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SOME BASIC RESULTS ON PRO-AFFINE ALGEBRAS AND IND-AFFINE SCHEMES

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Introduction

The theory of ind-affine varieties was first introduced by Shafarevich, who then employed it to elucidate the structure of the automorphism group of the affine space. (see [3], [4].) More recently we made certain revisions on the theory and applied it to study the Jacobian Problem on the endomorphisms of the complex affine space. (see [2].)

Since these pieces of work appeared, there has not been much progress made. This state may be due, in part, to the fact that the basic theory of these ind-affine or pro-affine objects as presented by us was still *ad hoc* and was rather rudimentary. So, we have embarked on building a theory of pro-affine algebras and ind-affine schemes anew and from the ground up. The outcome of our effort is the contents of the present paper. As we worked on the material we encountered a number of subtle examples, as shown in the main text below. It would seem that these examples perhaps suggest richness and mystery that this theory holds.

We mention a piece of specific result we have of our theory: The set of all morphisms of an affine variety over a field K to another may be identified with the K -rational point set of an appropriately constructed ind-affine scheme over K . This was proven using the theory of Gröbner bases over K , and is expected to be published in the near future along with certain related results about automorphisms of the affine space.

1. Pro-affine algebras

1.1. Definitions and Notations. Throughout we work over a ground field K of any characteristic. A commutative topological K -algebra A is said to be a *pro-affine algebra* if

1. A is *complete* and *separated*.
2. A base of open neighborhoods of 0 is given by a family of *countably many* ideals $\subseteq A$.

Let $\{\mathfrak{a}_i : i \in \mathbb{N}\}$ be a countable base referred to just above. Here, as elsewhere throughout the present paper, \mathbb{N} represents the set of all *nonnegative* integers. We may,

and shall always, assume that $\mathfrak{a}_i \supseteq \mathfrak{a}_j$ whenever $i \leq j$. The condition 1 then implies that

$$(1) \quad \bigcap_{i \in \mathbb{N}} \mathfrak{a}_i = \{0\} \quad \text{and} \quad A \simeq \varprojlim_{i \in \mathbb{N}} (A_i),$$

where, for each $i \in \mathbb{N}$, $A_i := A/\mathfrak{a}_i$ is a *discrete* K -algebra, with all maps $\mu_i: A_i \rightarrow A_{i-1}$ being *surjective*. Conversely, a K -algebra given as the limit of a *countable, surjective* inverse system of *discrete* K -algebras in the form of (1) is evidently pro-affine in our sense.

One recognizes then that a pro-affine K -algebra as above is the same thing as a “filtered commutative K -algebra which is complete and separated” in the sense of Northcott [5, Chap. 9].

Proposition 1.1.1. *Let A and B be pro-affine algebras. Then, the product $A \times B$ and the complete tensor product $A \hat{\otimes}_K B$ are both pro-affine K -algebras.*

Proof seems hardly necessary. If $\{\mathfrak{a}_i : i \in \mathbb{N}\}$ and $\{\mathfrak{b}_j : j \in \mathbb{N}\}$ are bases of open neighborhoods of 0 for A and B , respectively, then one adopts for $A \times B$ the ideals $\{\mathfrak{a}_k \times \mathfrak{b}_k : k \in \mathbb{N}\}$ as a base of open neighborhoods of 0. As for $A \hat{\otimes}_K B$, take the ideals $\{\mathfrak{a}_k \otimes B + A \otimes \mathfrak{b}_k : k \in \mathbb{N}\}$ as a base of open neighborhoods of $A \otimes_K B$, and then take its completion. □

A pro-affine algebra A is said to be *algebraic over K* , or *K -algebraic*, if A can be represented as in (1) where *all* A/\mathfrak{a}_i are *finitely generated over K* .

Let A, B be pro-affine K -algebras. A *morphism of A to B* is defined to be a *continuous K -algebra map* $\phi: A \rightarrow B$. Suppose that A and B are represented as $A = \varprojlim (A/\mathfrak{a}_i)$, $B = \varprojlim (B/\mathfrak{b}_j)$, respectively. Then, the morphism $\phi: A \rightarrow B$ gives rise to a commutative diagram

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \pi_i^A \downarrow & & \downarrow \pi_j^B \\ A/\mathfrak{a}_i & \xrightarrow{\phi_{ji}} & B/\mathfrak{b}_j \end{array}$$

standing valid for each given $j \in \mathbb{N}$ and for some corresponding $i = i(j) \in \mathbb{N}$ for which $\phi(\mathfrak{a}_i) \subseteq \mathfrak{b}_j$. Here, π_i^A and π_j^B denote the canonical residue-class maps, and $\phi_{ji}(x + \mathfrak{a}_i) \stackrel{\text{def.}}{=} \phi(x) + \mathfrak{b}_j$ for all $x \in A$.

NOTATIONS. Let us fix some notations we shall be using throughout this paper:

(a) Let $A = \varprojlim(A_i)$ be a pro-affine algebra, where we have put $A_i := A/\mathfrak{a}_i$ as before. The canonical surjective maps $A \rightarrow A_i$ and $A_j \rightarrow A_i$ for $i \leq j$ shall be denoted as follows:

$$(3) \quad \pi_i : A \longrightarrow A_i ; \quad \mu_{ij} : A_j \longrightarrow A_i,$$

with $\text{Ker}(\pi_i) = \mathfrak{a}_i$, and $\mu_{ii} = \text{Id}_{A_i}$. We abbreviate $\mu_{i-1,i}$ as μ_i .

(b) As a rule, for any subset $E \subseteq A$ or any element $a \in A$, we denote $\pi_i(E)$ by ${}_iE$ and $\pi_i(a)$ by ${}_ia$. (A notable exception is $\pi_i(A) = A/\mathfrak{a}_i$ which we denote by A_i and *not* by ${}_iA$.) When no fear of confusion is present, we often skip the left suffix and simply write a for ${}_ia$, so that $a = (\cdots \leftarrow {}_{i-1}a \leftarrow {}_ia \leftarrow \cdots)$ is expressed as $(\cdots \leftarrow a \leftarrow a \leftarrow \cdots)$. A sequence $\sigma := (\cdots \leftarrow s_{i-1} \leftarrow s_i \leftarrow \cdots)$ with $s_j \in A_j$ for all $j \in \mathbb{N}$ represents an element of A and thus $\sigma \in A$ if and only if $\mu_j(s_j) = s_{j-1}$ for all j , in which case we shall say σ is *coherent*.

In the notations above, it is then clear that the closure \overline{E} of E may be identified with $\varprojlim({}_iE)$. Thus, $E \subseteq A$ is *closed* if and only if every coherent sequence $\epsilon = (\cdots \leftarrow e_i \leftarrow \cdots)$ belongs to E as soon as all $e_i \in {}_iE$ for $i \in \mathbb{N}$.

Proposition 1.1.2. *The group of units $U(A)$ of a pro-affine algebra A is closed.*

Proof. Let $u = (\cdots \leftarrow u_{i-1} \leftarrow u_i \leftarrow \cdots) \in \overline{U(A)}$. For each i there is a unique $v_i \in A_i$ with $u_i \cdot v_i = 1_{A_i}$. Then, $v = (\cdots \leftarrow v_{i-1} \leftarrow v_i \leftarrow \cdots)$ is clearly coherent and satisfies $u \cdot v = 1$ so that $u \in U(A)$. □

EXAMPLE 1.1-A (cf. [2, (1.1), p. 482]). For each $n \in \mathbb{N}$, let $K^{[n]} := K[X_1, \dots, X_n]$ if $n > 0$, and $K^{[0]} := K$. Define $\mu_n : K^{[n]} \rightarrow K^{[n-1]}$ by setting $\mu_n(X_i) := X_i$ for all $1 \leq i \leq n-1$, and $\mu_n(X_n) := 0$. Denote

$$K^{[\infty]} := \varprojlim(K^{[n]})$$

and call it *the pro-affine polynomial algebra (over K)*. This algebra may be characterized as the set of those formal power-series on X_1, \dots, X_m, \dots which become polynomials when reduced modulo all but finitely many X_i 's.

1.2. The ideals in a pro-affine algebra.

Proposition 1.2.1. *Let \mathfrak{h} be a closed ideal in $A = \varprojlim(A_i)$. Then,*

$$A/\mathfrak{h} \simeq \varprojlim(A_i)/\varprojlim({}_i\mathfrak{h}) \simeq \varprojlim(A_i/{}_i\mathfrak{h}).$$

[This implies that A/\mathfrak{h} is a pro-affine algebra for any closed ideal \mathfrak{h} .]

Proof. Since \mathfrak{h} is closed, $\mathfrak{h} \simeq \lim_{\leftarrow} ({}_i\mathfrak{h})$ and all maps ${}_i\mathfrak{h} \rightarrow {}_{i-1}\mathfrak{h}$ are surjective. So, in the diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & {}_i\mathfrak{h} & \longrightarrow & A_i & \longrightarrow & A_i/{}_i\mathfrak{h} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & {}_{i-1}\mathfrak{h} & \longrightarrow & A_{i-1} & \longrightarrow & A_{i-1}/{}_{i-1}\mathfrak{h} & \longrightarrow & 0
 \end{array}$$

all vertical maps are surjective. One now applies the functor \lim_{\leftarrow} to this diagram, remembering the Mittag-Leffler condition which holds here. □

EXAMPLE 1.2-B. In the same notations as in Ex. 1.1-A, define an ideal $J_n \subset K^{[n]}$ by $J_n := \langle X_i X_j \mid 1 \leq i < j \leq n \rangle$, so geometrically the locus of J_n is the union of all coordinate axes in the affine n -space \mathbb{A}^n over K . Let $B_n := K^{[n]}/J_n = K[x_1, \dots, x_n]$. Consider the exact sequence

$$0 \longrightarrow J_n \longrightarrow K^{[n]} \longrightarrow B_n \longrightarrow 0$$

and take the \lim_{\leftarrow} of this sequence on all $n \in \mathbb{N}$. Since, for all n , $\mu_n: K^{[n]} \rightarrow K^{[n-1]}$ causes a surjection of J_n to J_{n-1} , there results a surjective K -map $K^{[\infty]} \rightarrow B := \lim_{\leftarrow} B_n$, and its kernel $J := \lim_{\leftarrow} J_n$ gives an example of a closed ideal in $K^{[\infty]}$. [In the subsequent B will be viewed as the coordinate algebra $\mathcal{O}(Y)$ of the closed subscheme Y of all coordinate axes in the ind-affine space \mathbb{A}^∞ .]

EXAMPLE 1.2-C. In Example 1.2-B replace each J_n by $J'_n := \langle X_1 \cdots X_n \rangle$, whose locus in \mathbb{A}^n is then the union of all coordinate hyperplanes in \mathbb{A}^n . Since the surjections $\mu_n: K^{[n]} \rightarrow K^{[n-1]}$ all cause zero maps of J'_n into J'_{n-1} , the Mittag-Leffler condition is trivially satisfied, and $J' := \lim_{\leftarrow, n} J'_n = \{0\}$ (which is a closed ideal in $K^{[\infty]}$). It follows that $K^{[\infty]} \simeq \lim_{\leftarrow, n} (K^{[n]}/J'_n)$. [So, the union of all coordinate hyperplanes in \mathbb{A}^n , as $n \rightarrow \infty$, is isomorphic to the whole ind-affine space \mathbb{A}^∞ .]

Proposition 1.2.2. For any maximal ideal $\mathfrak{m} \subset A$, the following conditions are equivalent to one another:

- (i) \mathfrak{m} is closed;
- (ii) For some i , $\pi_i(\mathfrak{m}) = {}_i\mathfrak{m} \subsetneq A_i$;
- (iii) For some i , $\mathfrak{a}_i \subseteq \mathfrak{m}$;
- (iv) For some i , $\mathfrak{m} = \pi_i^{-1}(\text{some maximal ideal in } A_i)$;
- (v) \mathfrak{m} is open.

Proof. (i) \Rightarrow (ii) : If ${}_i\mathfrak{m} = A_i$ for all i , then $(1 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots) \in \overline{\mathfrak{m}} = \mathfrak{m}$, so that $\mathfrak{m} = A$.

(ii) \Rightarrow (iii) : Let ${}_i\mathfrak{m} \subset A_i$ for a particular i . Then, ${}_i\mathfrak{m}$ must be a maximal ideal in

A_i , and $\pi_i^{-1}(i\mathfrak{m}) = \mathfrak{m} + \mathfrak{a}_i = \mathfrak{m}$, so $\mathfrak{a}_i \subseteq \mathfrak{m}$.

The implications (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i) are obvious. □

The same argument as used in (i) \Rightarrow (ii) above shows the following:

Corollary 1.2.3. *Every closed proper ideal in a pro-affine algebra A is contained in a closed maximal ideal.*

Proposition 1.2.4. *For any prime ideal $\mathfrak{p} \subset A$, the following conditions are equivalent to one another:*

1. \mathfrak{p} is open;
2. For some i , $\mathfrak{p} = \pi_i^{-1}(i\mathfrak{p})$;
3. For some j and a prime ideal $\mathfrak{q}_j \subset A_j$, $\mathfrak{p} = \pi_j^{-1}(\mathfrak{q}_j)$.

The proof of this obvious proposition is omitted.

Note that, in view of the two preceding propositions, the *open prime* (resp. *open maximal*) ideals of a pro-affine algebra A are precisely the inverse images of the *prime* (resp. *maximal*) ideals of the A_i 's for any $i \in \mathbb{N}$.

Proposition 1.2.5. *Let \mathfrak{a} be a finitely generated proper ideal in a pro-affine algebra A . Then, there exists an open maximal ideal \mathfrak{m} such that $\mathfrak{a} \subseteq \bar{\mathfrak{a}} \subseteq \mathfrak{m}$.*

We first prove the following key lemma due to N. Mohan Kumar:

Lemma 1.2.6 (N. Mohan Kumar). *Let $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$ be a finitely-generated ideal, and let $\bar{\mathfrak{a}}$ be its closure. For any $z \in A$, if $z \in \bar{\mathfrak{a}}$ then $z^{2^n} \in \mathfrak{a}$.*

Proof. The proof goes by induction on the number of generators n . First, take any $x \in A$ and let $z \in \overline{\langle x \rangle} = \lim_{\leftarrow} (A_i \cdot i x)$. Write

$$z = (a_0x \leftarrow a_1x \leftarrow \dots \leftarrow a_i x \leftarrow \dots), a_j \in A_j \quad \text{for all } j \in \mathbb{N},$$

where the coherence condition

$$(4) \quad \mu_i(a_i x) - a_{i-1}x = \mu_i(a_i \cdot i x) - a_{i-1} \cdot i_{i-1}x = (\mu_i(a_i) - a_{i-1}) \cdot i_{i-1}x = 0$$

is satisfied. Then, $\eta \stackrel{\text{def.}}{=} (a_0^2x \leftarrow a_1^2x \leftarrow \dots \leftarrow a_i^2x \leftarrow \dots)$ is coherent, as one sees from (4) that

$$\begin{aligned} \mu_i(a_i^2 \cdot i x) - a_{i-1}^2 \cdot i_{i-1}x &= (\mu_i(a_i)^2 - a_{i-1}^2) \cdot i_{i-1}x \\ &= (\mu_i(a_i) + a_{i-1})(\mu_i(a_i) - a_{i-1}) \cdot i_{i-1}x = 0. \end{aligned}$$

So $\eta \in A$. It follows that $z^2 = x\eta \in \langle x \rangle \subseteq A$.

Turning now to the next induction step, we let $z \in \overline{\langle x_1, \dots, x_n \rangle}$. Set $A' \stackrel{\text{def.}}{=} A/\overline{\langle x_1 \rangle}$, and consider its ideal $\overline{\langle x'_2, \dots, x'_n \rangle}$, where x'_2, \dots, x'_n denote the canonical images of x_2, \dots, x_n , respectively, in A' . Let $z' := z \text{-mod } \overline{\langle x_1 \rangle} \in \overline{\langle x'_2, \dots, x'_n \rangle}$. By induction hypothesis, $z'^{2^{n-1}} \in \langle x'_2, \dots, x'_n \rangle$. This implies that one can write $z'^{2^{n-1}} = z_1 + z_2$, where

$$z_1 \in \overline{\langle x_1 \rangle} \text{ and } z_2 \in \langle x_2, \dots, x_n \rangle.$$

But we saw just above that $z_1 \in \overline{\langle x_1 \rangle}$ gives $z_1^2 \in \langle x_1 \rangle$. Therefore,

$$z^{2^n} = (z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2 \in \langle x_1 \rangle + \langle x_2, \dots, x_n \rangle,$$

and we find $z^{2^n} \in \langle x_1, x_2, \dots, x_n \rangle$, as desired. \square

Proof of Proposition 1.2.5. now follows immediately from this lemma. Indeed, if a finitely-generated ideal \mathfrak{a} is such that $\bar{\mathfrak{a}} = A$, then $1 \in \bar{\mathfrak{a}}$, which implies $1 = 1^{2^n} \in \mathfrak{a}$ for some n . So, if \mathfrak{a} is proper, then $\bar{\mathfrak{a}}$ is proper; and one now applies Cor. 1.2.3. \square

REMARK. Proposition 1.2.5 fails to hold for ideals *not* finitely generated, as will be shown in §3 below (see Ex. 3-G). Also note that a finitely generated ideal need not be closed. In fact, even a principal ideal can be non-closed, as the following example shows:

EXAMPLE 1.2-D (N. Mohan Kumar). Let $K^{[2]} := K[X, Y]$ be a polynomial ring in X and Y , and for each $i \in \mathbb{N}$ let $A_i := K^{[2]}/\langle XY^{i+1} \rangle = K[x, y]$, with x, y standing for the canonical images of X, Y , respectively, in A_i . Let our pro-affine algebra A be $\lim_{\leftarrow} A_i$. Consider

$$\zeta := (x \leftarrow x(1+y) \leftarrow x(1+y+y^2) \leftarrow x(1+y+y^2+y^3) \leftarrow \dots).$$

Clearly, $\zeta \in \overline{\langle x \rangle}$. However, $\zeta \notin \langle x \rangle$. To see this, assume $\zeta \in \langle x \rangle$, and write $\zeta = x\eta$ for some $\eta \in A$. Then, η has to equal

$$(1+y p_1(y) \leftarrow 1+y+y^2 p_2(y) \leftarrow 1+y+y^2+y^3 p_3(y) \leftarrow \dots),$$

where $p_1(y), p_2(y), p_3(y), \dots$ are polynomials in y only. Now let, for each $i \in \mathbb{N}$, $f_i: A_i \rightarrow A_i/\langle x \rangle \simeq K[y]$ be the canonical mod- x map. Then,

$$f := \lim_{\leftarrow} f_i: \lim_{\leftarrow} A_i = A \longrightarrow K[y]$$

should map η to a polynomial in $K[y]$ of a certain degree, say of degree d . Since $f(\eta) = f_{d+1}(1+y+\dots+y^{d+1}+y^{d+2}p_{d+2}(y)) = 1+y+\dots+y^{d+1}+y^{d+2}p_{d+2}(y) \in K[y]$ is of degree at least $d+1$, there results a contradiction.

1.3. The radicals and Nullstellensatz. The radical $\mathcal{R}(A)$ and the nilradical $\mathcal{N}(A)$ of a pro-affine algebra A are defined as follows:

$$(5) \quad \mathcal{N}(A) = \bigcap_{\forall \mathfrak{p}} \mathfrak{p} \quad \text{and} \quad \mathcal{R}(A) = \bigcap_{\forall \mathfrak{m}} \mathfrak{m},$$

where the \mathfrak{p} 's and the \mathfrak{m} 's range over *all open prime* and *all open maximal* ideals, respectively.

Given an ideal $\mathfrak{a} \subseteq A$, the radical of \mathfrak{a} is defined as

$$(6) \quad \mathcal{N}(\mathfrak{a}) \stackrel{\text{def.}}{=} \bigcap_{\forall \mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$$

with \mathfrak{p} again ranging over *all open prime* ideals containing \mathfrak{a} .

As done in [2], for a pro-affine algebra $A = \lim_{\leftarrow} A_i$ we define two kinds of its reductions relative to the radicals:

$$(7) \quad A_{\text{red}} \stackrel{\text{def.}}{=} A/\mathcal{N}(A) \quad \text{and} \quad A_{\text{RED}} \stackrel{\text{def.}}{=} \lim_{\leftarrow} ((A_i)_{\text{RED}}) = \lim_{\leftarrow} ((A_i)_{\text{red}}),$$

where $(A_i)_{\text{RED}} := A_i/\mathcal{N}(A_i) = (A_i)_{\text{red}}$ is the usual residue-class ring *modulo* the nilradical of A_i . A is said to be *reduced* or *strongly reduced*, respectively, if $A = A_{\text{red}}$ or $A = A_{\text{RED}}$. One may define likewise two more radicals using the Jacobson radicals $\mathcal{R}(A)$'s and $\mathcal{R}(A_i)$'s, and these were actually what we dealt with in [2, (1.2), (1.3), pp. 483–484]. Just the same, the following counterpart of [2, Prop. (1.2), *loc. cit.*] stands valid, and we state it without proof:

Theorem 1.3.1. *For the canonical map $\rho: A = \lim_{\leftarrow} (A_i) \longrightarrow A_{\text{RED}}$, we have*

- (a) $\text{Ker}(\rho) = \mathcal{N}(A)$;
- (b) *The sequence $0 \longrightarrow \mathcal{N}(A) \longrightarrow A \longrightarrow A_{\text{RED}}$ is exact with $\text{Im}(\rho)$ dense in A_{RED} ;*
- (c) $\mathcal{N}(A) = \{f \in A : \lim_{N \rightarrow \infty} f^N = 0\}$ = *topologically nilpotent elements of A .*

REMARKS. **1.** We note that, even in the special context of the theorem above, the exactness of the sequence in (b) at the right-most end fails in general, or ρ is not surjective as a rule. Counter-examples are offered in Section 3 below (see Examples 3-E and 3-F). This point bears critically on the Jacobian Problem (cf. [2, (5.3), (5.4), pp. 497–498]).

2. Since $\mathcal{N}(A)$ is a closed ideal $\subset A$, we deduce from Prop. 1.2.1 that, whereas $\rho: A \rightarrow \lim_{\leftarrow} (A_i/\mathcal{N}(A_i))$ may not be surjective, the map $A \rightarrow \lim_{\leftarrow} (A_i/i\mathcal{N}(A))$ is surjective.

Theorem 1.3.1 and the Jacobson-radical version of it [2, (1.2), p. 483] coincide with each other in the K -algebraic case as seen just below:

Theorem 1.3.2 (Nullstellensatz). *If a pro-affine K -algebra A is algebraic over K , then $\mathcal{R}(A) = \mathcal{N}(A)$.*

Proof. In view of Props. 1.2.2 & 1.2.4, the remarks following these two and the algebraicity, we have

$$\mathcal{R}(A) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}(\mathcal{R}(A_i)) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}(\mathcal{N}(A_i)) = \mathcal{N}(A),$$

where the traditional Nullstellensatz $\mathcal{R}(A_i) = \mathcal{N}(A_i)$ has been applied. \square

2. Ind-affine schemes and ind-affine varieties

2.1. The spectra of pro-affine algebras and their topology. For any pro-affine algebra A , define its *prime spectrum* $\mathfrak{Sp}(A)$ and *maximal spectrum* $\mathfrak{Spm}(A)$, respectively, as

$$(8) \quad \begin{cases} \mathfrak{Sp}(A) & = \text{the set of all open, prime ideals } \subset A, \text{ and} \\ \mathfrak{Spm}(A) & = \text{the set of all open, maximal ideals } \subset A. \end{cases}$$

Then, in view of Prop. 1.2.2, $\mathfrak{Spm}(A)$ is the same as the set of all *closed* maximal ideals. Let us now introduce topology on $\mathfrak{Sp}(A)$ and $\mathfrak{Spm}(A)$ by extending Zariski topology: The *closed* sets $\subseteq \mathfrak{Sp}(A)$ are, by definition, those subsets of $\mathfrak{Sp}(A)$ in the form of

$$V(E) \stackrel{\text{def.}}{=} \{\mathfrak{p} \in \mathfrak{Sp}(A) : \mathfrak{p} \supseteq E\} \text{ for some set } E \subseteq A.$$

Likewise, the closed sets $\subseteq \mathfrak{Spm}(A)$ are defined to be precisely the $V_o(E)$'s where $V_o(E) \stackrel{\text{def.}}{=} V(E) \cap \mathfrak{Spm}(A)$.

The following proposition which should require no proofs shows that the preceding definition of the topologies on $\mathfrak{Sp}(A)$ and on $\mathfrak{Spm}(A)$ is valid:

Proposition 2.1.1. (i) *Let $\mathfrak{a} := \langle E \rangle$, the ideal generated by E , and let $\mathcal{N}(\mathfrak{a})$ be the radical of \mathfrak{a} . Then,*

$$V(\mathfrak{a}) = V(E) = V(\mathcal{N}(\mathfrak{a})).$$

(ii) $V(0) = \mathfrak{Sp}(A)$, $V(1) = \emptyset$.

(iii) *Given a family $\{E_i : i \in I\}$ of subsets of A , we have*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

(iv) *For ideals \mathfrak{b} and \mathfrak{c} , $V(\mathfrak{b} \cap \mathfrak{c}) = V(\mathfrak{b}\mathfrak{c}) = V(\mathfrak{b}) \cup V(\mathfrak{c})$.*

Next we define, for each $f \in A$, the *basic open set* $D(f) \subseteq \mathfrak{Sp}(A)$:

$$D(f) \stackrel{\text{def.}}{=} V(f)^c = \{ \mathfrak{p} \in \mathfrak{Sp}(A) : f \notin \mathfrak{p} \}.$$

Proposition 2.1.2. *Let f, g, f_α ($\alpha \in I$) be elements of A . Then,*

- (i) $D(f) \cap D(g) = D(f \cdot g)$.
- (ii) $\bigcup_{\alpha \in I} D(f_\alpha) = V(\langle f_\alpha : \alpha \in I \rangle)^c$.
- (iii) $D(f) = \emptyset \iff f \in \mathcal{N}(A) \iff f$ is topologically nilpotent.
- (iv) $D(f) = \mathfrak{Sp}(A) \iff f$ is a unit.
- (v) $D(g) \subseteq D(f) \iff g \in \mathcal{N}(\langle f \rangle)$.

Proof. Parts (i), (ii), (iii) immediately follow from relevant definitions. As for (iv), if $f \notin$ any open prime, then by Prop. 1.2.5 $\langle f \rangle$ must equal the unit ideal $\langle 1 \rangle$. Therefore, f must be a unit.

As for part (v), $D(g) \subseteq D(f) \iff \forall \mathfrak{p} \in \mathfrak{Sp}(A) [f \in \mathfrak{p} \Rightarrow g \in \mathfrak{p}]$, clearly, and this last condition is equivalent to $\mathcal{N}(\langle g \rangle) \subseteq \mathcal{N}(\langle f \rangle)$, or $g \in \mathcal{N}(\langle f \rangle)$. □

REMARK. Proposition 2.1.2 goes to show that the $D(f)$'s for all $f \in A$ form a base of open sets in our topology on $\mathfrak{Sp}(A)$, just as in the more traditional theory of affine schemes. Note, however, that in our theory here the open sets $D(f)$'s are *not quasi-compact* in general. This is due to the existence of infinitely-generated proper ideals whose closures are the unit ideal $\langle 1 \rangle$. See Ex. 3-G in §3 below.

2.2. Localization in pro-affine algebras and structure sheaves of ind-affine schemes. Let S be a *multiplicatively closed set* in a pro-affine algebra A . It will be assumed always that $1 \in S$ and $0 \notin \overline{S} = \lim_{\leftarrow} ({}_i S)$ for such an S . The localization $S^{-1}A$ can be defined in the standard manner, and this K -algebra naturally inherits its uniform topology from A . We shall adopt the completion of $S^{-1}A$ as our definition of A_S . Namely,

DEFINITION. For A and S as above, the localization A_S of A by S is defined to be

$$A_S \stackrel{\text{def.}}{=} \varprojlim ({}_i S^{-1}A_i).$$

Clearly, $A_S \simeq \overline{A_S}$, so one may assume from the beginning that S is closed. For useful examples of S one may mention $(f) \stackrel{\text{def.}}{=}} \{ f^n \mid n \in \mathbb{N} \}$ where f is not topologically nilpotent, and the complement $A - \mathfrak{p}$ of an open prime ideal \mathfrak{p} . In these instances, we shall denote $A_{(f)}$, $A_{A-\mathfrak{p}}$ by A_f , $A_{\mathfrak{p}}$, respectively.

Proposition 2.2.1. *Let $f, g \in A$, $U := D(f)$, $V := D(g)$, and let A_f, A_g be as just above. Let $A(U) := A_f$ and $A(V) := A_g$. Then,*

- (i) If $U = V$, then $A(U) \simeq A(V)$. (Thus $A(U)$ depends only on U , not on f .)
- (ii) If $V \subseteq U$, then there is a canonical homomorphism of pro-affine K -algebras $\rho_V^U: A(U) \rightarrow A(V)$, which depends only on U and V . (The ρ_V^U will be called the restriction homomorphism from U to V .)
- (iii) Let U, V be as above and $W = D(h)$ for $h \in A$. If $U \supseteq V \supseteq W$, we have

$$\rho_U^U = \text{Id}_{A(U)}, \quad \rho_W^V \circ \rho_V^U = \rho_W^U.$$

Proof. (ii) Assume $V \subseteq U$, or $D(g) \subseteq D(f)$. So, by Props. 1.2.2 & 1.2.4, $g \in \mathcal{N}(\langle f \rangle) = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}$ (the radical of $\langle if \rangle$ in A_i). This means that, for every $i \in \mathbb{N}$, there is an n_i such that $ig^{n_i} \in \langle if \rangle \subseteq A_i$. So, for each i there is an element $u_i \in A_i$ such that

$$(9) \quad (ig)^{n_i} = u_i \cdot if.$$

Now let $s \in A(U) = A_f = \lim_{\leftarrow, i} ((if)^{-1}A_i)$. Write s as a coherent sequence $s = (\dots \leftarrow a_{i-1}/(i-1f)^{m_{i-1}} \leftarrow a_i/(if)^{m_i} \leftarrow \dots)$. Define $\rho_V^U(s)$ to be equal to $(\dots \leftarrow s'_i \leftarrow \dots)$, where

$$(10) \quad s'_i \stackrel{\text{def.}}{=} a_i \cdot u_i^{m_i} / (ig)^{n_i m_i}.$$

If another pair (n'_i, u'_i) is chosen to make (9) stand, as $(ig)^{n'_i} = u'_i \cdot if$, then s'_i in (10) will have to be replaced by $s''_i = a_i \cdot u_i'^{m_i} / (ig)^{n'_i m_i}$. But one can check out easily that $s'_i = s''_i$ inside $(ig)^{-1}A_i$. So, $\rho_V^U(s)$ is well-defined provided that $s' := (\dots \leftarrow s'_{i-1} \leftarrow s'_i \leftarrow \dots)$ given by (10) just above is coherent.

Let us now check the coherence of s' . Since s is given coherent, one knows

$$(11) \quad [(i-1f)^{m_i} a_{i-1} - (i-1f)^{m_{i-1}} \mu_i(a_i)] \cdot (i-1f)^{\text{some power}} = 0,$$

and one need to verify

$$(12) \quad [a_{i-1}(u_{i-1})^{m_{i-1}}(i-1g)^{m_i n_i} - (i-1g)^{m_{i-1} n_{i-1}} \mu_i(a_i) \mu_i(u_i)^{m_i}] \cdot (i-1g)^{\text{some power}} = 0.$$

Applying μ_i to both sides of (9) and then raising them to the m_i -th power, one obtains $i-1g^{m_i n_i} = \mu_i(u_i)^{m_i} (i-1f)^{m_i}$; also, (9) for $i := i - 1$ gives $(i-1g)^{n_{i-1}} = u_{i-1} \cdot i-1f$. Substituting the right-hand sides of these two equalities for the appropriate terms inside the “[]” of (12), we find the said contents of [] to be

$$(13) \quad \begin{aligned} & a_{i-1} u_{i-1}^{m_{i-1}} \mu_i(u_i)^{m_i} (i-1f)^{m_i} - u_{i-1}^{m_{i-1}} (i-1f)^{m_{i-1}} \mu_i(a_i) \mu_i(u_i)^{m_i} \\ & = u_{i-1}^{m_{i-1}} \mu_i(u_i)^{m_i} [a_{i-1} (i-1f)^{m_i} - (i-1f)^{m_{i-1}} \mu_i(a_i)]. \end{aligned}$$

The expression inside the “[]” of (13) equals that of (11) and, consequently, gets killed by some power of $i-1f$. It follows that either side of (13) will be killed by

some power of ${}_{i-1}g$ because $({}_{i-1}g)^{n_{i-1}} = u_{i-1} \cdot {}_{i-1}f$. The proof of (ii) will be complete after (iii) and then (i) are established below.

(iii) That $\rho_U^U = \text{Id}_{A(U)}$ is clear in view of the preceding reasoning. As for the transitivity, we have

$$\forall i \in \mathbb{N} \exists n_i \exists ! i \in \mathbb{N} : ({}_i g)^{n_i} = u_i \cdot {}_i f \text{ and } ({}_i h)^{l_i} = v_i \cdot {}_i g, \text{ with } u_i, v_i \in A_i.$$

It follows that, for each i , $({}_i h)^{l_i m_i} = v_i^{m_i} u_i \cdot {}_i f$ holds, which implies that the composition $\rho_W^V \circ \rho_V^U$ maps $s = (\dots \leftarrow a_i / ({}_i f)^{m_i} \leftarrow \dots) \in A_f$ to

$$\rho_W^V \circ \rho_V^U(s) = (\dots \leftarrow a_i u_i^{m_i} v_i^{m_i n_i} / ({}_i h)^{m_i n_i l_i} \leftarrow \dots).$$

On the other hand, the relations $({}_i h)^{k_i} = w_i \cdot {}_i f$ for all $i \in \mathbb{N}$ corresponding to $W \subseteq U$ indicates $\rho_W^U(s) = (\dots \leftarrow a_i w_i^{m_i} / ({}_i h)^{m_i k_i} \leftarrow \dots)$. We already saw above that such coherent sequences are the same in A_h . Therefore, $\rho_W^V \circ \rho_V^U = \rho_W^U$.

(i) If $U = V$ or $D(f) = D(g)$, we have maps $\rho_V^U : A(U) \rightarrow A(V)$ and $\rho_U^V : A(V) \rightarrow A(U)$. As we just saw, $\rho_U^V \circ \rho_V^U = \rho_U^U = \text{Id}_{A(U)}$, and likewise for $\rho_V^U \circ \rho_U^V$. Hence $A(U) \simeq A(V)$. With (i) proven now, the proof of (ii) is complete. \square

It follows from Prop. 2.2.1 that the assignments $U = D(f) \mapsto A(U) = A_f$ and $[V = D(g) \hookrightarrow U = D(f)] \mapsto \rho_V^U$ produce a presheaf \mathcal{A} of pro-affine K -algebras on the base $\mathcal{B} = \{D(f) : f \in A\}$ of open sets of the topological space $\mathfrak{Sp}(A)$. (see [1, Chap. 0, §3.2, p. 25ff.] .)

Proposition 2.2.2. *Let A be a pro-affine algebra, and let \mathcal{A} be the presheaf over the base \mathcal{B} of open sets on $\mathfrak{Sp}(A)$ introduced just above. Let $U = D(g) \in \mathcal{B}$ be any basic open set, and let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ be a covering of U with each $U_\lambda = D(f_\lambda)$, $f_\lambda \in A_g$. Suppose given for each $\kappa \in \Lambda$ an element $s_\kappa \in \mathcal{A}(U_\kappa)$ such that $\rho_{U_{\lambda\nu}}^{U_\lambda}(s_\lambda) = \rho_{U_{\lambda\nu}}^{U_\nu}(s_\nu)$ for any $\lambda, \nu \in \Lambda$, where $U_{\lambda\nu}$ denotes $U_\lambda \cap U_\nu$. Then, there is one and only one $s \in \mathcal{A}(U)$ such that $\rho_{U_\kappa}^U(s) = s_\kappa$ for all $\kappa \in \Lambda$.*

Proof. The proof is based on the well-established fact that the proposition holds true in case of the affine schemes. (cf. [1, Th. (1.3.7), p. 86].)

It is clearly enough to prove the proposition in case $U = \mathfrak{Sp}(A)$ and $\mathcal{A}(U) = A$. Assume so and write $A = \lim_{\leftarrow} A_i$, $X_i = \text{Spec}(A_i)$. For each $\lambda \in \Lambda$ and each $i \in \mathbb{N}$, write $f_\lambda = (\dots \leftarrow {}_i f_\lambda \leftarrow \dots)$ and let

$$(14) \quad U_{\lambda,i} := \{\pi_i^{-1}(P) : P \in X_i \text{ and } {}_i f_\lambda \notin P\} = \pi_i^{-1}(D({}_i f_\lambda))$$

where $D({}_i f_\lambda)$ is the basic open set in $X_i = \text{Spec } A_i$. We then have two types of open coverings for each λ and each i , i. e.:

$$(15) \quad U_\lambda = \bigcup_{i \in \mathbb{N}} U_{\lambda,i} \quad \text{and} \quad X_i = \bigcup_{\lambda \in \Lambda} D({}_i f_\lambda).$$

[Uniqueness] Let $s', s'' \in A = \mathcal{A}(U)$ be such that $\rho_{U_\kappa}^U(s') = s_\kappa, \rho_{U_\kappa}^U(s'') = s_\kappa$ for all $\kappa \in \Lambda$. So, one may write $s' = (\cdots \leftarrow {}_i s' \leftarrow \cdots)$ and $s'' = (\cdots \leftarrow {}_i s'' \leftarrow \cdots)$, with ${}_i s' \in A_i, {}_i s'' \in A_i$ for each i . Now, since $\rho_{U_\kappa}^U(s') = \rho_{U_\kappa}^U(s'')$ for all κ , these agree on $U_{\kappa,i}$ for all i in the first covering of (15), or ${}_i(\rho_{U_\kappa}^U(s')) = {}_i(\rho_{U_\kappa}^U(s''))$. This means that ${}_i s'$ and ${}_i s''$ agree on each piece $D({}_i f_\kappa)$ of the second covering of (15) for each κ . It follows that ${}_i s' = {}_i s''$ on X_i for each i , because of the fact pointed out at the beginning of the proof. Therefore, we have $s' = s''$.

[Existence] We are locally given s_κ on U_κ for all κ such that s_λ and s_ν agree on $U_\lambda \cap U_\nu$ whenever the intersection is nonempty. The data will then induce, at each finite level i , the data of $\{ {}_i(s_\kappa) : \kappa \in \Lambda \}$ locally on each open piece $D({}_i f_\kappa)$ of the covering $X_i = \bigcup_{\lambda \in \Lambda} D({}_i f_\lambda)$. We can patch up the local data of ${}_i(s_\kappa)$'s on the affine scheme X_i so as to obtain $s_i \in A_i$. What remains to be checked out is that $(\cdots \leftarrow s_i \leftarrow s_{i+1} \leftarrow \cdots)$ is coherently defined. So, let $s'_i := \mu_{i+1}(s_{i+1})$, and we will show that $s_i = s'_i$. Now, denote the restriction map of X_i to $D({}_i f_\kappa)$ by $\rho_{i,\kappa}$. We have thus $\rho_{i,\kappa} : A_i \rightarrow (A_i)_{f_\kappa}$. By construction, $\rho_{i,\kappa}(s_i) = {}_i(s_\kappa)$ and $\rho_{i+1,\kappa}(s_{i+1}) = {}_{i+1}(s_\kappa)$. It follows that

$$(16) \quad \rho_{i,\kappa}(s'_i) = \rho_{i,\kappa}(\mu_{i+1}(s_{i+1})) = \mu'_{i+1}(\rho_{i+1,\kappa}(s_{i+1})) = \mu'_{i+1}({}_{i+1}(s_\kappa)) = {}_i(s_\kappa),$$

with $\mu'_{i+1} : (A_{f_\kappa})_{i+1} \rightarrow (A_{f_\kappa})_i$ standing for the map induced by $\mu_{i+1} : A_{i+1} \rightarrow A_i$. It is now shown that $\rho_{i,\kappa}(s_i) = \rho_{i,\kappa}(s'_i)$ for all $\kappa \in \Lambda$. Once again one draws upon the uniqueness in the affine-scheme case to conclude that $s_i = s'_i$. □

We now extend the presheaf \mathcal{A} to a presheaf over the topological space $\mathfrak{Sp}(A)$ by defining, for any open set $U \subseteq \mathfrak{Sp}(A)$, $\mathcal{A}(U) \stackrel{\text{def.}}{=} \lim_{\leftarrow} \mathcal{A}(V)$ where the \lim_{\leftarrow} is taken over all *basic* V 's for which $V = D(g) \subseteq U$ [1, chap. 0-3.2, pp. 25ff]. The extended presheaf will be denoted by \mathcal{A} , too. The next theorem follows immediately from Prop. 2.2.2. (cf. [1, *loc. cit.*].)

Theorem 2.2.3. *The presheaf \mathcal{A} is a sheaf.*

From here on, the topological space $\mathfrak{Sp}(A)$ endowed with the sheaf \mathcal{A} as above will be referred to as the *ind-affine scheme associated with A* and will be denoted by \mathcal{X}_A . \mathcal{A} is then, by definition, the *structure sheaf of \mathcal{X}_A* . In conformity with standard practice in scheme theory we shall also write $\mathcal{A} = \mathcal{O}(A)$. Similarly, the topological space $\mathfrak{Spm}(A)$ together with the sheaf induced on it from \mathcal{A} is called the *ind-affine variety associated with A* , and this variety will be denoted by \mathcal{V}_A .

We next address the issue of stalks of the sheaf \mathcal{A} . Let \mathcal{X}_A be an ind-affine scheme, with $A = \lim_{\leftarrow} A_i$. Let \mathfrak{p} be a point on \mathcal{X}_A , and let $\Lambda :=$ the filter of all basic open sets containing the point \mathfrak{p} , so $\Lambda = \{ D(g_\alpha) : \mathfrak{p} \in D(g_\alpha) \}$. Let us write $A_{i,\alpha} := (A_i)_{ig_\alpha}$ for all $i \in \mathbb{N}$ and all g_α for which $D(g_\alpha) \in \Lambda$. We then have the following commutative diagram in which all horizontal arrows represent surjections and vertical ones are restrictions occurring whenever $D(g_\alpha) \supseteq D(g_\beta)$, each column thus

forming a direct system:

$$(17) \quad \begin{array}{ccccccc} & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longleftarrow & A_{i-1,\alpha} & \longleftarrow & A_{i,\alpha} & \longleftarrow \cdots & \longleftarrow \lim_{\leftarrow,n} A_{n,\alpha} \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longleftarrow & A_{i-1,\beta} & \longleftarrow & A_{i,\beta} & \longleftarrow \cdots & \longleftarrow \lim_{\leftarrow,n} A_{n,\beta} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \cdot & & \cdot & & \cdot \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \cdot & & \cdot & & \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) \\ & & \downarrow & & \downarrow & & \downarrow \Phi \\ \cdots & \longleftarrow & \lim_{\rightarrow,\gamma} A_{i-1,\gamma} & \longleftarrow & \lim_{\rightarrow,\gamma} A_{i,\gamma} & \longleftarrow \cdots & \longleftarrow \lim_{\leftarrow,m}(\lim_{\rightarrow,\gamma} A_{m,\gamma}) \end{array}$$

In the diagram (17) one should recognize that $\lim_{\leftarrow,n} A_{n,\alpha} = A_{g_\alpha} = \mathcal{A}(D(g_\alpha))$, and $\lim_{\rightarrow,\gamma} A_{m,\gamma} = (A_m)_{m\mathfrak{p}}$. So, the map Φ on the lower right corner of (17) amounts to $\lim_{\rightarrow,\alpha}(\mathcal{A}(D(g_\alpha))) = \lim_{\rightarrow,\alpha} A_{g_\alpha} \xrightarrow{\Phi} \lim_{\leftarrow,m}((A_m)_{m\mathfrak{p}})$, and Φ gets induced as follows: (i) First, for each α there is a map $A_{g_\alpha} \rightarrow \lim_{\rightarrow,\gamma} A_{i,\gamma}$ for all i with appropriate commutativity of arrow paths; (ii) as a consequence there is a map $A_{g_\alpha} \rightarrow \lim_{\leftarrow,j}(\lim_{\rightarrow,\lambda} A_{j,\lambda}) = \lim_{\leftarrow,j}((A_j)_{j\mathfrak{p}})$; and finally (iii) the desired map $\lim_{\rightarrow,\gamma} A_{g_\gamma} \rightarrow \lim_{\leftarrow,j}((A_j)_{j\mathfrak{p}})$ again because of the appropriate commutativity.

We now come to study the map Φ . In order to describe its kernel, we need to introduce the notion of *elements infinitely near 0* in the ring $\lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha})$ and, before that, a new *ad hoc* notation: If $a_i \in A_{i,\alpha}$ then $[a_i]$ denotes the equivalence class represented by a_i in the direct limit $\lim_{\rightarrow,\gamma} A_{i,\gamma} = (A_i)_{i\mathfrak{p}}$. Likewise, if $(\cdots \leftarrow a_{q-1,\gamma} \leftarrow a_{q,\gamma} \leftarrow \cdots) \in \lim_{\leftarrow,n} A_{n,\gamma} = \mathcal{A}(D_{g_\gamma})$, then $[\cdots \leftarrow a_{q-1,\gamma} \leftarrow a_{q,\gamma} \leftarrow \cdots]$ is to mean the corresponding equivalence class $\in \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) = \lim_{\rightarrow,\alpha} \mathcal{A}(D(g_\alpha))$. Now, let

$$(18) \quad u := [\cdots \leftarrow u_{n-1,\alpha} \leftarrow u_{n,\alpha} \leftarrow \cdots] \in \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) = \lim_{\rightarrow,\alpha} \mathcal{A}(D(g_\alpha)).$$

We shall say that u is *infinitely near 0* if $\forall u_{n,\alpha} \exists \beta = \beta(n, \alpha) \geq \alpha$ such that $u_{n,\alpha} \mapsto u_{n,\beta} = 0$ under the restriction map due to the inclusion $D(n g_\beta) \subseteq D(n g_\alpha)$. The terminology is appropriate because, for such u , $[u_{n,\alpha}] = [0]$ for every n , yet u may not be 0.

It is easy to see that the set of all elements of $R := \lim_{\rightarrow,\alpha}(\lim_{\leftarrow,n} A_{n,\alpha}) = \lim_{\rightarrow,\alpha}(\mathcal{A}(D(g_\alpha)))$ that are infinitely near 0 form an ideal of the ring R .

Theorem 2.2.4. (a) *Let R be as just above. Then, the kernel of the map $\Phi: R \rightarrow \lim_{\leftarrow, m}(A_m)_{m\mathfrak{p}}$ is the ideal of all elements infinitely near 0 in R .*
 (b) *The image of Φ is everywhere dense in $\lim_{\leftarrow, m}(A_m)_{m\mathfrak{p}}$.*

Proof. (a) If $\Phi(u) = 0$ for u as in (18), that means $(\cdots \leftarrow [u_{n-1, \alpha}] \leftarrow [u_{n, \alpha}] \leftarrow \cdots) = (\cdots \leftarrow 0 \leftarrow 0 \leftarrow \cdots)$ inside $\lim_{\leftarrow, m}((A_m)_{m\mathfrak{p}})$, or $\forall n, [u_{n, \alpha}] = 0$. So, u is infinitely near 0. The converse clearly holds also.

(b) Given $\eta = (\cdots \leftarrow [u_{i-1, \alpha_{i-1}}] \leftarrow [u_{i, \alpha_i}] \leftarrow \cdots) \in \lim_{\leftarrow, m}((A_m)_{m\mathfrak{p}})$, write $\eta = (\cdots \leftarrow r_{i-1} \leftarrow r_i \leftarrow \cdots)$ with each $r_i \in (A_i)_{i\mathfrak{p}}$. For an arbitrary high $N > 0$, let $w_N := u_{N, \alpha_N} \in A_{N, \alpha_N}$. Clearly, one can complete w_N to an element

$$w = (\cdots \leftarrow w_{N-1} \leftarrow w_N \leftarrow w_{N+1} \leftarrow \cdots) \in \lim_{\leftarrow, n} A_{n, \alpha_n}$$

such that

$$[w_0] = [u_{0, \alpha_0}], [w_1] = [u_{1, \alpha_1}], \dots, [w_{N-1}] = [u_{N-1, \alpha_{N-1}}], [w_N] = [u_{N, \alpha_N}].$$

So, $[w] := [\cdots \leftarrow w_{N-1} \leftarrow w_N \leftarrow w_{N+1} \leftarrow \cdots] \in R$ is such that $\Phi([w])$ and η agree with each other up to the N -th place from the left. Since N was arbitrary, this shows the density of the image of Φ . □

In view of Th. 2.2.4, we define the *local ring of* a point \mathfrak{p} on an ind-affine scheme \mathcal{X}_A , $A = \lim_{\leftarrow} A_m$, to be $\lim_{\leftarrow, m}(A_m)_{m\mathfrak{p}}$. It is a pro-affine K -algebra, and a surjective inverse limit of local rings of the more traditional type.

3. Comments and Examples

(A) The *reduction* A_{red} and the *strong reduction* A_{RED} (see §1.3-(7) above):

In [2] we raised the question as to whether or not $A_{\text{red}} = A_{\text{RED}}$ for the types of pro-affine algebras A of interest to us, and we indicated how this issue bears upon the Jacobian Problem (cf. [2, (1.3), p. 484, and (5.4), p. 498]). As expected, this question is easily answered in the negative, as follows:

EXAMPLE 3-E. For all $i \in \mathbb{N}$ consider the same algebras A_i as occurred in [2, Ex. (1.4), p. 484] but with different connecting maps μ_i . Namely, let

$$A_i := K[T_1, \dots, T_{i-1}, T_i, T_{i+1}] / \langle T_{i+1}^2 \rangle = K[T_1, \dots, T_{i-1}, T_i, \tau_{i+1}],$$

and define $\mu_i: A_i \rightarrow A_{i-1}$ by stipulating

$$\mu_i(T_j) := T_j \text{ for } j < i; \mu_i(T_i) := \tau_i; \mu_i(\tau_{i+1}) := \tau_i \cdot T_1.$$

Then, in the exact sequence

$$(19) \quad 0 \longrightarrow \langle \tau_{i+1} \rangle \longrightarrow A_i \longrightarrow K[T_1, \dots, T_i] \longrightarrow 0$$

for all $i > 0$, the Mittag-Leffler condition fails to hold, so that the sequence

$$(20) \quad 0 \longrightarrow \mathcal{N}(A) \longrightarrow A \longrightarrow A_{\text{RED}} \longrightarrow 0, \text{ where } A_{\text{RED}} = K^{[[\infty]]}$$

obtained by applying $\lim_{\leftarrow, i}$ to (19), is expected to be nonexact on the right.

We can actually exhibit where the map $A \rightarrow A_{\text{RED}}$ fails to be surjective. In fact, let

$$f_i := T_1 + \dots + T_{i-1} + T_i \quad \text{for all } i \in \mathbb{N},$$

and consider $f := (f_1 \leftarrow \dots \leftarrow f_{i-1} \leftarrow f_i \leftarrow \dots) \in A_{\text{RED}}$. Suppose that there existed some $g \in A$ such that $g = (g_1 \leftarrow \dots \leftarrow g_i \leftarrow \dots) \mapsto f \in A_{\text{RED}}$. Then, for each $i \in \mathbb{N}$, it must hold that $g_i = f_i + \tau_{i+1} \cdot h_i = f_{i-1} + T_i + \tau_{i+1} \cdot h_i$ for a suitable $h_i \in K[T_1, \dots, T_{i-1}, T_i]$. On the other hand, $\mu_i(g_i) = g_{i-1}$, or

$$(21) \quad \begin{aligned} f_{i-1} + \tau_i + \tau_i \cdot T_1 \cdot h_i(T_1, \dots, T_{i-1}, \tau_i) \\ = f_{i-1} + \tau_i + \tau_i \cdot T_1 \cdot h_i(T_1, \dots, T_{i-1}, 0) \\ = f_{i-1} + \tau_i \cdot h_{i-1}, \end{aligned}$$

which implies that

$$(22) \quad h_{i-1} = 1 + T_1 \cdot h_i(T_1, \dots, T_{i-1}, 0) \text{ for all } i \in \mathbb{N}.$$

Using this last equation recursively, one would get

$$(23) \quad \begin{aligned} h_1(T_1) &= 1 + T_1 \cdot h_2(T_1, 0) \\ &= 1 + T_1(1 + T_1 \cdot h_3(T_1, 0, 0)) = 1 + T_1 + T_1^2 \cdot h_3(T_1, 0, 0) \\ &= \dots = 1 + T_1^2 + \dots + T_1^{k-1} \cdot h_k(T_1, 0, \dots, 0) = \dots \text{ (ad infin.)}. \end{aligned}$$

This lends an arbitrarily high T_1 -degree to the polynomial $h_1(T_1)$, an absurdity.

(B) Closed Embedding and Topology of Ind-affine schemes:

Let A, B be pro-affine algebras, and $X := \mathcal{X}_A, Y := \mathcal{X}_B$. A morphism of ind-affine schemes $f: Y \rightarrow X$ defined by a continuous K -map $\phi: A \rightarrow B$ is said to be a *closed embedding* if ϕ is open and surjective. When that is so, through appropriate representations $A = \lim_{\leftarrow} A_i, B = \lim_{\leftarrow} B_i$ of A and B as inverse limits, one may see to it that ϕ is induced by surjections $A_i \rightarrow B_i$ for all $i \in \mathbb{N}$. One can then say that the closed embedding $Y \rightarrow X$ is the direct limit of the closed embeddings $Y_i \rightarrow X_i$

for all i . The converse is inexact. Namely, if $\phi: A \rightarrow B$ comes as the inverse limit of surjective K -maps $A_i \rightarrow B_i$ for all i , ϕ need not be surjective. In other words, if $f: Y \rightarrow X$ is induced by closed embeddings $Y_i \rightarrow X_i$ ($\forall i$) of finite-dimensional affine K -schemes $X_i = \text{Spec}(A_i)$, $Y_i = \text{Spec}(B_i)$, f need not be a closed embedding of ind-affine schemes. This point is illustrated by the following example:

EXAMPLE 3-F (Burt Totaro). Let $X := \mathbb{A}^\infty = \mathcal{X}_{K^{[\infty]}}$, so $X = \bigcup_{i=1}^\infty X_i$ with $X_i = \mathbb{A}^i$. Define a subscheme $Y = \bigcup_{i=1}^\infty Y_i$ of X inductively, as follows: (a) $Y_1 := X_1 = \mathbb{A}^1$. (b) Having built Y_{i-1} , define Y_i to be the union of Y_{i-1} and a finite set of lines through the origin in X_i such that every polynomial function on X_i of degree $\leq i$ which vanishes on these lines must be 0 altogether on X_i . [Just take enough number of lines on X_i through the origin and in general position.]

Now consider the morphism $f: Y \rightarrow X$ arising as the dual of the natural map, $\mathcal{O}(X) := \lim_{\leftarrow, i} \mathcal{O}(X_i) \rightarrow \mathcal{O}(Y) := \lim_{\leftarrow, i} \mathcal{O}(Y_i)$, where the maps $\mathcal{O}(X_i) \rightarrow \mathcal{O}(Y_i)$ are surjections associated with the closed embeddings $Y_i \rightarrow X_i$ for all i . This f exhibits some pathological characters, as shall be seen now.

(a) First, let $J_i := \text{Ker}(\mathcal{O}(X_i) \rightarrow \mathcal{O}(Y_i))$. Then, J_i is a homogeneous ideal in $K^{[i]}$ whose generators may be taken to be forms of degree $> i$. This shows that the exact sequences $0 \rightarrow J_i \rightarrow \mathcal{O}(X_i) \rightarrow \mathcal{O}(Y_i) \rightarrow 0$ taken for all $i \in \mathbb{N}$ do not satisfy the Mittag-Leffler condition, and the non-surjectiveness of $\mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is strongly indicated.

(b) Second, let \mathfrak{m}_{X_i} , \mathfrak{m}_{Y_i} be the maximal ideals of the origin (0) on X_i , Y_i in the rings $\mathcal{O}(X_i)$, $\mathcal{O}(Y_i)$, respectively. Then, for every pair of r and i with $0 < r \leq i$, the natural surjection

$$\psi_{r,i}: \mathcal{O}(X_i)/\mathfrak{m}_{X_i}^r \longrightarrow \mathcal{O}(Y_i)/\mathfrak{m}_{Y_i}^r$$

is also injective because of the make-up of Y_i , so that $\psi_{r,i}$ is an isomorphism. It follows that $\psi_r := \lim_{i \rightarrow \infty} (\psi_{r,i})$ gives an isomorphism $\mathcal{O}(X)/\mathfrak{m}_X^{(r)} \simeq \mathcal{O}(Y)/\mathfrak{m}_Y^{(r)}$ for all $r > 0$. Consequently, $\mathfrak{m}_X/\mathfrak{m}_X^{(2)} \simeq \mathfrak{m}_Y/\mathfrak{m}_Y^{(2)}$ and $\mathfrak{m}_X^{(r)}/\mathfrak{m}_X^{(r+1)} \simeq \mathfrak{m}_Y^{(r)}/\mathfrak{m}_Y^{(r+1)}$. Since the point (0) on X satisfies the smoothness condition that $\hat{S}^n(\mathfrak{m}_X/\mathfrak{m}_X^{(2)}) \rightarrow \mathfrak{m}_X^{(n)}/\mathfrak{m}_X^{(n+1)}$ be an isomorphism for all $n > 0$ (see [2, p. 488]), so does (0) on Y , or Y is smooth at (0).

We can see that this creates a serious problem for the notion of smoothness of ind-affine varieties, as calling the point (0) a simple point on Y goes against our intuition. It appears that the notion of smoothness (or of simple point) should be reworked (see [2, p. 488], [3, p. 187ff]). We will not, however, go into this issue in this paper. Turning to the more immediate question on Totaro's example at hand, we find it impossible that the K -map $\mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ in (a) just above should be surjective, or that the morphism $Y \rightarrow X$ should be a closed immersion. For, were this the case, then the embedding theorem [2, (2.6), p. 488] would imply that Y is isomorphic to X as ind-

affine scheme. It follows that, for every i , Y_i is a closed subscheme of X_i but $Y \rightarrow X$ is not a closed immersion.

(C) Example of a proper ideal whose closure is the unit ideal:

We follow up on Example 1.2-D and Remark that precedes it.

EXAMPLE 3-G. Let

$$\begin{aligned} w_1 &:= (1 \leftarrow 1 + x_1 \leftarrow 1 + x_1 + x_2 \leftarrow \cdots \leftarrow 1 + x_1 + x_2 + \cdots + x_n \leftarrow \cdots) \\ w_2 &:= (1 \leftarrow 1 \leftarrow 1 + x_2 \leftarrow 1 + x_2 + x_3 \leftarrow \cdots \leftarrow 1 + x_2 + \cdots + x_n \leftarrow \cdots) \\ &\vdots \\ w_n &:= (1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow 1 + x_n \leftarrow 1 + x_n + x_{n+1} \leftarrow \cdots) \\ &\vdots \end{aligned}$$

be a sequence of elements in $K^{[\infty]}$. So, $w_n - w_{n+1} = (0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow x_n \leftarrow x_n \leftarrow \cdots)$ and $w_n - 1 = (0 \leftarrow \cdots \leftarrow 0 \leftarrow x_n \leftarrow x_n + x_{n+1} \leftarrow \cdots)$. It follows that $\lim_{n \rightarrow \infty} w_n = 1$ and $\overline{\langle w_1, \dots, w_n, \dots \rangle} = \langle 1 \rangle$. On the other hand, $\langle w_1, \dots, w_n, \dots \rangle \subsetneq \langle 1 \rangle$ because no finite linear combination of the w_i 's can equal 1. To be more specific, suppose $L = 1$ for an $K^{[\infty]}$ -linear combination L of w_k, w_l, \dots, w_m ($k < l < \cdots < m$), or $\langle w_k, w_l, \dots, w_m \rangle = \langle 1 \rangle$. Then, $\langle w_1, \dots, w_m \rangle = \langle w_1, w_1 - w_2, \dots, w_{m-1} - w_m \rangle = \langle 1 \rangle$. This implies that an $K^{[\infty]}$ -linear combination of

$$\begin{aligned} w_1 &= (1 \leftarrow 1 + x_1 \leftarrow \cdots \leftarrow 1 + x_1 + \cdots + x_m \leftarrow \cdots) \\ w_1 - w_2 &= (0 \leftarrow x_1 \leftarrow x_1 \leftarrow \cdots \leftarrow x_1 \leftarrow \cdots) \\ w_2 - w_3 &= (0 \leftarrow 0 \leftarrow x_2 \leftarrow \cdots \leftarrow x_2 \leftarrow \cdots) \\ &\vdots \\ w_{m-1} - w_m &= (0 \leftarrow 0 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_{m-1} \leftarrow \cdots) \end{aligned}$$

should produce $(1 \leftarrow 1 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots)$. Clearly, this is impossible.

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