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SIMPLE BIRATIONAL EXTENSIONS OF A POLYNOMIAL RING $k[x, y]$

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Introduction. Let k be an algebraically closed field of characteristic zero and let $k[x, y]$ be a polynomial ring over k in two variables x and y . Let f and g be two elements of $k[x, y]$ without common nonconstant factors, and let $A = k[x, y, f/g]$. In the present article we consider the structures of the affine k -domain A under an assumption that $V := \text{Spec}(A)$ has only isolated singularities.

In the first section we describe how V is obtained from $A^2 := \text{Spec}(k[x, y])$ and we see that if V has only isolated singularities V is a normal surface whose singular points (if any) are rational double points. The divisor class group $Cl(V)$ can be explicitly determined (cf. Theorem 1.9); we obtain, therefore, necessary and sufficient conditions for A to be a unique factorization domain. If g is irreducible and if the curves $f=0$ and $g=0$ on A^2 meet each other then A is a unique factorization domain if and only if the curves $f=0$ and $g=0$ meet in only one point where both curves intersect transversally. We consider, in the same section, a problem: When is every invertible element of A constant?

In the second section we prove the following:

Theorem. *Assume that V has only isolated singularities. Then A has a nonzero locally nilpotent k -derivation if and only if we have $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$.*

An affine k -domain of type A as above was studied by Russell [8] and Sathaye [9] in connection with the following result:

Assume that A is isomorphic to a polynomial ring over k in two variables. In a polynomial ring $k[x, y, z]$ over k in three variables x, y and z , let $u = gz - f$. Then there exist two elements v, w of $k[x, y, z]$ such that $k[x, y, z] = k[u, v, w]$.

Our terminology and notation are as follows:

k : an algebraically closed field of characteristic zero which we fix throughout the paper.

A^* : the group of all invertible elements of a ring A .

$Cl(V)$: the divisor class group of a normal surface V .

$\varphi'(C)$: the proper transform of a curve C on a normal surface Y by a birational morphism $\varphi: X \rightarrow Y$ from a normal surface X to Y .

$(t)_X$: the divisor of a function t on a normal surface X .

$p_a(D)$: the arithmetic genus of a divisor D on a nonsingular projective surface.

$(C^2), (C \cdot C')$: the intersection multiplicity.

A^n : the n -dimensional affine space.

P^n : the n -dimensional projective space.

1. The structures of the affine domain $k[x, y, f/g]$

1.1. Let $k[x, y, z]$ be a polynomial ring over k in three variables x, y and z , and let $A^3 := \text{Spec}(k[x, y, z])$. Let V be an affine hypersurface on A^3 defined by $gz - f = 0$, and let $\pi: V \rightarrow A^2 := \text{Spec}(k[x, y])$ be the projection $\pi: (x, y, z) = (x, y)$. Let F and G be respectively the curves $f=0$ and $g=0$ on A^2 . Then we have:

Lemma. (1) For each point $P \in F \cap G$, $\pi^{-1}(P)$ is isomorphic to the affine line A^1 .

(2) If Q is a point on G but not on F , then $\pi^{-1}(Q) = \emptyset$.

Proof. Straightforward.

1.2. The Jacobian criterion of singularity applied to the hypersurface V shows us the following:

Lemma. Let P be a point on F and G . Then the following assertions hold:

(1) If P is a singular point for both F and G then every point of $\pi^{-1}(P)$ is a singular point of V .

(2) If P is a singular point of F but not a singular point of G then the point $(P, z=0)$ is the unique singular point of V lying on $\pi^{-1}(P)$.

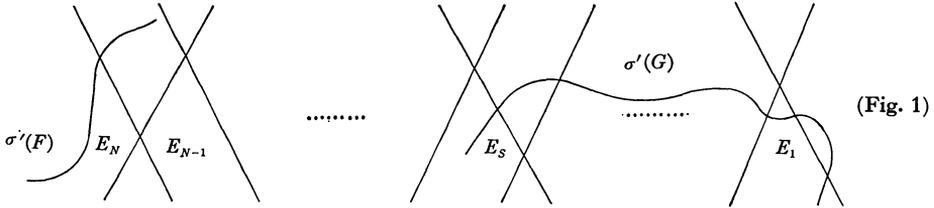
(3) If P is a singular point of G but not a singular point of F then V is nonsingular at every point of $\pi^{-1}(P)$.

(4) If P is a nonsingular point of both F and G and if $i(F, G; P) \geq 2$ then the point $(P, z=\alpha)$ is the unique singular point of V lying on $\pi^{-1}(P)$, where $\alpha \in k$ satisfies: $\frac{\partial f}{\partial x}(P) = \frac{\partial g}{\partial x}(P)\alpha$ and $\frac{\partial f}{\partial y}(P) = \frac{\partial g}{\partial y}(P)\alpha$. If $i(F, G; P) = 1$ then V is nonsingular at every point of $\pi^{-1}(P)$.

We assume, from now on, that V has only isolated singularities. Hence, if $P \in F \cap G$, either F or G is nonsingular at P . Furthermore, we assume that $F \cap G \neq \emptyset$. When $F \cap G = \emptyset$ then $A = k[x, y, 1/g]$ and A is a unique factorization domain.

1.3. Let P be a point on F and G . We first consider the case where F is nonsingular at P but G is not. Let $P_1 := P$ and let ν_1 be the multiplicity of G at P_1 . Let $\sigma_1: V_1 \rightarrow V_0 := A^2$ be the quadratic transformation with center at P_1 , let

$P_2 := \sigma'_1(F) \cap \sigma_1^{-1}(P_1)$ and let ν_2 be the multiplicity of $\sigma'_1(G)$ at P_2 . For $i \geq 1$ we define a surface V_i , a point P_{i+1} on V_i and an integer ν_{i+1} inductively as follows: When V_{i-1} , P_i and ν_i are defined, let $\sigma_i: V_i \rightarrow V_{i-1}$ be the quadratic transformation of V_{i-1} with center at P_i , let $P_{i+1} := (\sigma_1 \cdots \sigma_i)'(F) \cap \sigma_i^{-1}(P_i)$ and let ν_{i+1} be the multiplicity of $(\sigma_1 \cdots \sigma_i)'(G)$ at P_{i+1} . Let s be the smallest integer such that $\nu_{s+1} = 0$, and let $N := \nu_1 + \cdots + \nu_s$. We may simply say that P_1, \dots, P_s are all points of G on the curve F over P_1 and ν_1, \dots, ν_s are the multiplicities of G at P_1, \dots, P_s , respectively. Let $\sigma: V_N \rightarrow V_0$ be the composition of quadratic transformations $\sigma := \sigma_1 \cdots \sigma_N$ and let $E_i := (\sigma_{i+1} \cdots \sigma_N)' \sigma_i^{-1}(P_i)$ for $1 \leq i \leq N$. In a neighborhood of $\sigma^{-1}(P_1)$, $\sigma^{-1}(F \cup G)$ has the following configuration:



If $g = c g_1^{\beta_1} \cdots g_n^{\beta_n}$ ($c \in k^*$) is a decomposition of g into n distinct irreducible factors, let G_j be the curve $g_j = 0$ on $V_0 = A^2$ for $1 \leq j \leq n$. Let $\nu_i(j)$ be the multiplicity of G_j at the point P_i for $1 \leq i \leq s$ and $1 \leq j \leq n$. Then it is clear that $\nu_i = \beta_1 \nu_i(1) + \cdots + \beta_n \nu_i(n)$ for $1 \leq i \leq s$.

1.4. We prove the following:

Lemma. *With the same assumption and notations as in 1.3, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to V_N with the curves E_1, \dots, E_{N-1} and $\sigma'(G)$ deleted off.*

Proof. Let $\mathcal{O} := \mathcal{O}_{V_0, P_1}$, $\tilde{V}_0 := \text{Spec}(\mathcal{O})$ and $\tilde{V} = V \times_{V_0} \tilde{V}_0$. Since the curve F is nonsingular at P_1 there exist local parameters u, v of V_0 at P_1 such that $v = f$. Let $g(u, v) = 0$ be a local equation of G at P_1 . Then $\tilde{V} = \text{Spec}(\mathcal{O}[v/g(u, v)])$. Note that V is nonsingular in a neighborhood of $\pi^{-1}(P_1)$ (cf. 1.2). Hence there exist a nonsingular projective surface \tilde{V} and a birational mapping $\varphi: V \rightarrow \tilde{V}$ such that φ is an open immersion in a neighborhood of $\pi^{-1}(P_1)$ and a birational mapping $\bar{\pi} = \pi \circ \varphi^{-1}: \tilde{V} \rightarrow P^2$ is a morphism, where V_0 is embedded canonically into the projective plane P^2 as an open set. Since $\pi(\pi^{-1}(P_1)) = P_1$ we know that $\bar{\pi}$ is factored by the quadratic transformation of P^2 at P_1 . Hence we know that $\pi: V \rightarrow V_0$ is factored by $\sigma_1: V_1 \rightarrow V_0$, i.e., $\pi: V \xrightarrow{\pi_1} V_1 \xrightarrow{\sigma_1} V_0$.

Set $v = uv_1, u = vu_1, g(u, uv_1) = u^{\nu_1} g_1(u, v_1)$ and $g(vu_1, v) = v^{\nu_1} g'_1(u_1, v)$. Then $V_1 \times_{V_0} \tilde{V}_0 = \text{Spec}(\mathcal{O}[v_1]) \cup \text{Spec}(\mathcal{O}[u_1])$; $\sigma_1^{-1}(P_1)$ and $\sigma'_1(G)$ are respectively defined

by $u=0$ and $g_1(u, v_1)=0$ on $\text{Spec}(\mathcal{O}[v_1])$, and by $v=0$ and $g'_1(u_1, v)=0$ on $\text{Spec}(\mathcal{O}[u_1])$. Since $\tilde{V}:=V \times_{\tilde{V}_0} \tilde{V}_0 = V \times_{\tilde{V}_1} (V_1 \times_{\tilde{V}_0} \tilde{V}_0) = V \times_{\tilde{V}_1} \text{Spec}(\mathcal{O}[v_1]) \cup V \times_{\tilde{V}_1} \text{Spec}(\mathcal{O}[u_1]) = \text{Spec}(\mathcal{O}[v_1, v_1/u^{v_1-1}g_1(u, v_1)]) \cup \text{Spec}(\mathcal{O}[u_1, 1/v^{v_1-1}g'_1(u_1, v)])$ and since v is an invertible function on $\text{Spec}(\mathcal{O}[u_1, 1/v^{v_1-1}g'_1(u_1, v)])$, we know that:

- (i) $\tilde{V} = \text{Spec}(\mathcal{O}[v_1, v_1/u^{v_1-1}g_1(u, v_1)])$,
- (ii) $\tilde{\pi} := \pi \times_{\tilde{V}_0} \tilde{V}_0: \tilde{V} \rightarrow \tilde{V}_0$ is a composition of $\tilde{\pi}_1 := \pi_1 \times_{\tilde{V}_0} \tilde{V}_0: \tilde{V} \rightarrow \tilde{V}_1 :=$

$\text{Spec}(\mathcal{O}[v_1])$ and $\tilde{\sigma}_1 := \sigma_1|_{\tilde{V}_1}: \tilde{V}_1 \rightarrow \tilde{V}_0$,

- (iii) if $Q \in (\sigma_1^{-1}(P_1) \cup \sigma'_1(G)) - \sigma'_1(F)$ then $\tilde{\pi}_1^{-1}(Q) = \emptyset$.

Set $v_1 = uv_2, \dots, v_{s-1} = uv_s$ and $g_1(u, v_1) = u^{v_2}g_2(u, v_2), \dots, g_{s-1}(u, v_{s-1}) = u^{v_s}g_s(u, v_s)$. Set $\tilde{V}_2 = \text{Spec}(\mathcal{O}[v_2]), \dots, \tilde{V}_s = \text{Spec}(\mathcal{O}[v_s])$. Then, by the same argument as above, we know that the following assertions hold for $2 \leq i \leq s$:

- (i) $\tilde{V} = \text{Spec}(\mathcal{O}[v_i, v_i/u^{v_1+\dots+v_i-i}g_i(u, v_i)])$;
- (ii) $\tilde{\pi}: \tilde{V} \rightarrow \tilde{V}_0$ is a composition of a morphism $\tilde{\pi}_i: \tilde{V} \rightarrow \tilde{V}_i$ and $\tilde{\sigma}_1\tilde{\sigma}_2 \dots \tilde{\sigma}_i: \tilde{V}_i \rightarrow \tilde{V}_0$, where $\tilde{\sigma}_i := \sigma_i|_{\tilde{V}_i}: \tilde{V}_i \rightarrow \tilde{V}_{i-1}$; moreover, $\tilde{\pi}_{i-1} = \tilde{\sigma}_i \cdot \tilde{\pi}_i$;

- (iii) if $Q \in (\sigma_i^{-1}(P_i) \cup (\sigma_1 \dots \sigma_i)'(G)) - (\sigma_1 \dots \sigma_i)'(F)$ then $\tilde{\pi}_i^{-1}(Q) = \emptyset$.

When $i=s$, the proper transform $(\sigma_1 \dots \sigma_s)'(G)$ of G on V_s does not meet the proper transform $(\sigma_1 \dots \sigma_s)'(F)$ of F on \tilde{V}_s (cf. the definition of s in (1.3)). Therefore, in virtue of (iii) above, we know that $g_s(u, v_s)$ is an invertible function on \tilde{V} , where $g_s(u, v_s) = 0$ is the equation of the proper transform $(\sigma_1 \dots \sigma_s)'(G)$ of G on \tilde{V}_s . Thus, $\tilde{V} = \text{Spec}(\mathcal{O}[v_s, v_s/u^{N-s}])$.

Furthermore, set $v_s = uv_{s+1}, \dots, v_{N-1} = uv_N$ and $\tilde{V}_{s+1} = \text{Spec}(\mathcal{O}[v_{s+1}]), \dots, \tilde{V}_N = \text{Spec}(\mathcal{O}[v_N])$. Then it is easy to see that the following assertions hold for $s+1 \leq i \leq N$:

- (i) $\tilde{V} = \text{Spec}(\mathcal{O}[v_i, v_i/u^{N-i}])$,
- (ii) $\tilde{\pi}_s: \tilde{V} \rightarrow \tilde{V}_s$ is a composition of a morphism $\tilde{\pi}_i: \tilde{V} \rightarrow \tilde{V}_i$ and $\tilde{\sigma}_{s+1} \dots \tilde{\sigma}_i: \tilde{V}_i \rightarrow \tilde{V}_s$, where $\tilde{\sigma}_i = \sigma_i|_{\tilde{V}_i}: \tilde{V}_i \rightarrow \tilde{V}_{i-1}$ and $\tilde{\pi}_{i-1} = \tilde{\sigma}_i \cdot \tilde{\pi}_i$.

Then $\tilde{V} = \tilde{V}_N = \text{Spec}(\mathcal{O}[v_N])$. Hence, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to V_N with the curves E_1, \dots, E_{N-1} and $\sigma'(G)$ deleted off. In particular, $\pi^{-1}(P_1) = \mathcal{E} := E_N - E_N \cap E_{N-1}$. Q.E.D.

1.5. Assume that we are given two curves (not necessarily irreducible) F, G on a nonsingular surface V_0 and a point $P_1 \in F \cap G$ at which one of F and G , say F , is nonsingular. Let P_1, P_2, \dots, P_s be all points of G on F over P_1 , and let ν_1, \dots, ν_s be the multiplicities of G at P_1, \dots, P_s , respectively. Let $N = \nu_1 + \dots + \nu_s$. As explained in 1.3, define $\sigma: V_N \rightarrow V_0$ as a composition of quadratic transformations with centers at N points P_1, \dots, P_N on F , each P_i ($2 \leq i \leq N$) being infinitely near to P_{i-1} . We call $\sigma: V_N \rightarrow V_0$ the *standard transformation of V_0 with respect to a triplet (P_1, F, G)* . The configuration of $\sigma^{-1}(F \cup G)$ in a neighborhood of $\sigma^{-1}(P_1)$ is given by the Figure 1. With the notations in the Figure 1, we have a new surface V by deleting E_1, \dots, E_{N-1} from V_N . We then say that V is obtained from V_0 by the *standard process of the first kind with respect to (P_1, F, G)* . On

the other hand, note that $(E_i^2) = -2$ for $1 \leq i \leq N-1$. Hence we obtain a new normal surface V' from V_N by contracting E_1, \dots, E_{N-1} to a point Q_1 on V' which is a rational double point (cf. Artin [2; Theorem 2.7]). We then say that V' is obtained from V_0 by the standard process of the second kind with respect to (P_1, F, G) .

1.6. We next consider the case where, at a point $P_1 \in F \cap G$, the curve G is nonsingular. Indeed, we prove the following:

Lemma. *With the assumption as above, let V' be the surface obtained from $V_0 := A^2$ by the standard process of the second kind with respect to (P_1, G, F) . Then, in a neighborhood of $\pi^{-1}(P_1)$, V is isomorphic to V' with the proper transform of G deleted off. If either F is singular at P_1 or $i(F, G; P_1) \geq 2$, V has a unique rational double point on $\pi^{-1}(P_1)$.*

Proof. Let P_1, P_2, \dots, P_r be all points of F on G over P_1 , and let μ_1, \dots, μ_r be the multiplicities of F at P_1, \dots, P_r , respectively. Let $M := \mu_1 + \dots + \mu_r$. We prove the assertions by induction on M . Note that $M=1$ if and only if $i(F, G; P_1)=1$. It is then easy to see that V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to a surface V'_1 obtained as follows: Let $\sigma_1: V_1 \rightarrow V_0$ be the quadratic transformation of $V_0 := A^2$ with center at P_1 , and let $V'_1 := V_1 - \sigma_1^{-1}(G)$. Now, assume that $M > 1$. Since G is nonsingular at P_1 there exist local parameters u, v of V_0 at P_1 such that $v=g$. Let $f(u, v)=0$ be a local equation of F at P_1 . Then, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to an affine hypersurface $uz=f(u, v)$ in the affine 3-space A^3 . There exists only one singular point $Q'_1: (u, v, z)=(0, 0, 0)$ of V lying on $\pi^{-1}(P_1)$. Let $\rho_1: W_1 \rightarrow A^3$ be the blowing-up of A^3 with center the curve $\pi^{-1}(P_1): u=v=0$, let V'_1 be the proper transform of V on W_1 , and let $\tau_1 := \rho_1|_{V'_1}: V'_1 \rightarrow V$ be the restriction of ρ_1 onto V'_1 .

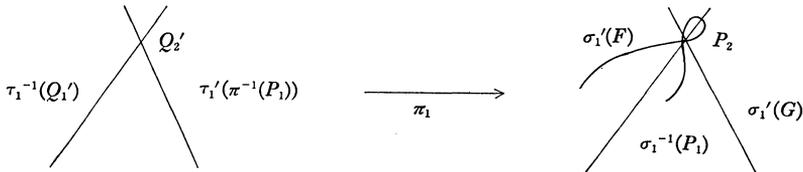
Set $v=uv_1, u=uv_1$ and $f(u, uv_1)=u^{\mu_1}f_1(u, v_1), f(vu_1, v)=v^{\mu_1}f_1(u_1, v)$. Then V'_1 is given by $v_1z=u^{\mu_1-1}f_1(u, v_1)$ with respect to the coordinate system (u, v_1, z) and by $z=v^{\mu_1-1}f_1(u_1, v)$ with respect to the coordinate system (u_1, v, z) . By construction of V'_1, V'_1 dominates the surface V_1 obtained from V_0 by the quadratic transformation σ_1 with center at P_1 ;

$$\begin{array}{ccc} V'_1 & \xrightarrow{\tau_1} & V \\ \downarrow \pi_1 & & \downarrow \pi \\ V_1 & \xrightarrow{\sigma_1} & V_0 \end{array} .$$

The proper transform $\tau_1^{-1}(\pi^{-1}(P_1))$ of $\pi^{-1}(P_1)$ on V'_1 is given by $u=v_1=0$; the curve $\tau_1^{-1}(Q'_1)$ is given by $u=z=0$; $\tau_1: V'_1 - \tau_1^{-1}(Q'_1) \xrightarrow{\sim} V - \{Q'_1\}$; the singular point of V'_1 is possibly $Q'_2: (u, v_1, z)=(0, 0, 0)$.

The morphism $\pi_1: V'_1 \rightarrow V_1$ is isomorphic at every point of $\tau_1^{-1}(Q'_1) - \{Q'_2\}$.

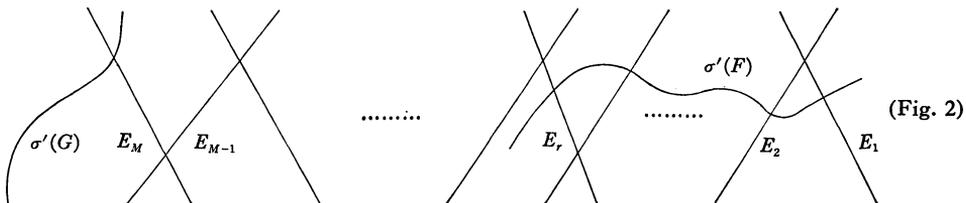
Indeed, if $v_1 \neq 0$ or ∞ , π_1 is given by $(u, v_1, z) = (u, v_1, u^{\mu_1-1}f_1(u, v_1)/v_1) \mapsto (u, v_1)$ which is clearly isomorphic; if $v_1 = \infty$, π_1 is given by $(u_1, v, v^{\mu_1-1}\tilde{f}_1(u_1, v)) \mapsto (u_1, v)$ which is isomorphic as well. Under this isomorphism, $\tau_1^{-1}(Q'_1)$ corresponds to $\sigma_1^{-1}(P_1)$:



Note that the following assertions hold:

- (i) V'_1 is isomorphic, in a neighborhood of $\pi_1^{-1}(P_2)$, to an affine hyper-surface $v_1z = u^{\mu_1-1}f_1(u, v_1)$ on A^3 ;
- (ii) in a neighborhood of P_2 , $\sigma'_1(G)$ is defined by $v_1 = 0$ and $\sigma'_1(F)$ is defined by $f_1(u, v_1) = 0$;
- (iii) P_2, \dots, P_r are all points of the curve $F_1: u^{\mu_1-1}f_1(u, v_1) = 0$ on $\sigma'_1(G)$ over P_2 , and the sum of multiplicities of the curve F_1 at P_2, \dots, P_r is $M-1$.

Then, by the assumption of induction applied to V'_1 , we obtain V''_1 from the surface V'_1 , which is obtained from V_1 by the standard process of the second kind with respect to a triplet $(P_2, \sigma'_1(G), F_1)$, by deleting the proper transform of $\sigma'_1(G)$ on V''_1 :



where the surface V''_1 is obtained by contracting E_2, \dots, E_{M-1} . Then it is easy to see that V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to the surface V' , which is obtained from V''_1 by contracting E_1, \dots, E_{M-1} , with the proper transform of $\sigma'(G)$ deleted off. Hence, the unique singular point of V lying on $\pi^{-1}(P_1)$ is a rational double point. Q.E.D.

1.7. Let $P_1 \in F \cap G$, and assume that G is nonsingular at P_1 . Let P_1, P_2, \dots, P_r be all points of F on G over P_1 , and let μ_1, \dots, μ_r be the multiplicities of F at P_1, \dots, P_r , respectively. If $f = cf_1^{\alpha_1} \dots f_m^{\alpha_m}$ ($c \in k^*$) is a decomposition of f into distinct irreducible factors, let F_j ($1 \leq j \leq m$) be the curve on V_0 defined by $f_j = 0$. Let $\mu_i(j)$ be the multiplicity of F_j at P_i for $1 \leq i \leq r$ and $1 \leq j \leq m$. Then it is clear that $\mu_i = \alpha_1 \mu_i(1) + \dots + \alpha_m \mu_i(m)$ for $1 \leq i \leq r$.

1.8. As a consequence of Lemmas 1.4 and 1.6, we have the following:

Theorem. *Assume that V has only isolated singularities. Let W be the surface obtained from $V_0 := A^2$ by the standard processes of the first (or the second) kind at every point of $F \cap G$. Then V is isomorphic to the surface W with the proper transform of G on W deleted off. The surface V is, therefore, a normal surface whose singular points (if any) are rational double points.*

1.9. In the paragraphs 1.9~1.11 we shall study the divisor class group $Cl(V)$. Let $g = cg_1^{e_1} \cdots g_n^{e_n}$ ($c \in k^*$) be a decomposition of g into distinct irreducible factors, and let G_j be the curve $g_j = 0$ on V_0 for $1 \leq j \leq n$. Assume that $F \cap G \neq \emptyset$. Let $F \cap G = \{P_1^{(l)}, \dots, P_l^{(e)}\}$. For $1 \leq l \leq e$, either F is nonsingular at $P_1^{(l)}$ but G is not, or G is nonsingular at $P_1^{(l)}$. We may assume that F is nonsingular at $P_1^{(1)}, \dots, P_1^{(a)}$ but G is not, and G is nonsingular at $P_1^{(a+1)}, \dots, P_1^{(e)}$. (The number a may be 0.) For $l \leq a$, let $P_1^{(l)}, \dots, P_{s_l}^{(l)}$ be all points of G on F over $P_1^{(l)}$, and let $\nu_i^{(l)}(j)$ be the multiplicity of G_j at $P_i^{(l)}$ for $1 \leq i \leq s_l$ and $1 \leq j \leq n$; let $N^{(l)}(j) = \nu_1^{(l)}(j) + \dots + \nu_{s_l}^{(l)}(j)$, let $\nu_i^{(l)} = \beta_1 \nu_i^{(l)}(1) + \dots + \beta_n \nu_i^{(l)}(n)$ and let $N^{(l)} = \beta_1 N^{(l)}(1) + \dots + \beta_n N^{(l)}(n)$. For $a+1 \leq l \leq e$, let $P_1^{(l)}, \dots, P_{r_l}^{(l)}$ be all points of F on G over $P_1^{(l)}$, and let $\mu_i^{(l)}$ be the multiplicity of F at $P_i^{(l)}$ for $1 \leq i \leq r_l$. Let $M^{(l)} = \mu_1^{(l)} + \dots + \mu_{r_l}^{(l)}$. Since G is nonsingular at $P_1^{(l)}$, there exists a unique G_j ($1 \leq j \leq n$) such that $P_1^{(l)}, \dots, P_{r_l}^{(l)}$ lie on G_j . Then we set $M^{(l)}(j) = M^{(l)}$ and $M^{(l)}(j') = 0$ for $j' \neq j$. Let $\mathcal{E}^{(l)} = \pi^{-1}(P_1^{(l)})$ for $1 \leq l \leq e$.

1.10. The structure of the divisor class group $Cl(V)$ is given by the following:

Theorem. *With the notations as above, the divisor class group $Cl(V)$ is isomorphic to:*

$$\{Z\mathcal{E}^{(1)} + \dots + Z\mathcal{E}^{(e)}\} / \left\{ \sum_{l=1}^a N^{(l)}(j)\mathcal{E}^{(l)} + \sum_{l=a+1}^e M^{(l)}(j)\mathcal{E}^{(l)}; 1 \leq j \leq n \right\}.$$

Proof. Embed $V_0 := A^2$ into the projective plane P^2 in a canonical way as an open set, and let $l_\infty := P^2 - V_0$. For $1 \leq l \leq e$, let $E_1^{(l)}, \dots, E_q^{(l)}$ be all exceptional curves which arise by the standard transformation of V_0 with respect to a triplet $(P_1^{(l)}, F, G)$ (or $(P_1^{(l)}, G, F)$) where $q = N^{(l)}$ (or $M^{(l)}$). Let $\tau: W \rightarrow P^2$ be a composition of standard transformations of P^2 with respect to triplets $(P_1^{(l)}, F, G)$ for $1 \leq l \leq a$ and triplets $(P_1^{(l)}, G, F)$ for $a+1 \leq l \leq e$. Then it is easy to see that the divisor

$$(g_j)_W - \left\{ \sum_{l=1}^a N^{(l)}(j)E_{x^{(l)}}^{(l)} + \sum_{l=a+1}^e M^{(l)}(j)E_{y^{(l)}}^{(l)} \right\} \quad (1 \leq j \leq n)$$

has support on $\tau^{-1}(G_j)$, $\tau^{-1}(l_\infty)$, $E_1^{(l)}, \dots, E_{q-1}^{(l)}$ ($q = N^{(l)}$ or $M^{(l)}$) for $1 \leq l \leq e$. Hence we have:

$$\sum_{l=1}^a N^{(l)}(j)\mathcal{E}^{(l)} + \sum_{l=a+1}^e M^{(l)}(j)\mathcal{E}^{(l)} \sim 0 \quad (1 \leq j \leq n).$$

Now, let C be an irreducible curve on V such that $\pi(C)$ is not a point, and

let the closure of $\pi(C)$ be defined by $h=0$ with $h \in k[x, y]$. Then, by considering the divisor $(h)_W$ on W , we easily see that C is linearly equivalent to an integral combination of $\varepsilon^{(1)}, \dots, \varepsilon^{(e)}$. Hence, by setting

$$\mathcal{Q} := \{ \mathbf{Z}\varepsilon^{(1)} + \dots + \mathbf{Z}\varepsilon^{(e)} \} / \{ \sum_{j=1}^e N^{(j)}(j)\varepsilon^{(j)} + \sum_{l=a+1}^e M^{(l)}(j)\varepsilon^{(l)}; 1 \leq j \leq n \},$$

we have a surjective homomorphism:

$$\theta: \mathcal{Q} \rightarrow Cl(V); \theta(\varepsilon^{(l)}) = \varepsilon^{(l)} \quad (1 \leq l \leq e).$$

Assume that $\text{Ker } \theta \neq (0)$, and let $d_1\varepsilon^{(1)} + \dots + d_e\varepsilon^{(e)} = (t)_V$ on V , where $d_i \in \mathbf{Z}$ ($1 \leq l \leq e$) and $t \in k(V)$. Let $(t)_{V_0} = \sum m_i C_i$ with irreducible curves C_i and $m_i \in \mathbf{Z}$. Let $t_i \in k[x, y]$ be such that C_i is given by $t_i = 0$, and write:

$$(t_i)_V = \pi'(C_i) + \sum_{l=1}^e b_{il}\varepsilon^{(l)} \quad \text{with } b_{il} \in \mathbf{Z}.$$

Then we have:

$$(t)_V = \sum_i \{ m_i \pi'(C_i) + \sum_{l=1}^e m_i b_{il} \varepsilon^{(l)} \} = \sum_{l=1}^e d_l \varepsilon^{(l)}.$$

Therefore, either $m_i = 0$ for every i , or $\pi'(C_i) = \phi$ for every i . In the first case, t is a constant $\in k$, whence $d_l = 0$ for $1 \leq l \leq e$. In the second case, C_i must coincide with one of G_j 's ($1 \leq j \leq n$). Then $d_1\varepsilon^{(1)} + \dots + d_e\varepsilon^{(e)} = 0$ in \mathcal{Q} . This is a contradiction. Therefore, θ is an isomorphism. Q.E.D.

1.11. The affine domain $A = k[x, y, f/g]$ is a unique factorization domain if and only if $Cl(V) = (0)$. We have the following two consequences of 1.10.

1.11.1. **Corollary.** *With the notation of 1.9, if $e > n$ then A is not a unique factorization domain.*

1.11.2. **Corollary.** *Assume that g is irreducible and that $F \cap G \neq \phi$. Then A is a unique factorization domain if and only if the curves F and G meet each other in only one point where they intersect each other transversally.*

1.12. Let A^* be the group of all invertible elements of $A = k[x, y, f/g]$. Then A^* contains $k^* = k - (0)$ as a subgroup. By virtue of ([4], Remark 2, p. 174) we know that A^*/k^* is a torsion-free \mathbf{Z} -module of finite rank and A^* is isomorphic to a direct product of k^* and A^*/k^* . The purpose of this paragraph is to determine the group A^*/k^* . Let H be the subgroup of $\mathbf{Z}\varepsilon^{(1)} + \dots + \mathbf{Z}\varepsilon^{(e)}$ generated by

$$\{ \sum_{j=1}^e N^{(j)}(j)\varepsilon^{(j)} + \sum_{l=a+1}^e M^{(l)}(j)\varepsilon^{(l)}; 1 \leq j \leq n \}.$$

Let T_1, \dots, T_n be n -indeterminates, and let $\eta: \mathbf{Z}^{(n)} := \mathbf{Z}T_1 + \dots + \mathbf{Z}T_n \rightarrow H$ be a homomorphism such that, for $1 \leq j \leq n$,

$$\eta(T_j) = \sum_{l=1}^e N^{(l)}(j)\epsilon^{(l)} + \sum_{l=a+1}^e M^{(l)}(j)\epsilon^{(l)}.$$

Let L be the kernel of η . Since $N^{(l)}(j)$ and $M^{(l)}(j)$ are non-negative integers for $1 \leq l \leq e$ and $1 \leq j \leq n$, each nonzero element of L is written in the form: $\gamma_1 T_1 + \dots + \gamma_n T_n$ ($\gamma_i \in \mathbf{Z}$), where some of γ_i 's are negative. Define a homomorphism $\xi: L \rightarrow K^*$ (where $K = k(x, y)$) by $\xi(\gamma_1 T_1 + \dots + \gamma_n T_n) = g_1^{\gamma_1} \dots g_n^{\gamma_n}$. Then we have the following:

Lemma. *The homomorphism ξ induces an isomorphism $\xi: L \xrightarrow{\sim} A^*/k^*$.*

Proof. (1) Since $(g_j)_v = \sum_{l=1}^a N^{(l)}(j)\epsilon^{(l)} + \sum_{l=a+1}^e M^{(l)}(j)\epsilon^{(l)} = \eta(T_j)$ for $1 \leq j \leq n$, we have:

$$\eta(\gamma_1 T_1 + \dots + \gamma_n T_n) = (g_1^{\gamma_1} \dots g_n^{\gamma_n})_v.$$

Therefore, if $\gamma_1 T_1 + \dots + \gamma_n T_n \in L$ then $g_1^{\gamma_1} \dots g_n^{\gamma_n}$ is an invertible element of A , which is a constant if and only if $\gamma_1 = \dots = \gamma_n = 0$. Thus, ξ is a monomorphism from L into A^*/k^* .

(2) Let t be a non-constant invertible element of A . Write $(t)_v = \sum_i m_i C_i$ with irreducible curves C_i and $m_i \in \mathbf{Z}$. Let C_i be defined by $t_i = 0$ with $t_i \in k[x, y]$. As in the proof of 1.10, write:

$$(t)_v = \pi'(C_i) + \sum_{l=1}^e b_{il} \epsilon^{(l)} \quad \text{with } b_{il} \in \mathbf{Z}.$$

Then we have:

$$(t)_v = \sum_i \{m_i \pi'(C_i) + \sum_{l=1}^e m_i b_{il} \epsilon^{(l)}\} = 0.$$

Therefore, either $m_i = 0$ for every i , or $\pi'(C_i) = \phi$ for every i . The first case does not occur because, if otherwise, t is a constant. In the second case, C_i must coincide with one of G_j 's. Hence we could write:

$$(t)_{v_0} = \sum_{i=1}^n m_i G_i.$$

Then $t = c g_1^{m_1} \dots g_n^{m_n}$ with $c \in k^*$. It is then clear that $m_1 T_1 + \dots + m_n T_n \in L$ and $\xi(m_1 T_1 + \dots + m_n T_n) = t$. Therefore, $\xi: L \rightarrow A^*/k^*$ is an isomorphism. Q.E.D.

1.13. By virtue of 1.10 and 1.12, we have the following:

Theorem. *Assume that V has only isolated singularities. Then we have the following exact sequence of \mathbf{Z} -modules:*

$$0 \rightarrow A^*/k^* \rightarrow \mathbf{Z}^{(n)} \rightarrow \mathbf{Z}^{(e)} \rightarrow Cl(V) \rightarrow 0,$$

where $\mathbf{Z}^{(r)}$ stands for a free \mathbf{Z} -module of rank r ; n is the number of distinct irreducible

factors of g ; e is the number of distinct points of $F \cap G$.

- 1.14. REMARKS. (1) It is clear from 1.13 that if g is irreducible then $A^* = k^*$.
 (2) $\text{rank}(Cl(V)) - \text{rank}(A^*/k^*) = e - n$.
 (3) Though we proved Theorem 1.13 under the assumption that $F \cap G \neq \emptyset$ it is clear that the theorem is valid also in the case where $F \cap G = \emptyset$.

2. Locally nilpotent derivation on $k[x, y, f/g]$

2.1. Let A be an affine k -domain. A k -derivation D on A is said to be *locally nilpotent* if, for every element a of A , $D^n(a) = 0$ for sufficiently large n . If D is a locally nilpotent k -derivation on A , we define a k -algebra homomorphism

$$\Delta: A \rightarrow A[X] \quad (\text{with an indeterminate } X)$$

$$\text{by} \quad \Delta(a) = \sum_{n \geq 0} (1/n!) D^n(a) X^n.$$

Then it is known (cf. [6]) that Δ gives rise to an action of the additive group scheme G_a on $\text{Spec}(A)$. Conversely, every action of G_a on $\text{Spec}(A)$ is expressed in the above-mentioned way with some locally nilpotent k -derivation on A . We set $A_0 := \{a \in A \mid D(a) = 0\}$. Then A_0 is an inert subring of A , and A_0 is, in fact, the ring of G_a -invariant elements of A with respect to the corresponding G_a -action on $\text{Spec}(A)$. For other relevant results on these materials, the readers are referred to [4] and [6].

2.2. In this section, we set $A = k[x, y, f/g]$, and assume that A is normal. Assume that A has a nonzero locally nilpotent k -derivation D . Then we assert the following:

Lemma. *The subring A_0 of D -constants is a finitely generated, normal, rational k -domain of dimension 1.*

Proof. Since A_0 is the ring of G_a -invariants in a normal domain A , A_0 is integrally closed in the quotient field $Q(A_0)$ of A_0 and $A_0 = A \cap Q(A_0)$, where $Q(A_0)$ is the field of G_a -invariants in the quotient field $Q(A)$ of A . Then, by virtue of Zariski's Theorem (cf. Nagata [7; p. 52]), A_0 is a finitely generated normal k -domain of dimension 1. Besides, A_0 is rational over k by Lüroth's Theorem because A is rational over k . Q.E.D.

2.3. Let $V := \text{Spec}(A)$. Then V has a nontrivial G_a -action corresponding to the derivation D on A . Let $U := \text{Spec}(A_0)$; U is isomorphic to an open set of the affine line A^1 (cf. 2.2). Let $q: V \rightarrow U$ be the morphism defined by the canonical inclusion $A_0 \hookrightarrow A$. By 2.2, we know that $A_0 = k[t, 1/h(t)]$ with $h(t) \in k[t]$. For almost all elements α of k such that $h(\alpha) \neq 0$, the fibre $q^{-1}(\alpha)$ is a G_a -orbit and is, therefore, isomorphic to the affine line. Let $\rho: V' \rightarrow V$ be the minimal resolution of singularities of V . As we saw in 1.8, singular points of V are

rational double points. Hence, ρ is a composition of quadratic transformations with centers at singular points. Let $q' := q \cdot \rho: V' \rightarrow U$. Almost all fibres of q' are therefore isomorphic to the affine line. Now we shall prove the following:

Lemma. *There exist a nonsingular projective surface W and a surjective morphism $p: W \rightarrow \mathbf{P}^1$ satisfying the following conditions:*

- (1) *Almost all fibres of p are isomorphic to \mathbf{P}^1 .*
 - (2) *There exists an open immersion $\iota: V' \rightarrow W$ such that $p \cdot \iota = \tau \cdot q'$, where $\tau: U \hookrightarrow \mathbf{P}^1$ is the canonical open immersion via $U \hookrightarrow \mathbf{A}^1 := \text{Spec}(k[t])$.*
- Then the fibration p has a cross-section S such that $S \subset W - \iota(V')$.*

Proof. Let \bar{V} be a nonsingular projective surface containing V' as an open set. Then, a subfield $k(t)$ of $k(V') = k(\bar{V})$ defines a linear pencil $\bar{\Lambda}$ of effective divisors on \bar{V} such that a general member of $\bar{\Lambda}$ cuts out a general fibre of q' on V' . The base points of $\bar{\Lambda}$ are situated on $\bar{V} - V'$. Let $\theta: W \rightarrow \bar{V}$ be the shortest succession of quadratic transformations of \bar{V} with centers at the base points of $\bar{\Lambda}$ such that the proper transform Λ of $\bar{\Lambda}$ by θ has no base points, and let $p: W \rightarrow \mathbf{P}^1$ be the morphism defined by Λ . Since V' is canonically embedded into W as an open set, let $\iota: V' \rightarrow W$ be the canonical immersion. Then it is not hard to see that $p: W \rightarrow \mathbf{P}^1$ and $\iota: V' \rightarrow W$ satisfy the conditions (1), (2) of Lemma. Q.E.D.

2.4. We shall prove the following:

Lemma. (cf. [3]). *Let $p: W \rightarrow \mathbf{P}^1$ be a surjective morphism from a nonsingular projective surface W onto \mathbf{P}^1 such that almost all fibres are isomorphic to \mathbf{P}^1 . Let $F = n_1 C_1 + \dots + n_r C_r$ be a reducible fibre of p , where C_i is an irreducible curve and $n_i > 0$. Then we have:*

- (1) *For $1 \leq i \leq r$, C_i is isomorphic to \mathbf{P}^1 and $(C_i^2) < 0$.*
- (2) *For $i \neq j$, C_i and C_j do not intersect or intersect transversally at a single point.*
- (3) *For distinct indices i, j and l , $C_i \cap C_j \cap C_l = \emptyset$.*
- (4) *One of C_i 's, say C_1 , is an exceptional curve of the first kind. If $\tau: W \rightarrow W_1$ is the contraction of C_1 , then p factors as $p: W \xrightarrow{\tau} W_1 \xrightarrow{p_1} \mathbf{P}^1$, where $p_1: W_1 \rightarrow \mathbf{P}^1$ is a fibration by \mathbf{P}^1 .*

Proof. For each i , $n_i(C_i^2) + \sum_{j \neq i} n_j(C_i \cdot C_j) = 0$, where $(C_i \cdot C_j) > 0$ for some j because F is connected. Hence $(C_i^2) < 0$. To prove the remaining assertions we have only to show that one of C_i 's is an exceptional curve of the first kind. Let K be the canonical divisor of W . Then $(F \cdot K) = -2$ because $p_a(F) = 0$. Hence, $-2 = (F \cdot K) = \sum_i n_i(C_i \cdot K) = \sum_i n_i(2p_a(C_i) - 2 - (C_i^2))$, where $2p_a(C_i) - 2 - (C_i^2) \geq -1$ and the equality holds if and only if C_i is an exceptional curve of the

first kind. However, it is impossible that $2p_a(C_i) - 2 - (C_i^2) \geq 0$ for every i . Therefore, $2p_a(C_i) - 2 - (C_i^2) = -1$ for some i . Q.E.D.

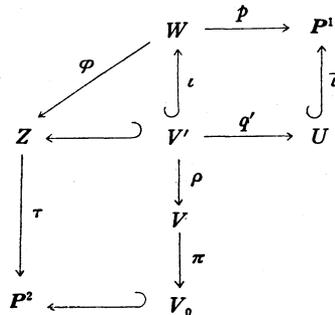
2.5. With the notations of 2.3, Lemma 2.4 implies:

Lemma. Write $W - \iota(V') = \bigcup_{i=1}^r C_i$ with irreducible curves C_i . Then we have:

- (1) Every C_i is isomorphic to P^1 .
- (2) For $i \neq j$, C_i and C_j meet each other (if at all) in a single point where they intersect transversally.
- (3) For distinct indices i, j and l , $C_i \cap C_j \cap C_l = \emptyset$.
- (4) $\bigcup_{i=1}^r C_i$ does not contain any cyclic chains.

Proof. Note that one of C_i 's is the cross-section S and the other components are contained in the fibres of p . Then the above assertions follow from 2.4. Q.E.D.

2.6. Let $V_0 := \text{Spec}(k[x, y])$, and let F, G be as in 1.1. Let $G_i (1 \leq i \leq n)$ be as in 1.9. Embed V_0 into P^2 in a canonical way. Let $l_\infty := P^2 - V_0$ and let $\bar{F}, \bar{G}, \bar{G}_j (1 \leq j \leq n)$ be the closures of F, G, G_j in P^2 , respectively. Let $\tau: Z \rightarrow P^2$ be a composition of the standard transformations of P^2 with respect to triplets (P, F, G) (or (P, G, F)), where P runs over all points of $F \cap G$. Then we know that V' is embedded into Z as an open set. We may assume, by replacing W if necessary by a surface which is obtained from W by a succession of the quadratic transformations, that there exists a birational morphism $\varphi: W \rightarrow Z$ such that we have the following commutative diagram:



2.7. Let $P_1 \in F \cap G$. Assume that F is nonsingular at P_1 but G is not. Then, in a neighborhood of $\tau^{-1}(P_1)$, $\tau^{-1}(F \cup G)$ has the configuration as in the Figure 1. With the notations of the Figure 1, we can show the following assertions:

- (1) A general fibre λ of p may intersect $\varphi'(E_N)$.
- (2) $\varphi'(E_1), \dots, \varphi'(E_{N-1})$ are contained in one and only one fibre of p .

Indeed, $\lambda_{V'} = \lambda \cap V'$ is isomorphic to the affine line, and $\tau\varphi(\lambda - \lambda_{V'})$ lies on l_∞ . Hence λ does not meet any of $\varphi(E_1), \dots, \varphi(E_{N-1})$. This proves the second

assertion. By the same reason, we have:

(3) For $1 \leq j \leq n$, $(\tau\varphi)'(\bar{G}_j)$ is contained in a fibre of p . In particular, $(\tau\varphi)'(\bar{G}_j)$ is isomorphic to P^1 .

2.8. We have the following:

Lemma. (1) For $1 \leq j \leq n$, G_j has one place at infinity; every singular point of G_j is a one-place point.

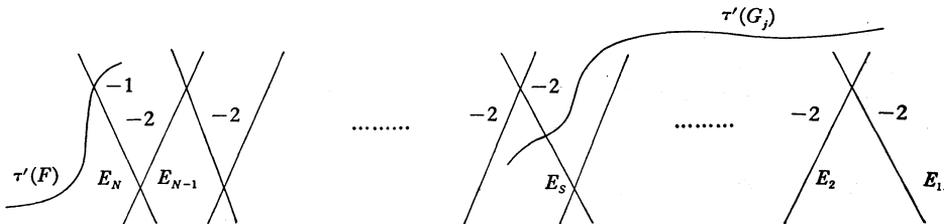
(2) For distinct i, j ($1 \leq i, j \leq n$), $G_i \cap G_j = \emptyset$.

Proof. Note that if φ is not an isomorphism φ is a composition of quadratic transformations of Z with centers at a point on $\tau'(l_\infty)$ and its infinitely near points. Then, both assertions follow from 2.5 (cf. 2.7). Q.E.D.

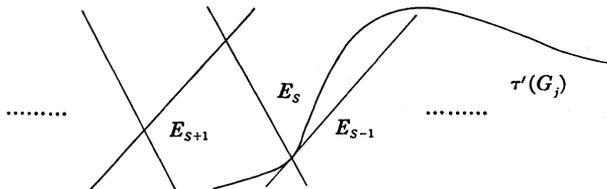
2.9. We prove the following:

Lemma. For $1 \leq j \leq n$, the curve G_j is nonsingular.

Proof. Note that if P is a singular point of G_j , then P is a point of $F \cap G_j$ (cf. 2.7, (3)). Assume that $P \in F \cap G_j$. Then, in a neighborhood of $\tau^{-1}(P)$, $\tau^{-1}(F \cap G_j)$ must have the following configuration as in the Figure 1:



where $\varphi'(E_1), \dots, \varphi'(E_{N-1})$ and $(\tau\varphi)'(G_j)$ belong to the same fibre of p . Note that $(\tau\varphi)'(G_j)$ intersects $\varphi'(E_s)$ transversally in one point if $N \geq s+1$ (cf. Lemma 2.4). Assume that $\nu_b \geq 2$ and $\nu_{b+1} = \dots = \nu_s = 1$. (For the notations, see 1.3.) Such b exists because we assume that P is a singular point of G_j . Then $N \geq s+1$, and it is not hard to see that $s=b+1$, and that we have the configuration:



where $\tau'(G_j)$ touches E_{s-1} with $(\tau'(G_j) \cdot E_{s-1}) = \nu_b - 1$. This contradicts Lemma 2.4, (3). Therefore, the curve G_j is nonsingular. Q.E.D.

2.10. Now we can prove:

Theorem. *Assume that V has only isolated singularities. Then A has a nonzero locally nilpotent k -derivation if and only if we have $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$.*

Proof. Assume that $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$. Then $D = g \frac{\partial}{\partial x}$ is a nonzero locally nilpotent k -derivation on A . We prove the converse. With the notations of 2.1~2.9, G_j ($1 \leq j \leq n$) is a nonsingular rational curve with one place at infinity (cf. 2.8, (1) and 2.9). [Note that G_j is a rational curve because $(\tau\varphi)'(\bar{G}_j)$ is a component of a fibre of p (cf. 2.4).] Hence, G_j is isomorphic to the affine line A^1 . By virtue of the Embedding Theorem of Abhyankar-Moh (cf. [1], [5]), we may assume that $g_1 = y$ after a suitable change of coordinates x, y of $k[x, y]$. Then, for $2 \leq j \leq n$, g_j is written in the form: $g_j = c_j + yh_j$, with $c_j \in k$ and $h_j \in k[x, y]$ because $G_j \cap G_1 = \emptyset$ (cf. 2.8, (2)). On the other hand, the fact that G_j has only one place at infinity implies that the curve $g_j = \alpha$ on A^2 is irreducible for every $\alpha \in k$ (cf. [5]). Therefore, h_j is a constant $\in k$. Thus, $g \in k[y]$. Q.E.D.

2.11. We know by [4; Theorem 1] that A is isomorphic to a polynomial ring over k if and only if A satisfies the following conditions:

- (1) A is a unique factorization domain,
- (2) $A^* = k^*$,
- (3) A has a nonzero locally nilpotent k -derivation.

The condition (1) above can be described as follows:

Lemma. *Assume that A satisfies the conditions (2) and (3) above. We may assume that $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$. Write: $f(x, y) = a_0(y) + a_1(y)x + \cdots + a_r(y)x^r$ with $a_i(y) \in k[y]$ ($0 \leq i \leq r$). Then A is a unique factorization domain if and only if $a_1(y)$ is a unit modulo $gk[x, y]$ and $a_i(y)$ is nilpotent modulo $gk[x, y]$ for $2 \leq i \leq r$.*

Proof. Assume that A is a unique factorization domain. With the notations of 1.9, $a = 0$ because every G_j ($1 \leq j \leq n$) is nonsingular and $G_i \cap G_j = \emptyset$ if $i \neq j$. By virtue of 1.13, we have: $e = n$. Theorem 1.10 then implies that every G_i intersects F transversally. This is easily seen to be equivalent to the condition on $f(x, y)$ in the above statement. The "if" part of Lemma will be clear by the above argument and Theorem 1.10.

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