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SOME BASIC RESULTS ON PRO-AFFINE ALGEBRAS
AND IND-AFFINE SCHEMES

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Introduction

The theory of ind-affine varieties was first introduced by Shafarevich, who then employed it to elucidate the structure of the automorphism group of the affine space. (see [3], [4].) More recently we made certain revisions on the theory and applied it to study the Jacobian Problem on the endomorphisms of the complex affine space. (see [2].)

Since these pieces of work appeared, there has not been much progress made. This state may be due, in part, to the fact that the basic theory of these ind-affine or pro-affine objects as presented by us was still ad hoc and was rather rudimentary. So, we have embarked on building a theory of pro-affine algebras and ind-affine schemes anew and from the ground up. The outcome of our effort is the contents of the present paper. As we worked on the material we encountered a number of subtle examples, as shown in the main text below. It would seem that these examples perhaps suggest richness and mystery that this theory holds.

We mention a piece of specific result we have of our theory: The set of all morphisms of an affine variety over a field $K$ to another may be identified with the $K$-rational point set of an appropriately constructed ind-affine scheme over $K$. This was proven using the theory of Gröbner bases over $K$, and is expected to be published in the near future along with certain related results about automorphisms of the affine space.

1. Pro-affine algebras

1.1. Definitions and Notations. Throughout we work over a ground field $K$ of any characteristic. A commutative topological $K$-algebra $A$ is said to be a pro-affine algebra if

1. $A$ is complete and separated.
2. A base of open neighborhoods of 0 is given by a family of countably many ideals $\subseteq A$.

Let $\{a_i : i \in \mathbb{N}\}$ be a countable base referred to just above. Here, as elsewhere throughout the present paper, $\mathbb{N}$ represents the set of all nonnegative integers. We may,
and shall always, assume that $a_i \supseteq a_j$ whenever $i \leq j$. The condition 1 then implies that

$$(1) \quad \bigcap_{i \in \mathbb{N}} a_i = \{0\} \quad \text{and} \quad A \simeq \lim_{i \in \mathbb{N}} (A_i),$$

where, for each $i \in \mathbb{N}$, $A_i := A/a_i$ is a discrete $K$-algebra, with all maps $\mu_i : A_i \to A_{i-1}$ being surjective. Conversely, a $K$-algebra given as the limit of a countable, surjective inverse system of discrete $K$-algebras in the form of (1) is evidently pro-affine in our sense.

One recognizes then that a pro-affine $K$-algebra as above is the same thing as a “filtered commutative $K$-algebra which is complete and separated” in the sense of Northcott [5, Chap. 9].

**Proposition 1.1.1.** Let $A$ and $B$ be pro-affine algebras. Then, the product $A \times B$ and the complete tensor product $A \hat{\otimes}_K B$ are both pro-affine $K$-algebras.

Proof seems hardly necessary. If $\{a_i : i \in \mathbb{N}\}$ and $\{b_j : j \in \mathbb{N}\}$ are bases of open neighborhoods of 0 for $A$ and $B$, respectively, then one adopts for $A \times B$ the ideals $\{a_k \times b_k : k \in \mathbb{N}\}$ as a base of open neighborhoods of 0. As for $A \hat{\otimes}_K B$, take the ideals $\{a_k \otimes B + A \otimes b_k : k \in \mathbb{N}\}$ as a base of open neighborhoods of $A \otimes_K B$, and then take its completion. □

A pro-affine algebra $A$ is said to be algebraic over $K$, or $K$-algebraic, if $A$ can be represented as in (1) where all $A/a_i$ are finitely generated over $K$.

Let $A$, $B$ be pro-affine $K$-algebras. A morphism of $A$ to $B$ is defined to be a continuous $K$-algebra map $\phi : A \to B$. Suppose that $A$ and $B$ are represented as $A = \lim_\rightarrow (A/a_i)$, $B = \lim_\rightarrow (B/b_j)$, respectively. Then, the morphism $\phi : A \to B$ gives rise to a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\pi_i^A} & & \downarrow{\pi_j^B} \\
A/a_i & \xrightarrow{\phi_i} & B/b_j
\end{array}
$$

standing valid for each given $j \in \mathbb{N}$ and for some corresponding $i = i(j) \in \mathbb{N}$ for which $\phi(a_i) \subseteq b_j$. Here, $\pi_i^A$ and $\pi_j^B$ denote the canonical residue-class maps, and $\phi_i(x + a_i) \overset{\text{def.}}{=} \phi(x) + b_j$ for all $x \in A$. 

Notations. Let us fix some notations we shall be using throughout this paper:

(a) Let \( A = \lim_{\rightarrow}(A_i) \) be a pro-affine algebra, where we have put \( A_i := A/\alpha_i \) as before. The canonical surjective maps \( A \rightarrow A_i \) and \( A_j \rightarrow A_i \) for \( i \leq j \) shall be denoted as follows:

\[
\pi_i : A \rightarrow A_i ; \quad \mu_{ij} : A_j \rightarrow A_i,
\]

with \( \ker(\pi_i) = \alpha_i \), and \( \mu_{ij} = \text{Id}_{A_i} \). We abbreviate \( \mu_{i-1,i} \) as \( \mu_i \).

(b) As a rule, for any subset \( E \subseteq A \) or any element \( a \in A \), we denote \( \pi_i(E) \) by \( iE \) and \( \pi_i(a) \) by \( ia \). (A notable exception is \( \pi_i(A) = A/\alpha_i \) which we denote by \( A_i \) and not by \( iA_i \).) When no fear of confusion is present, we often skip the left suffix and simply write \( a \) for \( ia \), so that \( a = (\cdots \leftarrow i_{1-1}a \leftarrow ia \leftarrow \cdots) \) is expressed as \( (\cdots \leftarrow a \leftarrow a \leftarrow \cdots) \). A sequence \( \sigma := (\cdots \leftarrow s_{j-1} \leftarrow s_j \leftarrow \cdots) \) with \( s_j \in A_j \) for all \( j \in \mathbb{N} \) represents an element of \( A \) and thus \( \sigma \in A \) if and only if \( \mu_j(s_j) = sj \) for all \( j \), in which case we shall say \( \sigma \) is coherent.

In the notations above, it is then clear that the closure \( \overline{E} \) of \( E \) may be identified with \( \lim_{\rightarrow}(iE) \). Thus, \( E \subseteq A \) is closed if and only if every coherent sequence \( \epsilon = (\cdots \leftarrow e_i \leftarrow \cdots) \) belongs to \( E \) as soon as all \( e_i \in iE \) for \( i \in \mathbb{N} \).

**Proposition 1.1.2.** The group of units \( U(A) \) of a pro-affine algebra \( A \) is closed.

Proof. Let \( u = (\cdots \leftarrow u_{i-1} \leftarrow u_i \leftarrow \cdots) \in \overline{U(A)} \). For each \( i \) there is a unique \( v_i \in A_i \) with \( u_i \cdot v_i = 1_{A_i} \). Then, \( v := (\cdots \leftarrow v_{i-1} \leftarrow v_i \leftarrow \cdots) \) is clearly coherent and satisfies \( u \cdot v = 1 \) so that \( u \in U(A) \). \( \square \)

**Example 1.1-A** (cf. [2, (1.1), p. 482]). For each \( n \in \mathbb{N} \), let \( K^{[n]} := K[X_1, \ldots, X_n] \) if \( n > 0 \), and \( K^{[0]} := K \). Define \( \mu_n : K^{[n]} \rightarrow K^{[n-1]} \) by setting \( \mu_n(X_i) := X_i \) for all \( 1 \leq i \leq n-1 \), and \( \mu_n(X_n) := 0 \). Denote

\[
K^{[\infty]} := \lim_{\rightarrow}(K^{[n]})
\]

and call it the pro-affine polynomial algebra (over \( K \)). This algebra may be characterized as the set of those formal power-series on \( X_1, \ldots, X_m, \ldots \) which become polynomials when reduced modulo all but finitely many \( X_i \)'s.

1.2. The ideals in a pro-affine algebra.

**Proposition 1.2.1.** Let \( \mathfrak{a} \) be a closed ideal in \( A = \lim_{\rightarrow}(A_i) \). Then,

\[
A/\mathfrak{a} \simeq \lim_{\rightarrow}(A_i)/\lim_{\rightarrow}(\mathfrak{a}) \simeq \lim_{\rightarrow}(A_i/\mathfrak{a}).
\]
[This implies that $A/\mathfrak{h}$ is a pro-affine algebra for any closed ideal $\mathfrak{h}$.]

Proof. Since $\mathfrak{h}$ is closed, $\mathfrak{h} \simeq \lim_\to (\mathfrak{h})$ and all maps $i\mathfrak{h} \to i^{-1}\mathfrak{h}$ are surjective. So, in the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & i\mathfrak{h} & \longrightarrow & A_i & \longrightarrow & A_i/i\mathfrak{h} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i^{-1}\mathfrak{h} & \longrightarrow & A_{i^{-1}} & \longrightarrow & A_{i^{-1}}/i^{-1}\mathfrak{h} & \longrightarrow & 0
\end{array}
$$

all vertical maps are surjective. One now applies the functor $\lim_\to$ to this diagram, remembering the Mittag-Leffler condition which holds here. \hfill \square

**Example 1.2-B.** In the same notations as in Ex. 1.1-A, define an ideal $J_n \subset K^{[n]}$ by $J_n := \langle X_iX_j \mid 1 \leq i < j \leq n \rangle$, so geometrically the locus of $J_n$ is the union of all coordinate axes in the affine $n$-space $\mathbb{A}^n$ over $K$. Let $B_n := K^{[n]}/J_n = K[x_1, \ldots, x_n]$. Consider the exact sequence

$$
0 \longrightarrow J_n \longrightarrow K^{[n]} \longrightarrow B_n \longrightarrow 0
$$

and take the $\lim_\to$ of this sequence on all $n \in \mathbb{N}$. Since, for all $n$, $\mu_n: K^{[n]} \to K^{[n-1]}$ causes a *surjection* of $J_n$ to $J_{n-1}$, there results a surjective $K$-map $K^{[\infty]} \to B := \lim_\to B_n$, and its kernel $J := \lim_\to J_n$ gives an example of a *closed ideal* in $K^{[\infty]}$. [In the subsequent $B$ will be viewed as the coordinate algebra $\mathcal{O}(Y)$ of the closed subscheme $Y$ of all coordinate axes in the ind-affine space $\mathbb{A}^{\infty}$.]

**Example 1.2-C.** In Example 1.2-B replace each $J_n$ by $J_n' := \langle X_1 \cdots X_n \rangle$, whose locus in $\mathbb{A}^n$ is then the union of all coordinate hyperplanes in $\mathbb{A}^n$. Since the surjections $\mu_n: K^{[n]} \to K^{[n-1]}$ all cause zero maps of $J_n'$ into $J_{n-1}'$, the Mittag-Leffler condition is trivially satisfied, and $J' := \lim_\to J_n = \{0\}$ (which is a closed ideal in $K^{[\infty]}$). It follows that $K^{[\infty]} \simeq \lim_\to (K^{[n]}/J_n')$. [So, the union of all coordinate hyperplanes in $\mathbb{A}^n$, as $n \to \infty$, is isomorphic to the whole ind-affine space $\mathbb{A}^{\infty}$.]

**Proposition 1.2.2.** For any maximal ideal $m \subset A$, the following conditions are equivalent to one another:

(i) $m$ is closed;

(ii) For some $i$, $\pi_i(m) = i m \subsetneq A_i$;

(iii) For some $i$, $a_i \subseteq m$;

(iv) For some $i$, $m = \pi_i^{-1}(\text{some maximal ideal in } A_i)$;

(v) $m$ is open.

Proof. (i) $\Rightarrow$ (ii) : If $i m = A_i$ for all $i$, then $(0 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots) \in \overline{m} = m$, so that $m = A$.

(ii) $\Rightarrow$ (iii) : Let $im \subset A_i$ for a particular $i$. Then, $im$ must be a maximal ideal in
$A_i$, and $\pi_i^{-1}(\mathfrak{m}) = \mathfrak{m} + \mathfrak{a}_i = \mathfrak{m}$, so $\mathfrak{a}_i \subseteq \mathfrak{m}$.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are obvious. \qed

The same argument as used in (i) $\Rightarrow$ (ii) above shows the following:

**Corollary 1.2.3.** Every closed proper ideal in a pro-affine algebra $A$ is contained in a closed maximal ideal.

**Proposition 1.2.4.** For any prime ideal $\mathfrak{p} \subset A$, the following conditions are equivalent to one another:
1. $\mathfrak{p}$ is open;
2. For some $i$, $\mathfrak{p} = \pi_i^{-1}(\mathfrak{p})$;
3. For some $j$ and a prime ideal $\mathfrak{q}_j \subset A_j$, $\mathfrak{p} = \pi_j^{-1}(\mathfrak{q}_j)$.

The proof of this obvious proposition is omitted.

Note that, in view of the two preceding propositions, the open prime (resp. open maximal) ideals of a pro-affine algebra $A$ are precisely the inverse images of the prime (resp. maximal) ideals of the $A_i$’s for any $i \in \mathbb{N}$.

**Proposition 1.2.5.** Let $\mathfrak{a}$ be a finitely generated proper ideal in a pro-affine algebra $A$. Then, there exists an open maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} \subseteq \overline{\mathfrak{a}} \subseteq \mathfrak{m}$.

We first prove the following key lemma due to N. Mohan Kumar:

**Lemma 1.2.6** (N. Mohan Kumar). Let $\mathfrak{a} = \langle x_1, \ldots, x_n \rangle$ be a finitely-generated ideal, and let $\overline{\mathfrak{a}}$ be its closure. For any $z \in A$, if $z \in \overline{\mathfrak{a}}$ then $z^n \in \mathfrak{a}$.

**Proof.** The proof goes by induction on the number of generators $n$. First, take any $x \in A$ and let $z \in \langle x \rangle = \lim_{\to} (A_i \cdot ix)$. Write

$z = (a_0x \leftarrow a_1x \leftarrow \cdots \leftarrow a_tx \leftarrow \cdots), a_j \in A_j$ for all $j \in \mathbb{N}$,

where the coherence condition

\begin{equation}
\mu_i(a_ix) - a_{i-1} \cdot i_{-1}x = \mu_i(a_i \cdot i_{-1}x) = (\mu_i(a_i) - a_{i-1}) \cdot i_{-1}x = 0
\end{equation}

is satisfied. Then, $\eta \overset{\text{def.}}{=} (a_0^2x \leftarrow a_1^2x \leftarrow \cdots \leftarrow a_t^2x \leftarrow \cdots)$ is coherent, as one sees from (4) that

$\mu_i(a_i^2 \cdot i_{-1}x) - a_{i-1}^2 \cdot i_{-1}x = (\mu_i(a_i)^2 - a_{i-1}^2) \cdot i_{-1}x = (\mu_i(a_i) + a_{i-1}) (\mu_i(a_i) - a_{i-1}) \cdot i_{-1}x = 0.
$

So $\eta \in A$. It follows that $z^2 = x\eta \in \langle x \rangle \subseteq A$. 

Turning now to the next induction step, we let \( z \in \langle x_1, \ldots, x_n \rangle \). Set \( A' \overset{\text{def}}{=} A/(x_1) \), and consider its ideal \( \langle x_1', \ldots, x_n' \rangle \), where \( x_1', \ldots, x_n' \) denote the canonical images of \( x_1, \ldots, x_n \), respectively, in \( A' \). Let \( z' := z-\text{mod}(x_1) \in \langle x_1', \ldots, x_n' \rangle \). By induction hypothesis, \( z'^{2^{n-1}} \in \langle x_1', \ldots, x_n' \rangle \). This implies that one can write \( z'^{2^{n-1}} = z_1 + z_2 \), where

\[
\begin{align*}
  z_1 &\in \langle x_1 \rangle & &\text{and} & & z_2 \in \langle x_2, \ldots, x_n \rangle.
\end{align*}
\]

But we saw just above that \( z_1 \in \langle x_1 \rangle \) gives \( z_1^2 \in \langle x_1 \rangle \). Therefore,

\[
\begin{align*}
  z^{2^n} = (z_1 + z_2)^2 = z_1^2 + 2z_1 z_2 + z_2^2 \in \langle x_1 \rangle + \langle x_2, \ldots, x_n \rangle,
\end{align*}
\]

and we find \( z^{2^n} \in \langle x_1, x_2, \ldots, x_n \rangle \), as desired. \( \square \)

Proof of Proposition 1.2.5. now follows immediately from this lemma. Indeed, if a finitely-generated ideal \( \mathfrak{a} \) is such that \( \mathfrak{a} = A \), then \( 1 \in \mathfrak{a} \), which implies \( 1 = 1^{2^n} \in \mathfrak{a} \) for some \( n \). So, if \( \mathfrak{a} \) is proper, then \( \mathfrak{a} \) is proper; and one now applies Cor. 1.2.3. \( \square \)

Remark. Proposition 1.2.5 fails to hold for ideals not finitely generated, as will be shown in §3 below (see Ex. 3-G). Also note that a finitely generated ideal need not be closed. In fact, even a principal ideal can be non-closed, as the following example shows:

Example 1.2-D (N. Mohan Kumar). Let \( K[2] := K[X, Y] \) be a polynomial ring in \( X \) and \( Y \), and for each \( i \in \mathbb{N} \) let \( A_i := K[2]/(XY^{i+1}) = K[x, y] \), with \( x, y \) standing for the canonical images of \( X, Y \), respectively, in \( A_i \). Let our pro-affine algebra \( A \) be \( \lim_{\longrightarrow} A_i \). Consider

\[
\zeta := (x \leftarrow x(1 + y) \leftarrow x(1 + y + y^2) \leftarrow x(1 + y + y^2 + y^3) \leftarrow \cdots),
\]

Clearly, \( \zeta \in \langle x \rangle \). However, \( \zeta \notin \langle x \rangle \). To see this, assume \( \zeta \in \langle x \rangle \), and write \( \zeta = x\eta \) for some \( \eta \in A \). Then, \( \eta \) has to equal

\[
(1 + y) p_1(y) \leftarrow 1 + y + y^2 p_2(y) \leftarrow 1 + y + y^2 + y^3 p_3(y) \leftarrow \cdots,
\]

where \( p_1(y), p_2(y), p_3(y), \ldots \) are polynomials in \( y \) only. Now let, for each \( i \in \mathbb{N} \), \( f_i : A_i \to A_i/\langle x \rangle \simeq K[y] \) be the canonical mod-\( X \) map. Then,

\[
f := \lim_{\longrightarrow} f_i : \lim_{\longrightarrow} A_i = A \longrightarrow K[y]
\]

should map \( \eta \) to a polynomial in \( K[y] \) of a certain degree, say of degree \( d \). Since \( f(\eta) = f_{d+1}(1 + y + \cdots + y^{d+1} + y^{d+2} p_{d+2}(y)) = 1 + y + \cdots + y^{d+1} + y^{d+2} p_{d+2}(y) \in K[y] \) is of degree at least \( d + 1 \), there results a contradiction.
1.3. The radicals and Nullstellensatz. The radical $\mathcal{R}(A)$ and the nilradical $\mathcal{N}(A)$ of a pro-affine algebra $A$ are defined as follows:

$$\mathcal{N}(A) = \bigcap \mathfrak{p} \quad \text{and} \quad \mathcal{R}(A) = \bigcap \mathfrak{m},$$

where the $\mathfrak{p}$’s and the $\mathfrak{m}$’s range over all open prime and all open maximal ideals, respectively.

Given an ideal $\mathfrak{a} \subseteq A$, the radical of $\mathfrak{a}$ is defined as

$$\mathcal{N}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$$

with $\mathfrak{p}$ again ranging over all open prime ideals containing $\mathfrak{a}$.

As done in [2], for a pro-affine algebra $A = \lim_i A_i$ we define two kinds of its reductions relative to the radicals:

$$A_{\text{red}} \overset{\text{def}}{=} A/\mathcal{N}(A) \quad \text{and} \quad A_{\text{RED}} \overset{\text{def}}{=} \lim_i (A_i)_{\text{RED}} = \lim_i (A_i)_{\text{red}},$$

where $(A_i)_{\text{RED}} := A_i/\mathcal{N}(A_i) = (A_i)_{\text{red}}$ is the usual residue-class ring modulo the nilradical of $A_i$. $A$ is said to be reduced or strongly reduced, respectively, if $A = A_{\text{red}}$ or $A = A_{\text{RED}}$. One may define likewise two more radicals using the Jacobson radicals $\mathcal{R}(A)$’s and $\mathcal{R}(A_i)$’s, and these were actually what we dealt with in [2, (1.2), (1.3), pp. 483–484]. Just the same, the following counterpart of [2, Prop. (1.2), loc. cit.] stands valid, and we state it without proof:

**Theorem 1.3.1.** For the canonical map $\rho: A = \lim_i (A_i) \rightarrow A_{\text{RED}}$, we have

(a) $\text{Ker}(\rho) = \mathcal{N}(A)$;
(b) The sequence $0 \rightarrow \mathcal{N}(A) \rightarrow A \rightarrow A_{\text{RED}}$ is exact with $\text{Im}(\rho)$ dense in $A_{\text{RED}}$;
(c) $\mathcal{N}(A) = \{ f \in A : \lim_{N \rightarrow \infty} f^N = 0 \}$ = topologically nilpotent elements of $A$.

**Remarks.** 1. We note that, even in the special context of the theorem above, the exactness of the sequence in (b) at the right-most end fails in general, or $\rho$ is not surjective as a rule. Counter-examples are offered in Section 3 below (see Examples 3-E and 3-F). This point bears critically on the Jacobian Problem (cf. [2, (5.3), (5.4), pp. 497–498]).

2. Since $\mathcal{N}(A)$ is a closed ideal $\subseteq A$, we deduce from Prop. 1.2.1 that, whereas $\rho: A \rightarrow \lim_i (A_i/\mathcal{N}(A_i))$ may not be surjective, the map $A \rightarrow \lim_i (A_i/\mathcal{N}(A))$ is surjective.

Theorem 1.3.1 and the Jacobson-radical version of it [2, (1.2), p. 483] coincide with each other in the $K$-algebraic case as seen just below:
Theorem 1.3.2 (Nullstellensatz). If a pro-affine $K$-algebra $A$ is algebraic over $K$, then $\mathcal{R}(A) = \mathcal{N}(A)$.

Proof. In view of Props. 1.2.2 & 1.2.4, the remarks following these two and the algebraicity, we have

$$\mathcal{R}(A) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}(\mathcal{R}(A_i)) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}(\mathcal{N}(A_i)) = \mathcal{N}(A),$$

where the traditional Nullstellensatz $\mathcal{R}(A_i) = \mathcal{N}(A_i)$ has been applied. \qed

2. Ind-affine schemes and ind-affine varieties

2.1. The spectra of pro-affine algebras and their topology. For any pro-affine algebra $A$, define its prime spectrum $\mathfrak{Sp}(A)$ and maximal spectrum $\mathfrak{Spm}(A)$, respectively, as

$$\left\{ \begin{array}{ll}
\mathfrak{Sp}(A) & = \text{the set of all open, prime ideals } \subset A, \text{ and } \\
\mathfrak{Spm}(A) & = \text{the set of all open, maximal ideals } \subset A.
\end{array} \right. \hspace{1cm} (8)$$

Then, in view of Prop. 1.2.2, $\mathfrak{Spm}(A)$ is the same as the set of all closed maximal ideals. Let us now introduce topology on $\mathfrak{Sp}(A)$ and $\mathfrak{Spm}(A)$ by extending Zariski topology: The closed sets $\subseteq \mathfrak{Sp}(A)$ are, by definition, those subsets of $\mathfrak{Sp}(A)$ in the form of

$$\mathcal{V}(E) \overset{\text{def.}}{=} \{ p \in \mathfrak{Sp}(A) : p \supseteq E \} \text{ for some set } E \subseteq A.$$ 

Likewise, the closed sets $\subseteq \mathfrak{Spm}(A)$ are defined to be precisely the $\mathcal{V}_0(E)$'s where $\mathcal{V}_0(E) \overset{\text{def.}}{=} \mathcal{V}(E) \cap \mathfrak{Spm}(A)$.

The following proposition which should require no proofs shows that the preceding definition of the topologies on $\mathfrak{Sp}(A)$ and on $\mathfrak{Spm}(A)$ is valid:

Proposition 2.1.1. (i) Let $a := (E)$, the ideal generated by $E$, and let $\mathcal{N}(a)$ be the radical of $a$. Then,

$$\mathcal{V}(a) = \mathcal{V}(E) = \mathcal{V}(\mathcal{N}(a)).$$

(ii) $\mathcal{V}(0) = \mathfrak{Sp}(A), \mathcal{V}(1) = \emptyset$.

(iii) Given a family $\{E_i : i \in I\}$ of subsets of $A$, we have

$$\mathcal{V}\left( \bigcup_{i \in I} E_i \right) = \bigcap_{i \in I} \mathcal{V}(E_i).$$

(iv) For ideals $b$ and $c$, $\mathcal{V}(b \cap c) = \mathcal{V}(bc) = \mathcal{V}(b) \cup \mathcal{V}(c)$. 

Next we define, for each $f \in A$, the basic open set $D(f) \subseteq \mathfrak{Sp}(A)$:

$$D(f) \overset{\text{def}}{=} V(f^c) = \{ p \in \mathfrak{Sp}(A) : f \not\in p \}.$$ 

**Proposition 2.1.2.** Let $f, g, f_{\alpha}$ ($\alpha \in I$) be elements of $A$. Then,

(i) $D(f) \cap D(g) = D(f : g)$.
(ii) $\bigcup_{\alpha \in I} D(f_{\alpha}) = V((f_{\alpha} : \alpha \in I))^c$.
(iii) $D(f) = \emptyset \iff f \in \mathfrak{N}(A) \iff f$ is topologically nilpotent.
(iv) $D(f) = \mathfrak{Sp}(A) \iff f$ is a unit.
(v) $D(g) \subseteq D(f) \iff g \in \mathfrak{N}(f)$.

Proof. Parts (i), (ii), (iii) immediately follow from relevant definitions. As for (iv), if $f \not\in$ any open prime, then by Prop. 1.2.5 $\langle f \rangle$ must equal the unit ideal $\langle 1 \rangle$. Therefore, $f$ must be a unit.

As for part (v), $D(g) \subseteq D(f) \iff \forall p \in \mathfrak{Sp}(A) \ [ f \in p \Rightarrow g \in p ]$, clearly, and this last condition is equivalent to $\mathfrak{N}(\langle g \rangle) \subseteq \mathfrak{N}(\langle f \rangle)$, or $g \in \mathfrak{N}(\langle f \rangle)$.

**Remark.** Proposition 2.1.2 goes to show that the $D(f)$’s for all $f \in A$ form a base of open sets in our topology on $\mathfrak{Sp}(A)$, just as in the more traditional theory of affine schemes. Note, however, that in our theory here the open sets $D(f)$’s are not quasi-compact in general. This is due to the existence of infinitely-generated proper ideals whose closures are the unit ideal $\langle 1 \rangle$. See Ex. 3-G in §3 below.

### 2.2. Localization in pro-affine algebras and structure sheaves of ind-affine schemes.

Let $S$ be a multiplicatively closed set in a pro-affine algebra $A$. It will be assumed always that $1 \in S$ and $0 \not\in \overline{S} = \lim_{\leftarrow} \langle i \rangle S$ for such an $S$. The localization $S^{-1}A$ can be defined in the standard manner, and this $K$-algebra naturally inherits its uniform topology from $A$. We shall adopt the completion of $S^{-1}A$ as our definition of $A_S$. Namely,

**Definition.** For $A$ and $S$ as above, the localization $A_S$ of $A$ by $S$ is defined to be

$$A_S \overset{\text{def}}{=} \lim_{\leftarrow} \langle i \rangle S^{-1}A_f.$$ 

Clearly, $A_S \simeq A_S$, so one may assume from the beginning that $S$ is closed. For useful examples of $S$ one may mention $(f)^c \overset{\text{def}}{=} \{ f^n \mid n \in \mathbb{N} \}$ where $f$ is not topologically nilpotent, and the complement $A - p$ of an open prime ideal $p$. In these instances, we shall denote $A(f)$, $A_{A - p}$ by $A_f$, $A_p$, respectively.

**Proposition 2.2.1.** Let $f, g \in A$. $U := D(f)$, $V := D(g)$, and let $A_f, A_g$ be as just above. Let $A(U) := A_f$ and $A(V) := A_g$. Then,
(i) If \( U = V \), then \( A(U) \simeq A(V) \). (Thus \( A(U) \) depends only on \( U \), not on \( f \).)

(ii) If \( V \subseteq U \), then there is a canonical homomorphism of pro-affine \( K \)-algebras \( \rho^U_V : A(U) \longrightarrow A(V) \), which depends only on \( U \) and \( V \). (The \( \rho^U_V \) will be called the restriction homomorphism from \( U \) to \( V \).)

(iii) Let \( U, V \) be as above and \( W = D(h) \) for \( h \in A \). If \( U \supseteq V \supseteq W \), we have

\[
\rho^U_V = \text{Id}_{A(U)} , \quad \rho^V_W \circ \rho^U_V = \rho^U_W .
\]

Proof. (ii) Assume \( V \subseteq U \), or \( D(g) \subseteq D(f) \). So, by Props. 1.2.2 & 1.2.4, \( g \in N((f)) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}(\text{the radical of } \langle f \rangle \text{ in } A_i) \). This means that, for every \( i \in \mathbb{N} \), there is an \( n_i \) such that \( ig^{n_i} \in \langle f \rangle \subseteq A_i \). So, for each \( i \) there is an element \( u_i \in A_i \) such that

\[
(i) = u_i \cdot if .
\]

Now let \( s \in A(U) = A_f = \lim_{\leftarrow} (A_i) \). Write \( s \) as a coherent sequence \( s = (\cdots a_{i-1} / (i-1)f)^{n_{i-1}} \leftarrow a_i / (i)f^{n_i} \leftarrow \cdots) \). Define \( \rho^U_V(s) \) to be equal to \( (\cdots a_{i-1} \leftarrow \cdots) \), where

\[
(10) \quad s'_i \text{ def. } = a_i \cdot u_i^{n_i} / (i)g^{m_i} .
\]

If another pair \( (n'_i, u'_i) \) is chosen to make (9) stand, as \( (i)g^{n'_i} = u'_i \cdot i.f \), then \( s'_i \) in (10) will have to be replaced by \( s''_i = a_i \cdot u_i^{n_i} / (i)g^{m_i} \). But one can check out easily that \( s'_i = s''_i \) inside \( (i)g^{-1}A_i \). So, \( \rho^U_V(s) \) is well-defined provided that \( s' := (\cdots a'_{i-1} \leftarrow s'_{i-1} \leftarrow \cdots) \) given by (10) just above is coherent.

Let us now check the coherence of \( s' \). Since \( s \) is given coherent, one knows

\[
[(i-1)f]^{m_i}a_{i-1} - (i-1)f^{m_i-1}\mu_i(a_i) \cdot (i-1)f \text{some power } = 0 ,
\]

and one need to verify

\[
[(i-1)f]^{m_i}a_{i-1} - (i-1)f^{m_i-1}\mu_i(a_i) \cdot (i-1)g \text{some power } = 0 .
\]

Applying \( \mu_i \) to both sides of (9) and then raising them to the \( m_i \)-th power, one obtains \( i^{-1}g^{m_i} = \mu_i(a_i) \cdot (i-1)f^{m_i} \); also, (9) for \( i := i - 1 \) gives \( (i-1)g = u_{i-1} \cdot i-1f \). Substituting the right-hand sides of these two equalities for the appropriate terms inside the “[ ]” of (12), we find the said contents of [ ] to be

\[
a_{i-1} - \mu_i(a_i)u_{i-1}^{m_i} = \mu_i(a_i)u_{i-1}^{m_i} - u_{i-1}^{m_i+(i-1)f} - \mu_i(a_i)u_{i-1}^{m_i} .
\]

The expression inside the “[ ]” of (13) equals that of (11) and, consequently, gets killed by some power of \( i-1f \). It follows that either side of (13) will be killed by
some power of \( i_{-1}g \) because \((i_{-1}g)^{n_{i-1}} = u_{i-1} \cdot i_1f\). The proof of (ii) will be complete after (iii) and then (i) are established below.

(iii) That \( \rho_U^V = \text{Id}_{A(U)} \) is clear in view of the preceding reasoning. As for the transitivity, we have

\[ \forall i \in \mathbb{N} \exists n_i \exists l_i \in \mathbb{N} : (g)^{n_i} = u_i \cdot i_1f \quad \text{and} \quad (h)^{l_i} = v_i \cdot i_1g, \text{with} \ u_i, v_i \in A_i. \]

It follows that, for each \( i \), \((h)^{l_i} (g)^{n_i} = u_i \cdot i_1f\) holds, which implies that the composition \( \rho_W^V \circ \rho_V^U \) maps \( s = (\cdots \leftarrow a_i/(i_1f)^{m_i} \leftarrow \cdots) \in A_f \) to

\[ \rho_W^V \circ \rho_V^U (s) = (\cdots \leftarrow a_i u_i^{m_i} v_i^{m_i} / (h)^{m_i n_i} \leftarrow \cdots). \]

On the other hand, the relations \((h)^{k_i} = u_i \cdot i_1f\) for all \( i \in \mathbb{N} \) corresponding to \( W \subseteq U \) indicates \( \rho_W^V (s) = (\cdots \leftarrow a_i u_i^{m_i} / (h)^{m_i k_i} \leftarrow \cdots) \). We already saw above that such coherent sequences are the same in \( A_n \). Therefore, \( \rho_W^V \circ \rho_V^U = \rho_W^U \).

(i) If \( U = V \) or \( D(f) = D(g) \), we have maps \( \rho_U^V : A(U) \to A(V) \) and \( \rho_V^U : A(V) \to A(U) \). As we just saw, \( \rho_V^U \circ \rho_U^V = \rho_U^V = \text{Id}_{A(U)} \), and likewise for \( \rho_U^V \circ \rho_V^U \). Hence \( A(U) \cong A(V) \) if \( U \) is proven now, the proof of (ii) is complete.

It follows from Prop. 2.2.1 that the assignments \( U = D(f) \mapsto A(U) = A_f \) and \( V = D(g) \mapsto U = D(f) \mapsto \rho_U^V \) produce a presheaf \( \mathcal{A} \) of pro-affine \( K \)-algebras on the base \( \mathcal{B} = \{ D(f) : f \in A \} \) of open sets of the topological space \( \mathfrak{S}p(A) \). (see [1, Chap. 0, §3.2, p. 25ff.].)

**Proposition 2.2.2.** Let \( A \) be a pro-affine algebra, and let \( \mathcal{A} \) be the presheaf over the base \( \mathcal{B} \) of open sets on \( \mathfrak{S}p(A) \) introduced just above. Let \( U = D(g) \in \mathcal{B} \) be any basic open set, and let \( U = \bigcup_{\lambda \in \Lambda} U_\lambda \) be a covering of \( U \) with each \( U_\lambda = D(f_\lambda) \), \( f_\lambda \in A_g \). Suppose given for each \( \kappa \in \Lambda \) an element \( s_\kappa \in A(U_\kappa) \) such that \( \rho_{U_\lambda}^{U_\kappa} (s_\lambda) = \rho_{U_\nu}^{U_\kappa} (s_\nu) \) for any \( \lambda, \nu \in \Lambda \), where \( U_{\lambda \nu} \) denotes \( U_\lambda \cap U_\nu \). Then, there is one and only one \( s \in A(U) \) such that \( \rho_U^{U_\kappa} (s) = s_\kappa \) for all \( \kappa \in \Lambda \).

Proof. The proof is based on the well-established fact that the proposition holds true in case of the affine schemes. (cf. [1, Th. (1.3.7), p. 86].)

It is clearly enough to prove the proposition in case \( U = \mathfrak{S}p(A) \) and \( A(U) = A \). Assume so and write \( A = \lim \_ \_ A_i, \ X_i = \text{Spec}(A_i) \). For each \( \lambda \in \Lambda \) and each \( i \in \mathbb{N} \), write \( f_\lambda = (\cdots \leftarrow i f_\lambda \leftarrow \cdots) \) and let

\[ U_{\lambda, i} := \{ \pi_i^{-1}(P) : P \in X_i \text{ and } i f_\lambda \notin P \} = \pi_i^{-1} (D(if_\lambda)) \]

where \( D(if_\lambda) \) is the basic open set in \( X_i = \text{Spec} A_i \). We then have two types of open coverings for each \( \lambda \) and each \( i \), i.e.,

\[ U_\lambda = \bigcup_{i \in \mathbb{N}} U_{\lambda, i} \quad \text{and} \quad X_i = \bigcup_{\lambda \in \Lambda} D(if_\lambda). \]
[Uniqueness] Let \( s', s'' \in A = A(U) \) be such that \( p^{U}_{\kappa}(s') = s_{\kappa} \), \( p^{U}_{\kappa}(s'') = s_{\kappa} \) for all \( \kappa \in \Lambda \). So, one may write \( s' = (\cdots \leftarrow i_{s'} \leftarrow \cdots) \) and \( s'' = (\cdots \leftarrow i_{s''} \leftarrow \cdots) \), with \( i_{s'} \in A_{i} \), \( i_{s''} \in A_{i} \) for each \( i \). Now, since \( p^{U}_{\kappa}(s') = p^{U}_{\kappa}(s'') \) for all \( \kappa \), these agree on \( U_{\kappa;i} \) for all \( i \) in the first covering of (15), or \( i(p^{U}_{\kappa}(s')) = i(p^{U}_{\kappa}(s'')) \). This means that \( i_{s'} \) and \( i_{s''} \) agree on each piece \( D(i_{\kappa}) \) of the second covering of (15) for each \( \kappa \). It follows that \( i_{s'} = i_{s''} \) on \( X_{i} \) for each \( i \), because of the fact pointed out at the beginning of the proof. Therefore, we have \( s' = s'' \).

[Existence] We are locally given \( s_{\kappa} \) on \( U_{\kappa} \) for all \( \kappa \) such that \( s_{\lambda} \) and \( s_{\nu} \) agree on \( U_{\lambda} \cap U_{\nu} \) whenever the intersection is nonempty. The data will then induce, at each finite level \( i \), the data of \( \{ i(s_{\kappa}) : \kappa \in \Lambda \} \) locally on each open piece \( D(i_{\kappa}) \) of the covering \( X_{i} = \bigcup_{\kappa \in \Lambda} D(i_{\kappa}) \). We can patch up the local data of \( i(s_{\kappa})'s \) on the affine scheme \( X_{i} \) so as to obtain \( s_{i} \in A_{i} \). What remains to be checked out is that \( (\cdots \leftarrow s_{i} \leftarrow s_{i+1} \leftarrow \cdots) \) is coherently defined. So, let \( s'_{i} := i_{i+1}(s_{i+1}) \), and we will show that \( s_{i} = s'_{i} \). Now, denote the restriction map of \( X_{i} \) to \( D(i_{\kappa}) \) by \( \rho_{i;\kappa} \). We have thus \( \rho_{i;\kappa} : A_{i} \rightarrow (A_{i})^{\kappa} \). By construction, \( \rho_{i;\kappa}(s_{i}) = i(s_{\kappa}) \) and \( \rho_{i+1;\kappa}(s_{i+1}) = i_{i+1}(s_{\kappa}) \). It follows that

\[
\rho_{i;\kappa}(s'_{i}) = i_{i+1}(\rho_{i}(s_{i})) = \rho_{i+1}(\rho_{i+1;\kappa}(s_{i+1})) = \rho_{i+1}(i_{i+1}(s_{\kappa})) = i(s_{\kappa}),
\]

with \( \rho_{i+1} : (A_{p_{i}})_{i_{i+1}} \rightarrow (A_{p_{i}})^{i_{i+1}} \) standing for the map induced by \( \rho_{i+1} : A_{i+1} \rightarrow A_{i} \). It is now shown that \( \rho_{i;\kappa}(s_{i}) = \rho_{i;\kappa}(s'_{i}) \) for all \( \kappa \in \Lambda \). Once again one draws upon the uniqueness in the affine-scheme case to conclude that \( s_{i} = s'_{i} \).

We now extend the presheaf \( A \) to a presheaf over the topological space \( \mathcal{S}p(A) \) by defining, for any open set \( U \subseteq \mathcal{S}p(A) \), \( A(U) \overset{\text{def.}}{=} \lim_{\longrightarrow} A(V) \) where the \( \lim_{\longrightarrow} \) is taken over all basic \( V \)'s for which \( V = D(g) \subseteq U \) [1, chap. 0-3.2, pp. 25ff]. The extended presheaf will be denoted by \( \mathcal{A} \), too. The next theorem follows immediately from Prop. 2.2.2. (cf. [1, loc. cit.].)

**Theorem 2.2.3.** The presheaf \( A \) is a sheaf.

From here on, the topological space \( \mathcal{S}p(A) \) endowed with the sheaf \( A \) as above will be referred to as the **ind-affine scheme associated with** \( A \) and will be denoted by \( \mathcal{X}_{A} \). \( A \) is then, by definition, the **structure sheaf** of \( \mathcal{X}_{A} \). In conformity with standard practice in scheme theory we shall also write \( A = \mathcal{O}(A) \). Similarly, the topological space \( \mathcal{S}p_{m}(A) \) together with the sheaf induced on it from \( A \) is called the **ind-affine variety associated with** \( A \), and this variety will be denoted by \( \mathcal{V}_{A} \).

We next address the issue of stalks of the sheaf \( A \). Let \( \mathcal{X}_{A} \) be an ind-affine scheme, with \( A = \lim_{\longrightarrow} A_{i} \). Let \( p \) be a point on \( \mathcal{X}_{A} \), and let \( \Lambda := \) the filter of all basic open sets containing the point \( p \), so \( \Lambda = \{ D(g_{\alpha}) : \alpha \in D(g_{\alpha}) \} \). Let us write \( A_{i;\alpha} := (A_{i})_{g_{\alpha}} \) for all \( i \in \mathbb{N} \) and all \( g_{\alpha} \) for which \( D(g_{\alpha}) \in \Lambda \). We then have the following commutative diagram in which all horizontal arrows represent surjections and vertical ones are restrictions occurring whenever \( D(g_{\alpha}) \subseteq D(g_{\beta}) \), each column thus...
forming a direct system:

(17)

\[
\begin{array}{ccccccccc}
\cdots & \leftarrow & A_{i_1}^{-1, \alpha} & \leftarrow & A_{i_2}^{\alpha} & \leftarrow & \cdots & \leftarrow & \lim_{\rightarrow \gamma} A_{n, \alpha} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
\cdots & \leftarrow & A_{i_1}^{-1, \beta} & \leftarrow & A_{i_2}^{\beta} & \leftarrow & \cdots & \leftarrow & \lim_{\rightarrow \gamma} A_{n, \beta} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
& & \cdots & \downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow \\
\cdots & \leftarrow & \lim_{\rightarrow \gamma} A_{i_1}^{-1, \gamma} & \leftarrow & \lim_{\rightarrow \gamma} A_{i_2}^{\gamma} & \leftarrow & \cdots & \leftarrow & \lim_{\rightarrow \gamma} \lim_{\rightarrow \gamma} A_{n, \gamma} \\
\end{array}
\]

In the diagram (17) one should recognize that \(\lim_{\rightarrow i} A_{n, \alpha} = A_{g, \alpha} = \mathcal{A}(D(g, \alpha))\), and \(\lim_{\rightarrow \gamma} A_{n, \gamma} = (A_{m})_{g, \beta}\). So, the map \(\Phi\) on the lower right corner of (17) amounts to \(\lim_{\rightarrow \gamma} \mathcal{A}(D(g, \alpha)) = \lim_{\rightarrow \gamma} A_{g, \gamma} \phi \lim_{\rightarrow m} (A_{m})_{g, \beta}\), and \(\Phi\) gets induced as follows: (i) First, for each \(\alpha\) there is a map \(A_{g, \gamma} \rightarrow \lim_{\rightarrow \gamma} A_{i, \gamma}\) for all \(i\) with appropriate commutativity of arrow paths; (ii) as a consequence there is a map \(A_{g, \gamma} \rightarrow \lim_{\rightarrow \gamma} (\lim_{\rightarrow \gamma} A_{j, \lambda}) = \lim_{\rightarrow \gamma} (\lim_{\rightarrow \gamma} (A_{j})_{g, \beta})\); and finally (iii) the desired map \(\lim_{\rightarrow \gamma} A_{g, \gamma} \rightarrow \lim_{\rightarrow \gamma} (\lim_{\rightarrow \gamma} (A_{j})_{g, \beta})\) again because of the appropriate commutativity.

We now come to study the map \(\Phi\). In order to describe its kernel, we need to introduce the notion of \textit{elements infinitely near 0} in the ring \(\lim_{\rightarrow, \alpha} (\lim_{\rightarrow, \gamma} A_{n, \alpha})\) and, before that, a new \textit{ad hoc} notation: If \(a_l \in A_{i, \alpha}\) then \([a_l]\) denotes the equivalence class represented by \(a_l\) in the direct limit \(\lim_{\rightarrow, \gamma} A_{i, \gamma} = (A_l)_g\). Likewise, if \((\cdots \leftarrow a_{q-1, \gamma} \leftarrow a_{q, \gamma} \leftarrow \cdots) \in \lim_{\rightarrow, \gamma} A_{n, \gamma} = \mathcal{A}(D(g, \gamma))\), then \((\cdots \leftarrow a_{q-1, \gamma} \leftarrow a_{q, \gamma} \leftarrow \cdots)\) is to mean the corresponding equivalence class \((\lim_{\rightarrow, \gamma} \lim_{\rightarrow, \alpha} A_{n, \gamma}) = \lim_{\rightarrow, \alpha} \mathcal{A}(D(g, \alpha))\). Now, let

(18) \(u := ([\cdots \leftarrow u_{n-1, \alpha} \leftarrow u_{n, \alpha} \leftarrow \cdots]) \in \lim_{\rightarrow, \alpha} (\lim_{\rightarrow, \gamma} A_{n, \alpha}) = \lim_{\rightarrow, \alpha} \mathcal{A}(D(g, \alpha))\),

We shall say that \(u\) is \textit{infinitely near 0} if \(\forall u_{n, \alpha} \exists \beta = \beta(n, \alpha) \geq \alpha\) such that \(u_{n, \beta} = 0\) under the restriction map due to the inclusion \(D_{g, \beta} \subseteq D_{g, \alpha}\). The terminology is appropriate because, for such \(u\), \([u_{n, \alpha}] = [0]\) for every \(n\), yet \(u\) may not be 0.

It is easy to see that the set of all elements of \(R := \lim_{\rightarrow, \alpha} (\lim_{\rightarrow, \gamma} A_{n, \alpha}) = \lim_{\rightarrow, \alpha} \mathcal{A}(D(g, \alpha))\) that are infinitely near 0 form an ideal of the ring \(R\).
Theorem 2.2.4. (a) Let $R$ be as just above. Then, the kernel of the map $\Phi: R \to \lim_{\to jn}(A_{m,jp})$ is the ideal of all elements infinitely near 0 in $R$.
(b) The image of $\Phi$ is everywhere dense in $\lim_{\to jn}(A_{m,jp})$.

Proof. (a) If $\Phi(\mu) = 0$ for $\mu$ as in (18), that means $(\cdots \leftarrow [u_{n-1,\alpha}] \leftarrow [u_{n,\alpha}] \leftarrow \cdots) = (\cdots \leftarrow 0 \leftarrow 0 \leftarrow \cdots)$ inside $\lim_{\to jn}(A_{m,jp})$, or $\forall n$, $[u_{n,\alpha}] = 0$. So, $\mu$ is infinitely near 0. The converse clearly holds also.

(b) Given $\eta = (\cdots \leftarrow [u_{i-1,\alpha_{i-1}}] \leftarrow [u_{i,\alpha_i}] \leftarrow \cdots) \in \lim_{\to jn}(A_{m,jp})$, write $\eta = (\cdots \leftarrow r_{i-1} \leftarrow r_i \leftarrow \cdots)$ with each $r_i \in (A_i)_{jp}$. For an arbitrary high $N > 0$, let $w_N := u_{N,\alpha_N} \in A_{N,\alpha_N}$. Clearly, one can complete $w_N$ to an element

$$w = (\cdots \leftarrow w_{N-1} \leftarrow w_N \leftarrow w_{N+1} \leftarrow \cdots) \in \lim_{\to jn} A_{N,\alpha_N}$$

such that

$$[w_0] = [u_{0,\alpha_0}], \quad [w_1] = [u_{1,\alpha_1}], \ldots, [w_{N-1}] = [u_{N-1,\alpha_{N-1}}], \quad [w_N] = [u_{N,\alpha_N}].$$

So, $[w] := [\cdots \leftarrow w_{N-1} \leftarrow w_N \leftarrow w_{N+1} \leftarrow \cdots] \in R$ is such that $\Phi([w])$ and $\eta$ agree with each other up to the $N$-th place from the left. Since $N$ was arbitrary, this shows the density of the image of $\Phi$. 

In view of Th. 2.2.4, we define the local ring of a point $p$ on an ind-affine scheme $X_A$, $A = \lim_{\to jn} A_n$, to be $\lim_{\to jn}(A_{m,jp})$. It is a pro-affine $K$-algebra, and a surjective inverse limit of local rings of the more traditional type.

3. Comments and Examples

(A) The reduction $A_{\text{red}}$ and the strong reduction $A_{\text{RED}}$ (see §1.3-(7) above):

In [2] we raised the question as to whether or not $A_{\text{red}} = A_{\text{RED}}$ for the types of pro-affine algebras $A$ of interest to us, and we indicated how this issue bears upon the Jacobian Problem (cf. [2, (1.3), p. 484, and (5.4), p. 498]). As expected, this question is easily answered in the negative, as follows:

Example 3-E. For all $i \in \mathbb{N}$ consider the same algebras $A_i$ as occurred in [2, Ex. (1.4), p. 484] but with different connecting maps $\mu_i$. Namely, let

$$A_i := K[T_i, \cdots, T_{i-1}, T_i, T_{i+1}]/(T_i^2, T_{i+1}) = K[T_i, \cdots, T_{i-1}, T_i, r_{i+1}],$$

and define $\mu_i: A_i \to A_{i-1}$ by stipulating

$$\mu_i(T_j) := T_j \text{ for } j < i; \quad \mu_i(T_i) := r_i; \quad \mu_i(r_{i+1}) := \tau_i \cdot T_i.$$
Then, in the exact sequence
\[(19) \quad 0 \to \langle \tau_{i+1} \rangle \to A_i \to K[T_1, \ldots, T_i] \to 0\]
for all \(i > 0\), the Mittag-Leffler condition fails to hold, so that the sequence
\[(20) \quad 0 \to \mathcal{N}(A) \to A \to A_{\text{RED}} \to 0, \quad \text{where } A_{\text{RED}} = K^{[\langle \infty \rangle]}\]
onobtained by applying \(\lim_{\to i}\) to (19), is expected to be nonexact on the right.

We can actually exhibit where the map \(A \to A_{\text{RED}}\) fails to be surjective. In fact, let
\[f_i := T_1 + \cdots + T_{i-1} + T_i \quad \text{for all } i \in \mathbb{N},\]
and consider \(f := (f_1 \leftarrow \cdots \leftarrow f_{i-1} \leftarrow f_i \leftarrow \cdots) \in A_{\text{RED}}\). Suppose that there existed some \(g \in A\) such that \(g = (g_1 \leftarrow \cdots \leftarrow g_t \leftarrow \cdots) \mapsto f \in A_{\text{RED}}\). Then, for each \(i \in \mathbb{N}\), it must hold that \(g_i = f_i + \tau_{i+1} \cdot h_i = f_{i-1} + T_i + \tau_{i+1} \cdot h_i\) for a suitable \(h_i \in K[T_1, \ldots, T_{i-1}, T_i]\). On the other hand, \(\mu_i(g_i) = g_{i-1}\), or
\[(21) \quad f_{i-1} + \tau_i + \tau_{i+1} : T_i \cdot h_i(T_1, \cdots, T_{i-1}, \tau_i) = f_{i-1} + \tau_i + \tau_{i+1} : T_i \cdot h_{i-1}(T_1, \cdots, T_{i-1}, 0) = f_{i-1} + \tau_i \cdot h_{i-1},\]
which implies that
\[(22) \quad h_{i-1} = 1 + T_1 \cdot h_i(T_1, \cdots, T_{i-1}, 0) \quad \text{for all } i \in \mathbb{N}.\]

Using this last equation recursively, one would get
\[(23) \quad h_1(T_1) = 1 + T_1 \cdot h_2(T_1, 0) = 1 + T_1 + T_1^2 \cdot h_3(T_1, 0, 0) = \cdots = 1 + T_1^2 + \cdots + T_1^{k-1} \cdot h_k(T_1, 0, \ldots, 0) = \cdots (ad \ infin.).\]

This lends an arbitrarily high \(T_1\)-degree to the polynomial \(h_1(T_1)\), an absurdity.

(B) Closed Embedding and Topology of Ind-affine schemes:

Let \(A, B\) be pro-affine algebras, and \(X := \mathcal{X}_A, Y := \mathcal{X}_B\). A morphism of ind-affine schemes \(f : Y \to X\) defined by a continuous \(K\)-map \(\phi : A \to B\) is said to be a closed embedding if \(\phi\) is open and surjective. When that is so, through appropriate representations \(A = \lim_\to A_i, B = \lim_\to B_i\) of \(A\) and \(B\) as inverse limits, one may see to it that \(\phi\) is induced by surjections \(A_i \to B_i\) for all \(i \in \mathbb{N}\). One can then say that the closed embedding \(Y \to X\) is the direct limit of the closed embeddings \(Y_i \to X_i\) for all \(i \in \mathbb{N}\).
for all \( i \). The converse is inexact. Namely, if \( \phi: A \to B \) comes as the inverse limit of surjective \( K \)-maps \( A_i \to B_i \) for all \( i \), \( \phi \) need not be surjective. In other words, if \( f: Y \to X \) is induced by closed embeddings \( Y_i \to X_i (\forall i) \) of finite-dimensional affine \( K \)-schemes \( X_i = \text{Spec}(A_i), Y_i = \text{Spec}(B_i) \), \( f \) need not be a closed embedding of ind-affine schemes. This point is illustrated by the following example:

**Example 3-F** (Burt Totaro). Let \( X := \mathbb{A}^{\infty} = \mathbb{A}^{K[\infty]} \), so \( X = \bigcup_{i=1}^{\infty} X_i \) with \( X_i = \mathbb{A}^i \). Define a subscheme \( Y = \bigcup_{i=1}^{\infty} Y_i \) of \( X \) inductively, as follows: (a) \( Y_1 := X_1 = \mathbb{A}^1 \).
(b) Having built \( Y_{i-1} \), define \( Y_i \) to be the union of \( Y_{i-1} \) and a finite set of lines through the origin in \( X_i \) such that every polynomial function on \( X_i \) of degree \( \leq i \) which vanishes on these lines must be 0 altogether on \( X_i \). [Just take enough number of lines on \( X_i \) through the origin and in general position.]

Now consider the morphism \( f: Y \to X \) arising as the dual of the natural map, \( \mathcal{O}(X) := \varprojlim_{i} \mathcal{O}(X_i) \to \mathcal{O}(Y) := \varprojlim_{i} \mathcal{O}(Y_i) \), where the maps \( \mathcal{O}(X_i) \to \mathcal{O}(Y_i) \) are surjections associated with the closed embeddings \( Y_i \to X_i \) for all \( i \). This \( f \) exhibits some pathological characters, as shall be seen now.

(a) First, let \( J_i := \ker(\mathcal{O}(X_i) \to \mathcal{O}(Y_i)) \). Then, \( J_i \) is a homogeneous ideal in \( K[i] \) whose generators may be taken to be forms of degree \( \geq i \). This shows that the exact sequences \( 0 \to J_i \to \mathcal{O}(X_i) \to \mathcal{O}(Y_i) \to 0 \) taken for all \( i \in \mathbb{N} \) do not satisfy the Mittag-Leffler condition, and the non-surjectiveness of \( \mathcal{O}(X) \to \mathcal{O}(Y) \) is strongly indicated.

(b) Second, let \( m_{x_i}, m_{y_i} \) be the maximal ideals of the origin \((0)\) on \( X_i, Y_i \) in the rings \( \mathcal{O}(X_i), \mathcal{O}(Y_i) \), respectively. Then, for every pair of \( r \) and \( i \) with \( 0 < r \leq i \), the natural surjection

\[
\psi_{r,i}: \mathcal{O}(X_i)/m_X^r \to \mathcal{O}(Y_i)/m_Y^r
\]

is also injective because of the make-up of \( Y_i \), so that \( \psi_{r,i} \) is an isomorphism. It follows that \( \psi_r := \lim_{i \to \infty}(\psi_{r,i}) \) gives an isomorphism \( \mathcal{O}(X)/m_X^r \simeq \mathcal{O}(Y)/m_Y^r \) for all \( r > 0 \). Consequently, \( m_X/m_X^{(2)} \simeq m_Y/m_Y^{(2)} \) and \( m_X/m_X^{(r+1)} \simeq m_Y/m_Y^{(r+1)} \). Since the point \((0)\) on \( X \) satisfies the smoothness condition that \( \mathcal{H}^0(m_X/m_X^{(n)}) \to \mathcal{H}^0(m_Y/m_Y^{(n)}) \) be an isomorphism for all \( n > 0 \) (see [2, p. 488]), so does \((0)\) on \( Y \), or \( Y \) is smooth at \((0)\).

We can see that this creates a serious problem for the notion of smoothness of ind-affine varieties, as calling the point \((0)\) a simple point on \( Y \) goes against our intuition. It appears that the notion of smoothness (or of simple point) should be reworked (see [2, p. 488], [3, p. 187ff]). We will not, however, go into this issue in this paper. Turning to the more immediate question on Totaro’s example at hand, we find it impossible that the \( K \)-map \( \mathcal{O}(X) \to \mathcal{O}(Y) \) in (a) just above should be surjective, or that the morphism \( Y \to X \) should be a closed immersion. For, were this the case, then the embedding theorem [2, (2.6), p. 488] would imply that \( Y \) is isomorphic to \( X \) as ind-
affine scheme. It follows that, for every \( i \), \( Y_i \) is a closed subscheme of \( X_i \) but \( Y \to X \) is not a closed immersion.

(C) Example of a proper ideal whose closure is the unit ideal:

We follow up on Example 1.2-D and Remark that precedes it.

**Example 3-G.** Let

\[
w_1 := (1 \leftarrow 1 + x_1 \leftarrow 1 + x_1 + x_2 \leftarrow \cdots \leftarrow 1 + x_1 + x_2 + \cdots + x_n \leftarrow \cdots)
\]

\[
w_2 := (1 \leftarrow 1 \leftarrow 1 + x_2 \leftarrow 1 + x_2 + x_3 \leftarrow \cdots \leftarrow 1 + x_2 + \cdots + x_n \leftarrow \cdots)
\]

\[
\vdots
\]

\[
w_n := (1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow 1 + x_n \leftarrow 1 + x_n + x_{n+1} \leftarrow \cdots)
\]

\[
\vdots
\]

be a sequence of elements in \( K^{[\infty]} \). So, \( w_n - w_{n+1} = (0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow x_n \leftarrow x_n \leftarrow \cdots) \) and \( w_n - 1 = (0 \leftarrow \cdots \leftarrow 0 \leftarrow x_n \leftarrow x_n + x_{n+1} \leftarrow \cdots) \). It follows that \( \lim_{n \to \infty} w_n = 1 \) and \( \langle w_1, \ldots, w_n, \ldots \rangle = \langle 1 \rangle \). On the other hand, \( \langle w_1, \ldots, w_n, \ldots \rangle \subseteq \langle 1 \rangle \) because no finite linear combination of the \( w_i \)'s can equal \( 1 \). To be more specific, suppose \( L = 1 \) for an \( K^{[\infty]} \)-linear combination \( L \) of \( w_k, w_l, \ldots, w_m \) (\( k < l < \cdots < m \)), or \( \langle w_k, w_l, \ldots, w_m \rangle = \langle 1 \rangle \). Then, \( \langle w_1, \ldots, w_m \rangle = \langle w_1, w_1 - w_2, \ldots, w_{m-1} - w_m \rangle = \langle 1 \rangle \). This implies that an \( K^{[\infty]} \)-linear combination of

\[
w_1 = (1 \leftarrow 1 + x_1 \leftarrow \cdots \leftarrow 1 + x_1 + \cdots + x_m \leftarrow \cdots)
\]

\[
w_1 - w_2 = (0 \leftarrow x_1 \leftarrow x_1 \leftarrow \cdots \leftarrow x_1 \leftarrow \cdots)
\]

\[
w_2 - w_3 = (0 \leftarrow 0 \leftarrow x_2 \leftarrow \cdots \leftarrow x_2 \leftarrow \cdots)
\]

\[
\vdots
\]

\[
w_{m-1} - w_m = (0 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_{m-1} \leftarrow \cdots)
\]

should produce \( (1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots) \). Clearly, this is impossible.

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References


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