' \qquad \omega(\mu) = (1+(\mu^2/2))^{1/2}.$$

$$(3.7)' \qquad r(z,\lambda,\mu) = \frac{(2+\mu^2)\mu^2 z^2}{2(2-\lambda^2\mu^2 z^2)} + \frac{4\lambda^2\mu^2+8-4z^2+\lambda^4\mu^4 z^2}{4(2-\lambda^2\mu^2 z^2)^2}.$$

$$(3.8)' \qquad P_1(z) = \begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 1\\ i\omega & -i\omega\end{pmatrix}.$$

$$(3.10)' \qquad \lambda Y_1'(z) = A_1(z)Y_1(z).$$

$$(3.13)' \qquad \lambda q'(z) = 2i\omega q(z) + ir\lambda^2(1+q(z))^2/(2\omega).$$

$$(3.14)' \qquad \lambda\tilde q'(z) = -2i\omega\tilde q(z) - ir\lambda^2(1+\tilde q(z))^2/(2\omega).$$

$$(3.18)' \qquad a_1(z) = i\omega + ir\lambda^2(1+\tilde q(z))/(2\omega).$$

$$(3.19)' \qquad a_2(z) = -i\omega - ir\lambda^2(1+q(z))/(2\omega).$$

$$(3.21)' \qquad r_0(z,\mu) = (2+\mu^2)\mu^2 z^2/4 + (2-z^2)/4.$$

$$(3.25)' \qquad D(z) = 2i\omega(z-a^{(1)})/\lambda.$$

$$(3.26)' \qquad \lambda w'(z) = g(z, w(z)\cdot\exp(D(z)))\cdot\exp(-D(z)).$$

$$(3.28)' \qquad h = -\frac{r_0'}{4\omega^2} - \frac{ir_1\lambda}{2\omega}(1+q_2\lambda^2)^2 + \frac{ir_0^2\lambda}{8\omega^3}(2+q_2\lambda^2).$$

$$(3.31)' \qquad (Sw)(z_0,\lambda,\mu) = \int_{a^{(1)}}^{z_0}\frac{1}{\lambda}g(z, w(z)\cdot e^{D(z)})\cdot e^{-D(z)}dz.$$

$$(3.33)' \qquad \int_0^s \exp(-\mathcal{R}e\, D(z(t)))dt = \frac{|\lambda|(\exp(-\mathcal{R}e\, D(z(s)))-1)}{2|\omega|\cos(\psi-\theta)},$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{|\mu|^2\sin 2\beta}{2+|\mu|^2\cos 2\beta}\right) - \alpha.$$

Then we have the following

Proposition 3.2. *For given positive numbers* R, μ_0, *there exist positive numbers* k_0, M_2, K_2 *for which it holds that*

$$(3.42)' \qquad |C_k^\nu(x/\sqrt{2\nu})| \leqslant \frac{M_2}{\sqrt{k(2+\gamma)}}\left(\frac{(2+\gamma)^{2+\gamma}}{4\gamma^\gamma}\right)^{\nu/2} e^{K_2/k^2}, \quad (k\colon \text{even}),$$

$$(3.43)' \qquad |C_k^\nu(x/\sqrt{2\nu})| \leqslant \frac{M_2}{\sqrt{k}}\left(\frac{(2+\gamma)^{2+\gamma}}{4\gamma^\gamma}\right)^{\nu/2} e^{K_2/k^2}, \qquad (k\colon \text{odd}),$$

for $k \geqslant k_0$, $(k/\nu) \leqslant \mu_0$ *and* $-R \leqslant x \leqslant R$.

3.2. Proof of Construction Theorem.

In this subsection, we assume the condition (2.24) for the boundary function ψ. Now our first assertion is the following

Proposition 3.3 ("Tail part"). *For an arbitrarily fixed positive number* γ_0, *we have*

$$(3.45) \qquad \lim_{n\uparrow\infty} \sum_{k\geqslant n\gamma_0} |f_k^n(\pi_n x)| = 0 \qquad \textit{for a point} \quad x \in \mathring{D}_\infty .$$

Proof. Noticing Proposition 3.1 and the expression (2.22) of $f_k^n(\pi_n x)$, we have only to show the following with the aid of Stirling's formula:

$$(3.46) \qquad \lim_{n\uparrow\infty} \sum_{k\geqslant n\gamma_0} \frac{(1+\gamma)^{3/4}}{\sqrt{\gamma}} k^{(p-2)/2}\left(||x||_\infty \cdot e^{K_1/\nu^3}\right)^k \Phi(\gamma)^{\nu/2} = 0 .$$

Here we set $\nu = (n-1)/2$ and

$$(3.47) \qquad \Phi(\gamma) = \frac{(2+\gamma)^{2+\gamma}}{2(2+2\gamma)^{1+\gamma}} \qquad \text{for} \quad \gamma \in [0, \infty).$$

Since the function $\Phi(\gamma)$ is strictly monotone decreasing:

$$(3.48) \qquad \frac{d}{d\gamma} \log \Phi(\gamma) = \log\left(\frac{2+\gamma}{2+2\gamma}\right) \qquad \text{for} \quad \gamma \in (0, \infty)$$

and $\Phi(0)=1$, the formula (3.46) can be easily seen by observing $||x||_\infty < 1$. (Q.E.D.)

Second we show the following

Proposition 3.4 ("Middle part"). *For an arbitrarily fixed positive number* γ_0, *we have*

$$(3.49) \qquad \lim_{k_0\uparrow\infty} \overline{\lim_{n\uparrow\infty}} \sum_{k=k_0}^{n\gamma_0} |f_k^n(\pi_n x)| = 0 \qquad \textit{for a point} \quad x \in \mathring{D}_\infty .$$

Proof. We can easily prove this proposition in a similar manner to Proposition 3.3 with the aid of Proposition 3.2. (Q.E.D).

Now we notice the following Mehler's formula:

$$(3.50) \qquad \sum_K s^{|K|} H_K(x) H_K(y) = \exp\left[-((x^2+y^2)s^2 - 2sx\cdot y)/2\right]/(1-s^2)^{p/2},$$

where $-1<s<1$, $K=(k_1, \cdots, k_p)$, $(k_i \geqslant 0, i \geqslant 1)$, $x=(x_1, \cdots, x_p)$, $y=(y_1, \cdots, y_p) \in R^p$. Hence it holds that

$$\sum_K \frac{1}{(2\pi)^{p/2}} \int \tilde{\psi}(u) H_K(u) e^{-u^2/2} du \ ||x||_\infty^{|K|} H_K\left(\frac{\hat{x}}{||x||_\infty}\right) \tag{3.51}$$
$$= \int_{R_p} \tilde{\psi}(u) \left[\exp\left(-\frac{(u-\hat{x})^2}{2(1-||x||_\infty^2)}\right)\right] \Big/ (2\pi(1-||x||_\infty^2))^{p/2} du$$

for $\tilde{\psi} \in \mathcal{L}^2$ and $x \in \mathring{D}_\infty$, and the series of the left-hand side converges absolutely. On the other hand, we have the following uniform convergence formula

$$\lim_{\nu \uparrow \infty} \frac{1}{(2\nu)^{k/2}} C_k^\nu\left(\frac{x}{\sqrt{2\nu}}\right) = \frac{1}{k!} H_k(x) \tag{3.52}$$

with respect to bounded real numbers x, (k: fixed). Consequently for a fixed number k and a point $x \in \mathring{D}_\infty$, we have

$$\lim_{n \uparrow \infty} f_k^n(\pi_n x) = \sum_{|K|=k} \frac{1}{(2\pi)^{p/2}} \int \tilde{\psi}(u) H_K(u) e^{-u^2/2} du \ ||x||_\infty^k H_K\left(\frac{\hat{x}}{||x||_\infty}\right) \tag{3.53}$$

with the aid of Mehler's formula. Therefore we have

$$\lim_{n \uparrow \infty} f_n(\pi_n x) = \lim_{n \uparrow \infty} \sum_{k=0}^{\infty} f_k^n(\pi_n x) \tag{3.54}$$
$$= \int_{R_p} \tilde{\psi}(u) \frac{\exp\left[-(u-\hat{x})^2/(2(1-||x||_\infty^2))\right]}{(2\pi(1-||x||_\infty^2))^{p/2}} du$$

for $x=(x_1, \cdots, x_p, \cdots) \in \mathring{D}_\infty$.

Thus Construction Theorem has been proved by the asymptotic calculus. Concluding this paper, we notice the similarity between the limiting procedure in Construction Theorem and the one in the statistical mechanics, e.g., evaluation of specific free energy, (cf. Berlin and Kac [2], Dyson [6]).

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Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa-ku, Nagoya 466,
Japan