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UNIQUENESS OF POSITIVE SOLUTIONS OF THE HEAT EQUATION

Dedicated to Professor Masanori Kishi on his 60th birthday

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1. Introduction

Let D be an unbounded domain in $\boldsymbol{R}^2$ and u a nonnegative solution of the heat equation on D. We consider the property (U) for D:

$$\text{(U)}: \qquad u=0 \quad \text{on} \quad \partial_p D \Rightarrow u=0 \quad \text{on } D\,,$$

where $\partial_p D$ is the parabolic boundary of D. In the case of $D=\boldsymbol{R}\times(0, T)$ or $(0, \infty)\times(0, T)$, it is known that the property (U) holds (see [6]).

In this paper, by using a special form of the boundary Harnack principle, we shall show the following generalization.

Theorem. *For $T\in(-\infty, \infty]$ and an upper semicontinous function φ on $\boldsymbol{R}$, we set*

$$D(\varphi, T)=\{(x, t);\ \varphi(x)<t<T\}\ .$$

If φ is bounded below, then the property (U) *holds for $D(\varphi, T)$.*

By Theorem we obtain the following

Corollary. *Let φ, T and $D(\varphi, T)$ be as in Theorem, and assume that φ is bounded below. Let u, v be positive solutions of the heat equation on $D(\varphi, T)$. If $u-v$ vanishes continuously on $\partial_p D(\varphi, T)$, then $u=v$ on $D(\varphi, T)$.*

On the other hand, for $D=\{(x, t);\ mt<x\}$, the property (U) does not hold (see Lemma 7, Proposition 2). By using the Appell transformation, we shall show that this is critical (see Proposition 1).

2. Preliminaries

For a domain D in $\boldsymbol{R}^2$ we denote by $\partial_p D$ the set of $(x, t)\in\partial D$ (the boundary of D) satisfying $V\cap D\cap(\boldsymbol{R}\times(t, \infty))\neq\emptyset$ for every neighborhood V of (x, t) and call it the parabolic boundary of D. For $X=(x, t)\in D$, we denote by ω_D^X

the parabolic measure at X with respect to D. The parabolic measure ω_D^X is supported by $\partial_p D \cap \boldsymbol{R} \times (-\infty, t]$ and for any bounded continuous function f on $\partial_p D$, the function $\int f d\omega_D^X$ of X is the solution of the Dirichlet problem with boundary value f. A boundary point $Y \in \partial_p D$ is said to be regular if for every bounded continuous function f on $\partial_p D$,

$$\lim_{\substack{X \to Y \\ X \in D}} \int f d\omega_D^X = f(Y).$$

Remark 1. We note that every boundary point in $\partial_p D(\varphi, T)$ is regular, where φ, T and $D(\varphi, T)$ are as in Theorem (see for example [1]).

We say that u is a parabolic function on a domain D if

$$(\partial/\partial t - \partial^2/\partial x^2)\, u = 0 \quad \text{on } D.$$

Recalling the Perron-Wiener-Brelot method, we have

Lemma 1. *Let D be a domain in $\boldsymbol{R}^2$ and u a nonnegative parabolic function on D. If u is continuous on $D \cup \partial_p D$, we have*

$$u(X) \geq \int u d\omega_D^X$$

on D.

Let W be the fundamental solution of the heat equation, defined by

$$\begin{aligned} W(x, t; y, s) &= (4\pi(t-s))^{-1/2} \exp\left(-\frac{(x-y)^2}{4(t-s)}\right) \quad \text{for} \quad t > s \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad t \leq s. \end{aligned}$$

We put

$$G(x, t; y) = W(x, t; y, 0) - W(x, t; -y, 0)$$

and for $m \in \boldsymbol{R}$,

$$K_m(x, t; s) = \frac{x - mt}{t - s} W(x, t; ms, s).$$

Then the function $G(x, t; y)$ of y is the density of the parabolic measure at (x, t) with respect to $(0, \infty) \times (0, T)$ $(T > 0)$. By Lemma 1 we obtain the following

Lemma 2. *Let u be a nonnegative parabolic function on $D = (0, \infty) \times (0, T)$ for $T > 0$. If u is continuous on $\partial_p D \cup D$, then we have*

$$u(x, t) \geq \int_0^\infty G(x, t; y)\, u(y, 0)\, dy.$$

On the other hand, Kaufman and Wu showed in [2] that $K_m(x, t; s)$ is the density at (ms, s) of the parabolic measure at (x, t) with respect to $\Omega_m=\{(x, t); mt<x\}$. More precisely, we have

Lemma 3. *For m, T_0 and T_1 in $\boldsymbol{R}$, we set*

$$\Omega = \{(x, t);\ T_0<t<T_1,\ mt<x\} .$$

Then for $(x, t)\in\Omega$,

$$\omega_\Omega^{(x,t)} = K_m(x, t; s)\, ds$$

on $\{(ms, s);\ T_0\leq s\leq T_1\}$.

3. The boundary Harnack principle

For $\alpha>0$, we denote by τ_α the parabolic dilation defined by $\tau_\alpha(x, t)=(\alpha x, \alpha^2 t)$. The heat equation is stable for every parabolic dilation. This implies the following

REMARK 2. The parabolic measure $\omega_{\tau_\alpha(D)}^{\tau_\alpha(X)}$ is equal to the τ_α-image of ω_D^X.

For $Y\in\boldsymbol{R}^2$ and $r>0$, we put

$$V_r(Y) = \{Y+(x, t);\ -r^2<t<r^2,\ -r+tr^{-1}<x<r\} ,$$

and

$$V_r'(Y) = V_r(Y)\setminus\{Y+(x, t);\ x\leq 0,\ t\leq 0\} .$$

Since Y is a regular boundary point of $V_r'(Y)$, we have the following lemma (use Remark 2).

Lemma 4. *For any $\varepsilon>0$, there exists $0<\beta<1$ independent of r such that*

$$\omega_{V_r'(Y)}^X\,(\partial V_r'(Y)\setminus\{Y+(x, t);\ x\leq 0,\ t\leq 0\})<\varepsilon$$

for all $X\in V_{\beta r}'(Y)$.

To prove our main theorem, the following lemma plays an important role, which is a special form of the boundary Harnack principle.

Lemma 5. *Let φ be a decreasing upper semicontinuous function on $[0, \infty)$ satifying $\varphi(0)=2$ and $\varphi\geq 0$. Put*

$$D = \{(x, t);\ 0<x,\ \varphi(x)<t<2\} .$$

Then there exists a constant $C>0$ such that for any nonnegative parabolic function u on D vanishing continuously on $\partial_p D$

$$u(x, t)\leq Cu(x, 1)$$

for every (x, t) with $1<x<\infty$, $\varphi(x-1)<1/2$ and with $\varphi(x)<t<1/2$.

Proof. For each $x_0 \in (1, \infty)$ with $\varphi(x_0-1)<1/2$, we set

$$D_{x_0} = D \cap \{(x, t);\ 0<x<x_0+2,\ \varphi(x_0+1)<t<2\}\ .$$

First we shall show that for $Y_0=(y_0, \varphi(x_0+1))$ with $x_0+1<y_0<x_0+3/2$, there exists a constant $C_0>0$ independent of x_0 and y_0 such that for $r \leq s \leq 1/4$ and any $x \in D_{x_0} \setminus V_s(Y_0)$,

$$\omega_{D_{x_0}}^{X}(V_r(Y_0) \cap \partial D_{x_0}) \leq C_0\, \omega_{D_{x_0}}^{A_s(Y_0)}(V_r(Y_0) \cap \partial D_{x_0})\,, \tag{1}$$

where $A_r(Y_0)=Y_0+(r, 2r^2)$. Put $f(x)=\omega_{D_{x_0}}^{X}(V_r(Y_0) \cap \partial D_{x_0})$. It suffices to show that

$$f(X) \leq C_0 f(A_{2^k r}(Y_0)),\ X \in D_{x_0} \setminus V_{2^k r}(Y_0) \tag{2}$$

for every positive integer k with $2^k r<1/4$.

By the maximum principle and the Harnack inequality, we have

$$\begin{aligned} f(X) &\leq 1 = C_1\, \omega_{V'_r(Y_0)}^{A_{r/2}(Y_0)}(\partial V'_r(Y_0) \cap \{Y_0+(x, t);\ t \leq 0,\ x \leq 0\}) \\ &\leq C_1\, \omega_{V'_r(Y_0) \cap D_{x_0}}^{A_{r/2}(Y_0)}(\partial D_{x_0} \cap \partial V'_r(Y_0)) \\ &\leq C_1 f(A_{r/2}(Y_0)) \\ &\leq C_1 C_2 f(A_r(Y_0))\,, \end{aligned}$$

for some $C_1, C_2>0$ independent of r, which shows (2) for $k=0$. By Remark 2, we have

$$f(x) \leq C_1 C_2^2 f(A_{2r}(Y_0))\,.$$

By using Lemma 4 for $\varepsilon=C_2^{-1}$, there exists $0<\beta<1$ such that for any $Y \in \partial V_{2r}(Y_0) \cap \partial D_{x_0}$ and $X \in V_{\beta r}(Y) \cap D_{x_0}$,

$$\begin{aligned} f(X) &\leq C_1 C_2^2 f(A_{2r}(Y_0))\, \omega_{V'_r(Y)}^{X}(\partial V'_r(Y_0) \cap \{Y+(x, t);\ t \leq 0,\ x \leq 0\}^c) \\ &\leq C_1 C_2 f(A_{2r}(Y_0))\,, \end{aligned}$$

where $\{\cdot\}^c$ denotes the complement of a set $\{\cdot\}$. By the Harnack inequality, there exists $C_3>0$ such that

$$f(X) \leq C_3 f(A_{2r}(Y_0)),\ X \in \partial V_{2r}(Y_0) \cap D_{x_0} \bigcap_{Y \in \partial V_{2r} \cap D_{x_0}} V_{\beta r}(Y)^c\,,$$

so that by the maximum principle,

$$f(X) \leq C_0 f(A_{2r}(Y_0)),\ X \in D_{x_0} \setminus V_{2r}(Y_0)$$

with $C_0=\max(C_1 C_2, C_3)$, which shows (2) for $k=1$. Inductively we have (2) for every k with $2^k r<1/4$.

Similarly we see that for $Y_0=(y_0, t_0) \in \partial D_{x_0}$ with $y_0 \geq x_0+2/3$, $t_0 \leq 1/2$ and for $r \leq s \leq 1/4$,

(3) $\qquad \omega^X_{D_{x_0}}(U_r(Y_0)\cap\partial D_{x_0})\leq C'\omega^{B_s(Y_0)}_{D_{x_0}}(U_r(Y_0)\cap\partial D_{x_0}),\ X\in D_{x_0}\backslash U_s(Y_0)$

with some constant $C'>0$, where

$$U_r(Y)=\{Y+(x,t);\ -r^2<t<r^2,\ -r<x<r-tr^{-1}\}\ ,$$

and

$$B_r(Y)=Y+(-r,2r^2)\,.$$

From (1), (3) and the Harnack inequality, it follows that with some constant $C>0$,

$$\omega^{(x,t)}_{D_{x_0}}(V_r(Y_0)\cap\partial D_{x_0})\leq C\,\omega^{(x_0,1)}_{D_{x_0}}(V_r(Y_0)\cap\partial D_{x_0})\,,$$

for every $Y_0=(y_0,t_0)\in\partial D_{x_0}$ with $y_0\leq x_0+1$, $t_0\leq 1/2$ and every $(x,t)\in D_{x_0}$ with $x\leq x_0$, $t\leq 1/2$. Since for $t<1$, $\omega^{(x,t)}_{D_{x_0}}$ is absolutely continuous with respect to $\omega^{(x_0,1)}_{D_{x_0}}$, we have

$$\omega^{(x,t)}_{D_{x_0}}\leq C\,\omega^{(x_0,1)}_{D_{x_0}}\quad\text{on}\quad \partial D_{x_0}\cap[x_0+1,x_0+2]\times[\varphi(x_0+1),1/2]$$

for every $(x,t)\in D_{x_0}$ with $x\leq x_0$, $t\leq 1/2$, which proves Lemma 5.

4. The uniqueness of positive parabolic functions

First we show the following plain

Lemma 6. *Let φ be a decreasing upper semicontinuous function bounded below on $[a,\infty)$ for some $a\in\boldsymbol{R}$, and put*

$$D=\{(x,t);\ a<x,\ \varphi(x)<t<\varphi(a)\}\ .$$

If a nonnegative parabolic function u on D vanishes continuously on $\partial_p D$, then $u=0$ on D.

Proof. We may assume that $a=0$, $\varphi(0)=3$ and $\lim_{x\to\infty}\varphi(x)=0$. It suffices to show that $u(x,1)=0$ if $(x,1)\in D$.

Let $(x_0,1)\in D$ be fixed. We choose $y_0>x_0+1$ with $\varphi(y_0-1)<1/4$. For $y>y_0$, we put

$$J_y=\{(x,t);\ x<y,\ 0<t<2\}\ .$$

By the Harnack inequality and Lemma 5, there exists a constant $C>0$ such that

$$u(y,t)\leq Cu(y,1)$$

for $y>y_0$ and $\varphi(y)<t<1$. Hence we have

$$\begin{aligned}u(x_0,1)&=\int u(y,t)\,d\omega^{(x_0,1)}_{D\cap J_y}\\&\leq C\,u(y,1)\,\omega^{(x_0,1)}_{D\cap J_y}(\{y\}\times[0,1])\end{aligned}$$

$$\leq C u(y, 1)\, \omega_y^{(x_0, 1)}(\{y\} \times [0, 1])$$
$$= C u(y, 1) \int_0^1 K_0(y-x_0, 1; s)\, ds ,$$

for every $y>y_0$ (see Lemma 3). Since Lemma 2 gives

$$\int_{y_0}^{\infty} u(y, 1) \int_0^1 K_0(y-x_0, 1; s)\, ds dy < \infty ,$$

we have

$$u(x_0, 1) \leq C \liminf_{y \to \infty} (u(y, 1) \int_0^1 K_0(y-x_0, 1; s)\, ds) = 0 ,$$

which shows Lemma 6.

Now we shall prove our main theorem.

Theorem. *For $T \in (-\infty, \infty]$ and an upper semicontinuous function φ bounded below on $\boldsymbol{R}$, we set*

$$D(\varphi, T) = \{(x, t); \varphi(x) < t < T\} .$$

Let u be continuous on $D(\varphi, T) \cup \partial_p D(\varphi, T)$ and nonnegative parabolic on $D(\varphi, T)$. If $u=0$ on $\partial_p D(\varphi, T)$, then $u=0$ on $D(\varphi, T)$.

Proof. Put

$$\psi(x) = \min (\sup_{y \leq x} \varphi(y), \sup_{y \geq x} \varphi(y)) .$$

Then clearly $\varphi \leq \psi$. We can easily check that the upper semicontinuity of ψ follows from that of φ. Furthermore there exists $a \in [-\infty, \infty]$ such that ψ is increasing on $(-\infty, a)$ and decreasing on (a, ∞). From the definition of ψ, it follows that for a nonnegative parabolic function u on $D(\varphi, T)$ vanishing continuousy continuously on $\partial_p D(\varphi, T)$, $u(x, t)=0$ for every (x, t) with $\varphi(x) \leq t < \psi(x)$. Hence

$$u = 0 \quad \text{on} \quad \partial_p D(\psi, T).$$

Then Lemma 6 gives $u(x, t)=0$ for $(x, t) \in D(\psi, T)$ with $t < \sup \psi$, which yields Theorem.

Proof of Corollary. Let u, v be positive parabolic functions on $D(\varphi, T)$ such that $u-v$ vanishes continuously on $\partial_p D(\varphi, T)$. Then for any $\varepsilon>0$ and $T'<T$, we can find an upper semicontinuous function φ_ε such that $\partial_p D(\varphi_\varepsilon, T') \subset D(\varphi, T)$ and that

$$|u-v| < \varepsilon \quad \text{on} \quad D(\varphi, T') \backslash D(\varphi_\varepsilon, T') .$$

Lemma 1 gives

$$u(X)-\int u\, d\omega^X_{D(\varphi_\varepsilon,T')}\geq 0 \quad \text{for} \quad X\in D(\varphi_\varepsilon, T'),$$

so that by Theorem we have

$$u(X)=\int u\, d\omega^X_{D(\varphi_\varepsilon,T')} \quad \text{for} \quad X\in D(\varphi_\varepsilon, T')$$

(see Remark 1). The similar equality holds for v. Therefore for $X\in D(\varphi_\varepsilon, T')$,

$$|u(X)-v(X)|\leq\int |u-v|\, d\omega^X_{D(\varphi_\varepsilon,T')}<\varepsilon,$$

which shows Corollary.

5. Application

In this section, by using the Appell transformation, we shall give some application of our theorem.

For $(x, t)\in \boldsymbol{R}\times(-\infty, 0)$, we define

$$\tau(x, t)=(-t^{-1}x, -t^{-1})$$

and

$$Au(x, t)=W(x, t; 0, 0)\, u(t^{-1}x, -t^{-1}),$$

for a function u on a domain D in $\boldsymbol{R}\times(-\infty, 0)$. The Appell transformation gives a one-to-one correspondence of the parabolic functions on D to the parabolic functions on $\tau(D)$.

By using the Appell transformation, we have

Lemma 7. *For $a\in(-\infty, \infty]$ and an upper semicontinuous function ψ on $(-\infty, a)$, we set*

$$\Omega(\psi)=\{(x, t);\ t<a,\ \psi(t)<x\}.$$

If $\limsup_{t\downarrow 0} t\,\psi(-t^{-1})<\infty$, then the property (U) *does not hold.*

This and our main theorem imply the following

Proposition 1. *Let $\psi_\alpha(t)=(-t)^\alpha$ and $\Omega(\psi_\alpha)$ be as in Lemma 7. If $\alpha>1$, the property* (U) *holds for $\Omega(\psi_\alpha)$ but if $\alpha\leq 1$, then the property* (U) *does not hold.*

EXAMPLE. Let $\psi(t)=-t\log\log\log(-t)$ on $(-\infty, -e^e)$. Then the property (U) holds for $\Omega(\psi)$.

Finally we consider an integral representation of positive parabolic functions on $\Omega_m=\{(x, t);\ mt<x\}$.

Proposition 2. *For every positive parabolic function u on Ω_m, there exist* *positive Borel measures* μ, ν *and a constant* $C \geq 0$ *such that*

$$u(x, t) = \int_{-\infty}^{\infty} K_m(x, t; s)\, d\mu(s) + \int_{-m/2}^{\infty} k(x, t; \lambda)\, d\nu(\lambda) + C(x-mt) \exp(m^2t/4 - mx/2)\,,$$

where $k(x, t; \lambda) = \exp(\lambda^2 t + \lambda x) - \exp((\lambda+m)^2 t - (\lambda+m)\, x)$. *The measures* μ, ν *and the constant* C *are uniquely determined.*

This is a modification of the following proposition by the Appell transformation. Kaufman and Wu [2] and Mair [3] obtained the integral representation for $m=0$.

Proposition 3 (see [6]). *For every positive parabolic function* u *on* $(0, \infty)\times(0, \infty)$, *there exist unique positive measures* μ, ν *on* $\boldsymbol{R}$ *such that*

$$u(x, t) = \int_{-\infty}^{\infty} K_0(x, t; s)\, d\mu(s) + \int_0^{\infty} G(x, t; y)\, d\nu(y)\,.$$

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