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Citation	Osaka Journal of Mathematics. 2021, 58(4), p. 815-825
Version Type	VoR
URL	<a href="https://doi.org/10.18910/84951">https://doi.org/10.18910/84951</a>
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# THE TORSION GENERATING SET OF THE EXTENDED MAPPING CLASS GROUPS IN LOW GENUS CASES

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(Received February 17, 2020, revised June 10, 2020)

## Abstract

We prove that for genus  $g = 3, 4$ , the extended mapping class group  $\text{Mod}^\pm(S_g)$  can be generated by two elements of finite orders. But for  $g = 1$ ,  $\text{Mod}^\pm(S_1)$  cannot be generated by two elements of finite orders.

## 1. Introduction

Let  $S_g$  be a connected oriented closed surface of genus  $g$ . We denote by  $\text{Mod}(S_g)$  the mapping class group of  $S_g$ , the group of isotopy classes of orientation-preserving diffeomorphisms on  $S_g$ . We also denote by  $\text{Mod}^\pm(S_g)$  the extended mapping class group of  $S_g$ , the group of isotopy classes of all orientation-preserving and orientation-reversing diffeomorphisms on  $S_g$ .

Korkmaz has proved that the mapping class group  $\text{Mod}(S_g)$  can be generated by two elements of finite orders in [4]. Using the notation that  $\langle m, n \rangle$  ( $m, n$  are integers) to mean a group can be generated by two elements whose orders are  $m$  and  $n$  respectively, Korkmaz's result shows the orders of the generators are as in Table 1.

Table 1.

$\text{Mod}(S_g)$	torsion generating set consisting of two elements
$g = 1$	$\langle 4, 6 \rangle$
$g = 2$	$\langle 6, 10 \rangle$
$g \geq 3$	$\langle 4g + 2, 4g + 2 \rangle$

It is an open problem listed in [5] that whether the extended mapping class group  $\text{Mod}^\pm(S_g)$  can be generated by two torsion elements. In [1], the author partially solved such a problem when the genus  $g \geq 5$ . In this paper, we deal with  $g = 1, 3, 4$ .

When  $g = 3, 4$ , the method and idea in the process of calculation in this paper are mostly the same as those in [1] and [4]. The reason for  $g = 3$  and  $g = 4$  should be treated separately is as the follow. When the genus is high, there will be plenty of space to find a simple closed curve satisfying two conditions: (1) it lies in the periodic orbit; (2) it does not intersect with some given curves. When the genus is less than 5, we cannot do this. So we use other

treatment carefully. When  $g = 1$ , we use the presentation of  $GL(2, \mathbb{Z})$  to prove it cannot be generated by two elements of finite orders. So we can summarize the result as in Table 2.

Table 2.

$\text{Mod}^\pm(S_g)$	torsion generating set consisting of two elements
$g = 1$	impossible
$g = 2$	still unknown
$g \geq 3$	$\langle 2, 4g + 2 \rangle$

## 2. Preliminary

**2.1. Notations.** (a) We use the convention of functional notation, namely, elements of the mapping class group are applied right to left, i.e. the composition  $FG$  means that  $G$  is applied first.

(b) On an oriented surface, for each explicit two-sided simple closed curve, a *Dehn twist* means a right-handed Dehn twist along such a curve according to the orientation of the surface, and a left-handed Dehn twist is the inverse of a right-handed Dehn twist.

(c) We denote the curves by lower case letters  $a, b, c, d$  (possibly with subscripts) and the Dehn twists about them by the corresponding capital letters  $A, B, C, D$ . Notationally we do not distinguish a diffeomorphism/curve and its isotopy class.

**2.2. Basic relations between Dehn twists.** We recall the following results (see, for instance, section 3.3, 5.1, 7.5 of [2]):

**Lemma 2.1.** *For any  $\varphi \in \text{Mod}(S_g)$  and any isotopy classes  $a, b$  of simple closed curves in  $S_g$  satisfying  $\varphi(a) = b$ , we have:*

$$B = \varphi A \varphi^{-1}.$$

**Lemma 2.2.** *For any  $\varphi \in \text{Mod}^\pm(S_g) \setminus \text{Mod}(S_g)$  and any isotopy classes  $a, b$  of simple closed curves in  $S_g$  satisfying  $\varphi(a) = b$ , we have:*

$$B^{-1} = \varphi A \varphi^{-1}.$$

**Lemma 2.3.** *Let  $a, b$  be two simple closed curves on  $S_g$ . If  $a$  is disjoint from  $b$ , then*

$$AB = BA.$$

**Lemma 2.4** (Lantern relation). *Let  $a, b, c, d, x, y, z$  be the curves showed in Figure 1 on a genus zero surface with four boundaries. Then*

$$ABCD = XYZ.$$

*In other words, since  $a, b, c$  are disjoint from  $x, y, z$ , we have*

$$D = (XA^{-1})(YB^{-1})(ZC^{-1}).$$

**2.3. Humphries generators and the  $(4g + 2)$ -gon.** Humphries have proved the following theorem ([3]).

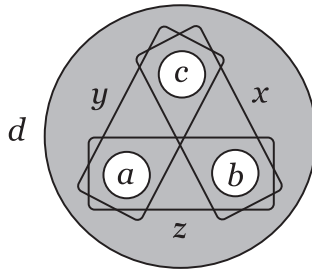


Fig.1

**Theorem 2.5.** *Let  $a_1, a_2, \dots, a_{2g}, b_0$  be the curves as on the left-hand side of Figure 2. Then the mapping class group  $\text{Mod}(S_g)$  is generated by  $A_i$ 's and  $B_0$ .*

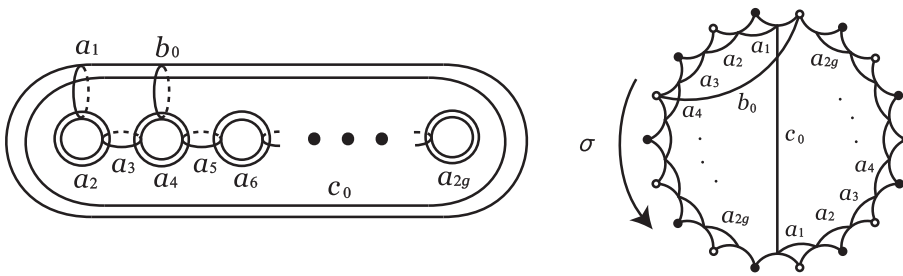


Fig.2

The genus  $g$  surface can be looked as a  $(4g + 2)$ -gon, whose opposite edges are glued together in pairs.  $(4g + 2)$  vertices of the  $(4g + 2)$ -gon are glued to be two vertices.

We can also draw the curves  $a_1, a_2, \dots, a_{2g}, b_0$  on the  $(4g + 2)$ -gon as the right-hand side of Figure 2. There is a natural rotation  $\sigma$  of the  $(4g + 2)$ -gon that sends  $a_i$  to  $a_{i+1}$ . In this paper, we will use the curve  $c_0$  as Figure 2 shows. Denote  $b_i = \sigma^i(b_0), c_i = \sigma^i(c_0)$ . They are also used in this paper.

We need the intersection numbers between the curves  $a_j, b_k, c_l$ . Consider the indexes  $i, j, k$  in modulo  $4g + 2$  classes. The intersection numbers between  $a_j, b_k, c_l$  are listed as follow:

- (1)  $i(a_j, a_k) = 0$  if and only if  $|j - k| \neq 1$ .
- (2)  $i(a_j, a_k) = 1$  if and only if  $|j - k| = 1$ .
- (3)  $i(b_j, b_k) = 0$  if and only if  $|j - k| \notin \{1, 2, 3, 2g - 2, 2g\}$ .
- (4)  $i(b_j, b_k) = 1$  if and only if  $|j - k| \in \{1, 3, 2g - 2, 2g\}$ .
- (5)  $i(b_j, b_k) = 2$  if and only if  $|j - k| = 2$ .
- (6)  $i(c_j, c_k) = 0$  if and only if  $j = k$ .
- (7)  $i(c_j, c_k) = 1$  if and only if  $j \neq k$ .
- (8)  $i(a_j, b_k) = 0$  if and only if  $j - k \notin \{0, 4\}$ .
- (9)  $i(a_j, b_k) = 1$  if and only if  $j - k \in \{0, 4\}$ .
- (10)  $i(a_j, c_k) = 0$  if and only if  $k - j \notin \{-1, 0\}$ .
- (11)  $i(a_j, c_k) = 1$  if and only if  $k - j \in \{-1, 0\}$ .
- (12)  $i(b_j, c_k) = 0$  if and only if  $k - j \notin \{0, 1, 2, 3\}$ .
- (13)  $i(b_j, c_k) = 1$  if and only if  $k - j \in \{0, 1, 2, 3\}$ .

Except (3), (4) and (5), the above intersection numbers can be verified directly on the  $(4g + 2)$ -gon, as shown by the right picture of Figure 2. For (3), (4) and (5), we can verify them from the Figure 3 in [1].

REMARK 2.6. In the calculation of (3), (4) and (5), when viewing these curves in the  $(4g + 2)$ -gon, we need to be careful. Sometimes though two such curves meet at the vertex of the  $(4g + 2)$ -gon, They do not really intersect. We can perturb them a little to cancel the intersection point.

**2.4. Some torsion elements.** Obviously we have  $\sigma^{4g+2} = 1$ . Take the reflection  $\tau$  of the regular  $(4g + 2)$ -gon satisfying  $\tau(b_0) = b_0$ . We can check  $(\tau B_0)^2 = 1$ . See Figure 3.

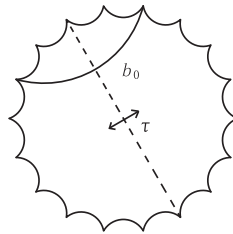


Fig.3

In [1] we know  $\text{Mod}^\pm(S_g) = \langle \sigma, \tau B_0 \rangle$  for  $g \geq 5$ . We will see it is also true for  $g = 3, 4$ .

### 3. The main result and the proof

**Theorem 3.1.** *Let  $\tau, \sigma, B_0$  as before. For  $g = 3, 4$ ,  $\text{Mod}^\pm(S_g) = \langle \sigma, \tau B_0 \rangle$ .*

Proof. Denote the subgroup generated by  $\tau B_0$  and  $\sigma$  as  $G$ . We will prove that  $G$  includes all the elements in  $\text{Mod}^\pm(S_g)$ . Similar to [1], The proof of the theorem has 4 steps.

Step 1. For every  $i, k$ , we prove  $B_i B_k^{-1}$  is in  $G$ .

Step 2. For every  $i, k$ , we prove  $B_i A_k^{-1}$  is in  $G$ .

Step 3. Using lantern relation, we prove that for every  $i$ ,  $A_i$  is in  $G$ .

Step 4.  $G = \text{Mod}^\pm(S_g)$ .

The motivation of step 2 and step 3 is as follow. There is a lantern on the surface where the curves in the lantern relation appear as  $a_1, a_3, a_5, b_0, b_2, e, f$  showed on the upper side of Figure 4. The lantern relation  $B_0 B_2 E = A_1 A_3 A_5 F$  can be also written as  $A_1 = (B_0 A_3^{-1})(B_2 A_5^{-1})(E F^{-1})$ . So one Dehn twist can be decomposed into the product of pairs of Dehn twists. Draw the lantern in the  $(4g + 2)$ -gon as on the lower side of Figure 4. We will find some of the pairs of Dehn twists we use can be expressed as the form  $B_k A_i^{-1}$ . When the  $g \leq 2$ , we cannot find a lantern on the surface.

The proof of Step 1:

We can check  $\sigma^j(\tau B_0)\sigma^j(\tau B_0) = B_j^{-1} B_0$ . Choosing  $j$  such that  $j$  is coprime to  $4g + 2$  and  $b_j$  do not intersect with  $b_0$ , we have  $B_j$  commutes with  $B_0$ , hence  $B_0 B_j^{-1}$  is in  $G$ . For every  $i$ , by conjugating  $B_0 B_j^{-1}$  with  $\sigma^i$ , we have  $B_i B_{i+j}^{-1}$  is in  $G$ . Since  $j$  is coprime to  $4g + 2$ , we have  $B_i B_k^{-1}$  is in  $G$  for every  $i, k$ .

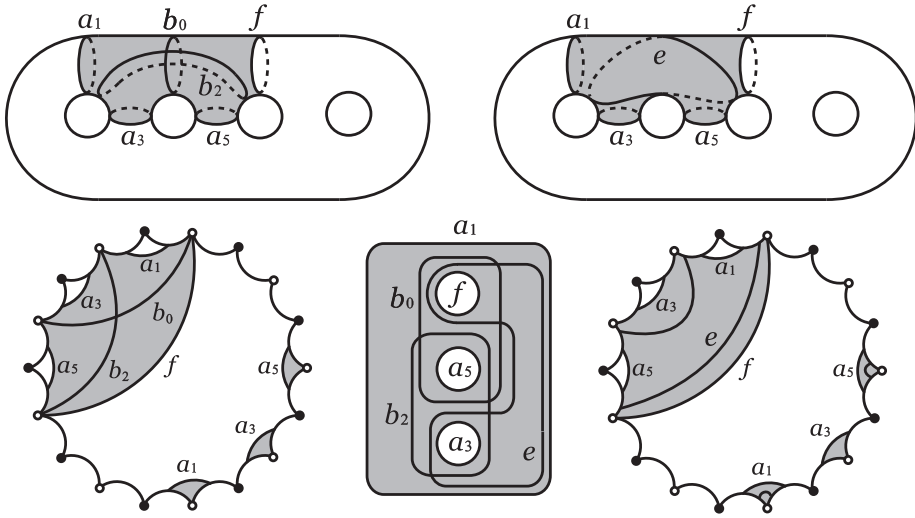


Fig.4

The proof of step 2:

Suppose the genus  $g = 4$ .

We already know  $b_{11}$  does not intersect with  $b_0$  or  $b_6$ . So  $B_{11}B_6^{-1}$  maps the pair of curves  $(b_{11}, b_0)$  to the pair of curves  $(b_{11}, B_6^{-1}(b_0))$ . Since  $B_{11}B_0^{-1}$  is in  $G$ ,  $B_{11}(B_6^{-1}B_0^{-1}B_6)$  is in  $G$ . We also have for every  $k$ ,  $B_k(B_6^{-1}B_0^{-1}B_6) = (B_kB_{11}^{-1})(B_{11}(B_6^{-1}B_0^{-1}B_6))$  is in  $G$ . See Figure 5.

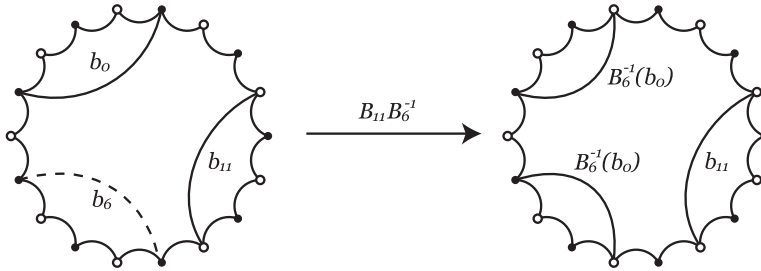


Fig.5

We know  $b_1$  does not intersect with  $b_5$ . We can check  $B_1B_5^{-1}B_6^{-1}(b_0) = a_5$ . So  $B_5^{-1}$  maps the pair of curves  $(b_5, B_6^{-1}(b_0))$  to the pair of curves  $(b_5, B_5^{-1}B_6^{-1}(b_0))$ ,  $B_1$  maps the pair of curves  $(b_5, B_5^{-1}B_6^{-1}(b_0))$  to the pair of curves  $(b_5, a_5)$ . This means  $B_1B_5^{-1}$  maps the pair of curves  $(b_5, B_6^{-1}(b_0))$  to the pair of curves  $(b_5, a_5)$ . See Figure 6.

Hence  $B_5A_5^{-1}$  is in  $G$ . After conjugating some power of  $\sigma$  and multiplying some  $B_iB_j^{-1}$ , we have for every  $i, j$ ,  $B_iA_j^{-1}$  is in  $G$ .

Suppose the genus  $g = 3$ .

We know that  $b_9$  does not intersect with  $b_0$  or  $b_4$ . So  $B_9B_4^{-1}$  maps the pair of curves  $(b_9, b_0)$  to the pair of curves  $(b_9, B_4^{-1}(b_0))$ . We can also check when the genus is 3,  $c_0 = B_4^{-1}(b_0)$ . So  $B_9C_0^{-1}$  is in  $G$ . See Figure 7.

After conjugating with some power of  $\sigma$  and multiplying some  $B_iB_j^{-1}$ , we have for every  $i, j$ ,  $B_iC_j^{-1}$  and  $C_iB_j^{-1}$  are in  $G$ . We also have for every  $i, j$ ,  $C_iC_j^{-1}$  is in  $G$ .

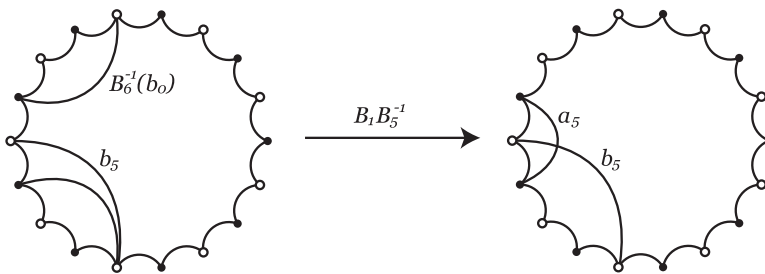


Fig.6

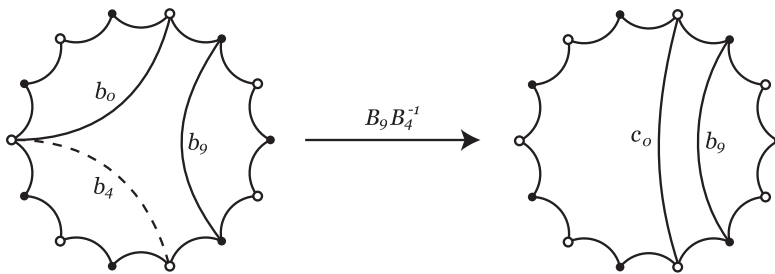


Fig.7

We know  $c_0$  does not intersect with  $b_1$  or  $b_2$ . So  $B_2 C_0^{-1}$  maps the pair of curves  $(c_0, b_1)$  to the pair of curves  $(c_0, B_2(b_1))$ . Then  $C_0(B_2 B_1^{-1} B_2^{-1})$  is in  $G$ . For every  $i$ ,  $C_i(B_2 B_1^{-1} B_2^{-1})$  is also in  $G$ . See Figure 8.

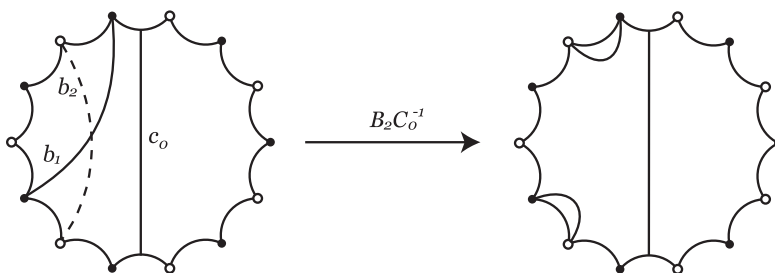


Fig.8

We know  $c_4$  does not intersect with  $b_6$  or  $B_2(b_1)$ . So  $C_4 B_6^{-1}$  maps the pair of curves  $(c_4, B_2(b_1))$  to the pair of curves  $(c_4, B_6^{-1} B_2(b_1))$ . Then  $C_0(B_6^{-1} B_2 B_1^{-1} B_2^{-1} B_6)$  is in  $G$ . See Figure 9.

We know  $c_4$  does not intersect with  $b_5$  or  $B_6^{-1} B_2(b_1)$ . So  $C_4 B_5^{-1}$  maps the pair of curves  $(c_4, B_6^{-1} B_2(b_1))$  to the pair of curves  $(c_4, B_5^{-1} B_6^{-1} B_2(b_1))$ . Then  $C_4 (B_5^{-1} B_6^{-1} B_2 B_1^{-1} B_2^{-1} B_6 B_5)$  is in  $G$ . See Figure 10.

We can check that  $B_5^{-1} B_6^{-1} B_2(b_1) = a_2$ . So  $C_4 A_2^{-1} = C_4 (B_5^{-1} B_6^{-1} B_2 B_1^{-1} B_2^{-1} B_6 B_5)$  is in  $G$ . Conjugating with some power of  $\sigma$  and multiplying  $C_j C_k^{-1}$ , we have for every  $j, k$ ,  $C_j A_k^{-1}$  is in  $G$ . Multiplying it by  $B_i C_j^{-1}$ , we have for every  $i, k$ ,  $B_i A_k^{-1}$  is in  $G$ .

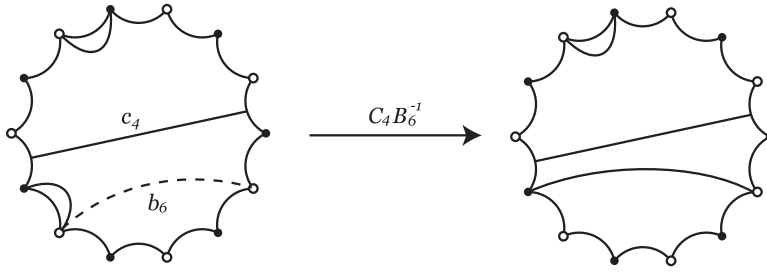


Fig.9

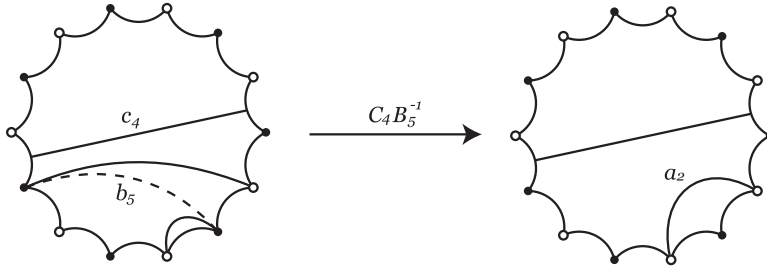


Fig.10

The proof of step 3:

We want to show for every  $i$ ,  $A_i$  is in  $G$ .

Recall lantern relation, we have  $B_0 B_2 E = A_1 A_3 A_5 F$ , or  $A_1 = (B_0 A_3^{-1}) (B_2 A_5^{-1}) (E F^{-1})$ , where  $e$  and  $f$  are the curves showed in Figure 4. By the result of step 2,  $B_0 A_3^{-1}$  and  $B_2 A_5^{-1}$  are in  $G$ . What we need is to prove  $E F^{-1}$  is also in  $G$ . Notice  $E F^{-1} = (E B_i^{-1})(B_i B_j^{-1})(B_j F^{-1})$ . We only need to show there exist some  $i, j$  such that  $E B_i^{-1}$  and  $B_j F^{-1}$  are in  $G$ .

Suppose  $g = 4$ .

We can check that  $f = B_3^{-1} A_6 A_5 A_4 (b_0)$ . We also know  $b_7$  does not intersect with  $a_4, a_5, a_6, b_3$ . So  $(B_7 B_3^{-1})(A_6 B_7^{-1})(A_5 B_7^{-1})(A_4 B_7^{-1})$  maps  $(b_7, b_0)$  to  $(b_7, f)$ . Hence  $B_7 F^{-1}$  is in  $G$ . See Figure 11.

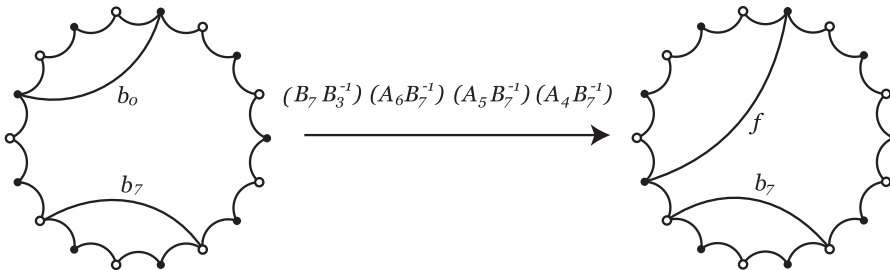


Fig.11

We can check  $e = A_2 A_1 A_4^{-1} B_1 (a_5)$ . Since  $b_{12}$  does not intersect with  $a_1, a_2, a_4, a_5, b_1, (A_2 B_{12}^{-1})(A_1 B_{12}^{-1})(B_{12} A_4^{-1})(B_1 B_{12}^{-1})$  maps  $(a_5, b_{12})$  to  $(e, b_{12})$ . Hence  $E B_{12}^{-1}$  is in  $G$ . See Figure 12.

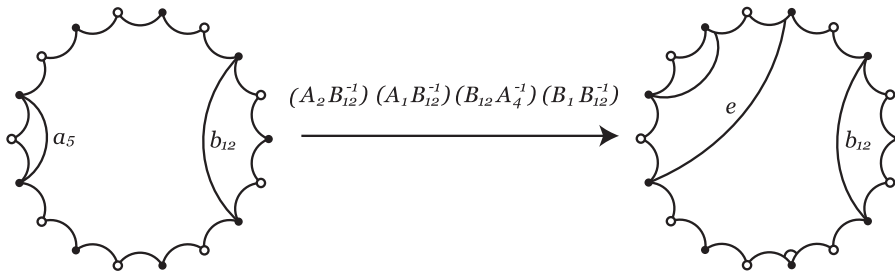


Fig. 12

Suppose  $g = 3$ .

The fact  $f = B_3^{-1}A_6A_5A_4(b_0)$  still holds. When  $g = 3$  we cannot find some  $b_i$  that does not intersect with  $a_4, a_5, a_6, b_3$  simultaneously. We use some curves  $c_i$  instead.

At first we find  $c_6$  does not intersect with  $a_4, a_5, b_0$ . So  $(A_5C_6^{-1})(A_4C_6^{-1})$  maps  $(c_6, b_0)$  to  $(c_6, A_5A_4(b_0))$ ,  $C_6(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1}$  is in  $G$ . See Figure 13.

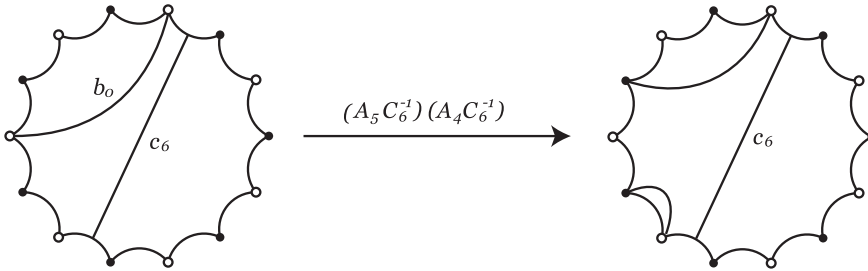


Fig. 13

$B_8(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1} = (B_8C_6^{-1})(C_6(A_5A_4B_0A_4^{-1}A_5^{-1})^{-1})$  is also in  $G$ . Then we find  $b_8$  does not intersect with  $a_6, b_3$  or  $A_5A_4(b_0)$ . So  $(B_8B_3^{-1})(B_8^{-1}A_6)$  maps  $(b_8, A_5A_4(b_0))$  to  $(b_8, B_3^{-1}A_6A_5A_4(b_0)) = (b_8, f)$ . Hence  $B_8F^{-1}$  is in  $G$ . See Figure 14.

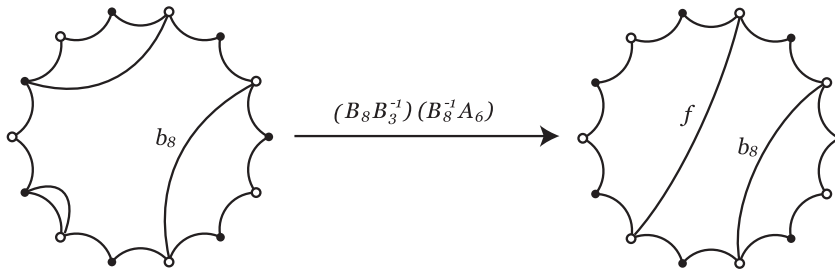


Fig. 14

Similarly, The fact  $e = A_2A_1A_4^{-1}B_1(a_5)$  still holds. When  $g = 3$ , we can find  $c_i$  does not intersect with  $a_1, a_2, a_4, a_5, b_1$ . So  $(A_2C_6^{-1})(A_1C_6^{-1})(C_6A_4^{-1})(B_1C_6^{-1})$  maps  $(a_5, c_6)$  to  $(e, c_6)$ . Hence  $EC_6^{-1}$  is in  $G$ . And then multiply  $C_6B_i^{-1}$ , we have  $EB_i^{-1}$  in  $G$ . See Figure 15.

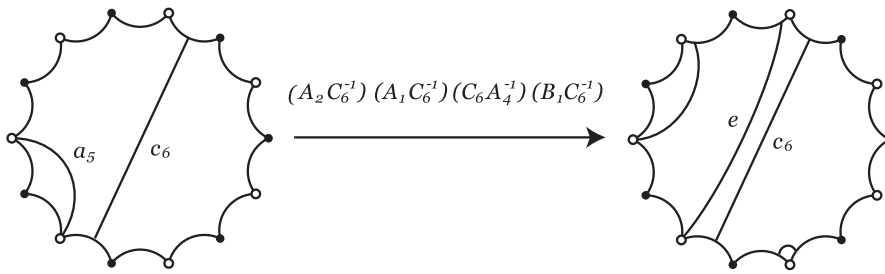


Fig. 15

The proof of step 4:

Since both  $B_i A_j^{-1}$  and  $A_j$  are in  $G$ , by Humphries’s result,  $G$  contains the mapping class group  $\text{Mod}(S_g)$ . Now  $\tau B_0 \in G$  is an orientation reversing element.  $\text{Mod}(S_g)$  is an index 2 subgroup of  $\text{Mod}^\pm(S_g)$ . So  $G = \text{Mod}^\pm(S_g)$  for  $g = 3, 4$ .  $\square$

**Theorem 3.2.** For  $g = 1$ ,  $\text{Mod}^\pm(S_1)$  is  $GL(2, \mathbb{Z})$ . It cannot be generated by two elements of finite orders.

Proof. We only need to prove that  $PGL(2, \mathbb{Z})$  cannot be generated by two elements of finite orders. The idea of the proof is:

- (1) use a presentation of  $PGL(2, \mathbb{Z})$  whose generators are elements of finite orders;
- (2) list the possible conjugacy classes of all finite order elements in  $PGL(2, \mathbb{Z})$ ;
- (3) give a homomorphism from  $PGL(2, \mathbb{Z})$  to a finite group  $D_6 \times \mathbb{Z}_2$ ;
- (4) check all the possible generating set of  $D_6 \times \mathbb{Z}_2$  consisting of two generators;
- (5) find the conjugacy classes of the pre-image of the generators of  $PGL(2, \mathbb{Z})$  in (4), prove either they cannot be the conjugacy classes of finite orders elements or they cannot generate  $PGL(2, \mathbb{Z})$ .

It is well known that  $PGL(2, \mathbb{Z}) \cong D_6 *_{\mathbb{Z}_2} D_4$ , where  $D_6$  and  $D_4$  are the dihedral group of order 6 and order 4 respectively (see, for example, [6] or [7]). It has a presentation as

$$PGL(2, \mathbb{Z}) = \langle a, b, t \mid a^3 = t^2 = b^2 = 1, at = ta^2, bt = tb \rangle.$$

Every element  $\alpha$  in  $PGL(2, \mathbb{Z})$  can be written as a reduced form in one of the following 8 types:

- |   |   |
|---|---|
| (1) $a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k}$ ,           | (2) $b^{j_1} a^{i_1} \dots b^{j_k} a^{i_k}$ ,           |
| (3) $a^{i_0} b^{j_1} a^{i_1} \dots b^{j_k} a^{i_k}$ ,   | (4) $b^{j_0} a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k}$ ,   |
| (5) $a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k} t$ ,         | (6) $b^{j_1} a^{i_1} \dots b^{j_k} a^{i_k} t$ ,         |
| (7) $a^{i_0} b^{j_1} a^{i_1} \dots b^{j_k} a^{i_k} t$ , | (8) $b^{j_0} a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k} t$ . |

Here each  $i_n \in \{1, 2\}$  and each  $j_n = 1$ .

- For an element in type (1), (2), (5) or (6), since its power will have a larger word length, it must not be of finite order.
- For an element in type (3) or (4), it can be conjugated to an element in (1) or (2), or it can be conjugated to an element in (4) or (3) with a shorter word length.
- For type (7),  $a^{i_0} b^{j_1} a^{i_1} \dots b^{j_k} a^{i_k} t$  is conjugated to  $b^{j_1} a^{i_1} \dots b^{j_k} a^{i_k+2i_0} t$ , which is in type (8) with a shorter word length or in type (6).

- For type (8),  $b^{j_0} a^{i_1} b^{j_1} \dots b^{j_{k-1}} a^{i_k} b^{j_k} t$  is conjugated to  $a^{i_1} b^{j_1} \dots b^{j_{k-1}} a^{i_k} t$ , which is in type (7) with a shorter word length.

For an element of finite order in  $PGL(2, \mathbb{Z})$ , we can conjugate it to an element with shortest word length. The only seven possible choices are as follow:  $a, a^2, t, at, a^2t, b, bt$ . About the order of the elements in their conjugacy classes, the elements conjugating to  $a$  or  $a^2$  have order 3, and the elements conjugating to  $t, at, a^2t, b$  or  $bt$  have order 2.

By adding a new relation  $ab = ba$ , we get a quotient group homomorphic to the Cartesian product  $D_6 \times \mathbb{Z}_2$ , which is a finite group with the presentation

$$\langle a_1, t_1, b_1 \mid a_1^3 = t_1^2 = b_1^2 = 1, a_1 t_1 = t_1 a_1^2, b_1 t_1 = t_1 b_1, a_1 b_1 = b_1 a_1 \rangle.$$

For the convenience of calculation, we can regard  $D_6 \times \mathbb{Z}_2$  as a permutation subgroup of the symmetric group  $\Sigma_5$ . Table 3 lists the images in  $\Sigma_5$  of all elements in  $D_6 \times \mathbb{Z}_2$ .

Table 3.

element in $D_6 \times \mathbb{Z}_2$	permutation	element in $D_6 \times \mathbb{Z}_2$	permutation
1	()	$b_1$	(45)
$a_1$	(123)	$a_1 b_1$	(123)(45)
$a_1^2$	(132)	$a_1^2 b_1$	(132)(45)
$t_1$	(12)	$b_1 t_1$	(12)(45)
$a_1 t_1$	(13)	$a_1 b_1 t_1$	(13)(45)
$a_1^2 t_1$	(23)	$a_1^2 b_1 t_1$	(23)(45)

As the permutations in  $\Sigma_5$ , both  $a_1$  and  $a_1^2$  have the same 3-cycle orbit type, they are in the same conjugacy class in  $D_6 \times \mathbb{Z}_2$ . All of  $t_1, a_1 t_1, a_1^2 t_1$  and  $b_1$  have the same 2-cycle orbit type. However,  $b_1$  generates the product factor  $\mathbb{Z}_2$  of  $D_6 \times \mathbb{Z}_2$ . Hence we can check that  $t_1, a_1 t_1, a_1^2 t_1$  belong to the same conjugacy class, and  $b_1$  belongs to another conjugacy class. All of  $b_1 t_1, a_1 b_1 t_1$  and  $a_1^2 b_1 t_1$  have the same orbit type as a product of two disjoint 2-cycles. For the conjugacy classes of the possible finite order elements in  $PGL(2, \mathbb{Z})$ , we can check the their possible images in  $D_6 \times \mathbb{Z}_2$  are as Table 4 shows.

Table 4.

Conjugacy classes in $PGL(2, \mathbb{Z})$	elements in $D_6 \times \mathbb{Z}_2$
$a, a^2$	$a_1, a_1^2$
$t, at, a^2t$	$t_1, a_1 t_1, a_1^2 t_1$
$b$	$b_1$
$bt$	$b_1 t_1, a_1 b_1 t_1, a_1^2 b_1 t_1$

Each finite order element in  $PGL(2, \mathbb{Z})$  will be mapped to an element in  $D_6 \times \mathbb{Z}_2$  which conjugates with an element in  $\{a_1, a_1^2, t_1, a_1 t_1, a_1^2 t_1, b_1, b_1 t_1, a_1 b_1 t_1, a_1^2 b_1 t_1\}$ .

By direct computation in the permutation subgroup of the symmetric group, we know that if two elements  $x, y$  generate  $D_6 \times \mathbb{Z}_2$ , there are only two possible cases:

- (1)  $\{x, y\}$  contain  $a_1 b_1$  or  $a_1^2 b_1$ ;
- (2) both  $x$  and  $y$  are elements of order 2.

Either  $a_1b_1$  or  $a_1^2b_1$  cannot conjugate to an element in  $\{a_1, a_1^2, t_1, a_1t_1, a_1^2t_1, b_1, b_1t_1, a_1b_1t_1, a_1^2b_1t_1\}$ , hence their pre-image in  $PGL(2, \mathbb{Z})$  cannot be elements of finite order.

The pre-images of the order 2 elements in  $D_6 \times \mathbb{Z}_2$  can be conjugated to the elements in  $\{t, at, a^2t, b, bt\}$  in  $PGL(2, \mathbb{Z})$ . They are still of order 2. But two elements of order 2 can only generate a dihedral group, not  $PGL(2, \mathbb{Z})$ .  $\square$

REMARK 3.3. Though  $GL(2, \mathbb{Z})$  cannot be generated by two torsion elements, it can be generated by two elements. In fact, since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

we have

$$GL(2, \mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

So the extended mapping class group  $GL(2, \mathbb{Z})$  for  $g = 1$  case: (1) can be generated by two elements; (2) cannot be generated by two elements of finite orders. This is different from the case  $g \geq 3$ .

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