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# CURVES IN A SPACELIKE HYPERSURFACE IN MINKOWSKI SPACE-TIME

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## Abstract

In mathematical physics, Minkowski space (or Minkowski space-time) is a combination of three-dimensional Euclidean space and time into a four-dimensional manifold.

The hyperbolic surface and de Sitter surface of a curve are defined in the spacelike hypersurface  $M$  in Minkowski 4-space and located, respectively, in hyperbolic 3-space and de Sitter 3-space. In this study, techniques from singularity theory were applied to obtain the generic shape of such surfaces and their singular value sets and the geometrical meanings of these singularities were investigated.

## 1. Introduction

Submanifolds in Lorentz-Minkowski space are investigated from various mathematical viewpoints and are of interest also in relativity theory. In recent years, the use of singularity theory has led to significant progress and many investigations have focused on the classification and characterisation of the singularity of submanifolds in both Euclidean spaces and semi-Euclidean spaces (see [1]-[8] and [10]). The results of the present study have complemented a whole study of the extrinsic geometry of curves in different ambient spaces, as mentioned above.

We considered a spacelike embedding  $X : U \rightarrow \mathbb{R}_1^4$  from an open subset  $U \subset \mathbb{R}^3$  and identified  $M$  and  $U$  through embedding  $X$ , where  $\mathbb{R}_1^4$  is the Minkowski 4-space. For a curve  $\gamma : I \rightarrow M$  with nowhere vanishing curvature, we defined a hyperbolic surface in hyperbolic space  $H^3(-1)$  and a de Sitter surface in de Sitter space  $S_1^3$  associated with curve  $\gamma$ . Singularity theory techniques, and in particular, the classical deformation theory, were applied for the study of the generic differential geometry of those surfaces and their singular sets.

This paper is organised as follows: Section 2 reviews some basic definitions of Minkowski 4-space, as well as the definition of  $A_k$ -singularities and discriminant sets, and reports the construction of a moving frame along  $\gamma$  together with Frenet-Serret type formulae; Sections 3 and 5 address the definition of two families of height functions on  $\gamma$ , namely timelike tangential height functions and spacelike tangential height functions, which measure the contact of curve  $t$  with special hyperplanes and whose differentiation yields invariants related to each surface. The hyperbolic surface of  $\gamma$  is described as the discriminant set of the family of timelike tangential height functions (Corollary 3.2) and de Sitter surface of  $\gamma$  is the

discriminant set of the family of spacelike tangential height functions (Corollary 5.2). The theory of deformations provides a classification and a characterisation of the diffeomorphisms type of such surfaces (Theorems 3.5 and 5.5). The sections also report on an investigation on the geometrical meaning of the invariants, and the results enable curve  $\gamma$  to be part of a slice surface (Propositions 3.6 and 5.6). When  $\gamma$  is not part of a slice surface, the contact of  $\gamma$  with a slice surface is characterised by the singularity types of both its hyperbolic surface (Proposition 3.7) and de Sitter surface (Proposition 5.7). Sections 4 and 6 provide examples of curves on spacelike hypersurface in  $\mathbb{R}_1^4$  and the surfaces studied in [3].

**2. Preliminaries**

*Minkowski space*  $\mathbb{R}_1^4$  is the vector space  $\mathbb{R}^4$  endowed with the pseudo-scalar product  $\langle x, y \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ , for any  $x = (x_0, x_1, x_2, x_3)$  and  $y = (y_0, y_1, y_2, y_3)$  in  $\mathbb{R}_1^4$  (see, e.g., [9]). A non-zero vector  $x \in \mathbb{R}_1^4$  is said to be *spacelike* if  $\langle x, x \rangle > 0$ , *lightlike* if  $\langle x, x \rangle = 0$  and *timelike* if  $\langle x, x \rangle < 0$ , respectively.  $\gamma : I \rightarrow \mathbb{R}_1^4$ , with  $I \subset \mathbb{R}$  open interval, is *spacelike* (resp. *timelike*) if tangent vector  $\gamma'(t)$  is a *spacelike* (resp. *timelike*) vector for any  $t \in I$ .

The norm of a vector  $x \in \mathbb{R}_1^4$  is defined by  $\|x\| = \sqrt{|\langle x, x \rangle|}$ . For a non-zero vector  $v \in \mathbb{R}_1^4$  and a real number  $c$ , *hyperplane* with *pseudo-normal*  $v$  is defined by

$$HP(v, c) = \{x \in \mathbb{R}_1^4 \mid \langle x, v \rangle = c\}.$$

We call  $HP(v, c)$  a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if  $v$  is timelike, spacelike or lightlike, respectively. Let us now consider the pseudo-spheres in  $\mathbb{R}_1^4$ : The *hyperbolic 3-space* is defined by

$$H^3(-1) = \{x \in \mathbb{R}_1^4 \mid \langle x, x \rangle = -1\},$$

and the *de Sitter 3-space* is denoted by

$$S_1^3 = \{x \in \mathbb{R}_1^4 \mid \langle x, x \rangle = 1\}.$$

For any  $x = (x_0, x_1, x_2, x_3)$ ,  $y = (y_0, y_1, y_2, y_3)$ ,  $z = (z_0, z_1, z_2, z_3) \in \mathbb{R}_1^4$ , the pseudo vector product of  $x$ ,  $y$  and  $z$  is defined as follows:

$$x \wedge y \wedge z = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where  $\{e_0, e_1, e_2, e_3\}$  is the canonical basis of  $\mathbb{R}^4$ .

Considering a spacelike embedding  $X : U \rightarrow \mathbb{R}_1^4$  from an open subset  $U \subset \mathbb{R}^3$ , we write  $M = X(U)$  and identify  $M$  and  $U$  through embedding  $X$ .  $X$  is said to be a *spacelike embedding* if the tangent space  $T_pM$  consists of spacelike vectors at any  $p = X(u)$ . Let  $\tilde{\gamma} : I \rightarrow U$  be a regular curve. Therefore, a curve  $\gamma : I \rightarrow M \subset \mathbb{R}_1^4$  is defined by  $\gamma(s) = X(\tilde{\gamma}(s))$ , and is a *curve in the spacelike hypersurface*  $M$ . Since  $\gamma$  is a spacelike curve, it can be reparametrized by the arc length  $s$ , which gives a unit tangent vector  $t(s) = \gamma'(s)$ . In this case, we call  $\gamma$  a *unit speed spacelike curve*. Since  $X$  is a spacelike embedding, a unit timelike normal vector field  $n$  along  $M = X(U)$  is defined by

$$n(p) = \frac{X_{u_1}(u) \wedge X_{u_2}(u) \wedge X_{u_3}(u)}{\|X_{u_1}(u) \wedge X_{u_2}(u) \wedge X_{u_3}(u)\|}$$

for  $p = X(u)$ , where  $X_{u_i} = \partial X / \partial u_i$ ,  $i = 1, 2, 3$ .  $n$  is *future directed* if  $\langle n, e_0 \rangle < 0$ . We chose the orientation of  $M$ , such that  $n$  is future directed and we defined  $n_\gamma(s) = n \circ \gamma(s)$ , to obtain a unit timelike normal vector field  $n_\gamma$  along  $\gamma$ . Under the assumption that  $\| \langle n_\gamma(s), t'(s) \rangle n_\gamma(s) + t'(s) \| \neq 0$ , we defined

$$n_1(s) = \frac{\langle n_\gamma(s), t'(s) \rangle n_\gamma(s) + t'(s)}{\| \langle n_\gamma(s), t'(s) \rangle n_\gamma(s) + t'(s) \|}.$$

It follows that  $\langle t, n_1 \rangle = 0$  and  $\langle n_\gamma, n_1 \rangle = 0$ . Therefore, a spacelike unit vector is defined by  $n_2(s) = n_\gamma \wedge t(s) \wedge n_1(s)$ , and a pseudo-orthonormal frame  $\{n_\gamma, t(s), n_1(s), n_2(s)\}$  is called a *Lorentzian Darboux frame* along  $\gamma$ . By standard arguments, the Frenet-Serret type formulae for the above frame are given by

$$\begin{cases} n'_\gamma(s) = k_n(s) t(s) + \tau_1(s) n_1(s) + \tau_2(s) n_2(s), \\ t'(s) = k_n(s) n_\gamma(s) + k_g(s) n_1(s), \\ n'_1(s) = \tau_1(s) n_\gamma(s) - k_g(s) t(s) + \tau_g(s) n_2(s), \\ n'_2(s) = \tau_2(s) n_\gamma(s) - \tau_g(s) n_1(s), \end{cases}$$

where  $k_n(s) = -\langle n_\gamma(s), t'(s) \rangle$ ,  $\tau_1(s) = \langle n_1(s), n'_\gamma(s) \rangle$ ,  $\tau_2(s) = \langle n_2(s), n'_\gamma(s) \rangle$ ,  $k_g(s) = \| \langle n_\gamma(s), t'(s) \rangle n_\gamma(s) + t'(s) \| = \| -k_n(s) n_\gamma(s) + t'(s) \|$  and  $\tau_g(s) = \langle -n'_2(s), n_1(s) \rangle$ . The invariant  $k_n$  is called a normal curvature,  $\tau_1$  is a first normal torsion,  $\tau_2$  is a second normal torsion,  $k_g$  is a geodesic curvature, and  $\tau_g$  is a geodesic torsion.

By assumption,  $k_g(s) = \| \langle n_\gamma(s), t'(s) \rangle n_\gamma(s) + t'(s) \| \neq 0$ , so that  $k_g(s) > 0$ .

**DEFINITION 2.1.** Let  $F : \mathbb{R}_1^4 \rightarrow \mathbb{R}$  be a submersion and  $\gamma : I \rightarrow M$  a regular curve.  $\gamma$  and  $F^{-1}(0)$  have contact of order  $k$  at  $s_0$  if function  $g(s) = F \circ \gamma(s)$  satisfies  $g(s_0) = g'(s_0) = \dots = g^{(k)}(s_0) = 0$  and  $g^{(k+1)}(s_0) \neq 0$ , i.e.,  $g$  has an  $A_k$ -singularity at  $s_0$ .

Let  $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \rightarrow \mathbb{R}$  be a family of germs of functions. We call  $G$  an  $r$ -parameter deformation of  $f$  if  $f(s) = G_{x_0}(s)$ . Supposing  $f$  has an  $A_k$ -singularity ( $k \geq 1$ ) at  $s_0$ , we write

$$j^{(k-1)} \left( \frac{\partial G}{\partial x_i}(s, x_0) \right) (s_0) = \sum_{j=0}^{k-1} \alpha_{ji} (s - s_0)^j,$$

for  $i = 1, \dots, r$ . Then,  $G$  is a *versal deformation* if the  $k \times r$  matrix of coefficients  $(\alpha_{ji})$  has rank  $k$  ( $k \leq r$ ) (see [1]).

The *discriminant set* of  $G$  is

$$D_G = \left\{ x \in (\mathbb{R}^r, x_0) \mid G = \frac{\partial G}{\partial s} = 0 \text{ at } (s, x) \text{ for some } s \in (\mathbb{R}, s_0) \right\}$$

and the *bifurcation set* of  $G$  is

$$B_G = \left\{ x \in (\mathbb{R}^r, x_0) \mid \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = 0 \text{ at } (s, x) \text{ for some } s \in (\mathbb{R}, s_0) \right\}.$$

The next result is from reference [1].

**Theorem 2.2.** *Let  $G : \mathbb{R} \times \mathbb{R}^r, (s_0, x_0) \rightarrow \mathbb{R}$  be an  $r$ -parameter deformation of  $f$ , such that  $f$  has an  $A_k$ -singularity at  $s_0$ . Supposing  $G$  is a versal deformation,  $\mathcal{D}_G$  is locally diffeomorphic to*

- (1)  $C \times \mathbb{R}^{r-2}$  if  $k = 2$ ,
- (2)  $SW \times \mathbb{R}^{r-3}$  if  $k = 3$ ,

where  $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$  is the ordinary cusp and  $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$  is the swallowtail surface.

In Sections 3 and 5, special families of functions on curves in  $M$  were used for the study of the hyperbolic surface and de Sitter surface. In fact, such surfaces are the discriminant sets of those families.

### 3. Timelike tangential height functions

This section introduces the family of timelike tangential height functions on a curve in a spacelike hypersurface  $M$ , and addresses the definition and study of the hyperbolic surface given by the discriminant set of this family.

A family of functions on a curve  $\gamma : I \rightarrow M \subset \mathbb{R}_1^4$  is defined as

$$H_t^T : I \times H^3(-1) \rightarrow \mathbb{R}; \quad (s, v) \mapsto \langle t(s), v \rangle.$$

We call  $H_t^T$  a family of timelike tangential height functions of  $\gamma$ , and  $(h_t^T)_v(s) = H_t^T(s, v)$  is denoted for any fixed  $v \in H^3(-1)$ . The family  $H_t^T$  measures the contact of the curve  $t$  with spacelike hyperplanes in  $\mathbb{R}_1^4$ , which is, generically, of order  $k, k = 1, 2, 3$ .

The conditions that characterise the  $A_k$ -singularity,  $k = 1, 2, 3$  can be obtained in Proposition 3.1.

The proof of (2) in the following proposition leads to  $k_g^2(s) > k_n^2(s)$ , therefore, we can assume that there exists an interval  $I$ , such that  $k_g^2(s) > k_n^2(s)$  for  $s \in I$ . Towards avoiding complicated situations, we have assumed  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$  for any  $s \in I$ .

**Proposition 3.1.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve with  $k_g(s) \neq 0$  and  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ . Therefore,*

- (1)  $(h_t^T)_v(s) = 0$  if and only if there exist  $\mu, \lambda, \eta \in \mathbb{R}$ , such that  $-\mu^2 + \lambda^2 + \eta^2 = -1$  and  $v = \mu n_\gamma(s) + \lambda n_1(s) + \eta n_2(s)$ .
- (2)  $(h_t^T)_v(s) = (h_t^T)'_v(s) = 0$  if and only if there exists  $\theta \in \mathbb{R}$ , such that

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s).$$

- (3)  $(h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = 0$  if and only if

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s),$$

$$\tanh \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s).$$

- (4)  $(h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = (h_t^T)'''_v(s) = 0$  if and only if

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s),$$

$$\tanh \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s) \text{ and } \rho(s) = 0, \text{ where}$$

$$\rho(s) = \left( (-k_g k''_n - k_g k_n \tau_2^2 - 2k_g k'_g \tau_1 - k_g^2 \tau_1' - k_g^2 \tau_g \tau_2 + 2k_n k'_n \tau_1 + k_n^2 \tau_1' - k_n^2 k_g \tau_2 + k'_g k_n - k_g k_n \tau_g^2)(k_n \tau_2 + k_g \tau_g) + (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(2k'_n \tau_2 + k_n \tau_1 \tau_g + k_n \tau_2' + 2k'_g \tau_g + k_g \tau_1 \tau_2 + k_g \tau_g') \right)(s).$$

(5)  $(h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = (h_t^T)'''_v(s) = (h_t^T)^{(4)}_v(s) = 0$  if and only if

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s),$$

$$\tanh \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s) \text{ and } \rho(s) = \rho'(s) = 0.$$

Proof. By definition,  $(h_t^T)_v(s) = 0$  if and only if  $\langle t(s), v \rangle = 0$ . This is equivalent to  $v = \mu n_\gamma(s) + \lambda n_1(s) + \eta n_2(s)$ , where  $\mu, \lambda, \eta \in \mathbb{R}$  and  $-\mu^2 + \lambda^2 + \eta^2 = -1$  so that (1) follows. For (2),  $(h_t^T)_v(s) = (h_t^T)'_v(s) = 0$  if and only if  $v = \mu n_\gamma(s) + \lambda n_1(s) + \eta n_2(s)$  with  $-\mu^2 + \lambda^2 + \eta^2 = -1$  and  $\langle t'(s), v \rangle = -\mu k_n + \lambda k_g = 0$ . This is equivalent to

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s).$$

For (3),  $(h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = 0$  if and only if

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s) \text{ and } \langle t''(s), v \rangle = 0.$$

Since  $t''(s) = (k_n^2(s) - k_g^2(s))t(s) + (k'_n(s) + k_g(s)\tau_1(s))n_\gamma(s) + (k_n(s)\tau_1(s) + k'_g(s))n_1(s) + (k_n(s)\tau_2(s) + k_g(s)\tau_g(s))n_2(s)$ , the previous assertion is equivalent to

$$v = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s)$$

and  $\tanh \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s).$

Items (4) and (5) were calculated by Frenet-Serret type formulae of  $\gamma$ . Since such calculations are laborious and long, details have been omitted. □

Following Proposition 3.1, we defined the invariant

$$\rho(s) = \left( (-k_g k''_n - k_g k_n \tau_2^2 - 2k_g k'_g \tau_1 - k_g^2 \tau_1' - k_g^2 \tau_g \tau_2 + 2k_n k'_n \tau_1 + k_n^2 \tau_1' - k_n^2 k_g \tau_2 + k'_g k_n - k_g k_n \tau_g^2)(k_n \tau_2 + k_g \tau_g) + (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(2k'_n \tau_2 + k_n \tau_1 \tau_g + k_n \tau_2' + 2k'_g \tau_g + k_g \tau_1 \tau_2 + k_g \tau_g') \right)(s)$$

of the curve  $\gamma$ . The geometrical meaning of this invariant will be studied.

Motivated by the calculations of this proposition, we defined a surface and its singular locus. Let  $\gamma : I \rightarrow M$  be a unit speed curve with  $k_g(s) \neq 0$  and  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$ . A surface  $S_\gamma : I \times \mathbb{R} \rightarrow H^3(-1)$  is defined by

$$S_\gamma(s, \theta) = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} \left( k_g(s)n_\gamma(s) + k_n(s)n_1(s) \right) + \sinh \theta n_2(s).$$

We call  $S_\gamma$  a *hyperbolic surface* of  $\gamma$ . Since we have assumed  $k_g^2(s) > k_n^2(s)$  for any  $s \in I$ , the hyperbolic surface exists. We now define  $CH_\gamma = S_\gamma(s, \theta(s))$ , where  $\tanh \theta(s) = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s)$ , which is generically a curve. We call  $CH_\gamma$  a *hyperbolic curve* of  $\gamma$ . By Theorem 3.5 (1), this curve is the locus of the singular points of the hyperbolic surface of  $\gamma$ .

**Corollary 3.2.** *The hyperbolic surface of  $\gamma$  is the discriminant set  $D_{H_t^T}$  of the family of timelike tangential height functions  $H_t^T$ .*

Proof. The proof follows from the definition of the discriminant set given in Section 2 and Proposition 3.1 (2). □

In the following proposition, we show the family of timelike tangential height functions on a curve in  $M$  is a versal deformation of an  $A_k$ -singularity,  $k = 2, 3$ , of its members. Furthermore, we will study the geometric meaning of the invariant  $\rho$ . We write  $\lambda_0(s) = (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(s)$ .

**Proposition 3.3.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve with  $k_g(s) \neq 0$  and  $(k_n\tau_2+k_g\tau_g)(s) \neq 0$ .*

- (a) *If  $(h_t^T)_{v_0}$  has an  $A_2$ -singularity at  $s_0$ , then  $H_t^T$  is a versal deformation of  $(h_t^T)_{v_0}$ .*
- (b) *If  $(h_t^T)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  and  $\lambda_0(s_0) \neq 0$  (which is a generic condition), then  $H_t^T$  is a versal deformation of  $(h_t^T)_{v_0}$ .*

Proof. The family of timelike tangential height functions is given by

$$H_t^T(s, v) = -v_0 x'_0(s) + v_1 x'_1(s) + v_2 x'_2(s) + v_3 x'_3(s),$$

where  $v = (v_0, v_1, v_2, v_3)$ ,  $t(s) = (x'_0(s), x'_1(s), x'_2(s), x'_3(s))$  and  $v_0 = \sqrt{1 + v_1^2 + v_2^2 + v_3^2}$ .

Thus

$$\frac{\partial H_t^T}{\partial v_i}(s, v) = x'_i(s) - \frac{v_i}{v_0} x'_0(s),$$

for  $i = 1, 2, 3$ . Therefore, the 1-jet of  $\frac{\partial H_t^T}{\partial v_i}(s, v)$  at  $s_0$  is given by

$$x'_i(s_0) - \frac{v_i}{v_0} x'_0(s_0) + \left( x''_i(s_0) - \frac{v_i}{v_0} x''_0(s_0) \right) (s - s_0)$$

and the 2-jet of  $\frac{\partial H_t^T}{\partial v_i}(s, v)$  at  $s_0$  is given by

$$x'_i(s_0) - \frac{v_i}{v_0} x'_0(s_0) + \left( x''_i(s_0) - \frac{v_i}{v_0} x''_0(s_0) \right) (s - s_0) + \frac{1}{2} \left( x'''_i(s_0) - \frac{v_i}{v_0} x'''_0(s_0) \right) (s - s_0)^2.$$

First, we assumed that  $(h_i^T)_v$  has an  $A_2$ -singularity at  $s = s_0$ , and show that the rank of the matrix

$$B = \begin{pmatrix} x'_1(s_0) - \frac{v_1}{v_0} x'_0(s_0) & x'_2(s_0) - \frac{v_2}{v_0} x'_0(s_0) & x'_3(s_0) - \frac{v_3}{v_0} x'_0(s_0) \\ x''_1(s_0) - \frac{v_1}{v_0} x''_0(s_0) & x''_2(s_0) - \frac{v_2}{v_0} x''_0(s_0) & x''_3(s_0) - \frac{v_3}{v_0} x''_0(s_0) \end{pmatrix}$$

is two.

We calculated the Gram-Schmidt matrix of  $\tilde{B} = v_0 B$ , and denoted the lines of  $\tilde{B}$  by

$$F = (x'_1(s_0)v_0 - x'_0(s_0)v_1, x'_2(s_0)v_0 - x'_0(s_0)v_2, x'_3(s_0)v_0 - x'_0(s_0)v_3),$$

$$G = (x''_1(s_0)v_0 - x''_0(s_0)v_1, x''_2(s_0)v_0 - x''_0(s_0)v_2, x''_3(s_0)v_0 - x''_0(s_0)v_3).$$

Since  $\langle v, v \rangle = -1$ ,  $\langle t(s), t(s) \rangle = 1$ ,  $\langle t(s), v \rangle = 0$ ,  $\langle t'(s), v \rangle = 0$  and  $\langle t'(s), t'(s) \rangle = k_g^2(s) - k_n^2(s)$ , we have the following Euclidean inner product

$$F.F = v_0^2 - (x'_0)^2, \quad F.G = -x'_0 x''_0 \quad \text{and} \quad G.G = v_0^2(k_g^2(s) - k_n^2(s)) - (x''_0)^2.$$

Therefore, the Gram-Schmidt matrix of  $\tilde{B}$  is

$$G_{\tilde{B}} = \begin{pmatrix} v_0^2 - (x'_0)^2 & -x'_0 x''_0 \\ -x'_0 x''_0 & v_0^2(k_g^2(s) - k_n^2(s)) - (x''_0)^2 \end{pmatrix}.$$

Through a Lorentzian motion of the curve, we can assume  $n_\gamma(s_0) = (1, 0, 0, 0)$ . In this case,  $x'_0(s_0) = 0$ ,  $x''_0(s_0) = k_n(s_0)$  and  $v_0 = \frac{k_g(s_0) \cosh \theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}}$ . Therefore, the determinant of  $G_{\tilde{B}}$  is

$$v_0^2 (k_g^2(s_0) - k_n^2(s_0)) (v_0^2 - (x'_0)^2) - v_0^2 (x''_0)^2 = \frac{k_g^2(s_0) \cosh^2 \theta_0}{k_g^2(s_0) - k_n^2(s_0)} (k_g^2(s_0) \cosh^2 \theta_0 - k_n^2(s_0)),$$

which is different from zero, since  $k_g^2(s_0) > k_n^2(s_0)$ . Consequently, the rank of the matrix B is two, and assertion (a) follows.

We now assume  $(h_i^T)_v$  has an  $A_3$ -singularity at  $s = s_0$ . In this case, the determinant of the  $3 \times 3$  matrix

$$A = \begin{pmatrix} x'_1(s_0) - \frac{v_1}{v_0} x'_0(s_0) & x'_2(s_0) - \frac{v_2}{v_0} x'_0(s_0) & x'_3(s_0) - \frac{v_3}{v_0} x'_0(s_0) \\ x''_1(s_0) - \frac{v_1}{v_0} x''_0(s_0) & x''_2(s_0) - \frac{v_2}{v_0} x''_0(s_0) & x''_3(s_0) - \frac{v_3}{v_0} x''_0(s_0) \\ x'''_1(s_0) - \frac{v_1}{v_0} x'''_0(s_0) & x'''_2(s_0) - \frac{v_2}{v_0} x'''_0(s_0) & x'''_3(s_0) - \frac{v_3}{v_0} x'''_0(s_0) \end{pmatrix}$$

is nonzero. Denoting

$$a = \begin{pmatrix} x'_0(s_0) \\ x''_0(s_0) \\ x'''_0(s_0) \end{pmatrix}, \quad b_i = \begin{pmatrix} x'_i(s_0) \\ x''_i(s_0) \\ x'''_i(s_0) \end{pmatrix},$$

for  $i = 1, 2, 3$ , then

$$\det A = \frac{v_0}{v_0} \det(b_1 \ b_2 \ b_3) - \frac{v_1}{v_0} \det(a \ b_2 \ b_3) - \frac{v_2}{v_0} \det(b_1 \ a \ b_3) - \frac{v_3}{v_0} \det(b_1 \ b_2 \ a).$$

On the other hand,

$$(\gamma' \wedge \gamma'' \wedge \gamma''')(s_0) = (-\det(b_1 \ b_2 \ b_3), -\det(a \ b_2 \ b_3), -\det(b_1 \ a \ b_3), -\det(b_1 \ b_2 \ a)).$$

Therefore,

$$\det A = \left\langle \left( \frac{v_0}{v_0}, \frac{v_1}{v_0}, \frac{v_2}{v_0}, \frac{v_3}{v_0} \right), (\gamma' \wedge \gamma'' \wedge \gamma''')(s_0) \right\rangle = \frac{\cosh \theta_0 \left( k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g \right)^2}{v_0 \sqrt{k_g^2 - k_n^2 (k_n \tau_2 + k_g \tau_g)}}(s_0).$$

If  $(h_i^T)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  and  $\lambda_0(s_0) \neq 0$ , then  $\det A \neq 0$  and  $H_i^T$  is a versal deformation of  $(h_i^T)_{v_0}$ , which completes the proof.  $\square$

According to Proposition 3.3, if  $(h_i^T)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  and  $\lambda_0(s_0) \neq 0$ , then  $H_i^T$  is a versal deformation of  $(h_i^T)_{v_0}$ . Let us now investigate what occurs if  $\lambda_0(s_0) = 0$ .

First, we must define a new deformation of  $(h_i^T)_{v_0}$  and prove it is a versal deformation. Then, the Recognition Lemma is applied for cuspidal beaks, or cuspidal lips given in [6].

Using Proposition 3.1 with  $\lambda_0(s_0) = 0$ ,  $(h_i^T)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  if and only if

$$\theta = 0, v(s_0) = \frac{1}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}} (k_g n_\gamma + k_n n_1)(s_0), \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0, \text{ where}$$

$$\rho'(s_0) = \frac{-1}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}} (-3\lambda'_0(s_0)\lambda_1(s_0) + \lambda_2(s_0)) \neq 0,$$

$$\lambda_1(s) = (k'_n \tau_2 + k_n \tau'_2 + k'_g \tau_g + k_g \tau'_g)(s),$$

$$\begin{aligned} \lambda_2(s) = & (k_g k_n''' + 3k'_g k_g \tau_1 + 3k'_g \tau'_1 k_g + k_g^2 \tau_1' + k_g^2 \tau_g \tau_2' - k_g^2 \tau_g^2 \tau_1 - k_n \tau_1 \tau_2 k_g^2 + k_n \tau_1 \tau_2 k_g \tau_g \\ & - 3k_n k_n'' \tau_1 - 3k_n k_n' \tau_1' - k_n^2 \tau_1'' + k_n k_n''' + k_n^2 \tau_1 \tau_g^2 + k_g^2 \tau_1 \tau_2^2 - k_n^2 \tau_1 \tau_2^2 + k_n^2 \tau_2 \tau_g' - k_n^2 \tau_2' \tau_g \\ & - k_g^2 \tau_g' \tau_2 + 2\tau_1^2 k_n k_g - 2\tau_1^2 k'_g k_n)(s). \end{aligned}$$

We now define a deformation  $\widetilde{H} : I \times H^3(-1) \times \mathbb{R} \rightarrow \mathbb{R}$  by  $\widetilde{H}(s, v, u) = H_i^T(s, v) + u(s - s_0)^2 = \langle t(s), v \rangle + u(s - s_0)^2$ . The germ at  $(s_0, v_0, 0)$  represented by  $\widetilde{H}$  is considered.

**Proposition 3.4.** *If  $(h_i^T)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  and  $\lambda_0(s_0) = 0$ , then  $\widetilde{H}$  is a versal deformation of  $(h_i^T)_{v_0}$ .*

Proof.

$$\widetilde{H}(s, v, u) = H_i^T(s, v) + u(s - s_0)^2 = -v_0 x'_0(s) + v_1 x'_1(s) + v_2 x'_2(s) + v_3 x'_3(s) + u(s - s_0)^2,$$

where  $v = (v_0, v_1, v_2, v_3)$ ,  $t(s) = (x'_0(s), x'_1(s), x'_2(s), x'_3(s))$  and  $v_0 = \sqrt{1 + v_1^2 + v_2^2 + v_3^2}$ .

Therefore,

$$\frac{\partial \widetilde{H}}{\partial v_i}(s, v, 0) = x'_i(s) - \frac{v_i}{v_0} x'_0(s),$$

for  $i = 1, 2, 3$ . Therefore, the 2-jet of  $\frac{\partial \widetilde{H}}{\partial v_i}(s, v, 0)$  at  $s_0$  is

$$x'_i(s_0) - \frac{v_i}{v_0}x'_0(s_0) + \left(x''_i(s_0) - \frac{v_i}{v_0}x''_0(s_0)\right)(s - s_0) + \frac{1}{2}\left(x'''_i(s_0) - \frac{v_i}{v_0}x'''_0(s_0)\right)(s - s_0)^2,$$

and the 2-jet of  $\frac{\partial \widetilde{H}}{\partial u}(s, v, 0)$  at  $s_0$  is  $(s - s_0)^2$ .

We assume  $(h^T_i)_v$  has an  $A_3$ -singularity at  $s = s_0$ , and it is enough to show

$$\begin{aligned} \text{rank} & \begin{pmatrix} x'_1(s_0) - \frac{v_1}{v_0}x'_0(s_0) & x'_2(s_0) - \frac{v_2}{v_0}x'_0(s_0) & x'_3(s_0) - \frac{v_3}{v_0}x'_0(s_0) & 0 \\ x''_1(s_0) - \frac{v_1}{v_0}x''_0(s_0) & x''_2(s_0) - \frac{v_2}{v_0}x''_0(s_0) & x''_3(s_0) - \frac{v_3}{v_0}x''_0(s_0) & 0 \\ x'''_1(s_0) - \frac{v_1}{v_0}x'''_0(s_0) & x'''_2(s_0) - \frac{v_2}{v_0}x'''_0(s_0) & x'''_3(s_0) - \frac{v_3}{v_0}x'''_0(s_0) & 1 \end{pmatrix} \\ = \text{rank} & \begin{pmatrix} 0 & 0 & 1 \\ x'_1(s_0) - \frac{v_1}{v_0}x'_0(s_0) & x''_1(s_0) - \frac{v_1}{v_0}x''_0(s_0) & 0 \\ x'_2(s_0) - \frac{v_2}{v_0}x'_0(s_0) & x''_2(s_0) - \frac{v_2}{v_0}x''_0(s_0) & 0 \\ x'_3(s_0) - \frac{v_3}{v_0}x'_0(s_0) & x''_3(s_0) - \frac{v_3}{v_0}x''_0(s_0) & 0 \end{pmatrix} = 3. \end{aligned}$$

The rank of the last matrix has the same value of the rank of

$$\begin{pmatrix} 1 & 0 & 1 \\ x'_1(s_0) - \frac{v_1}{v_0}x'_0(s_0) & x''_1(s_0) - \frac{v_1}{v_0}x''_0(s_0) & 0 \\ x'_2(s_0) - \frac{v_2}{v_0}x'_0(s_0) & x''_2(s_0) - \frac{v_2}{v_0}x''_0(s_0) & 0 \\ x'_3(s_0) - \frac{v_3}{v_0}x'_0(s_0) & x''_3(s_0) - \frac{v_3}{v_0}x''_0(s_0) & 0 \end{pmatrix}.$$

Let us consider

$$a(s_0) = \left(1, x'_1(s_0) - \frac{v_1}{v_0}x'_0(s_0), x'_2(s_0) - \frac{v_2}{v_0}x'_0(s_0), x'_3(s_0) - \frac{v_3}{v_0}x'_0(s_0)\right),$$

$$b(s_0) = \left(0, x''_1(s_0) - \frac{v_1}{v_0}x''_0(s_0), x''_2(s_0) - \frac{v_2}{v_0}x''_0(s_0), x''_3(s_0) - \frac{v_3}{v_0}x''_0(s_0)\right)$$

and  $c(s_0) = (1, 0, 0, 0)$ .  $a(s_0), b(s_0), c(s_0)$  are linearly independent. Indeed, if  $a(s_0), b(s_0), c(s_0)$  are linearly dependent, then  $x'_1(s_0) = \frac{v_1}{v_0}x'_0(s_0)$ ,  $x'_2(s_0) = \frac{v_2}{v_0}x'_0(s_0)$  and  $x'_3(s_0) = \frac{v_3}{v_0}x'_0(s_0)$ , that is,  $t(s_0)$  and  $v$  are parallel, which leads to a contradiction, since  $t$  is spacelike and  $v$  is timelike. □

The cuspidal beaks are defined to be a germ of surface diffeomorphic to  $CBK = \{(x_1, x_2, x_3) | x_1 = v, x_2 = -2u_3 + v_2u, x_3 = 3u_4 - v_2u_2\}$  (see picture in [6]). Using Theorem 2.2, Propositions 3.3 and 3.4, we can obtain the diffeomorphism type of the hyperbolic surface in the following theorem.

**Theorem 3.5.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve with  $k_g(s) \neq 0$ ,  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$  and  $k_g^2(s) > k_n^2(s)$ . Let  $S_\gamma$  be the hyperbolic surface of  $\gamma$ . Then*

- (1)  $S_\gamma$  is singular at  $(s_0, \theta_0)$  if and only if

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{\sqrt{k_g^2 - k_n^2} (k_n \tau_2 + k_g \tau_g)}(s_0).$$

The singular points of the hyperbolic surface are given by  $S_\gamma(s) = S_\gamma(s, \theta(s))$ , where  $\tanh \theta(s)$  satisfies the above equation.

- (2) The germ of  $S_\gamma$  at  $(s_0, \theta_0)$  is diffeomorphic to a cuspidal edge if

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0) \text{ and } \rho(s_0) \neq 0.$$

- (3) The germ of  $S_\gamma$  at  $(s_0, \theta_0)$  is diffeomorphic to a swallowtail if

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0), \lambda_0(s_0) \neq 0, \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

- (4) The germ of  $S_\gamma$  at  $(s_0, \theta_0)$  is diffeomorphic to cuspidal beaks if

$$\lambda_0(s_0) = 0, \lambda_1(s_0) \neq 0, \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

- (5) Cuspidal lips do not appear.

Proof. Let us consider the hyperbolic surface

$$S_\gamma(s, \theta) = \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s).$$

Therefore, we have

$$\begin{aligned} \frac{\partial S_\gamma}{\partial s}(s, \theta) &= \left( \frac{\cosh \theta (-k'_g k_n^2 + k_g k_n k'_n + k_n \tau_1 k_g^2 - k_n^3 \tau_1) + \sinh \theta \tau_2 (k_g^2 - k_n^2) \sqrt{k_g^2 - k_n^2}}{(k_g^2 - k_n^2) \sqrt{k_g^2 - k_n^2}} \right) (s)n_\gamma(s) \\ &+ \left( \frac{\cosh \theta (k_g^3 \tau_1 - k_g \tau_1 k_n^2 + k'_n k_g^2 - k_n k_g k'_g) - \sinh \theta \tau_g (k_g^2 - k_n^2) \sqrt{k_g^2 - k_n^2}}{(k_g^2 - k_n^2) \sqrt{k_g^2 - k_n^2}} \right) (s)n_1(s) \\ &+ \left( \frac{\cosh \theta (k_g \tau_2 + k_n \tau_g)}{\sqrt{k_g^2 - k_n^2}} \right) (s)n_2(s) \text{ and} \\ \frac{\partial S_\gamma}{\partial \theta}(s, \theta) &= \frac{\sinh \theta k_g(s)}{\sqrt{k_g^2(s) - k_n^2(s)}} n_\gamma(s) + \frac{\sinh \theta k_n(s)}{\sqrt{k_g^2(s) - k_n^2(s)}} n_1(s) + \cosh \theta n_2(s). \end{aligned}$$

Therefore, the vectors  $\left\{ \frac{\partial S_\gamma}{\partial s}(s_0, \theta_0), \frac{\partial S_\gamma}{\partial \theta}(s_0, \theta_0) \right\}$  are linearly dependent if and only if

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0) \text{ and assertion (1) holds.}$$

By Corollary 3.2, the discriminant set  $\mathcal{D}_{H_t^T}$  of the family of timelike tangential height functions  $H_t^T$  of  $\gamma$  is the hyperbolic surface  $S_\gamma$ . It also follows from assertions (4) and (5) of Proposition 3.1 that  $(h_t^T)_{v_0}$  has an  $A_2$ -singularity (respectively, an  $A_3$ -singularity) at  $s = s_0$  if and only if

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0) \text{ and } \rho(s_0) \neq 0$$

(respectively,  $\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0), \rho(s_0) = 0$  and  $\rho'(s_0) \neq 0$ ). Therefore,

by Proposition 3.3, we have assertions (2) and (3).

By Proposition 7.5 in [6] and previous Proposition 3.4,  $H_t^T$  is a Morse family of hypersurfaces.

Calculating  $\varphi = (\partial^2 H_t^T / \partial s^2) |_{\mathcal{D}_{H_t^T}}$ , we have

$$\begin{aligned} \frac{\partial^2 H_t^T}{\partial s^2}(s, \theta) &= \left( t''(s), \frac{\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sinh \theta n_2(s) \right) \\ &= \frac{-\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(s) + \sinh \theta (k_n \tau_2 + k_g \tau_g)(s). \end{aligned}$$

The Hessian matrix of  $\varphi(s, \theta) = \frac{-\cosh \theta}{\sqrt{k_g^2(s) - k_n^2(s)}} (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(s) + \sinh \theta (k_n \tau_2 + k_g \tau_g)(s)$  is

$$\text{Hess}(\varphi)(s_0, 0) = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial s^2}(s_0, 0) & \lambda_1(s_0) \\ \lambda_1(s_0) & 0 \end{pmatrix}.$$

Since  $\lambda_1(s_0) \neq 0$ ,  $\det \text{Hess}(\varphi)(s_0, 0) \neq 0$ . By Lemma 7.7 in [6],  $H_t^T$  is  $P$ - $\mathcal{K}$ -equivalent to  $t^4 \pm v_1^2 t^2 + v_2 t + v^3$  (the notion of generating families, Legendrian equivalence and  $P$ - $\mathcal{K}$ -equivalent are given in [6] page 30). The singular set of  $\mathcal{D}_{H_t^T}$  is given by  $\varphi(s, \theta) = 0$ . Therefore it consists of two curves that transversally intersect at  $(s_0, 0)$ . Therefore, the normal form is  $t^4 - v_1^2 t^2 + v_2 t + v^3$ , the surface is diffeomorphic to cuspidal beaks, and we have assertions (4) and (5). □

We have three types of models of surfaces in  $M$ , which are given by intersections of  $M$  with hyperplanes in  $\mathbb{R}^4$ . We call a surface  $M \cap HP(v, c)$  a *timelike slice* if  $v$  is spacelike, a *spacelike slice* if  $v$  is timelike, or a *lightlike slice* if  $v$  is lightlike.

In the following proposition, the curve  $\gamma$  of the hyperbolic surface is related to the invariant  $\rho$  and a slice surface. In this case, the singular locus of the hyperbolic surface of  $\gamma$  is a point.

**Proposition 3.6.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve, such that  $k_g(s) \neq 0$ ,  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$  and  $k_g^2(s) > k_n^2(s)$  for any  $s \in I$ . Let  $S_\gamma(s, \theta(s))$  be the singular points of the hyperbolic surface of  $\gamma$ . Then, the following conditions are equivalent:*

- (1)  $S_\gamma(s, \theta(s))$  is a constant timelike vector;
- (2)  $\rho(s) \equiv 0$ ;
- (3) there exist a timelike vector  $v$  and a real number  $c$ , such that  $Im(\gamma) \subset M \cap HP(v, c)$ .

Proof. By definition

$$S_\gamma(s, \theta(s)) = \frac{\cosh \theta(s)}{\sqrt{k_g^2 - k_n^2}(s)} \left( (k_g n_\gamma)(s) + (k_n n_1)(s) + \frac{(k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(s)}{(k_n \tau_2 + k_g \tau_g)(s)} n_2(s) \right).$$

Thus,

$$\begin{aligned} \frac{dS_\gamma(s, \theta(s))}{ds} = & \left( \frac{\cosh \theta(s)}{\sqrt{k_g^2(s) - k_n^2(s)}} \right)' \left( k_g(s)n_\gamma(s) + k_n(s)n_1(s) + \frac{(k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)}{(k_n \tau_2 + k_g \tau_g)}(s)n_2(s) \right) \\ & + \left( \frac{\cosh \theta(s)}{\sqrt{k_g^2(s) - k_n^2(s)}} \right) \left( k_g(s)n_\gamma(s) + k_n(s)n_1(s) + \frac{(k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)}{(k_n \tau_2 + k_g \tau_g)}(s)n_2(s) \right)'. \end{aligned}$$

Furthermore,

$$\theta'(s) = \frac{X(s)}{\sqrt{(k_g^2 - k_n^2)(s)((k_g^2 - k_n^2)(k_n \tau_2 + k_g \tau_g)^2 - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2)(s)}},$$

where  $X(s) = (k_g k''_n + 2k_g k'_g \tau_1 + k_g^2 \tau'_1 - 2k_n k'_n \tau_1 - k_n^2 \tau'_1 - k_n k''_g)(k_g^2 - k_n^2)(k_n \tau_2 + k_g \tau_g) - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)((k_g k'_g - k_n k'_n)(k_n \tau_2 + k_g \tau_g) + (k_g^2 - k_n^2)(k'_n \tau_2 + k_n \tau'_2 + k'_g \tau_g + k_g \tau'_g))(s)$ .

Using the Frenet-Serret type formulae, replacing  $\theta'(s)$  in the previous expression of the derivative and performing some calculations, we have

$$\frac{dS_\gamma(s, \theta(s))}{ds} = \frac{-\cosh \theta(an_\gamma + bn_1 + cn_2)\rho}{\sqrt{k_g^2 - k_n^2}(k_n \tau_2 + k_g \tau_g)((k_g^2 - k_n^2)(k_n \tau_2 + k_g \tau_g)^2 - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2)}(s),$$

where  $a(s) = k_g(k'_n k_g + k_g^2 \tau_1 - k_n^2 \tau_1 - k'_g k_n)(s)$ ,  $b(s) = k_n(k'_n k_g + k_g^2 \tau_1 - k_n^2 \tau_1 - k'_g k_n)(s)$ ,  $c(s) = (k_g^2 - k_n^2)(k_n \tau_2 + k_g \tau_g)(s)$  and  $\rho(s)$  is the invariant.

Therefore,  $\frac{dS_\gamma}{ds} \equiv 0$  if and only if  $\rho(s) \equiv 0$ . Therefore, statements (1) and (2) are equivalent. We now assume statement (1) holds and has

$$\langle \gamma(s), S_\gamma(s, \theta(s)) \rangle = \frac{\cosh \theta}{\sqrt{k_g^2 - k_n^2}} \left( k_g \langle \gamma, n_\gamma \rangle + k_n \langle \gamma, n_1 \rangle + \frac{(k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)}{k_n \tau_2 + k_g \tau_g} \langle \gamma, n_2 \rangle \right) (s).$$

Let  $g(s) = \langle \gamma(s), S_\gamma(s, \theta(s)) \rangle$ . Deriving, by Frenet-Serret type formulae and making long calculations, we show

$$g'(s) = g_1(s)\langle \gamma(s), n_\gamma(s) \rangle + g_2(s)\langle \gamma(s), n_1(s) \rangle + g_3(s)\langle \gamma(s), n_2(s) \rangle,$$

where  $g_1(s) = \frac{A(s) \cosh \theta(s)}{D(s)}$ ,  $g_2(s) = \frac{B(s) \cosh \theta(s)}{D(s)}$  and  $g_3(s) = \frac{C(s) \cosh \theta(s)}{D_1(s)}$  with

$$A(s) = \left( k_g(k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g) \left[ (k_g k''_n + 2k_g k'_g \tau_1 + k_g^2 \tau'_1 - 2k_n k'_n \tau_1 - k_n^2 \tau'_1 - k_n k''_g) \right. \right. \\ \left. \left. (k_g^2 - k_n^2)(k_n \tau_2 + k_g \tau_g) - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g) (k_g k'_g - k_n k'_n) (k_n \tau_2 + k_g \tau_g) \right. \right. \\ \left. \left. + (k_g^2 - k_n^2)(k'_n \tau_2 + k_n \tau'_2 + k'_g \tau_g + k_g \tau'_g) \right] - k_g(k_g k'_g - k_n k'_n) (k_n \tau_2 + k_g \tau_g)^3 (k_g^2 - k_n^2) \right. \\ \left. + k_g(k_g k'_g - k_n k'_n) (k_n \tau_2 + k_g \tau_g) (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2 + (k_n \tau_2 + k_g \tau_g) (k'_g + \right. \\ \left. k_n \tau_1) + \tau_2 k_g k'_n + \tau_2 k_g^2 \tau_1 - \tau_2 k_n^2 \tau_1 - \tau_2 k_n k'_g \right) (k_g^2 - k_n^2)^2 (k_n \tau_2 + k_g \tau_g)^2 - \left( (k'_g + k_n \tau_1) \right. \\ \left. (k_n \tau_2 + k_g \tau_g) + \tau_2 k_g k'_n + \tau_2 k_g^2 \tau_1 - \tau_2 k_n^2 \tau_1 - \tau_2 k_n k'_g \right) (k_g^2 - k_n^2) (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - \\ \left. k_n k'_g)^2 \right) (s),$$

$$D(s) = \left( (k_n \tau_2 + k_g \tau_g) \sqrt{(k_g^2 - k_n^2)^3 \left( (k_g^2 - k_n^2) (k_n \tau_2 + k_g \tau_g)^2 - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2 \right)} \right) (s),$$

$$B(s) = \left( k_n(k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g) \left[ (k_g k''_n + 2k_g k'_g \tau_1 + k_g^2 \tau'_1 - 2k_n k'_n \tau_1 - k_n^2 \tau'_1 - k_n k''_g) \right. \right. \\ \left. \left. (k_g^2 - k_n^2)(k_n \tau_2 + k_g \tau_g) - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g) (k_g k'_g - k_n k'_n) (k_n \tau_2 + k_g \tau_g) \right. \right. \\ \left. \left. + (k_g^2 - k_n^2)(k'_n \tau_2 + k_n \tau'_2 + k'_g \tau_g + k_g \tau'_g) \right] - k_n(k_g k'_g - k_n k'_n) (k_n \tau_2 + k_g \tau_g)^3 (k_g^2 - k_n^2) \right. \\ \left. + k_n(k_g k'_g - k_n k'_n) (k_n \tau_2 + k_g \tau_g) (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2 + (k_g k_n \tau_1 \tau_2 + k_n k'_n \tau_2 \right. \\ \left. + \tau_g k_n^2 \tau_1 + \tau_g k_n k'_g) (k_g^2 - k_n^2)^2 (k_n \tau_2 + k_g \tau_g) - (k_g k_n \tau_1 \tau_2 + k_n k'_n \tau_2 + \tau_g k_n^2 \tau_1 + \tau_g k_n k'_g) \right. \\ \left. (k_g^2 - k_n^2) (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2 \right) (s),$$

$$C(s) = \left( - (k_n \tau_2 + k_g \tau_g) (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2 (k_g \tau_2 + k_n \tau_g) - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 \right. \\ \left. - k_n k'_g) (k_n \tau_2 + k_g \tau_g)^2 (k_g k'_g - k_n k'_n) + (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g) (k'_n \tau_2 + k_n \tau_1 \tau_g + \right. \\ \left. k'_g \tau_g + k_g \tau_1 \tau_2) (k_g^2 - k_n^2) (k_n \tau_2 + k_g \tau_g) \right) (s),$$

$$D_1(s) = \left( \sqrt{k_g^2 - k_n^2} (k_n \tau_2 + k_g \tau_g) \left( (k_n^2 - k_g^2) (k_n \tau_2 + k_g \tau_g)^2 - (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)^2 \right) \right) (s).$$

Furthermore, reorganising the calculations in  $A(s)$ ,  $B(s)$  and  $C(s)$ , we show  $A(s) = B(s) = C(s) = 0$  for all  $s \in I$ , therefore,  $g_i(s) = 0$ ,  $i = 1, 2, 3$  for all  $s \in I$ , (i.e.,  $g'(s) = 0$  for all  $s \in I$ ), so that  $g$  is constant and the statement (3) follows. For the converse, we assume  $\langle \gamma(s), v \rangle = c$  for a constant vector  $v$  and a real number  $c$ , therefore,  $\langle \gamma'(s), v \rangle = 0$ , that is,  $(h_t^T)_v(s) = 0$  for all  $s$ , and  $(h_t^T)'_v(s) = (h_t^T)''_v(s) = (h_t^T)'''_v(s) = 0$  for all  $s$ . By

Proposition 3.1,  $v = S_\gamma(s, \theta(s))$  and  $\rho(s) = 0$  for all  $s$ , and (1) follows. □

In Proposition 3.6, the invariant  $\rho \equiv 0$  means the curve  $\gamma$  is part of a spacelike slice surface. For the next result, we assume  $\rho \neq 0$ , i.e.,  $\gamma$  is not part of any spacelike slice surface  $M \cap HP(v_0, c)$ .

We now consider the hyperbolic curve  $CH_\gamma$  of  $\gamma$ , defined in Section 3. We have defined  $C(2, 3, 4) = \{(t^2, t^3, t^4) \mid t \in \mathbb{R}\}$ , which is called a  $(2, 3, 4)$ -cusp, and obtained the following result.

**Proposition 3.7.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve, such that  $k_g(s) \neq 0$ ,  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$  and  $k_g^2(s) > k_n^2(s)$  for any  $s \in I$ . Let  $v_0 = S_\gamma(s_0, \theta_0)$  and  $c = \langle \gamma(s_0), v_0 \rangle$ . Then we have*

- (1)  *$\gamma$  and the spacelike slice surface  $M \cap HP(v_0, c)$  have contact of at least order 3 at  $s_0$  if and only if  $(h_t^T)_{v_0}$  has  $A_k$ -singularity at  $s_0$ ,  $k \geq 2$ . Furthermore, if  $\gamma$  and the spacelike slice surface  $M \cap HP(v_0, c)$  have contact of order exactly 3 at  $s_0$ , then the hyperbolic curve  $CH_\gamma$  of  $\gamma$  is, at  $s_0$ , locally diffeomorphic to a line.*
- (2)  *$\gamma$  and the spacelike slice surface  $M \cap HP(v_0, c)$  have contact of order 4 at  $s_0$  if and only if  $(h_t^T)_{v_0}$  has  $A_3$ -singularity at  $s_0$ . In this case, if  $\lambda_0(s_0) \neq 0$  then, the hyperbolic curve  $CH_\gamma$  of  $\gamma$  is, at  $s_0$ , locally diffeomorphic to the  $(2, 3, 4)$ -cusp  $C(2, 3, 4)$ .*

Proof. Let us consider  $v_0 = S_\gamma(s_0, \theta_0)$  and  $c = \langle \gamma(s_0), v_0 \rangle$  and  $D_{v_0} : M \rightarrow \mathbb{R}$  a function defined by  $D_{v_0}(x) = \langle x, v_0 \rangle - c$ . Then,  $D_{v_0}^{-1}(0) = M \cap HP(v_0, c)$ , which is a spacelike slice surface. Furthermore,  $D_{v_0}^{-1}(0)$  and  $\gamma$  have contact of at least order 3 at  $s_0$  if and only if the function  $g(s) = D_{v_0} \circ \gamma(s) = \langle \gamma(s_0), v_0 \rangle - c$  satisfies  $g(s_0) = g'(s_0) = g''(s_0) = g'''(s_0) = 0$ . Such conditions are equivalent to  $g(s_0) = (h_t^T)_v(s) = (h_t^T)'_v(s) = (h_t^T)''_v(s) = 0$ . By Proposition 3.1, they are equivalent to condition

$$v_0 = \frac{\cosh \theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}} (k_g(s_0)n_\gamma(s_0) + k_n(s_0)n_1(s_0)) + \sinh \theta_0 n_2(s_0),$$

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0). \text{ If } \gamma \text{ and the spacelike slice surface } M \cap HP(v_0, c)$$

have contact of order 3 at  $s_0$ , then

$$v_0 = \frac{\cosh \theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}} (k_g(s_0)n_\gamma(s_0) + k_n(s_0)n_1(s_0)) + \sinh \theta_0 n_2(s_0),$$

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0) \text{ and } \rho(s_0) \neq 0. \text{ Furthermore, by Theorem 3.5, the}$$

germ of the image of the hyperbolic surface  $S_\gamma$  at  $(s_0, \theta_0)$  is locally diffeomorphic to the cuspidal edge. Since the locus of the singularities of cuspidal edge is locally diffeomorphic to a line, assertion (1) holds.

The first part of (2) follows from assertions (4) and (5) of Proposition 3.1. For the second part, if  $\gamma$  and the spacelike slice surface  $M \cap HP(v_0, c)$  have contact of order 4 at  $s_0$ , then

$$v_0 = \frac{\cosh \theta_0}{\sqrt{k_g^2(s_0) - k_n^2(s_0)}} (k_g(s_0)n_\gamma(s_0) + k_n(s_0)n_1(s_0)) + \sinh \theta_0 n_2(s_0),$$

$$\tanh \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_g^2 - k_n^2}}(s_0), \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

Furthermore, we have the assumption  $\lambda_0(s_0) \neq 0$ . By Theorem 3.5, the germ of the image of the hyperbolic surface  $S_\gamma$  at  $(s_0, \theta_0)$  is locally diffeomorphic to the swallowtail surface. Since the locus of singularities of the swallowtail surface is locally diffeomorphic to  $C(2, 3, 4)$ , assertion (2) holds.  $\square$

### 4. Examples

This section provides two examples of curves on spacelike hypersurface  $M$  in  $\mathbb{R}_1^4$ , namely  $M = \mathbb{R}^3$  and  $M = H^3(-1)$ , which is the hyperbolic space.

EXAMPLE 4.1. We consider  $M = \mathbb{R}^3 = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}_1^4 \mid x_0 = 0\}$ . For  $\gamma : I \rightarrow \mathbb{R}^3$ , we have  $n_\gamma = e_0$ ,  $t(s) = \gamma'(s)$ ,  $n_1(s) = n(s)$  and  $n_2(s) = b(s)$ . Here  $\{t, n, b\}$  is the ordinary Frenet frame, and  $k_n = \tau_1 = \tau_2 = 0$ ,  $k_g = k$  and  $\tau_g = \tau$ . The Frenet-Serret type formulae are the original Frenet-Serret formulae (see [1]):

$$\begin{cases} e'_0(s) = 0, \\ t'(s) = k(s)n(s), \\ n'(s) = -k(s)t(s) + \tau(s)b(s), \\ b'(s) = -\tau(s)n(s). \end{cases}$$

The hyperbolic surface of  $\gamma$  in  $H^3(-1) \subset \mathbb{R}_1^4$  is given by

$$S_\gamma(s, \theta) = \cosh \theta e_0 + \sinh \theta b(s)$$

and the hyperbolic curve of  $\gamma$  is given by  $CH_\gamma(s) = e_0$ , which is a constant point.

EXAMPLE 4.2. Let us consider  $M = H^3(-1)$ . For  $\gamma : I \rightarrow H^3(-1)$ , we have  $n_\gamma(s) = \gamma(s)$ ,  $t(s) = \gamma'(s)$ ,  $n_1(s)$  and  $n_2(s)$ . Here  $\{\gamma, t, n_1, n_2\}$  is the pseudo orthonormal frame, and  $k_n(s) = 1$ ,  $\tau_1(s) = \tau_2(s) = 0$ ,  $k_g(s) = k_h(s)$  and  $\tau_g(s) = \tau_h(s)$ .

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = \gamma(s) + k_h(s)n_1(s), \\ n'_1(s) = -k_h(s)t(s) + \tau_h(s)n_2(s), \\ n'_2(s) = -\tau_h(s)n_1(s). \end{cases}$$

Therefore, for  $k_h^2(s) > 1$ , the hyperbolic surface of  $\gamma$  is given by

$$S_\gamma(s, \theta) = \frac{\cosh \theta}{\sqrt{k_h^2(s) - 1}} (k_h(s)\gamma(s) + n_1(s)) + \sinh \theta n_2(s).$$

Therefore, the hyperbolic surface is precisely the hyperbolic focal surface of  $\gamma$  given in [3].

### 5. Spacelike tangential height functions

This section introduces the family of spacelike tangential height functions on a curve in a spacelike hypersurface  $M$  and addresses the definition and a study of the de Sitter surface, given by the discriminant set of the family. The arguments and results are analogous to those of Section 3, therefore the detailed arguments are not presented.

We define a family of functions on a curve,  $\gamma : I \rightarrow M \subset \mathbb{R}_1^4$  as follows:

$$H_t^S : I \times S_1^3 \rightarrow \mathbb{R}; \quad (s, v) \mapsto \langle t(s), v \rangle.$$

We call  $H_t^S$  the family of spacelike tangential height functions of  $\gamma$ , and denote  $(h_t^S)_v(s) = H_t^S(s, v)$  for any fixed  $v \in S_1^3$ . The family  $H_t^S$  measures the contact of the curve  $t$  with timelike hyperplanes in  $\mathbb{R}_1^4$ , which, generically, can be of order  $k, k = 1, 2, 3$ .

The conditions that characterise  $A_k$ -singularities,  $k = 1, 2, 3$ , can be obtained in Proposition 5.1.

We assume  $k_n^2(s) > k_g^2(s)$  for  $s \in I$ , and towards avoiding more complicated situations,  $(k_n\tau_2 + k_g\tau_g)(s) \neq 0$  for any  $s \in I$ .

**Proposition 5.1.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve, such that  $k_g(s) \neq 0, (k_n\tau_2 + k_g\tau_g)(s) \neq 0$  and  $k_n^2(s) > k_g^2(s)$ . Then,*

- (1)  $(h_t^S)_v(s) = 0$  if and only if there exist  $\mu, \lambda, \eta \in \mathbb{R}$ , such that  $-\mu^2 + \lambda^2 + \eta^2 = 1$  and  $v = \mu n_\gamma(s) + \lambda n_1(s) + \eta n_2(s)$ .
- (2)  $(h_t^S)_v(s) = (h_t^S)'_v(s) = 0$  if and only if there exists  $\theta \in \mathbb{R}$ , such that

$$v = \frac{\cos \theta}{\sqrt{k_n^2(s) - k_g^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin \theta n_2(s).$$

- (3)  $(h_t^S)_v(s) = (h_t^S)'_v(s) = (h_t^S)''_v(s) = 0$  if and only if

$$v = \frac{\cos \theta}{\sqrt{k_n^2(s) - k_g^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin \theta n_2(s),$$

$$\tan \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_n^2 - k_g^2}}(s).$$

- (4)  $(h_t^S)_v(s) = (h_t^S)'_v(s) = (h_t^S)''_v(s) = (h_t^S)'''_v(s) = 0$  if and only if

$$v = \frac{\cos \theta}{\sqrt{k_n^2(s) - k_g^2(s)}} (k_g(s)n_\gamma(s) + k_n(s)n_1(s)) + \sin \theta n_2(s),$$

$$\tan \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_n^2 - k_g^2}}(s) \text{ and } \rho(s) = 0, \text{ where}$$

$$\rho(s) = \left( (-k_g k''_n - k_g k_n \tau_2^2 - 2k_g k'_g \tau_1 - k_g^2 \tau_1' - k_g^2 \tau_g \tau_2 + 2k_n k'_n \tau_1 + k_n^2 \tau_1' - k_n^2 k_g \tau_2 + k'_g k_n - k_g k_n \tau_g^2)(k_n \tau_2 + k_g \tau_g) + (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(2k'_n \tau_2 + k_n \tau_1 \tau_g + k_n \tau_2' + 2k'_g \tau_g + k_g \tau_1 \tau_2 + k_g \tau_g') \right)(s).$$

(5)  $(h_t^S)_v(s) = (h_t^S)'_v(s) = (h_t^S)''_v(s) = (h_t^S)'''_v(s) = (h_t^S)^{(4)}_v(s) = 0$  if and only if

$$v = \frac{\cos \theta}{\sqrt{k_n^2(s) - k_g^2(s)}} \left( k_g(s)n_\gamma(s) + k_n(s)n_1(s) \right) + \sin \theta n_2(s),$$

$$\tan \theta = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_n^2 - k_g^2}}(s) \text{ and } \rho(s) = \rho'(s) = 0.$$

Following Proposition 5.1, we define the invariant

$$\rho(s) = \left( (-k_g k''_n - k_g k_n \tau_2^2 - 2k_g k'_g \tau_1 - k_g^2 \tau_1' - k_g^2 \tau_g \tau_2 + 2k_n k'_n \tau_1 + k_n^2 \tau_1' - k_n^2 k_g \tau_2 + k'_g k_n - k_g k_n \tau_g^2)(k_n \tau_2 + k_g \tau_g) + (k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g)(2k'_n \tau_2 + k_n \tau_1 \tau_g + k_n \tau_2' + 2k'_g \tau_g + k_g \tau_1 \tau_2 + k_g \tau_g') \right)(s)$$

of the curve  $\gamma$ , whose geometric meaning will be studied. Motivated by Proposition 5.1, we define the following surface and its singular locus. Let  $\gamma : I \rightarrow M$  be a unit speed curve with  $k_g(s) \neq 0$ ,  $k_n^2(s) > k_g^2(s)$  and  $(k_n \tau_2 + k_g \tau_g)(s) \neq 0$ . A surface  $DS_\gamma : I \times J \rightarrow S_1^3$  is defined by

$$DS_\gamma(s, \theta) = \frac{\cos \theta}{\sqrt{k_n^2(s) - k_g^2(s)}} \left( k_g(s)n_\gamma(s) + k_n(s)n_1(s) \right) + \sin \theta n_2(s),$$

where  $J = [0, 2\pi]$ . We call  $DS_\gamma$  a *de Sitter surface* of  $\gamma$ . We now define  $DC_\gamma = DS_\gamma(s, \theta(s))$ ,

where  $\tan \theta(s) = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{\sqrt{k_n^2 - k_g^2}(k_n \tau_2 + k_g \tau_g)}(s)$ . We call  $DC_\gamma$  a *de Sitter curve* of  $\gamma$ . By

Theorem 5.5 (1), this curve is the locus of the singular points of the de Sitter surface of  $\gamma$

**Corollary 5.2.** *The de Sitter surface of  $\gamma$  is the discriminant set  $D_{H_t^S}$  of the family of spacelike tangential height functions  $H_t^S$ .*

Proof. The proof follows from the definition of the discriminant set given in Section 2 and Proposition 5.1 (2). □

**Proposition 5.3.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve with  $k_g(s) \neq 0$  and  $(k_n \tau_2 + k_g \tau_g)(s) \neq 0$ .*

- (a) *If  $(h_t^S)_{v_0}$  has an  $A_2$ -singularity at  $s_0$ , then  $H_t^S$  is a versal deformation of  $(h_t^S)_{v_0}$ .*
- (b) *If  $(h_t^S)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  and  $\lambda_0(s_0) \neq 0$  (which is a generic condition), then  $H_t^S$  is a versal deformation of  $(h_t^S)_{v_0}$ .*

Regarding the de Sitter surface, the result is analogous to that of Proposition 3.4, considering the deformation  $\tilde{H} : I \times S_1^3 \times \mathbb{R} \rightarrow \mathbb{R}$  by  $\tilde{H}(s, v, u) = H_t^S(s, v) + u(s - s_0)^2 = \langle t(s), v \rangle + u(s - s_0)^2$ .

**Proposition 5.4.** *If  $(h_t^S)_{v_0}$  has an  $A_3$ -singularity at  $s_0$  and  $\lambda_0(s_0) = 0$ , then  $\tilde{H}$  is a versal deformation of  $(h_t^S)_{v_0}$ .*

Propositions 5.3 and 5.4 provided the following result.

**Theorem 5.5.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve, such that  $k_g(s) \neq 0$ ,  $k_n^2(s) > k_g^2(s)$  and  $(k_n \tau_2 + k_g \tau_g)(s) \neq 0$ , and  $DS_\gamma$  the de Sitter surface of  $\gamma$ . Therefore,*

(1)  $DS_\gamma$  is singular at  $(s_0, \theta_0)$  if and only if

$$\tan \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{\sqrt{k_n^2 - k_g^2}(k_n \tau_2 + k_g \tau_g)}(s_0),$$

i.e., the singular points of the de Sitter surface are given by  $DS_\gamma(s) = DS_\gamma(s, \theta(s))$ , where  $\tan \theta(s)$  satisfies the above equation.

(2) The germ of  $DS_\gamma$  at  $(s_0, \theta_0)$  is locally diffeomorphic to the cuspidal edge if

$$\tan \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_n^2 - k_g^2}}(s_0) \text{ and } \rho(s_0) \neq 0.$$

(3) The germ of  $DS_\gamma$  at  $(s_0, \theta_0)$  is locally diffeomorphic to the swallowtail if

$$\tan \theta_0 = \frac{k_g k'_n + k_g^2 \tau_1 - k_n^2 \tau_1 - k_n k'_g}{(k_n \tau_2 + k_g \tau_g) \sqrt{k_n^2 - k_g^2}}(s_0), \quad \lambda_0(s_0) \neq 0, \quad \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

(4) The germ of  $DS_\gamma$  at  $(s_0, \theta_0)$  is diffeomorphic to cuspidal beaks if

$$\lambda_0(s_0) = 0, \quad \lambda_1(s_0) \neq 0, \quad \rho(s_0) = 0 \text{ and } \rho'(s_0) \neq 0.$$

(5) Cuspidal lips do not appear.

In the next proposition, the curve  $\gamma$  of the de Sitter surface is related to the invariant  $\rho$  and a timelike slice surface. In this case, the singular locus of the de Sitter surface of  $\gamma$  is a point.

**Proposition 5.6.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve, such that  $k_g(s) \neq 0$ ,  $(k_n \tau_2 + k_g \tau_g)(s) \neq 0$  and  $k_n^2(s) > k_g^2(s)$  for any  $s \in I$ , and  $DS_\gamma(s, \theta(s))$  be the singular points of the de Sitter surface of  $\gamma$ . The following conditions are equivalent:*

- (1)  $DS_\gamma(s, \theta(s))$  is a constant spacelike vector;
- (2)  $\rho(s) \equiv 0$ ;
- (3) there exist a spacelike vector  $v$  and a real number  $c$ , such that  $Im(\gamma) \subset M \cap HP(v, c)$ .

In the previous result, the invariant  $\rho \equiv 0$  means the curve  $\gamma$  is part of a timelike slice surface. For the next results, we have assumed  $\rho \neq 0$ , i.e.,  $\gamma$  is not part of any timelike slice surface  $M \cap HP(v, c)$ .

**Proposition 5.7.** *Let  $\gamma : I \rightarrow M$  be a unit speed curve, such that  $k_g(s) \neq 0$ ,  $(k_n \tau_2 + k_g \tau_g)(s) \neq 0$  and  $k_n^2(s) > k_g^2(s)$  for any  $s \in I$ , and  $v_0 = DS_\gamma(s_0, \theta_0)$  and  $c = \langle \gamma(s_0), v_0 \rangle$ . Therefore, we have*

- (1)  $\gamma$  and the timelike slice surface  $M \cap HP(v_0, c)$  have contact of at least order 3 at  $s_0$  if and only if  $(h_t^S)_{v_0}$  has  $A_k$ -singularity at  $s_0$ ,  $k \geq 2$ . Furthermore, if  $\gamma$  and the timelike slice surface  $M \cap HP(v_0, c)$  have contact of order exactly 3 at  $s_0$ , then the de Sitter curve  $DC_\gamma$  of  $\gamma$  is, at  $s_0$ , locally diffeomorphic to a line at  $s_0$ .
- (2)  $\gamma$  and the timelike slice surface  $M \cap HP(v_0, c)$  have contact of order 4 at  $s_0$  if and only if  $(h_t^S)_{v_0}$  has  $A_3$ -singularity at  $s_0$ . In this case, if  $\lambda_0(s_0) \neq 0$ , then the de Sitter curve  $DC_\gamma$  of  $\gamma$  is, at  $s_0$ , locally diffeomorphic to  $(2, 3, 4)$ -cusp  $C(2, 3, 4)$ .

### 6. Examples

This section provides two examples of curves on spacelike hypersurface  $M$  in  $\mathbb{R}_1^4$ , namely  $M = \mathbb{R}^3$  and  $H^3(-1)$ .

EXAMPLE 6.1. We consider  $M = \mathbb{R}^3$ ,  $\gamma : I \rightarrow \mathbb{R}^3$ , the Frenet frame  $\{t, n, b\}$  and the Frenet-Serret formulae, as in Example 4.1.

$$\begin{cases} e'_0(s) = 0, \\ t'(s) = k(s)n(s), \\ n'(s) = -k(s)t(s) + \tau(s)b(s), \\ b'(s) = -\tau(s)n(s). \end{cases}$$

In this case, the de Sitter surface of  $\gamma$  in  $S_1^3 \subset \mathbb{R}_1^4$  cannot be defined.

EXAMPLE 6.2. We consider  $M = H^3(-1)$ ,  $\gamma : I \rightarrow H^3(-1)$  and the pseudo orthonormal frame  $\{\gamma, t, n_1, n_2\}$ , as in Example 4.2.

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = \gamma(s) + k_h(s)n_1(s), \\ n'_1(s) = -k_h(s)t(s) + \tau_h(s)n_2(s), \\ n'_2(s) = -\tau_h(s)n_1(s). \end{cases}$$

Therefore, for  $k_h^2(s) < 1$ , the de Sitter surface of  $\gamma$  is given by

$$DS_\gamma(s, \theta) = \frac{\cos \theta}{\sqrt{1 - k_h^2(s)}}(k_h(s)\gamma(s) + n_1(s)) + \sin \theta n_2(s).$$

It follows de Sitter surface is precisely the de Sitter focal surface of  $\gamma$  given in [3].

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