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Osaka University
A NOTE ON THE FIXED RING OF
A GALOIS EXTENSION

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(Received October 7, 1966)

M. Harada [5] showed that if \( A \) is a central separable \( C \)-algebra and a Galois extension of \( B \) with group \( G \), and \( B \) is a separable \( B \cap C \)-algebra, then the order of the subgroup of \( G \) which leaves \( C \) fixed is a unit in \( C \). In this note we obtain a partial converse to this result (Theorem 4 below). The method of approach is to use the modules \( J_\sigma \) associated with automorphisms \( \sigma \) of \( A \). These modules were discovered in [8] and their connection with Galois extensions was recognized in [7].

The author would like to thank the referee for pointing out the reference [4] for the proof of Proposition 2.

We begin by recalling the definition of \( J_\sigma \):

DEFINITION. Let \( A \) be a central separable \( C \)-algebra and \( \sigma \) a ring automorphism of \( A \). Then

\[
J_\sigma = \{ x \in A \mid \sigma(a)x = xa \text{ for all } a \in A \}.
\]

It was shown in [8] that if \( \sigma \) is a \( C \)-algebra automorphism of \( A \), then \( J_\sigma \) is a rank one projective \( C \)-module. The following useful fact, noted for Galois extensions in [7], can also be extracted from [8]: \( (\otimes \text{ means } \otimes_C) \)

Lemma 1. Let \( A \) be a central separable \( C \)-algebra, and \( \sigma, \tau \) be two \( C \)-algebra automorphisms of \( A \). Then the map \( \kappa: J_\sigma \otimes J_\tau \to J_{\sigma \tau} \) given by \( \kappa(x \otimes y) = xy, x \in J_\sigma, y \in J_\tau \), is an isomorphism.

It is easy to see that the image of \( \kappa \) is in \( J_{\sigma \tau} \), and [8], Lemma 5, shows that there exists an isomorphism from \( J_\sigma \otimes J_\tau \) onto \( J_{\sigma \tau} \); the proof of Lemma 1 consists, first, in verifying that the sequence of isomorphisms connecting \( A \otimes J_\sigma \otimes J_\tau \) and \( A \otimes J_{\sigma \tau} \) on the last line of page 1112 of [8] sends \( a \otimes x \otimes y \) to \( a \otimes xy \), and then, using this fact, noticing that the sequence of isomorphisms on the bottom of page 1111 of [8] which gives the isomorphism of \( J_\sigma \otimes J_\tau \) with \( J_{\sigma \tau} \) is

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1) This material is adapted from the author's Ph. D. thesis at Cornell University. The author would like to thank Professor Alex Rosenberg for his advice and encouragement.
We omit the tedious details.

**Proposition 2.** Let $A$ be a central separable $C$-algebra and $G$ a finite group of $C$-algebra automorphisms of $A$. Let $N=\sum J_\sigma$, and suppose that as a $C$-module, the sum is direct. Then $N$ is a separable $C$-algebra if $\vert G \vert$, the order of $G$, is a unit of $C$.

Proof. Since the kernel of the map from $N^e$ to $N$ given by $x \otimes y \mapsto \otimes xy$ is a finitely generated $C$-module, we have by [1], III, 2.10 that $N$ is a separable $C$-algebra if $N \otimes C_m = N_m$ is a separable $C_m$-algebra for all maximal ideals $m$ of $C$. Moreover, if $G'$ is $G$ acting on $A \otimes C_m = A_m$ via $\sigma' = \sigma \otimes 1$, and $N' = \sum \oplus J_\sigma'$, where $J_\sigma' = \{ x' \in A_m \mid \sigma'(y')x' = x'y' \text{ for all } y' \in A_m \}$, then $N' = N_m$; in fact $J_\sigma' = (J_\sigma)_m$. For

$$ (J_\sigma)_m = \left\{ \frac{x}{s} \in A_m \mid \sigma(y)x = xy \text{ for all } y \in A \right\}, $$

and

$$ J_\sigma' = \left\{ \frac{x}{s} \in A_m \mid t \text{ in } C-m \text{ so that } t(\sigma(y)x-xy) = 0 \right\} $$

for all $y$ in $A$, so clearly $(J_\sigma)_m \subseteq J_\sigma'$. On the other hand, if $\frac{x}{s} \in J_\sigma'$, let $y_1, \cdots, y_r$ generate $A$ over $C$, $t_i$ be in $C-m$ such that $t_i(\sigma(y_i)x-xy_i) = 0$, and $t = \prod t_i$. Then $tx \in J_\sigma$, so $\frac{x}{s} = \frac{tx}{ts}$ is in $(J_\sigma)_m$. Now, since $\vert G \vert$ is a unit of $C$ if $\vert G \vert$ is a unit of $C_m$ for all $m$, it suffices to prove the theorem assuming $C$ is local.

Assuming $C$ local, $\sigma \in G$ is inner, conjugation by an element $u$, and $J_\sigma = Cu_\sigma$ ([8]). Since $Cu_\sigma, Cu_{\sigma'} = Cu_{\sigma \sigma'}$, $u_\sigma u_{\tau} = a_{\sigma, \tau} u_{\sigma \tau}$, $a_{\sigma, \tau}$ a unit of $C$, so $N = \sum \oplus Cu_\sigma$ is a twisted group ring (i.e. a crossed product with factor set in the units of $C$, and with $G$ acting trivially on $C$). Thus we may apply [4], Lemma 4, to obtain that $N$ is separable over $C$ if $\vert G \vert$ is a unit of $C$, as desired.

**Lemma 3.** If $A$ is a central separable $C$-algebra, $G$ is a finite group of $C$-algebra automorphisms of $A$, and $N = \sum J_\sigma$, then the fixed ring of $G$ acting on $A$, $A^G$, is equal to $A^N$, the commutator of $N$ in $A$.

Proof. If $x$ is in $A^N$ then $x$ is in $A^I_\sigma$ for all $\sigma$ in $G$, so $xy_\sigma = y_\sigma x$ for all $y_\sigma$ in $J_\sigma$. But since for all $x$ in $A$, $y_\sigma$ in $J_\sigma$, we have $\sigma(x)y_\sigma = y_\sigma x$, it follows that if $x$ is in $A^N$, $(\sigma(x)-x)y_\sigma = 0$ for all $y_\sigma$ in $J_\sigma$ and all $\sigma$ in $G$. By Lemma 1 $J_\sigma \cdot J_{\sigma^{-1}} = C$, so there exist $y_{\sigma, \nu} \in J_\sigma$, and $z_{\sigma, \nu} \in J_{\sigma^{-1}}$ so that $\sum y_{\sigma, \nu} z_{\sigma, \nu} = 1$. Thus $0 = \sum (\sigma(x)-x)y_{\sigma, \nu} z_{\sigma, \nu} = (\sigma(x)-x) \cdot 1$, so $x$ is in $A^G$. The converse is trivial.
Now, using Kanzaki's result ([7], Proposition 1) which states that if $A$ is a Galois extension of $B$ with group $G$, then $N=\Sigma \oplus J_\sigma$, we obtain our main result.

**Theorem 4.** Let $A$ be a ring whose center $C$ has no idempotents but 0 and 1. Suppose $A$ is a Galois extension of $B$ with group $G$, and $A$ is separable over $B \cap C$. Let $H$ be the subgroup consisting of all elements of $G$ which are the identity on $C$. Then if the order of $H$ is a unit in $C$, $B$ is a separable $B \cap C$-algebra.

Proof. If $A$ is a Galois extension of $B$ with group $G$, then directly from the definition of Galois extension $A$ is a Galois extension of $A^H$, the fixed ring of $H$, with group $H$. Thus $N=\Sigma \oplus J_\sigma$ by [7], Prop. 1. By Proposition 2, $N$ is a separable $C$-algebra, so by Lemma 3 and [6], Theorem 2, $A^H$ is separable over $C$.

Now $H$ is a normal subgroup of $G$, $G$ restricted to $A^H$ is isomorphic to $G/H$, as is $G$ restricted to $C$, and $C^G=B \cap C$. Since $A$ is assumed separable over $B \cap C$, the center $C$ of $A$ is separable over $B \cap C$, so ([3], 1.3) $C$ is a Galois extension of $B \cap C$ with group $G/H$. Defining the action of $G/H$ on $B \otimes_{B \cap C} C$ via $\sigma(b \otimes c)=b \otimes \sigma(c)$, $B \otimes_{B \cap C} C$ becomes a Galois extension of $B$ with group $G/H$, just as in [3], 1.7. Also $A^H$ is a Galois extension of $B$ with group $G/H$. The map from $B \otimes_{B \cap C} C$ to $A^H$ given by $b \otimes c \mapsto bc$ is a $G/H$-module and $B$-algebra map, so by a trivial extension of [3], 3.4, it is an isomorphism: $B \otimes_{B \cap C} C \cong A^H$. Thus, since $B \cap C$ is a $B \cap C$-direct summand of $C$ by [3], 1.6, $B$ is a $B$-direct summand of $A^H$, so is separable over $B \cap C$ by [2], IX, 7.1 and the fact that $A^H$ is separable over $B \cap C$. This completes the proof.

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