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LAMINATED BEAMS WITH TIME-VARYING DELAY

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Abstract

This manuscript is concerned with long-time dynamics for a laminated beam which consists of two identical layers of uniform thickness, taking into account that an adhesive of small thickness is bonding the two surfaces thereby producing an interfacial slip. Using the variable norm technique of Kato, we prove the global well-posedness of solutions. For asymptotic behavior, we apply the Energy Method. Assuming the control through a time-varying delay just on the transverse displacement of the beam, we establish the exponential decay of energy to the system by using an appropriate Lyapunov functional.

1. Introduction

The dynamics of laminated beams is a relevant research subject due to the high applicability of such materials in the industry. Of particular interest is a mathematical model of laminated beam (1.1)-(1.3) based on the Timoshenko system proposed by Hansen and Spies [12, 13] for two-layered beams in which a slip can occur at the interface of contact

$$\varrho u_{tt} + G(\psi - u_x)_x = 0, \;\; x \in (0, L), \;\; t \geq 0, \tag{1.1}$$

$$I_\varrho(3S_{tt} - \psi_{tt}) - G(\psi - u_x) - D(3S_{xx} - \psi_{xx}) = 0, \;\; x \in (0, L), \;\; t \geq 0, \tag{1.2}$$

$$3I_\varrho S_{tt} + 3G(\psi - u_x) + 4\delta_0 S + 4\gamma_0 S_t - 3DS_{xx} = 0, \;\; x \in (0, L), \;\; t \geq 0, \tag{1.3}$$

where $u = u(x, t)$ denotes the transverse displacement, $\psi = \psi(x, t)$ represents the rotation angle, $S = S(x, t)$ is proportional to the amount of slip along with the interface at time t and longitudinal spatial variable x, respectively, $\varrho, G, I_\varrho, D, \delta_0, \gamma_0$ are the density of the beams, the shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping of the beams. In this model, we have a "glue" layer of negligible thickness that bonds the two adjoining surfaces and produce the restoring force S_t. In [43] was proved that the frictional damping S_t created by the interfacial slip alone is not enough to stabilize this system exponentially to its equilibrium state. Naturally, the question arises of studying the action of additional stabilizing mechanisms on this model.

In recent years, the control of Partial Differential Equations with time delay effects has become an attractive area of research. In fact, time delays so often arise in many physical, chemical, biological and economical phenomena, see [38] and references therein. Whenever energy is physically transmitted from one place to another, there is a delay associated with the transmission, see [37]. Time delay is the property of a physical system by which the response to an applied force is delayed in its effect, and the central question is that delays source can destabilize a system that is asymptotically stable in the absence of delays, see [7].

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Problem with delay as internal feedback was considered in [27], where was proved the exponential decay of solution by Energy Method. By semigroup approach in [29] was proved the well-posedness and exponential stability for a wave equation with frictional damping and nonlocal time-delayed condition. In [5] was proved the global existence and energy decay of solutions for a wave equation with non-constant delay and nonlinear weights.

The stability of a Timoshenko beam system with boundary time delays was studied in [14]. In [42], the authors considered the interior damping and boundary delay. The approach for boundary varying delay we cite [28]. In [19], the authors obtained the well-posedness and exponential stability for Timoshenko beam with delay on the frictional damping under the condition $\mu_1 > \mu_2 > 0$ and $\tau(t) = \tau$. In [16] was extended the result of [19] for $\tau(t)$ a time-varying function. For a transmission problem in the presence of history and delay terms, under appropriate hypothesis on the relaxation function and the relationship between the weight of the damping and the weight of the delay, in [22] was proved well-posedness by using the semigroup theory and a decay result by introducing a suitable Lyapunov functional. Timoshenko theory was started in 1921 with S. P. Timoshenko [40, 41] and since then, Timoshenko system has been extensively studied by several authors, with different kinds of stabilization mechanisms. In [9] was considered a Timoshenko beams with linear time delay terms τ. In absence of delay, the existence and energy decay of the Timoshenko system has been extensively studied by several authors, we can cite a few of them [1, 2, 9, 10, 18, 23, 24, 25, 32, 34, 35, 36]. For Timoshenko system with delay we cite [3, 31, 11].

Structures with interfacial slip have gained much in popularity and are known under the name of laminated beams. They are of considerable importance in engineering, for instance we cite, [6, 13, 20, 21, 30, 33, 39]. In [4] was considered the following laminated beam with a single control in form of a frictional damping in the second equation

$$\rho w_{tt} + G(\psi - w_x)_x = 0, \tag{1.4}$$

$$I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) + \delta(3s_t - \psi_t) = 0, \tag{1.5}$$

$$3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s = 0. \tag{1.6}$$

The authors proved that the unique dissipation through the frictional damping is strong enough to exponentially stabilize the system similar to the full damped Timoshenko system.

In [8] was considered the laminated Timoshenko beams with time delay terms τ. Using the notion of effective rotation angle $\xi = 3s - \psi$ in (1.1)-(1.3) with $\delta_0 = 0$ and $\gamma_0 = 0$ and assuming that the weights of the delay are small, was established the exponential decay of energy to the system (1.7)-(1.9) by using an appropriate Lyapunov functional,

$$\rho w_{tt} + G(3s - \xi - w_x)_x + \alpha_1 w_t(x, t - \tau) = 0, \tag{1.7}$$

$$I_\rho \xi_{tt} - D\xi_{xx} - G(3s - \xi - w_x) + \alpha_2 \xi_t(x, t - \tau) = 0, \tag{1.8}$$

$$I_\rho s_{tt} - Ds_{xx} + G(3s - \xi - w_x) + \alpha_3 s_t(x, t - \tau) = 0. \tag{1.9}$$

To the best of our knowledge, laminated Timoshenko beams with time-varying delay $\tau(t)$ was not considered previously. We consider the following damped system bellow where the time-varying delay act in the frictional damping on the transversal vibrations of the beam

$$\varrho u_{tt}(x,t) + G(\psi - u_x)_x(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t - \tau(t)) = 0, \tag{1.10}$$

$$I_\varrho(3S_{tt}-\psi_{tt})(x,t)-G(\psi-u_x)(x,t)-D(3S_{xx}-\psi_{xx})(x,t)+\beta(3S_t-\psi_t)(x,t)=0, \tag{1.11}$$

$$3I_\varrho S_{tt}(x,t)+3G(\psi-u_x)(x,t)+4\delta_0 S(x,t)+4\gamma_0 S_t(x,t)-3DS_{xx}(x,t)=0. \tag{1.12}$$

Our purpose in this paper is the asymptotic behavior of the solution. The plan of the paper is as follows. First, we present the well-posedness of the problem (1.10)-(1.12). Next, we use the direct method, see [17], that consists in the use of appropriated multiplies to build a functional of Lyapunov for the system and our challenge is to prove the exponential stability of the damped system (1.10)-(1.12) with time-varying delay.

2. The well-posedness

In this section, under the assumption

$$\mu_2 \le \sqrt{1-d}\mu_1 \tag{2.1}$$

we present a existence result similar to the one obtained in [26] for a simple wave equation.

We introduce as in [28] the following new variable

$$z(x,\rho,t)=u_t(x,t-\tau(t)\rho),\ \ x\in(0,L),\ \ \rho\in[0,1],\ \ t>0. \tag{2.2}$$

It is straight forward to check that z satisfies

$$\tau(t)z_t(x,\rho,t)+(1-\tau'(t)\rho)z_\rho(x,\rho,t)=0,$$

consequently, problem (1.10)-(1.12) is equivalent to

$$\varrho u_{tt}+G(\psi-u_x)_x+\mu_1 u_t+\mu_2 z(x,1,t)=0, \tag{2.3}$$

$$I_\varrho(3S_{tt}-\psi_{tt})-G(\psi-u_x)-D(3S_{xx}-\psi_{xx})+\beta(3S_t-\psi_t)=0, \tag{2.4}$$

$$3I_\varrho S_{tt}+3G(\psi-u_x)+4\delta_0 S+4\gamma_0 S_t-3DS_{xx}=0, \tag{2.5}$$

$$\tau(t)z_t(x,\rho,t)+(1-\tau'(t)\rho)z_\rho(x,\rho,t)=0. \tag{2.6}$$

The above system is subject to the initial data

$$(u(x,0),\psi(x,0),S(x,0))=(u_0(x),\psi_0(x),S_0(x)), \tag{2.7}$$

$$(u_t(x,0),\psi_t(x,0),S_t(x,0))=(u_1(x),\psi_1(x),S_1(x)), \tag{2.8}$$

$$z(x,\rho,0)=f_0(x,-\tau(0)\rho) \tag{2.9}$$

and Dirichlet boundary conditions

$$u(0,t)=\psi(0,t)=S(0,t)=0, \tag{2.10}$$

$$u(L,t)=\psi(L,t)=S(L,t)=0, \tag{2.11}$$

where $\tau(t)$ is a time-varying delay satisfying

$$0<\tau_0\le\tau(t)\le\tau_1,\quad \tau'(t)\le d<1\quad \text{and}\quad \tau\in W^{2,\infty}(0,T),\ \text{for all } T>0. \tag{2.12}$$

Now, we introduce the vector function

$$U=(u,u_t,\xi,\xi_t,S,S_t,z)^T,$$

where $\xi=3S-\psi$.

The system (2.3)-(2.11) can be written as

$$\begin{cases} U_t - \mathcal{A}(t)U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{2.13}$$

where the linear operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ u_t \\ \xi \\ \xi_t \\ S \\ S_t \\ z \end{pmatrix} = \begin{pmatrix} u_t \\ -\frac{1}{\varrho}\left[G(3S-\xi-u_x)_x + \mu_1 u_t + \mu_2 z(\cdot,\ 1)\right] \\ \xi_t \\ \frac{1}{I_\varrho}\left[G(3S-\xi-u_x) + D\xi_{xx} - \beta\xi_t\right] \\ S_t \\ \frac{1}{I_\varrho}\left[DS_{xx} - G(3S-\xi-u_x) - \frac{4\delta_0}{3}S - \frac{4\gamma_0}{3}S_t\right] \\ -\frac{(1-\tau'(t)\rho)}{\tau(t)} z_\rho(x,\rho,t) \end{pmatrix},$$

with energy space

$$\mathcal{H} = [H_0^1(0,L) \times L^2(0,L)]^3 \times L^2((0,L)\times(0,1))$$

and

$$D(\mathcal{A}(t)) = \left\{(u,u_t,\xi,\xi_t,S,S_t,z)^T \in H : u = z(\cdot,0) \text{ in } (0,L)\right\},$$

for $t > 0$, where

$$H = [H^2(0,L)\cap H_0^1(0,L) \times H_0^1(0,L)]^3 \times L^2\left((0,L); H_0^1(0,1)\right).$$

Note that $D(\mathcal{A}(t))$ is independent of time $t > 0$, i.e.,

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \text{for all } t > 0. \tag{2.14}$$

We denote the $L^2(0,L)$ inner product by

$$\langle f,g\rangle = \int_0^L f(x)g(x)dx \text{ for all } f,g \in L^2(0,L) \text{ and consequently } \langle f,f\rangle = ||f||^2.$$

The space $\mathcal{H}$ is a Hilbert space with the norm

$$\begin{aligned} ||U||_{\mathcal{H}}^2 &= \varrho||u_t||^2 + I_\varrho||\xi_t||^2 + D||\xi_x||^2 + 3D||S_x||^2 + 3I_\varrho||S_t||^2 \\ &\quad + G||3S-\xi-u_x||^2 + 4\delta_0||S||^2 + \int_0^L\int_0^1 z^2(x,\rho)\,d\rho\,dx, \end{aligned}$$

for $U = (u,u_t,\xi,\xi_t,S,S_t,z)^T$.

Our existence and uniqueness result is stated as follows:

Theorem 2.1. *For any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution U of problem* (2.13) *satisfying*

$$U \in C([0,\infty),\mathcal{H})$$

for the problem (2.13). *Moreover, if $U_0 \in D(\mathcal{A}(0))$ then*

$$U \in C([0,\infty); D(\mathcal{A}(0))) \cap C^1([0,\infty);\mathcal{H}).$$

In order to prove Theorem 2.1, we will use the variable norm technique developed by Kato in [15]. The following Theorem is proved in [15].

Theorem 2.2. *Assume that*

(1) $D(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$;
(2) $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, *for all* $t > 0$;
(3) *for all* $t \in [0,T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,T]\}$ *is stable with stability constants C and m independent of t, i.e., the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies*

$$\|S_t(s)(u)\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}, \text{ for all } u \in \mathcal{H}, s \geq 0;$$

(4) $\mathcal{A}'(t) \in L^\infty_*([0,T], B(D(\mathcal{A}(0)),\mathcal{H}))$, *where* $L^\infty_*([0,T], B(D(\mathcal{A}(0)),\mathcal{H}))$, *is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(D(\mathcal{A}(0)),\mathcal{H})$ *of bounded operators from* $D(\mathcal{A}(0))$ *into* $\mathcal{H}$.

Then problem (2.13) *has a unique solution*

$$U \in C([0,T); D(\mathcal{A}(0))) \cap C^1([0,T);\mathcal{H}),$$

for any initial datum in $D(\mathcal{A}(0))$.

Proof of Theorem 2.1. To prove Theorem 2.1, we will follow method used in [26] with the necessary modification imposed by the nature of our problem.

First, we show that $D(\mathcal{A}(0))$ is dense in $\mathcal{H}$. For, let $\hat{U} = (\hat{u}, \hat{u}_t, \hat{\xi}, \hat{\xi}_t, \hat{S}, \hat{S}_t, \hat{z})^T \in \mathcal{H}$ be orthogonal to all elements of $D(\mathcal{A}(0))$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$:

$$\begin{aligned} 0 = \langle U, \hat{U}\rangle_{\mathcal{H}} &= \varrho\langle u_t, \hat{u}_t\rangle + I_\varrho\langle \xi_t, \hat{\xi}_t\rangle + D\langle \xi_x, \hat{\xi}_x\rangle + 3D\langle S_x, \hat{S}_x\rangle + 3I_\varrho\langle S_t, \hat{S}_t\rangle \\ &\quad + G\langle 3S - \xi - u_x, 3\hat{S} - \hat{\xi} - \hat{u}_x\rangle + 4\delta_0\langle S, \hat{S}\rangle + \int_0^L\int_0^1 z(x,\rho)\hat{z}(x,\rho)\,d\rho\,dx, \end{aligned} \tag{2.15}$$

for all $U = (u, u_t, \xi, \xi_t, S, S_t, z)^T \in D(\mathcal{A}(0))$.

We first take $u = u_t = \xi = \xi_t = S = S_t = 0$ and $z \in D((0,L)\times(0,1))$. As the vector $U = (0,0,0,0,0,0,z)^T \in D(\mathcal{A}(0))$ and therefore, from (2.15), we deduce that

$$\int_0^L\int_0^1 z(x,\rho)\hat{z}(x,\rho)\,d\rho\,dx = 0.$$

Since $D((0,L)\times(0,1))$ is dense in $L^2((0,L)\times(0,1))$, it follows then that $\hat{z} = 0$.

Similarly, let $u_t \in D(0,L)$, then $U = (0, u_t, 0,0,0,0,0)^T \in D(\mathcal{A}(0))$, which implies from (2.15) that

$$\langle u_t, \hat{u}_t\rangle = 0,$$

so, as above, $\hat{u}_t = 0$.

Next, let $U = (u,0,0,0,0,0,0)^T$ then we obtain from (2.15) that

$$\langle u_x, \hat{u}_x\rangle = 0.$$

It is obvious that $(u,0,0,0,0,0,0)^T \in D(\mathcal{A}(0))$ if and only if $u \in H^2(0,L)\cap H^1_0(0,L)$ and since $H^2(0,L)\cap H^1_0(0,L)$ is dense in $H^1_0(0,L)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{H^1_0(0,L)}$, we get $\hat{u} = 0$. By the same ideas as above, we can also show that $\hat{\xi} = \hat{S} = 0$. Finally for $\xi_t, S_t \in D(0,L)$, we get from (2.15)

$$\langle \xi_t, \hat{\xi}_t\rangle = 0 \quad\text{and}\quad \langle S_t, \hat{S}_t\rangle = 0,$$

respectively, and by density of $D(0,L)$ in $L^2(0,L)$, we obtain $\hat{\xi}_t = \hat{S}_t = 0$.

We consequently obtain that

$$D(\mathcal{A}(0)) \text{ is dense in } \mathcal{H}. \tag{2.16}$$

Now, we show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in $\mathcal{H}$ for a fixed t. We define the time-dependent norm on $\mathcal{H}$ (which is equivalent to classical norm)

$$\begin{aligned} ||U||_t^2 = {} & \varrho||u_t||^2 + I_\varrho||\xi_t||^2 + D||\xi_x||^2 + 3D||S_x||^2 + 3I_\varrho||S_t||^2 \\ & + G||3S - \xi - u_x||^2 + 4\delta_0||S||^2 + \zeta\tau(t)\int_0^L\int_0^1 z^2(x,\rho)\,d\rho\,dx, \end{aligned} \tag{2.17}$$

where ζ satisfies

$$\frac{\mu_2}{\sqrt{1-d}} < \zeta < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{2.18}$$

thanks to hypothesis (2.1).

We calculate $\langle\mathcal{A}(t)U, U\rangle_t$ for a fixed t. Take $U = (u, u_t, \xi, \xi_t, S, S_t, z)^T \in D(\mathcal{A}(t))$, then

$$\begin{aligned} \langle\mathcal{A}(t)U, U\rangle_t = & -\mu_1||u_t||^2 - \mu_2\langle z(x,1), u_t\rangle - \beta||\xi_t|| - 4\gamma_0||S_t||^2 \\ & - \zeta\int_0^L\int_0^1 (1-\tau'(t)\rho)z(x,\rho)z_\rho(x,\rho)\,d\rho\,dx. \end{aligned}$$

Observe that

$$\begin{aligned} \int_0^L\int_0^1 (1-\tau'(t)\rho)z(x,\rho)z_\rho(x,\rho)\,d\rho\,dx &= \frac{1}{2}\int_0^L\int_0^1 (1-\tau'(t)\rho)z(x,\rho)\frac{\partial}{\partial\rho}z^2(x,\rho)\,d\rho\,dx \\ &= \frac{\tau'(t)}{2}\int_0^L\int_0^1 z^2(x,\rho)\,d\rho\,dx \\ &\quad + \frac{1}{2}\int_0^L \Big[(1-\tau'(t))z^2(x,1) - z^2(x,0)\Big]\,dx. \end{aligned}$$

Whereupon

$$\begin{aligned} \langle\mathcal{A}(t)U, U\rangle_t = & -\mu_1||u_t||^2 - \mu_2\langle z(x,1), u_t\rangle - \beta||\xi_t|| - 4\gamma_0||S_t||^2 \\ & - \frac{\zeta\tau'(t)}{2}\int_0^L\int_0^1 z^2(x,\rho)\,d\rho\,dx + \frac{\zeta}{2}||u_t||^2 - \frac{\zeta}{2}\int_0^L (1-\tau'(t))z^2(x,1)\,dx. \end{aligned}$$

Due Young's inequality , we have

$$\mu_2\langle z(x,1), u_t\rangle \le \frac{\mu_2}{2\sqrt{1-d}}||u_t||^2 + \frac{\mu_2\sqrt{1-d}}{2}||z(x,1)||^2,$$

then

$$\begin{aligned} \langle\mathcal{A}(t)U, U\rangle_t \le & -\left(\mu_1 - \frac{\zeta}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)||u_t||^2 \\ & - \left(\frac{\zeta}{2}(1-\tau'(t)) - \frac{\mu_2\sqrt{1-d}}{2}\right)||z(x,1)||^2 \\ & - \beta||\xi_t||^2 - 4\gamma_0||S_t||^2 + \kappa(t)\langle U, U\rangle_t, \end{aligned}$$

where

$$\kappa(t) = \frac{\sqrt{1 + \tau'(t)^2}}{2\tau(t)}.$$

From (2.12) and (2.18) we conclude that

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0, \tag{2.19}$$

which means that operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Now, we prove the surjectivity of the operator $\lambda I - \mathcal{A}(t)$ for fixed $t > 0$ and $\lambda > 0$. For this purpose, let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathcal{H}$, we seek $U = (u, u_t, \xi, \xi_t, S, S_t, z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t))\, U = F,$$

that is verifying following system of equations

$$\begin{cases} \lambda u - u_t = f_1, \\ \lambda \varrho u_t + G(3S - \xi - u_x)_x + \mu_1 u_t + \mu_2 z(\cdot, 1) = \varrho f_2, \\ \lambda \xi - \xi_t = f_3, \\ \lambda I_\varrho \xi_t - G(3S - \xi - u_x) - D\xi_{xx} + \beta \xi_t = I_\varrho f_4, \\ \lambda S - S_t = f_5, \\ 3\lambda I_\varrho S_t - 3DS_{xx} + 3G(3S - \xi - u_x) + 4\delta_0 S + 4\gamma_0 S_t = 3I_\varrho f_6, \\ \lambda \tau(t) z + (1 - \tau'(t)\rho)\, z_\rho = \tau(t) f_7. \end{cases} \tag{2.20}$$

Suppose that we have found u, ξ and S with the appropriated regularity. Therefore, for the first, third and the fifth equations in (2.20) give

$$\begin{cases} u_t = \lambda u - f_1, \\ \xi_t = \lambda \xi - f_3, \\ S_t = \lambda S - f_5. \end{cases} \tag{2.21}$$

It is clear that $u_t, \xi_t, S_t \in H_0^1(0, L)$. Furthermore, by (2.2) we can find z as

$$z(x, 0) = u_t(x), \;\; \text{for } x \in (0, L).$$

Following the same approach [26], we obtain, by using the last equation in (2.20),

$$z(x, \rho) = u_t(x) e^{-\vartheta(\rho, t)} + \tau(t) e^{-\vartheta(\rho, t)} \int_0^\rho f_7(x, s) e^{\vartheta(s, t)}\, ds,$$

if $\tau'(t) = 0$, where $\vartheta(\ell, t) = \lambda \ell \tau(t)$, and

$$z(x, \rho) = u_t(x) e^{\sigma(\rho, t)} + e^{\sigma(\rho, t)} \int_0^\rho \frac{\tau(t) f_7(x, s)}{1 - s\tau'(s)} e^{-\sigma(s, t)}\, ds,$$

otherwise, where $\sigma(\ell, t) = \lambda \frac{\tau(t)}{\tau'(t)} \ln(1 - \ell \tau'(t))$.

From (2.21), we obtain

$$z(x, \rho) = \lambda u(x) e^{-\vartheta(\rho, t)} - f_1(x, \rho) e^{-\vartheta(\rho, t)} + \tau(t) e^{-\vartheta(\rho, t)} \int_0^\rho f_7(x, s) e^{\vartheta(s, t)}\, ds, \tag{2.22}$$

if $\tau'(t) = 0$, and

$$z(x,\rho) = \lambda u(x)e^{\sigma(\rho,t)} - f_1(x,\rho)e^{\sigma(\rho,t)} + e^{\sigma(\rho,t)}\int_0^\rho \frac{\tau(t)f_7(x,s)}{1-s\tau'(s)}e^{-\sigma(s,t)}\,ds, \tag{2.23}$$

otherwise.

In particular, from (2.22) and (2.22), we have

$$z(x,1) = \lambda u(x)N_1 + N_2, \tag{2.24}$$

where

$$N_1 = \begin{cases} e^{-\vartheta(1,t)}, & \text{if} \quad \tau'(t) = 0,\\ e^{\sigma(1,t)}, & \text{if} \quad \tau'(t) \neq 0 \end{cases}$$

and

$$N_2 = \begin{cases} -f_1(x,1)e^{-\vartheta(1,t)} + \tau(t)e^{-\vartheta(1,t)}\displaystyle\int_0^1 f_7(x,s)e^{\vartheta(s,t)}\,ds, & \text{if} \quad \tau'(t) = 0,\\[2ex] -f_1(x,1)e^{\sigma(1,t)} + e^{\sigma(1,t)}\displaystyle\int_0^1 \frac{\tau(t)f_7(x,s)}{1-s\tau'(t)}e^{-\sigma(s,t)}\,ds, & \text{if} \quad \tau'(t) \neq 0. \end{cases}$$

By using (2.20) and (2.21), the functions u, ξ and S satisfying the following system

$$\begin{cases} \alpha u + G(3S - \xi - u_x)_x = g_1,\\ \eta\xi - D\xi_{xx} - G(3S - \xi - u_x) = g_2,\\ \rho S - 3DS_{xx} + 3G(3S - \xi - u_x) = g_3, \end{cases} \tag{2.25}$$

with

$$\alpha = \lambda^2\varrho + \lambda\mu_1 + \lambda\mu_2 N_1, \quad \eta = \lambda^2 I_\varrho + \lambda\beta, \quad \rho = 3\lambda^2 I_\varrho + 4\lambda\gamma_0 + 4\delta_0,$$
$$g_1 = \lambda\varrho f_1 + \varrho f_2 + \mu_1 f_1 - \mu_2 N_2, \quad g_2 = \lambda I_\varrho f_3 + I_\varrho f_4 + \beta f_3$$
$$\text{and} \quad g_3 = 3\lambda I_\varrho f_5 + 3I_\varrho f_6 + 4\gamma_0 f_5.$$

Solving the system (2.25) is equivalent to finding $(u,\xi,S) \in \left[H^2(0,L)\cap H_0^1(0,L)\right]^3$ such that

$$\begin{cases} \displaystyle\int_0^L \left[\alpha u\tilde{u} - G(3S-\xi-u_x)\tilde{u}_x\right]dx = \int_0^L g_1\tilde{u}\,dx,\\ \displaystyle\int_0^L \left[\eta\xi\tilde{\xi} - G(3S-\xi-u_x)\tilde{\xi} + D\xi_x\tilde{\xi}_x\right]dx = \int_0^L g_2\tilde{\xi}\,dx,\\ \displaystyle\int_0^L \left[\rho S\tilde{S} + 3DS_x\tilde{S}_x + 3G(3S-\xi-u_x)\tilde{S}\right]dx = \int_0^L g_3\tilde{S}\,dx, \end{cases} \tag{2.26}$$

for all $\left(\tilde{u},\tilde{\xi},\tilde{S}\right) \in H_0^1(0,L)^3$.

Consequently, the equation (2.26) is equivalent to the problem

$$\Upsilon\left((u,\xi,S),\left(\tilde{u},\tilde{\xi},\tilde{S}\right)\right) = L\left(\tilde{u},\tilde{\xi},\tilde{S}\right), \tag{2.27}$$

where the bilinear form

$$\Upsilon : [H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)]^2 \to \mathbb{R}$$

and the linear form

$$L : H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L) \to \mathbb{R}$$

are defined by

$$\Upsilon\left((u,\xi,S),\left(\tilde{u},\tilde{\xi},\tilde{S}\right)\right) =\alpha\int_0^L u\tilde{u}\,dx + G\int_0^L (3S-\xi-u_x)\left(3\tilde{S}-\tilde{\xi}-\tilde{u}_x\right)dx + \eta\int_0^L \xi\tilde{\xi}\,dx$$
$$+ D\int_0^L \xi_x\tilde{\xi}_x\,dx + \rho\int_0^L S\tilde{S}\,dx + 3D\int_0^L S_x\tilde{S}_x\,dx$$

and

$$L\left(\tilde{u},\tilde{\xi},\tilde{S}\right) = \int_0^L g_1\tilde{u}\,dx + \int_0^L g_2\tilde{\xi}\,dx + \int_0^L g_3\tilde{S}\,dx.$$

It is easy to verify that Υ is continuous and coercive, and L is continuous. So applying the Lax-Milgram Theorem, we deduce that for all $\left(\tilde{u},\tilde{\xi},\tilde{S}\right) \in H_0^1(0,L)^3$ the problem (2.27) admits a unique solution

$$(u,\xi,S) \in H_0^1(0,L)^3.$$

Applying the classical elliptic regularity, it follows from (2.26) that

$$(u,\xi,S) \in H^2(0,L)^3.$$

Therefore, the operator $\lambda I - \mathcal{A}(t)$ is surjective for any $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this prove that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))\,I - \mathcal{A}(t) \text{ is surjective} \tag{2.28}$$

for any $\lambda > 0$ and $t > 0$.

To complete the proof of (iii), it's suffices to show that

$$\frac{||\Phi||_t}{||\Phi||_s} \le e^{\frac{c}{2\tau_0}|t-s|}, \quad for\ all \quad t,s\in[0,T], \tag{2.29}$$

where $\Phi = (u,u_t,\xi,\xi_t,S,S_t,z)^T$, c is a positive constant and $||\cdot||_t$ is the norm defined in (2.17). For all $t,s\in[0,T]$, we have

$$||\Phi||_t^2 - ||\Phi||_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1-e^{\frac{c}{\tau_0}|t-s|}\right)(\varrho||u_t||^2 + I_\varrho||\xi_t||^2 + D||\xi_x||^2 + 3D||S_x||^2 + 3I_\varrho||S_t|||^2$$
$$+G||3S-\xi-u_x||^2 + 4\delta_0||S||^2) + \zeta\left(\tau(t)-\tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^L\int_0^1 z^2(x,\rho)\,d\rho\,dx.$$

It is clear that $1-e^{\frac{c}{\tau_0}|t-s|} \le 0$. Now we will prove $\tau(t)-\tau(s)e^{\frac{c}{\tau_0}|t-s|} \le 0$ for some $c > 0$. To do this , we have

$$\tau(t) = \tau(s) + \tau'(r)(t-s),$$

where $r\in(s,t)$ which implies

$$\frac{\tau(t)}{\tau(s)} \le 1 + \frac{|\tau'(r)|}{\tau(s)}|t-s|.$$

Using (2.12), we deduce that

$$\frac{\tau(t)}{\tau(s)} \le 1 + \frac{c}{\tau_0}|t-s| \le e^{\frac{c}{\tau_0}|t-s|},$$

which proves (2.29) and therefore (iii) follows.

Now, as $\kappa'(t) = \frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+\tau'(t)^2}} - \frac{\tau'(t)\sqrt{1+\tau'(t)^2}}{2\tau(t)^2}$ is bounded on $[0,T]$ for all $T > 0$ (by (2.12)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1)}{\tau(t)^2} z_\rho \end{pmatrix},$$

with $\frac{\tau''(t)\tau(t)\rho-\tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$ is bounded on $[0,T]$ by (2.12). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0,T], B(D(\mathcal{A}(0)), \mathcal{H})), \tag{2.30}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(\mathcal{A}(0)), \mathcal{H})$.

Then, (2.19), (2.28) and (2.29) imply that the family $\tilde{\mathcal{A}} = \left\{\tilde{\mathcal{A}}(t) : t \in [0,T]\right\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t, by Proposition 1.1 from [15]. Therefore, the assumptions $(i) - (iv)$ of Theorem 2.2 are verified by (2.14), (2.16), (2.19), (2.28), (2.29) and (2.30), and thus, the problem

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}(t)\tilde{U}, \\ \tilde{U}(0) = U_0 \end{cases} \tag{2.31}$$

has a unique solution $\tilde{U} \in C([0,+\infty), \mathcal{H})$ and

$$\tilde{U} \in C([0,+\infty), D(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{H}),$$

for $U_0 \in D(\mathcal{A}(0))$. The requested solution of (2.13) is then given by

$$U(t) = e^{\int_0^t \kappa(s)\,ds}\tilde{U}(t)$$

because

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\int_0^t \kappa(s)\,ds}\tilde{U}(t) + e^{\int_0^t \kappa(s)\,ds}\tilde{U}_t(t) \\ &= e^{\int_0^t \kappa(s)\,ds}\left(\kappa(t) + \tilde{\mathcal{A}}(t)\right)\tilde{U}(t) \\ &= \mathcal{A}(t)e^{\int_0^t \kappa(s)\,ds}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t) \end{aligned}$$

which concludes the proof. □

3. Exponential stability

In this section we deduce the full energy of the system (2.3)-(2.11) and prove its dissipative property and assumption (2.1) we show that the solution of problem (2.3)-(2.11) decays exponentially to the steady state with an exponential decay rate.

For a positive constant ζ satisfying

$$\frac{\mu_2}{\sqrt{1-d}} < \zeta < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{3.1}$$

we define the energy of the problem (2.3)-(2.11) as follows

$$\begin{aligned} E(t) = \frac{1}{2}(\varrho\|u_t\|^2 + I_\varrho\|\xi_t\|^2 + D\|\xi_x\|^2 + 3D\|S_x\|^2 + 3I_\varrho\|S_t\|^2 \\ + G\|3S - \xi - u_x\|^2 + 4\delta_0\|S\|^2) + \frac{\zeta\tau(t)}{2}\int_0^L\int_0^1 z^2(x,\rho)\,d\rho\,dx. \end{aligned} \tag{3.2}$$

To achieve one of our goals in this section, we have the following proposition:

Lemma 3.1. *Let* (u,ξ,S,z) *be the solution to the system* (2.3)-(2.11). *Then the energy functional, defined by* (3.2) *satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) \le &-\left(\mu_1 - \frac{\zeta}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t\|^2 \\ &-\left(\frac{\zeta}{2}(1-\tau'(t)) - \frac{\mu_2\sqrt{1-d}}{2}\right)\|z(x,1,t)\|^2 \\ &-\beta\|\xi_t\|^2 - 4\gamma_0\|S_t\|^2 \\ \le\ & 0. \end{aligned} \tag{3.3}$$

Proof. Multiplying (2.3) by u_t, (2.4) by ξ_t, (2.5) by S_t and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\varrho\|u_t\|^2 + I_\varrho\|\xi_t\|^2 + D\|\xi_x\|^2 + 3D\|S_x\|^2 + 3I_\varrho\|S_t\|\|^2 + G\|3S-\xi-u_x\|^2 + 4\delta_0\|S\|^2\right) \\ &= -\mu_1\|u_t\|^2 - \mu_2\int_0^L z(x,1,t)u_t\,dx - \beta\|\xi_t\|^2 - 4\gamma_0\|S_t\|^2. \end{aligned} \tag{3.4}$$

Now multiplying (2.6) by $\zeta z(x,\rho,t)$ and integrate the resulting equation over $(0,L)\times(0,1)$ with respect to ρ and x, respectively, to obtain

$$\begin{aligned} \frac{\zeta}{2}\frac{d}{dt}\int_0^L\int_0^1 \tau(t)z^2(x,\rho,t)\,d\rho\,dx &= -\zeta\int_0^L\int_0^1(1-\tau'(t)\rho)z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho\,dx \\ &\quad + \frac{\zeta\tau'(t)}{2}\int_0^L\int_0^1 z^2(x,\rho,t)\,d\rho\,dx \\ &= -\frac{\zeta}{2}\int_0^L\int_0^1\frac{\partial}{\partial\rho}\left((1-\tau'(t)\rho)z^2(x,\rho,t)\right)d\rho\,dx \\ &= \frac{\zeta}{2}\int_0^1\left(z^2(x,0,t) - z^2(x,1,t)\right)dx \\ &\quad + \frac{\zeta\tau'(t)}{2}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{3.5}$$

From (3.2), (3.4) and (3.5), we get

$$\begin{aligned} \frac{d}{dt}E(t) = &-\left(\mu_1 - \frac{\zeta}{2}\right)\|u_t\|^2 + \left(\frac{\zeta\tau'(t)}{2} - \frac{\zeta}{2}\right)\|z^2(x,1,t)\|^2 \\ &-\beta\|\xi_t\|^2 - 4\gamma_0\|S_t\|^2 - \mu_2\int_0^L z(x,1,t)u_t\,dx. \end{aligned} \tag{3.6}$$

Using Young inequality , we obtain

$$\begin{aligned}\frac{d}{dt}E(t) \leq & -\left(\mu_1 - \frac{\zeta}{2} - \frac{\mu_2}{2\sqrt{1-d}}\right)\|u_t\|^2 \\ & -\left(\frac{\zeta}{2}(1-\tau'(t)) - \frac{\mu_2\sqrt{1-d}}{2}\right)\|z(x,1,t)\|^2 \\ & -\beta\|\xi_t\|^2 - 4\gamma_0\|S_t\|^2.\end{aligned}$$

Then, by using (2.12) and (3.1) our conclusion holds. □

3.1. Technical lemmas. The main point is to construct a Lyapunov functional $\mathcal{L}$ satisfying

$$\begin{gathered}\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \\ \frac{d}{dt}\mathcal{L}(t) \leq -\beta_3\mathcal{L}(t),\end{gathered}$$

for all $t \geq 0$ and some positive constants β_1,β_2,β_3. To achieve this, first we consider the followings lemmas.

Lemma 3.2. *Let* (u, ξ, S, z) *be the solution to the system* (2.3) − (2.11) *and*

$$\mathbb{S}(x,t) = \int_0^x S(r,t)dr.$$

Defining the functional

$$L_1(t) = I_\varrho\langle S_t, S\rangle + \frac{2}{3}\gamma_0\|S\|^2 + \varrho\langle u_t, \mathbb{S}\rangle, \tag{3.7}$$

we have the following estimate

$$\frac{d}{dt}L_1(t) \leq -D\|S_x\|^2 - d_0\|S\|^2 + d_1\|u_t\|^2 + d_2\|S_t\|^2 + d_3\|z(x,1,t)\|^2. \tag{3.8}$$

Proof. We have that

$$\begin{aligned}\frac{d}{dt}I_\varrho\langle S_t, S\rangle &= I_\varrho\langle S_{tt}, S\rangle + I_\varrho\|S_t\|^2 \\ &= \left\langle\left[DS_{xx} - G(3S - \xi - u_x) - \frac{4}{3}\delta_0 S - \frac{4}{3}\gamma_0 S_t\right], S\right\rangle + I_\varrho\|S_t\|^2 \\ &= D\langle S_{xx}, S\rangle - G\langle 3S - \xi - u_x, S\rangle - \frac{4}{3}\delta_0\|S\|^2 - \frac{4}{3}\gamma_0\langle S_t, S\rangle + I_\varrho\|S_t\|^2.\end{aligned}$$

Performing integration by parts, we have

$$\frac{d}{dt}I_\varrho\langle S_t, S\rangle = -D\|S_x\|^2 - G\langle 3S - \xi - u_x, S\rangle - \frac{4}{3}\delta_0\|S\|^2 - \frac{2}{3}\gamma_0\frac{d}{dt}\|S\|^2 + I_\varrho\|S_t\|^2. \tag{3.9}$$

Note that

$$\begin{aligned}\frac{d}{dt}\varrho\langle u_t, \mathbb{S}\rangle &= \varrho\langle u_{tt}, \mathbb{S}\rangle + \varrho\langle u_t, \mathbb{S}_t\rangle \\ &= \langle[-G(3S - \xi - u_x)_x - \mu_1 u_t - \mu_2 z(x,1,t)], \mathbb{S}\rangle + \varrho\langle u_t, \mathbb{S}_t\rangle \\ &= -G\langle(3S - \xi - u_x)_x, \mathbb{S}\rangle - \mu_1\langle u_t, \mathbb{S}\rangle - \mu_2\langle z(x,1,t), \mathbb{S}\rangle + \varrho\langle u_t, \mathbb{S}_t\rangle.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}(3.10)\qquad \frac{d}{dt}\varrho\langle u_t, \mathbb{S}\rangle &= G\langle 3S - \xi - u_x, \mathbb{S}_x\rangle - \mu_1\langle u_t, \mathbb{S}\rangle - \mu_2\langle z(x,1,t), \mathbb{S}\rangle + \varrho\langle u_t, \mathbb{S}_t\rangle \\ &= G\langle 3S - \xi - u_x, S\rangle - \mu_1\langle u_t, \mathbb{S}\rangle - \mu_2\langle z(x,1,t), \mathbb{S}\rangle + \varrho\langle u_t, \mathbb{S}_t\rangle.\end{aligned}$$

From (3.7), (3.9) and (3.10) we get

$$\frac{d}{dt}L_1(t) = -D\|S_x\|^2 - \frac{4}{3}\delta_0\|S\|^2 + I_\varrho\|S_t\|^2 - \mu_1\langle u_t, \mathbb{S}\rangle - \mu_2\langle z(x,1,t), \mathbb{S}\rangle + \varrho\langle u_t, \mathbb{S}_t\rangle.$$

Using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}L_1(t) \leq &- D\|S_x\|^2 - \frac{4}{3}\delta_0\|S\|^2 + I_\varrho\|S_t\|^2 + \mu_1\frac{1}{2\epsilon_0}\|u_t\|^2 + \mu_1\frac{\epsilon_0}{2}\|\mathbb{S}\|^2 \\ &+ \mu_2\frac{1}{2\epsilon_0}\|z(x,1,t)\|^2 + \mu_2\frac{\epsilon_0}{2}\|\mathbb{S}\|^2 + \frac{\varrho}{2}\|u_t\|^2 + \frac{\varrho}{2}\|\mathbb{S}_t\|^2.\end{aligned}$$

Noting that $\|\mathbb{S}\|^2 \leq \|S\|^2$ and $\|\mathbb{S}_t\|^2 \leq \|S_t\|^2$ so we get

$$\begin{aligned}\frac{d}{dt}L_1(t) \leq &- D\|S_x\|^2 - \left[\frac{4}{3}\delta_0 - (\frac{\mu_1}{2} + \frac{\mu_2}{2})\epsilon_0\right]\|S\|^2 + \left(\mu_1\frac{1}{2\epsilon_0} + \frac{\varrho}{2}\right)\|u_t\|^2 \\ &+ \left(I_\varrho + \frac{\varrho}{2}\right)\|S_t\|^2 + \mu_2\frac{1}{2\epsilon_0}\|z(x,1,t)\|^2.\end{aligned}$$

Take ϵ_0 small enough such that

$$d_0 = \frac{4}{3}\delta_0 - \left(\frac{\mu_1}{2} + \frac{\mu_2}{2}\right)\epsilon_0 > 0$$

and denoting

$$d_1 = \mu_1\frac{1}{2\epsilon_0} + \frac{\varrho}{2},\ \ d_2 = I_\varrho + \frac{\varrho}{2}\ \text{and}\ d_3 = \mu_2\frac{1}{2\epsilon_0}$$

we conclude the lemma. □

Lemma 3.3. *Let* (u, ξ, S, z) *be the solution to the system* (2.3) − (2.11) *and*

$$\Psi(x,t) = -\int_0^x \psi(r,t)dr.$$

Introducing the functional

$$(3.11)\qquad L_2(t) = \varrho\langle u_t, u\rangle + \frac{\mu_1}{2}\|u\|^2 + \varrho\langle u_t, \Psi\rangle,$$

we have for all $\epsilon_1 > 0$ *that there exists a constant* $C(\epsilon_1)$ *such that*

$$(3.12)\quad \frac{d}{dt}L_2(t) \leq -G\|3S - \xi - u_x\|^2 + C(\epsilon_1)\left(\|u_t\|^2 + \|z(x,1,t)\|^2\right) + \epsilon_1\left(\|\Psi\|^2 + \|\Psi_t\|^2 + \|u\|^2\right).$$

Proof. The derivative of $\varrho\langle u_t, u\rangle$ satisfies

$$\begin{aligned}\frac{d}{dt}\varrho\langle u_t, u\rangle &= \varrho\langle u_{tt}, u\rangle + \varrho\|u_t\|^2 \\ &= \Big\langle [-G(3S - \xi - u_x)_x - \mu_1 u_t - \mu_2 z(x,1,t)]\,, u\Big\rangle + \varrho\|u_t\|^2 \\ &= -G\langle (3S - \xi - u_x)_x, u\rangle - \frac{\mu_1}{2}\frac{d}{dt}\|u\|^2 - \mu_2\langle z(x,1,t), u\rangle + \varrho\|u_t\|^2.\end{aligned}$$

Integrating by parts, we have

$$\frac{d}{dt}\varrho\langle u_t, u\rangle = G\langle 3S-\xi-u_x, u_x\rangle - \frac{\mu_1}{2}\frac{d}{dt}\|u\|^2 - \mu_2\langle z(x,1,t), u\rangle + \varrho\|u_t\|^2. \tag{3.13}$$

Observe that

$$\begin{aligned}\frac{d}{dt}\varrho\langle u_t, \Psi\rangle &= \varrho\langle u_{tt}, \Psi\rangle + \varrho\langle u_t, \Psi_t\rangle \\ &= \langle[-G(3S-\xi-u_x)_x - \mu_1 u_t - \mu_2 z(x,1,t)], \Psi\rangle + \varrho\langle u_t, \Psi_t\rangle \\ &= -G\langle(3S-\xi-u_x)_x, \Psi\rangle - \mu_1\langle u_t, \Psi\rangle - \mu_2\langle z(x,1,t), \Psi\rangle + \varrho\langle u_t, \Psi_t\rangle \\ &= G\langle 3S-\xi-u_x, \Psi_x\rangle - \mu_1\langle u_t, \Psi\rangle - \mu_2\langle z(x,1,t), \Psi\rangle + \varrho\langle u_t, \Psi_t\rangle \\ &= -G\langle 3S-\xi-u_x, \psi\rangle - \mu_1\langle u_t, \Psi\rangle - \mu_2\langle z(x,1,t), \Psi\rangle + \varrho\langle u_t, \Psi_t\rangle.\end{aligned} \tag{3.14}$$

From (3.11), (3.13) and (3.14), we get

$$\frac{d}{dt}L_2(t) = -G\|3S-\xi-u_x\|^2 - \mu_2\langle z(x,1,t), u\rangle + \varrho\|u_t\|^2 - \mu_1\langle u_t, \Psi\rangle - \mu_2\langle z(x,1,t), \Psi\rangle + \varrho\langle u_t, \Psi_t\rangle.$$

Using Young's inequality the proof is complete. □

Lemma 3.4. *Let (u, ξ, S, z) be the solution to the system* (2.3) − (2.11) *and*

$$\Phi(x,t) = -\int_0^x \xi(r,t)\,dr.$$

Considering the functional

$$L_3(t) = I_\varrho\langle \xi_t, \xi\rangle + I_\varrho\langle u_t, \Phi\rangle + \frac{\beta}{2}\|\xi\|^2, \tag{3.15}$$

we have for all $\epsilon_2 > 0$ that there exists a constant $C(\epsilon_2)$ such that

$$\frac{d}{dt}L_3(t) \le -D\|\xi_x\|^2 + I_\varrho\|\xi_t\|^2 + C(\epsilon_2)\left(\|u_t\|^2 + \|z(x,1,t)\|^2\right) + \epsilon_2\left(\|\Phi\|^2 + \|\Phi_t\|^2\right). \tag{3.16}$$

Proof. By derivative of $I_\varrho\langle \xi_t, \xi\rangle$ we obtain

$$\begin{aligned}\frac{d}{dt}I_\varrho\langle \xi_t, \xi\rangle &= I_\varrho\langle \xi_{tt}, \xi\rangle + I_\varrho\|\xi_t\|^2 \\ &= \Big\langle[G(3S-\xi-u_x) + D\xi_{xx} - \beta\xi_t], \xi\Big\rangle + I_\varrho\|\xi_t\|^2 \\ &= G\langle 3S-\xi-u_x, \xi\rangle + D\langle \xi_{xx}, \xi\rangle - \frac{\beta}{2}\frac{d}{dt}\|\xi\|^2 + I_\varrho\|\xi_t\|^2.\end{aligned}$$

Integrating by parts and using boundary conditions, we have

$$\frac{d}{dt}I_\varrho\langle \xi_t, \xi\rangle = G\langle(3S-\xi-u_x), \xi\rangle - D\|\xi_x\|^2 - \frac{\beta}{2}\frac{d}{dt}\|\xi\|^2 + I_\varrho\|\xi_t\|^2. \tag{3.17}$$

Now note that

$$\begin{aligned}\frac{d}{dt}\varrho\langle u_t, \Phi\rangle &= \varrho\langle u_{tt}, \Phi\rangle + \varrho\langle u_t, \Phi_t\rangle \\ &= \Big\langle[-G(3S-\xi-u_x)_x - \mu_1 u_t - \mu_2 z(x,1,t)], \Phi\Big\rangle + \varrho\langle u_t, \Phi_t\rangle \\ &= -G\langle(3S-\xi-u_x)_x, \Phi\rangle - \mu_1\langle u_t, \Phi\rangle - \mu_2\langle z(x,1,t), \Phi\rangle + \varrho\langle u_t, \Phi_t\rangle \\ &= G\langle 3S-\xi-u_x, \Phi_x\rangle - \mu_1\langle u_t, \Phi\rangle - \mu_2\langle z(x,1,t), \Phi\rangle + \varrho\langle u_t, \Phi_t\rangle\end{aligned}$$

$$= -G\langle 3S - \xi - u_x, \xi\rangle - \mu_1\langle u_t, \Phi\rangle - \mu_2\langle z(x,1,t), \Phi\rangle + \varrho\langle u_t, \Phi_t\rangle. \tag{3.18}$$

From (3.15), (3.17) and (3.18) we obtain

$$\frac{d}{dt}L_3(t) = -D\|\xi_x\|^2 + I_\varrho\|\xi_t\|^2 - \mu_1\langle u_t, \Phi\rangle - \mu_2\langle z(x,1,t), \Phi\rangle + \varrho\langle u_t, \Phi_t\rangle.$$

Using Young's inequality we concludes the last lemma. □

As in [16], taking into account the last lemma, we introduce the functional

$$L_4(t) = \zeta\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho}z^2(x,\rho,t)\,d\rho\,dx. \tag{3.19}$$

For this functional we have the following estimate.

Lemma 3.5 ([16])**.** *Let* (u, ξ, S, z) *be a solution of* (2.3)-(2.11). *Then the functional* $L_4(t)$ *satisfies*

$$\frac{d}{dt}L_4(t) \le -2L_4(t) + \zeta\|u_t\|^2. \tag{3.20}$$

Now we are in position to show the main result of this work.

Theorem 3.6. *The full energy of the system* (2.3)-(2.11) *decay exponentially, i.e., there are positive constants C and w such that*

$$E(t) \le CE(0)e^{-wt}, \ \textit{for all } t > 0.$$

Proof. Let us define the Lyapunov functional

$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{4} L_i(t),$$

where N is a positive real number. Using the estimates (3.3), (3.8), (3.12), (3.16) and (3.20) we conclude for ϵ_1 and ϵ_2 sufficiently small and N big enough that there exists $\beta_0 > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \le -\beta_0 E(t), \quad \text{for all } t \ge 0. \tag{3.21}$$

On the other hand, from (3.2), (3.7), (3.11), (3.15) and (3.19), we deduce that exists two positive constants β_1, β_2 such that

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \quad \text{for all } t \ge 0. \tag{3.22}$$

Now, combining (3.21) and (3.22), we obtain

$$\frac{d}{dt}\mathcal{L}(t) \le -\beta_3\mathcal{L}(t) \tag{3.23}$$

where $\beta_3 = \beta_0/\beta_2$ and finally, solving the last ODE we obtain for $C = \beta_2/\beta_1$ and $w = \beta_3$ that

$$E(t) \le CE(0)e^{-wt}, \ \text{for all } t > 0.$$

Thus, the proof of theorem is completed. □

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